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The value of the four values

Ofer Arieli *, Arnon Avron [1]

Department of Computer Science, School of Mathematical Sciences, Tel-Aviv University, Ramat-Aviv 69978, Israel

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Abstract

In his well-known paper "How computer should think" Belnap (1977) argues that four-valued semantics is a very suitable setting for computerized reasoning. In this paper we vindicate this thesis by showing that the *logical* role that the four-valued structure has among Ginsberg's *bilattices* is similar to the role that the two-valued algebra has among Boolean algebras. Specifically, we provide several theorems that show that the most useful bilattice-valued logics can actually be characterized as four-valued inference relations. In addition, we compare the use of three-valued logics with the use of four-valued logics, and show that at least for the task of handling inconsistent or uncertain information, the comparison is in favor of the latter. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

In [8,9] Belnap introduced a logic intended to deal in a useful way with inconsistent and incomplete information. This logic is based on a structure called *FOUR*, which has four truth values: the classical ones, t and f, and two new ones: $\bot$ that intuitively denotes lack of information (no knowledge), and $\top$ that indicates inconsistency ("over"-knowledge). Belnap gave quite convincing arguments why "the way a computer should think" should be based on these four values. In [26,27] Ginsberg proposed algebraic structures called *bilattices* that naturally generalize Belnap's *FOUR*. The idea is to consider an *arbitrary* number of truth values, and to arrange them (as in *FOUR*) in two closely related partial orders, each forming a lattice. The original motivation of Ginsberg for introducing bilattices was to provide a uniform approach for a diversity of applications

* Corresponding author. E-mail: ofera@math.tau.ac.il.
[1] Email: aa@math.tau.ac.il.

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in AI. Bilattices were further investigated by Fitting, who showed that they are useful also for providing semantics for logic programs [17,19–21]. In [2,3] we presented bilattice-based *logics* and corresponding proof systems. These logics turned out to have many desirable properties (like paraconsistency). In the present paper we proceed with this logical approach. In particular, we consider bilattice-based logics that are *preferential* in the sense of Shoham [42,43], i.e., they are based on the idea that inferences should be taken not according to all models of a given theory, but only with respect to a subset of them, determined according to certain preference criteria. We use here two main guidelines for making such preferences among bilattice-based models:

(1) Prefer models that assume as much consistency as possible. This approach reflects the intuition that contradictory data corresponds to inadequate information about the real world, and therefore should be minimized.
(2) Prefer models that assume a minimal amount of knowledge; The idea this time is that we should not assume anything that is not *really* known.

FOUR, the structure that corresponds to Belnap's four-valued logic, is the minimal bilattice, exactly as the structure that is based on the classical two values is the minimal Boolean algebra. The main goal of this paper is to show that the *logical* role of *FOUR* among bilattices is also very similar to that the two-valued algebra has among Boolean algebras. Indeed, it turned out that all the natural bilattice-valued logics that we had introduced for various purposes can be characterized using only the four basic values! This does not mean, of course, that from now on bilattices have no value (exactly as the fact, that Boolean algebras can be characterized in $\{t, f\}$, does not mean that Boolean algebras have no value). It does demonstrate, however, the fundamental role of the four values.

In an opposite direction to that taken by Ginsberg and Fitting, other authors tried to get along by using just three values for achieving the same (or similar) goals. We show, however, that the use of four values is preferable to the use of three even for tasks that can in principle be handled using only three values.

Taken together, the main import of our results is a strong vindication (so we believe) of Belnap's thesis concerning the fundamental importance of the four basic values for the goal of computerized reasoning.

The rest of this paper is organized as follows: In Section 2 we introduce a propositional language with four-valued semantics. Our language is based on the basic bilattice operators together with an appropriate implication connective. In Section 3 we show the adequacy of this language by exploring its expressive power as well as those of its fragments. Section 4 is devoted to introducing the most important consequence relations that are based on *FOUR*, and to an examination of their main properties. In Section 5 we compare four-valued formalisms with three-valued ones, and in Section 6 we generalize the four-valued logics of Section 4 to arbitrary bilattices. The main result of this section is that by doing so we do not get any new logic. Finally, in Section 7 we summarize the main results and conclusions of this work.

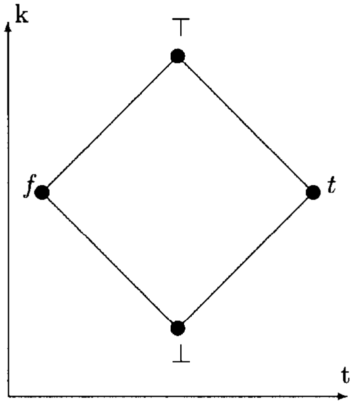

Fig. 1. *FOUR*.

2. The language and its four-valued semantics

2.1. The algebraic structure and its basic connectives

The truth values of Belnap's logic mentioned above have two natural orderings: First we have the standard logical partial order, $\leqslant_t$, which intuitively reflects differences in the "measure of truth" that every value represents. According to this order, f is the minimal element, t is the maximal one, and $\bot$, $\top$ are two intermediate values that are incomparable. $(\{t, f, \top, \bot\}, \leqslant_t)$ is a distributive lattice with an order reversing involution $\neg$, for which $\neg\top = \top$ and $\neg\bot = \bot$. We shall denote the meet and the join of this lattice by $\wedge$ and $\vee$, respectively.

The other partial order, $\leqslant_k$, is understood (again, intuitively) as reflecting differences in the amount of *knowledge* or *information* that each truth value exhibits. Again, $(\{t, f, \top, \bot\}, \leqslant_k)$ is a lattice where $\bot$ is its minimal element, $\top$—the maximal element, and t, f are incomparable. Following Fitting [17,18] we shall denote the meet and the join of the $\leqslant_k$-lattice by $\otimes$ and $\oplus$, respectively.

The two lattice orderings are closely related. The knowledge operators $\otimes$ and $\oplus$ are monotone with respect to the truth ordering $\leqslant_t$, and the truth operators $\wedge$, $\vee$, and $\neg$ (as well, of course, as $\otimes$ and $\oplus$) are monotone with respect to $\leqslant_k$. Moreover, all the 12 distributive laws hold, as well as De Morgan's laws. The structure that consists of these four elements and the five basic operators $(\wedge, \vee, \neg, \otimes, \oplus)$ is usually called *FOUR*. A double Hasse diagram of *FOUR* is given in Fig. 1.

2.2. Designated elements and models

The next step in using *FOUR* for reasoning is to choose its set of *designated* elements. The obvious choice is $\mathcal{D} = \{t, \top\}$, since both values intuitively represent formulae known to be true. The set $\mathcal{D}$ has the property that $a \wedge b \in \mathcal{D}$ iff $a \otimes b \in \mathcal{D}$ iff both a and b are in $\mathcal{D}$, while $a \vee b \in \mathcal{D}$ iff $a \oplus b \in \mathcal{D}$ iff either a or b is in $\mathcal{D}$. From this point the various semantic notions are defined on *FOUR* as natural generalizations of similar classical notions: A *valuation* ν is a function that assigns a truth value from *FOUR* to each atomic formula. Any valuation is extended to complex formulae in the obvious way. We will sometimes write $\psi : b \in \nu$ instead of $\nu(\psi) = b$. A valuation ν *satisfies* ψ iff $\nu(\psi) \in \mathcal{D}$. A valuation that satisfies every formula in a given set Γ of formulae is a *model* of Γ. The set of all models of Γ is denoted $mod(\Gamma)$. The structure *FOUR* together with $\mathcal{D}$ as the set of the designated elements will be denoted in the sequel by $\langle FOUR \rangle$.

2.3. Implication connectives

Unlike in the classical calculus, Belnap's logic has no tautologies. Thus, excluded middle is not valid in it. This implies that the definition of the material implication $\psi \mapsto \phi$ as $\neg\psi \vee \phi$ is not adequate there for representing entailments. We introduce therefore instead the following implications and equivalence operation on $\langle FOUR \rangle$:

Definition 1 (Arieli and Avron [2,6]).

$$a \supset b = \begin{cases} b & \text{if } a \in \mathcal{D}, \\ t & \text{if } a \notin \mathcal{D}, \end{cases}$$

$$a \to b = (a \supset b) \wedge (\neg b \supset \neg a),$$

$$a \leftrightarrow b = (a \to b) \wedge (b \to a).$$

Proposition 2.
(a) $\nu(\psi \to \phi)$ *is designated iff* $\nu(\psi) \leqslant_t \nu(\phi)$.
(b) $\nu(\psi \leftrightarrow \phi)$ *is designated iff* $\nu(\psi) = \nu(\phi)$.

Notes.
(1) Unlike the connectives of the basic language, the new connectives are *not* monotone with respect to $\leqslant_k$.
(2) On $\{t, f\}$ the material implication ($\mapsto$) and the two new implications are identical, so also the connectives of Definition 1 are generalizations of the classical implication.
(3) The sense in which $\supset$ is a true implication will be clarified in Proposition 20 below.

2.4. Canonical examples

Example 3 (*Tweety dilemma*). Consider the following well-known puzzle:

bird(Tweety) $\mapsto$ *fly(Tweety)*
penguin(Tweety) $\supset$ *bird(Tweety)*
penguin(Tweety) $\supset$ $\neg$*fly(Tweety)*
bird(Tweety)
penguin(Tweety)

Denote this set by Γ. The first assertion of Γ is formulated by the material "implication" $\mapsto$ (i.e., $\psi \mapsto \phi = \neg\psi \vee \phi$). This is an instance of a rule which is weaker than the other two

Model No.	*bird*(*Tweety*)	*fly*(*Tweety*)	*penguin*(*Tweety*)
M1–M2	$\top$	$\top$	$\top, t$
M3–M4	$\top$	f	$\top, t$
M5–M6	t	$\top$	$\top, t$

Fig. 2. The models of Γ.

Model No.	*quaker*(*Nixon*)	*republican*(*Nixon*)	*hawk*(*Nixon*)	*dove*(*Nixon*)
M1–M4	$\top, t$	$\top, t$	$\top$	$\top$
M5–M8	$\top, t$	$\top$	f	$\top, t$
M9–M12	$\top$	$\top, t$	$\top, t$	f

Fig. 3. The models of Δ.

rules, since it has exceptions. The rules without exceptions are formulated by a stronger implication, $\supset$, that is defined in Definition 1. The reason for choosing this connective (rather than $\rightarrow$, say) will become clear in Section 4.1. It is shown there that $\supset$ is the implication connective which corresponds to the basic consequence relation of the four-valued logic.

The six four-valued models of Γ are given in Fig. 2.

Example 4 (*Nixon diamond*). This is another famous example: Nixon was a republican and a quaker. Quakers are considered to be doves (however, there might be exceptions), and republicans are generally hawks. Hawks and doves represent two different political views, and each person is (roughly) either a hawk or a dove. A formulation of this puzzle is as follows:

$quaker(Nixon)$
$republican(Nixon)$
$quaker(Nixon) \mapsto dove(Nixon)$
$republican(Nixon) \mapsto hawk(Nixon)$
$dove(Nixon) \supset \neg hawk(Nixon)$
$hawk(Nixon) \supset \neg dove(Nixon)$
$hawk(Nixon) \vee dove(Nixon).$

Denote this set of assertions by Δ. Again, we use "$\mapsto$" for denoting the material implication, and "$\supset$" denotes the stronger implication defined in Definition 1. The twelve four-valued models of Δ are given in Fig. 3.

3. The expressive power of the language

In this section we examine the expressive power of the language we intoduced above. We do it from two different points of view (which happen to be equivalent in the two-valued case, but are not so in general).

3.1. Characterization of subsets of $FOUR^n$

Notation 5. For a set of formulae Γ denote by $\mathcal{A}(\Gamma)$ the set of atomic formulae that appear in some formula of Γ, and by $\mathcal{L}(\Gamma)$ the set of literals that appear in some formula of Γ.

Definition 6. Let ψ be a formula so that $\mathcal{A}(\psi) \subseteq \{p_1, \ldots, p_n\}$. S^n_ψ, the subset of $FOUR^n$ which is *characterized* by ψ, is:

$$S^n_\psi = \{(a_1, a_2, \ldots, a_n) \in FOUR^n \mid \forall \nu[(\forall 1 \leqslant i \leqslant n \ \nu(p_i) = a_i) \Rightarrow \nu(\psi) \in \mathcal{D}]\}.$$

Proposition 7. *A subset S of $FOUR^n$ is characterizable by some formula in the language of $\{\neg, \supset\}$ (or $\{\neg, \wedge, \vee, \otimes, \oplus, \supset, \top\}$) iff $(\top, \top, \ldots, \top) \in S$.*

Proof. If ψ is any formula in the language of $\{\neg, \wedge, \vee, \otimes, \oplus, \supset, \top\}$ s.t. $\mathcal{A}(\psi) \subseteq \{p_1, \ldots, p_n\}$ and $\nu(p_1) = \nu(p_2) = \cdots = \nu(p_n) = \top$, then $\nu(\psi) = \top$. Hence the condition is necessary. For the converse we introduce the following connectives:

$$p \bar{\wedge} q = \neg(p \supset \neg q), \quad p \bar{\vee} q = (p \supset q) \supset q,$$

$$f_n = p_1 \bar{\wedge} \neg p_1 \bar{\wedge} p_2 \bar{\wedge} \neg p_2 \bar{\wedge} \cdots \bar{\wedge} p_n \bar{\wedge} \neg p_n.$$

The following properties are easily verified:

(1) $\bar{\wedge}$ is associative. Moreover,

$$\nu(\psi_1 \bar{\wedge} \psi_2 \bar{\wedge} \cdots \bar{\wedge} \psi_n) = \begin{cases} f & \exists 1 \leqslant i \leqslant n-1 \ \nu(\psi_i) \notin \mathcal{D}, \\ \nu(\psi_n) & \forall 1 \leqslant i \leqslant n-1 \ \nu(\psi_i) \in \mathcal{D}. \end{cases}$$

(2) $\nu(\psi_1 \bar{\wedge} \psi_2 \bar{\wedge} \cdots \bar{\wedge} \psi_n) \in \mathcal{D}$ iff $\forall 1 \leqslant i \leqslant n \ \nu(\psi_i) \in \mathcal{D}$.

(3) $\bar{\vee}$ is associative. Moreover,

$$\nu(\psi_1 \bar{\vee} \psi_2 \bar{\vee} \cdots \bar{\vee} \psi_n) = \begin{cases} \nu(\psi_n) & \forall 1 \leqslant i \leqslant n-1 \ \nu(\psi_i) \notin \mathcal{D} \text{ or } \nu(\psi_n) = \top, \\ t & \text{otherwise.} \end{cases}$$

(4) $\nu(\psi_1 \bar{\vee} \psi_2 \bar{\vee} \cdots \bar{\vee} \psi_n) \in \mathcal{D}$ iff $\exists 1 \leqslant i \leqslant n \ \nu(\psi_i) \in \mathcal{D}$.

(5) f_n has the following property:

$$\nu(f_n) = \begin{cases} \top & \forall 1 \leqslant i \leqslant n \ \nu(p_i) = \top, \\ f & \text{otherwise.} \end{cases}$$

Now, by (2) and (4) it follows that:

$$\text{(i) } S^n_{\psi_1 \bar{\wedge} \cdots \bar{\wedge} \psi_m} = S^n_{\psi_1} \cap \cdots \cap S^n_{\psi_m}, \quad \text{(ii) } S^n_{\psi_1 \bar{\vee} \cdots \bar{\vee} \psi_m} = S^n_{\psi_1} \cup \cdots \cup S^n_{\psi_m}.$$

Let $\vec{a} = (a_1, \ldots, a_n) \in FOUR^n$. Define, for every $1 \leqslant i \leqslant n$,

$$\psi_i^{\vec{a}} = \begin{cases} p_i \bar{\wedge} \neg p_i & \text{if } a_i = \top \\ p_i \bar{\wedge} (\neg p_i \supset f_n) & \text{if } a_i = t \\ \neg p_i \bar{\wedge} (p_i \supset f_n) & \text{if } a_i = f \\ (\neg p_i \supset f_n) \bar{\wedge} (p_i \supset f_n) & \text{if } a_i = \bot. \end{cases}$$

Using the observations above, it is easy to see that $\psi_1^{\vec{a}} \bar{\wedge} \psi_2^{\vec{a}} \bar{\wedge} \cdots \bar{\wedge} \psi_n^{\vec{a}}$ characterizes $\{\vec{\top}, \vec{a}\}$, where $\vec{\top} = (\top, \top, \ldots, \top)$. This and (ii) above entail the proposition. □

Note. Obviously, the characterizing formula is much simpler in the $\{\neg, \wedge, \supset\}$-language, where we can use $\wedge$ instead of $\bar{\wedge}$ and $\vee$ instead of $\bar{\vee}$.

From Proposition 7 it follows that the language of $\{\neg, \supset\}$ should be extended in order to get full characterization of subsets of $FOUR^n$. One possibility is to add the propositional constant f:

Theorem 8. *Every subset of $FOUR^n$ is characterizable in the language of $\{\neg, \supset, f\}$.*

Proof. All we need to change in the proof of Proposition 7 is to use f instead of f_n in the definition of $\psi_i^{\vec{a}}$. After this change the $\bar{\wedge}$-conjunction of the new $\psi_i^{\vec{a}}$'s characterizes $\{\vec{a}\}$ and not $\{\vec{\top}, \vec{a}\}$. This suffices (using $\bar{\vee}$) for the characterization of every nonempty set. The empty set itself is characterized by f. □

Note. Since $f = \neg(\bot \supset \bot)$, the language of $\{\neg, \supset, \bot\}$ also suffices for representing all subsets of $FOUR^n$.

Proposition 7 entails that one cannot delete f from the set $\{\neg, \supset, f\}$ and retain the validity of Theorem 8. We next show that $\neg$ and $\supset$ cannot be deleted either:

Corollary 9. *$\supset$ is not definable in terms of the other connectives we consider here.*

Proof. By Theorem 8 it is sufficient to show that $\{\bot\}$ (for example) is not characterizable in the language $\{\neg, \wedge, \vee, \otimes, \oplus, t, f, \bot, \top\}$.[2] This follows from the fact that these connectives are all $\leqslant_k$-monotone. It follows that if $\mathcal{A}(\psi) \subseteq \{p_1\}$ and $\nu_1(p_1) \leqslant_k \nu_2(p_1)$ for some valuations ν_1, ν_2, then $\nu_1(\psi) \leqslant_k \nu_2(\psi)$. In particular if $\bot \in S_\psi^1$ then also $f, t, \top \in S_\psi^1$. □

Corollary 10. *$\neg$ is not definable in terms of the other connectives.*

Proof. Again, we show that without $\neg$ not all subsets of $FOUR$ are characterizable. For this it is sufficient to show that if ψ is a formula in the language of $\{\vee, \wedge, \oplus, \otimes, \supset, t, f, \bot, \top\}$ and $\mathcal{A}(\psi) \subseteq \{p_1\}$, then $\bot \in S_\psi^1$ iff $f \in S_\psi^1$. The proof of this fact is by an induction on the structure of ψ.

[2] Note that $\{\bot\}$ is not characterizable even though the use of the propositional constant $\bot$ is allowed.

- *Base step*: $S^1_t = S^1_\top = FOUR$, $S^1_f = S^1_\bot = \emptyset$, $S^1_{p_1} = \{t, \top\}$.
- *Induction step*:
  (1) $\bot \in S^1_{\psi\wedge\phi}$ iff $\bot \in S^1_\psi$ and $\bot \in S^1_\phi$, iff $f \in S^1_\psi$ and $f \in S^1_\phi$ (by the induction hypothesis), iff $f \in S^1_{\psi\wedge\phi}$.
  (2) $\bot \in S^1_{\psi\vee\phi}$ iff $\bot \in S^1_\psi$ or $\bot \in S^1_\phi$, iff $f \in S^1_\psi$ or $f \in S^1_\phi$ (by the induction hypothesis), iff $f \in S^1_{\psi\vee\phi}$.
  (3) $\bot \in S^1_{\psi\supset\phi}$ iff $\bot \notin S^1_\psi$ or $\bot \in S^1_\phi$, iff $f \notin S^1_\psi$ or $f \in S^1_\phi$ (by the induction hypothesis), iff $f \in S^1_{\psi\supset\phi}$.

The cases of $\otimes$ and $\oplus$ are similar to the cases for $\wedge$ and $\vee$, respectively. □

3.2. *Representation of operations on FOUR^n*

We turn now to the subject of functional completeness.

Definition 11. An operation $g : FOUR^n \to FOUR$ is *represented* by a formula ψ s.t. $\mathcal{A}(\psi) \subseteq \{p_1, \ldots, p_n\}$ if for every valuation ν we have $\nu(\psi) = g(\nu(p_1), \ldots, \nu(p_n))$.

The most important result of this section is the following:

Theorem 12. *The language $L^* = \{\neg, \wedge, \supset, \bot, \top\}$ is functionally complete for FOUR (i.e., every function from FOUR^n to FOUR is representable by some formula in L^*).*

Proof. Let $g : FOUR^n \to FOUR$. Since $f = \neg(\bot \supset \bot)$, by Theorem 8 every subset of $FOUR^n$ is characterizable in L^*. Let, accordingly, ψ^g_f, $\psi^g_\top$, and $\psi^g_\bot$ characterize $g^{-1}(\{f\})$, $g^{-1}(\{\top\})$, and $g^{-1}(\{\bot\})$, respectively. Define:

$$\Psi^g = (\psi^g_f \supset f) \wedge (\psi^g_\top \supset \top) \wedge (\psi^g_\bot \supset \bot).$$

It is easy to verify that Ψ^g represents g. □

Notes.

(1) If we follow the construction of Ψ^g step by step under the assumption that there are only two truth values (t and f), we shall get (with the help of trivial modifications, like replacing $p \supset f$ by $\neg p$ and $p \wedge \neg\neg p$ by p) the classical conjunctive normal form. Our construction is, therefore, a generalization of this normal form. It should be interesting to base four-valued logic programming on this type of clauses. In this paper, however, we still use the term "clause" in the usual sense.

(2) The functional completeness property for operations is completely independent, of course, of the choice of the designated values. It is remarkable that our choice of $\mathcal{D}$ has, nevertheless, a crucial role in its proof (through the notion of characterizability of subsets, which does depend on the choice of $\mathcal{D}$).

The ten connectives we use are not independent. Obviously, $\wedge$ and $\vee$ are definable in term of each other (using $\neg$), and so are t and f. There are, however, other dependencies. The following identities are particularly important: [3]

(1) $\top = (a \supset a) \oplus \neg(a \supset a)$,
(2) $a \oplus b = (a \wedge \top) \vee (b \wedge \top) \vee (a \wedge b)$,
(3) $\bot = f \otimes \neg f$,
(4) $f = \neg(\bot \supset \bot)$,
(5) $a \otimes b = (a \wedge \bot) \vee (b \wedge \bot) \vee (a \wedge b)$.

These identities mean that relative to the *basic classical language* $L = \{\neg, \wedge, \vee, \supset\}$ the connectives $\top$ and $\oplus$ are interdefinable, while $\bot$ is equivalent in expressive strength to the combination of $\otimes$ and f. It follows, for example, that the set $\{\neg, \wedge, \otimes, \oplus, \supset, f\}$ is also functionally complete. This set is obtained from the *full* classical language ($\{\neg, \wedge, \vee, \supset, t, f\}$) by adding to it the lattice operators of $\leqslant_k$ ($\otimes$ and $\oplus$).

Example 13 (*Kleene's three-valued logics and Fitting's guard connective*). The meet and the join in *FOUR* with respect to $\leqslant_t$ correspond to the conjunction and disjunction of strong Kleene's logic. In order to represent the connectives of the other Kleene's three-valued logics (weak-Kleene [4] and sequential-Kleene [5]), Fitting [21] introduces a new connective, called the *guard* connective. This connective is denoted $p : q$, and is evaluated as follows: if p is assigned a designated value (t or $\top$) the value of $p : q$ has the value of q, otherwise $p : q$ has the value $\bot$. The guard connective has the following simple and *natural* definition in our language: [6]

$$p : q = (p \supset q) \otimes \neg(p \supset \neg q).$$

We turn now to investigate the expressive power of the various fragments of our language which include at least the basic classical language $L = \{\neg, \wedge, \vee, \supset\}$. From the discussion before Example 13 it follows that there are at most eight such fragments, corresponding to extending L with some subset of (say) $\{\otimes, \oplus, f\}$. Our next theorem provides exact characterizations of the expressive power of each of these fragments, implying that they are all different from each other. We show that there is a correspondence between these eight fragments and the various possible combinations of the following three conditions:

(I) $g(\vec{\top}) = \top$.
(II) $g(\vec{x}) = \top \Rightarrow \exists 1 \leqslant i \leqslant n\ x_i = \top$.
(III) $g(\vec{x}) = \bot \Rightarrow \exists 1 \leqslant i \leqslant n\ x_i = \bot$.

Theorem 14. *Let $L = \{\neg, \wedge, \supset\}$ and suppose that Ξ is a subset of $\{\otimes, \oplus, f\}$. A function $g : FOUR^n \to FOUR$ is representable in $L \cup \Xi$ iff it satisfies those conditions from* (I)–(III) *that all the (functions that directly correspond to the) connectives in Ξ satisfy. In other words*:

[3] Definitions of $\vee$ and $\wedge$ in terms of $\oplus$, $\otimes$, t and f, which are dual to (2) and (5), have been given in [7].

[4] Also known as Bochvar's logic.

[5] Also known as McCarthy's logic.

[6] Fitting [21] also provides a definition for the guard connective, which is somewhat less straightforward, but does not require implication: $p : q = ((p \otimes t) \oplus \neg(p \otimes t)) \otimes q$.

- *g is representable in* $\{\neg, \wedge, \supset\}$ *iff it satisfies* (I), (II), *and* (III).
- *g is representable in* $\{\neg, \wedge, \supset, f\}$ *(the full classical language) iff it satisfies* (II) *and* (III).
- *g is representable in* $\{\neg, \wedge, \supset, \oplus\}$ *iff it satisfies* (I) *and* (III).
- *g is representable in* $\{\neg, \wedge, \supset, \otimes\}$ *iff it satisfies* (I) *and* (II).
- *g is representable in* $\{\neg, \wedge, \supset, \otimes, f\}$ *iff it satisfies* (II).
- *g is representable in* $\{\neg, \wedge, \supset, \oplus, \otimes\}$ *iff it satisfies* (I).
- *g is representable in* $\{\neg, \wedge, \supset, \oplus, f\}$ *iff it satisfies* (III).
- *g is representable in* $\{\neg, \wedge, \supset, \oplus, \otimes, f\}$.

Proof. The proof closely follows that of Theorem 12. The following changes should be made:

(1) If f is not available we use f_n as a substitute (see the proof of Proposition 7). In addition, instead of ψ^g_f, $\psi^g_\top$, and $\psi^g_\bot$ (which are not available in this case) we use ϕ^g_f, $\phi^g_\top$, and $\phi^g_\bot$—the formulae in the language of $\{\neg, \wedge, \supset\}$ which characterize $\{\vec{\top}\} \cup g^{-1}(\{f\})$, $\{\vec{\top}\} \cup g^{-1}(\{\top\})$ and $\{\vec{\top}\} \cup g^{-1}(\{\bot\})$, respectively (such formulae exist by Proposition 7).

(2) If $\top$ is not available (i.e., $\oplus \notin \Xi$) then we use the following sentence as a substitute:

$$\top_n = (p_1 \supset p_1) \wedge (p_2 \supset p_2) \wedge \cdots \wedge (p_n \supset p_n).$$

It is easy to verify that $\top_n$ has the following property:

$$\nu(\top_n) = \begin{cases} \top & \exists 1 \leqslant i \leqslant n \ \nu(p_i) = \top, \\ t & \text{otherwise.} \end{cases}$$

(3) If $\bot$ is not available (i.e., $\{\otimes, f\} \not\subseteq \Xi$) then if $\otimes \in \Xi$ we use as a substitute for $\bot$ the sentence

$$\bot_n = p_1 \otimes \neg p_1 \otimes p_2 \otimes \neg p_2 \otimes \cdots \otimes p_n \otimes \neg p_n.$$

If $\otimes \notin \Xi$ we use instead the following sentence:

$$\bot'_n = \bigvee_{i=1}^{n} \big(p_i \wedge ((p_i \vee \neg p_i) \supset f_n)\big).$$

These sentences have the following properties:

$$\nu(\bot_n) = \begin{cases} \top & \forall 1 \leqslant i \leqslant n \ \nu(p_i) = \top, \\ \bot & \text{otherwise,} \end{cases}$$

$$\exists 1 \leqslant i \leqslant n \ \nu\big(p_i\big) = \bot \Leftrightarrow \nu(\bot'_n) = \bot.$$

Following these guidelines, it is not difficult to prove the theorem. We show part (1) as an example, leaving the rest to the reader. Assume then that $g : FOUR^n \to FOUR$ satisfies (I)–(III). Define:

$$\Phi^g = \big(\phi^g_f \supset f_n\big) \wedge \big(\phi^g_\top \supset \top_n\big) \wedge \big(\phi^g_\bot \supset \bot'_n\big).$$

Φ^g is in the language of $\{\neg, \wedge, \supset\}$. We show that Φ^g represents g. Let $\vec{x} \in FOUR^n$ and assume that $\nu(p_i) = x_i$ for $i = 1, \ldots, n$.

Case 1: $g(\vec{x}) = t$. By condition (I), $\vec{x} \neq \vec{\top}$. Since $g(\vec{x}) \neq f$ this implies that

$$\vec{x} \notin \{\vec{\top}\} \cup g^{-1}(\{f\}).$$

Therefore, $\nu(\phi_f^g) \notin \{\top, t\}$ and so $\nu(\phi_f^g \supset f_n) = t$. The facts that $\nu(\phi_\top^g \supset \top_n) = t$ and $\nu(\phi_\bot^g \supset \bot_n') = t$ follows similarly. Hence $\nu(\Phi^g) = t = g(\vec{x})$.

Case 2: $g(\vec{x}) = f$. Again, by condition I $\vec{x} \neq \vec{\top}$, and so $\nu(f_n) = f$. In addition, $\nu(\phi_f^g) \in \{t, \top\}$ in this case, and so $\nu(\phi_f^g \supset f_n) = f$. It follows that $\nu(\Phi^g) = f = g(\vec{x})$.

Case 3a: $g(\vec{x}) = \top$ and $\vec{x} = \vec{\top}$. Since Φ^g is in the language of $\{\neg, \wedge, \supset\}$, also $\nu(\Phi^g) = \top = g(\vec{x})$.

Case 3b: $g(\vec{x}) = \top$ and $\vec{x} \neq \vec{\top}$. By condition (II) there exists $1 \leqslant i \leqslant n$ s.t. $x_i = \top$ and so $\nu(\top_n) = \top$. It follows that $\nu(\phi_\top^g \supset \top_n) = \top$ (since $\nu(\phi_\top^g) \in \{t, \top\}$ in this case). On the other hand, by the same arguments as in Case 1,

$$\nu(\phi_f^g \supset f_n) = \nu(\phi_\bot^g \supset \bot_n') = t.$$

Hence $\nu(\Phi^g) = \top = g(\vec{x})$.

Case 4: $g(\vec{x}) = \bot$. By (III) there exists $1 \leqslant i \leqslant n$ s.t. $x_i = \bot$ and so $\nu(\bot_n') = \bot$ and $\vec{x} \neq \vec{\top}$. Since in this case $\nu(\phi_\bot^g) \in \{t, \top\}$, it follows that

$$\nu(\phi_\bot^g \supset \bot_n') = \nu(\bot_n') = \bot.$$

Since the value of the other components is again t (as in Case 1), $\nu(\Phi^g) = \bot = g(\vec{x})$. □

Corollary 15. *The eight fragments above are different from each other.*

Proof. It is rather easy to construct for every subset of (I)–(III) a function from $FOUR^n$ to $FOUR$ that satisfies the conditions in this subset but not the rest. This easily implies the corollary. □

We conclude this section with a short discussion on the minimality of the set of connectives in each case. By Corollaries 9 and 10, neither $\neg$ nor $\supset$ can be deleted from any of the sets of connectives which we have provided in each case. Theorem 14 and Corollary 15 imply that none of the connectives in $\{\otimes, \oplus, f\}$ can be deleted in case it is included in the set we construct.[7] This leaves only the question of the necessity of $\wedge$. We shall content ourselves with an example in which this connective *is* necessary, and an example in which it is *not*.

Proposition 16. *The functionally complete set* $\{\neg, \wedge, \supset, \top, \bot\}$ *considered in Theorem* 12 *is minimal in the sense that no connective can be deleted from it without losing the functional completeness.*

Proof. We have discussed already the necessity of $\neg$, $\supset$, $\top$ and $\bot$ (again: $\bot$ takes here the role of $\otimes$ and f together). To show that $\wedge$ is also indispensable we prove, by induction on the structure of formulae, that no formula $\psi(p, q)$ in the language of $\{\neg, \supset, \top, \bot\}$ defines

[7] Although one can always replace $\oplus$ by $\top$, and the pair $\{\otimes, f\}$ by $\bot$.

a function g such that $g(t, \bot) = \bot$ while $g(\top, t) = \top$. In particular $\wedge$ itself is not definable in this language. □

The set $\{\neg, \wedge, \supset, \top, \bot\}$ is *not* minimal in the sense of the number of connectives in it. The next proposition shows that there is a smaller set which is functionally complete.

Proposition 17. *The set* $\{\neg, \oplus, \supset, \bot\}$ *is functionally complete for FOUR.*

Proof. $\top$ and f are definable from this set as shown in the discussion before Example 13. Now, define:

$$p \sqcap q = (p \bar{\wedge} q) \oplus ((\neg p \supset \neg q) \bar{\wedge} q).$$

The relevant properties of $\sqcap$ are the following:

$$\nu(p \sqcap q) = \begin{cases} t & \nu(p) = t,\ \nu(q) = t, \\ \bot & \nu(p) = t,\ \nu(q) = \bot, \\ \top & \nu(p) = \top,\ \nu(q) = t. \end{cases}$$

Now, given a function $g : FOUR^n \to FOUR$, define:

$$\Upsilon^g = (\psi^g_f \supset f) \bar{\wedge} ((\psi^g_\top \supset \top) \sqcap (\psi^g_\bot \supset \bot)).^{8}$$

It is easy now to check that Υ^g characterizes g. □

Notes.

(1) Using Theorem 14, Corollaries 9, 10, and Proposition 7, it is easy to show that no subset of $\{\neg, \wedge, \vee, \otimes, \oplus, \supset, t, f, \top, \bot\}$ with less than four connectives can be functionally complete.

(2) The fact that $\bot = f \otimes \neg f$ together with Proposition 17 imply that $\{\neg, \otimes, \oplus, \supset, f\}$ is functionally complete. Hence $\wedge$ can be deleted from the set provided by the last part of Theorem 14 (in contrast to that given in Theorem 12!).

4. Reasoning in $\langle FOUR \rangle$

4.1. The basic consequence relation

We start with the simplest consequence relation which naturally corresponds to *FOUR*.

Definition 18. Suppose that Γ and Δ are two sets of formulae. $\Gamma \models^4 \Delta$ if every model of Γ in $\langle FOUR \rangle$ is a model of some formula of Δ.

Proposition 19 (see [3]). $\models^4$ *is monotonic, compact, and paraconsistent.*

Proposition 20 (see [3]).

(a) $\supset$ *is an internal implication for* $\langle FOUR \rangle$, *i.e.*: $\Gamma, \psi \models^4 \phi, \Delta$ *iff* $\Gamma \models^4 \psi \supset \phi, \Delta$.

(b) $\leftrightarrow$ *is an equivalence operator for* $\langle FOUR \rangle$, *i.e.*: $\psi \leftrightarrow \phi \models^4 \Theta(\psi) \leftrightarrow \Theta(\phi)$.

[8] See the proof of Theorem 12 for the definition of ψ^g_f, $\psi^g_\top$, and $\psi^g_\bot$.

4.1.1. Canonical examples—revisited

Example 21 (*Tweety dilemma—continued*). Consider again the set Γ of Example 3. Although Γ is classically inconsistent, nontrivial conclusions about Tweety can be obtained by $\models^4$: Tweety is a penguin, a bird, and it cannot fly. The complementary conclusions *cannot* be obtained by $\models^4$, as expected.

Example 22 (*Nixon diamond—continued*). By using $\models^4$ on the assertions of Example 4 one cannot tell whether Nixon is a dove or a hawk (which seems reasonable given the conflicting defaults). One can still infer the explicit information about Nixon, i.e., that he is a republican and a quaker. However, unlike in the classical case, the negations of these assertions *cannot* be inferred, despite the inconsistency. What *can* be inferred is their disjunction: $\neg hawk(Nixon) \vee \neg dove(Nixon)$.

4.1.2. Proof system

One of the biggest advantages of $\models^4$ is that it has a corresponding proof system, which is both nice and efficient. It was denoted *GBL* in [2,3]:

Axioms. $\Gamma, \psi \Rightarrow \Delta, \psi$.

Rules. Exchange, Contraction, and the following logical rules:

$$[\neg\neg\Rightarrow]\ \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \neg\neg\psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \neg\neg\psi}\ [\Rightarrow\neg\neg]$$

$$[\wedge\Rightarrow]\ \frac{\Gamma, \psi, \phi \Rightarrow \Delta}{\Gamma, \psi \wedge \phi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \psi \wedge \phi}\ [\Rightarrow\wedge]$$

$$[\neg\wedge\Rightarrow]\ \frac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \wedge \phi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\psi, \neg\phi}{\Gamma \Rightarrow \Delta, \neg(\psi \wedge \phi)}\ [\Rightarrow\neg\wedge]$$

$$[\vee\Rightarrow]\ \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \phi \Rightarrow \Delta}{\Gamma, \psi \vee \phi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi, \phi}{\Gamma \Rightarrow \Delta, \psi \vee \phi}\ [\Rightarrow\vee]$$

$$[\neg\vee\Rightarrow]\ \frac{\Gamma, \neg\psi, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \vee \phi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\psi \quad \Gamma \Rightarrow \Delta, \neg\phi}{\Gamma \Rightarrow \Delta, \neg(\psi \vee \phi)}\ [\Rightarrow\neg\vee]$$

$$[\otimes\Rightarrow]\ \frac{\Gamma, \psi, \phi \Rightarrow \Delta}{\Gamma, \psi \otimes \phi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \psi \otimes \phi}\ [\Rightarrow\otimes]$$

$$[\neg\otimes\Rightarrow]\ \frac{\Gamma, \neg\psi, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \otimes \phi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\psi \quad \Gamma \Rightarrow \Delta, \neg\phi}{\Gamma \Rightarrow \Delta, \neg(\psi \otimes \phi)}\ [\Rightarrow\neg\otimes]$$

$$[\oplus\Rightarrow]\ \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \phi \Rightarrow \Delta}{\Gamma, \psi \oplus \phi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi, \phi}{\Gamma \Rightarrow \Delta, \psi \oplus \phi}\ [\Rightarrow\oplus]$$

$$[\neg\oplus\Rightarrow]\ \frac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \oplus \phi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\psi, \neg\phi}{\Gamma \Rightarrow \Delta, \neg(\psi \oplus \phi)}\ [\Rightarrow\neg\oplus]$$

$$[\supset\Rightarrow]\ \frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma, \phi \Rightarrow \Delta}{\Gamma, \psi \supset \phi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \psi \supset \phi, \Delta}\ [\Rightarrow\supset]$$

$$[\neg\supset\Rightarrow]\ \frac{\Gamma, \psi, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \supset \phi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma \Rightarrow \neg\phi, \Delta}{\Gamma \Rightarrow \neg(\psi \supset \phi), \Delta}\ [\Rightarrow\neg\supset]$$

$$[\neg t \Rightarrow]\ \Gamma, \neg t \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, t\ [\Rightarrow t]$$

$$[f \Rightarrow]\ \Gamma, f \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, \neg f\ [\Rightarrow \neg f]$$

$$[\bot \Rightarrow]\ \Gamma, \bot \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, \top\ [\Rightarrow \top]$$

$$[\neg\bot \Rightarrow]\ \Gamma, \neg\bot \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, \neg\top\ [\Rightarrow \neg\top]$$

It is easy to see that *GBL* is closed under weakening. We could, in fact, have taken weakening as a primitive rule.

Definition 23. We say that Δ *follows* from Γ in *GBL* ($\Gamma \vdash_{GBL} \Delta$) if there exist finite $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$ s.t. $\Gamma' \Rightarrow \Delta'$ is provable in *GBL*.

Theorem 24 (see [3]).

(a) (*Cut Elimination*) *If* $\Gamma_1 \vdash_{GBL} \Delta_1, \psi$ *and* $\Gamma_2, \psi \vdash_{GBL} \Delta_2$, *then* $\Gamma_1, \Gamma_2 \vdash_{GBL} \Delta_1, \Delta_2$.

(b) (*Soundness and Completeness*) $\Gamma \models^4 \Delta$ *iff* $\Gamma \vdash_{GBL} \Delta$.

Corollary 25. *The* $\{\wedge, \vee, \supset, t, f\}$*-fragment of* $\models^4$ *is identical to the corresponding fragment of classical logic.*

Note. This means that like modal logic, $\models^4$ can also be viewed as an *extension* of classical logic by new connectives (for example $\neg$). This is due to the fact that the classical negation of ψ can be translated into $\psi \supset f$. It is more useful, however, to view $\neg$ as the real counterpart of classical negation.

Corollary 26.

(a) *All the rules of GBL are reversible.*

(b) *Given any sequent* $\Gamma \Rightarrow \Delta$, *one can construct a finite set S of clauses such that*

$$\vdash_{GBL} \Gamma \Rightarrow \Delta \text{ iff } \vdash_{GBL} s$$

for every $s \in S$. [9]

Proof.

(a) This follows easily from Cut Elimination. For example, the rule $[\Rightarrow \neg\supset]$ is reversible because both $\neg(\psi \supset \phi) \Rightarrow \psi$ and $\neg(\psi \supset \phi) \Rightarrow \neg\phi$ are easily derivable, using $[\neg\supset \Rightarrow]$.

[9] By a "clause" we mean here a sequent which contains only literals.

(b) This is immediate from (a). □

Note. The last corollary together with the equivalence of $\vdash_{GBL}$ and $\models^4$ mean that we can develop a tableaux proof system for $\models^4$, which is almost identical to that of classical logic. [10] The main difference is that unlike in classical logic, here a clause $\Gamma \Rightarrow \Delta$ is valid iff $\Gamma \cap \Delta \neq \emptyset$. One should note also that it is impossible here to translate a clause $\Gamma \Rightarrow \Delta$ in which $\Gamma \neq \emptyset$ into a sentence of the language without using the implication connective $\supset$!

As we have seen, $\models^4$ has a lot of nice properties. Still, it has some serious drawbacks as well: It is too restrictive and "overcautious". Thus it is strictly weaker than classical logic even for *consistent* theories (a case in which one might prefer to use classical logic). Moreover, it *totally* rejects some very useful (and intuitively justified) inference rules, like the Disjunctive Syllogism: From $\neg p$ and $p \vee q$ one can *never* infer q by using $\models^4$. Under normal circumstances we would certainly like to be able to use this rule!

In the next subsections we consider several possibilities of refining $\models^4$. The main theme is to restrict the set of models we take into account, using some *preference* criteria. This is the idea behind the notion of a *preferential logic* considered in [42,43]. This idea has recently received a considerable attention (see, e.g., [28,29,31,33,34,38,40]).

4.2. *Taking advantage of the other partial order*

A natural approach for reducing the set of models which are used for drawing conclusions is to consider only the k-minimal models. The idea behind this approach is that we should not assume anything that is not really known. Keeping the amount of knowledge as minimal as possible may also be captured, at least in $\langle FOUR \rangle$, as a kind of consistency preserving method: As long as one keeps the redundant information as minimal as possible the tendency of getting into conflicts decreases.

Definition 27. Let ν_1, ν_2 be two four-valued valuations, and Γ—a set of formulae.

(a) ν_1 is *k-smaller* than ν_2 ($\nu_1 \leqslant_k \nu_2$) if for every atomic p, $\nu_1(p) \leqslant_k \nu_2(p)$;

(b) ν is a *k-minimal model* of Γ if ν is a $\leqslant_k$-minimal element of $mod(\Gamma)$.

Definition 28. $\Gamma \models^4_k \Delta$ iff every k-minimal model of Γ in $\langle FOUR \rangle$ is a model of some $\delta \in \Delta$.

Note. Obviously, if $\Gamma \models^4 \Delta$ then $\Gamma \models^4_k \Delta$.

Example 29 (*Tweety dilemma—continued*). Consider again Examples 3 and 21. Among the six models of Γ (see Fig. 2), two are k-minimal:

$$\text{M4} = \{\mathit{bird(Tweety)} : \top,\ \mathit{penguin(Tweety)} : t,\ \mathit{fly(Tweety)} : f\},$$

$$\text{M6} = \{\mathit{bird(Tweety)} : t,\ \mathit{penguin(Tweety)} : t,\ \mathit{fly(Tweety)} : \top\}.$$

[10] Such a system was introduced in [16,17], but only *validity* of *signed* formulae is considered there and not the consequence relation. Moreover, only k-monotonic operators are dealt with in those papers.

Using these models we reach the same conclusions as in $\models^4$:

$$\Gamma \models^4_k \mathit{bird(Tweety)}, \qquad \Gamma \models^4_k \mathit{penguin(Tweety)}, \qquad \Gamma \models^4_k \neg \mathit{fly(Tweety)},$$

$$\Gamma \not\models^4_k \neg \mathit{bird(Tweety)}, \qquad \Gamma \not\models^4_k \neg \mathit{penguin(Tweety)}, \qquad \Gamma \not\models^4_k \mathit{fly(Tweety)}.$$

Example 30 (*Nixon diamond—continued*). Consider again Examples 4 and 22. Among the twelve models of Δ listed in Fig. 3, three are k-minimal:

$$\begin{aligned} \mathrm{M4} = \big\{ & \mathit{quaker(Nixon)} : t,\ \mathit{republican(Nixon)} : t, \\ & \mathit{hawk(Nixon)} : \top,\ \mathit{dove(Nixon)} : \top \big\}, \\ \mathrm{M8} = \big\{ & \mathit{quaker(Nixon)} : t,\ \mathit{republican(Nixon)} : \top, \\ & \mathit{hawk(Nixon)} : f,\ \mathit{dove(Nixon)} : t \big\}, \\ \mathrm{M12} = \big\{ & \mathit{quaker(Nixon)} : \top,\ \mathit{republican(Nixon)} : t, \\ & \mathit{hawk(Nixon)} : t,\ \mathit{dove(Nixon)} : f \}. \end{aligned}$$

Again, using these models we reach the same conclusions as in $\models^4$, among which:

$$\Delta \models^4_k \mathit{quaker(Nixon)}, \qquad \Delta \models^4_k \mathit{republican(Nixon)},$$

$$\Delta \not\models^4_k \neg \mathit{quaker(Nixon)}, \qquad \Delta \not\models^4_k \neg \mathit{republican(Nixon)}.$$

The fact that in the last two examples we reached the same conclusions (at least with respect to the literals) as in $\models^4$ is not accidental. It is an instance of the following general proposition:

Proposition 31. *If Δ does not include $\supset$, then $\Gamma \models^4 \Delta$ iff $\Gamma \models^4_k \Delta$.*

Proof. For the proof we need the following lemma:

Lemma. *For every model M of Γ there exists a k-minimal model N of Γ s.t. $N \leqslant_k M$.* [11]

Proof. Suppose that M is some model of Γ, and let $S_M = \{M_i \mid M_i \in \mathit{mod}(\Gamma),\ M_i \leqslant_k M\}$. Let $C \subseteq S_M$ be a descending chain with respect to $\leqslant_k$. We shall show that C is bounded in S_M, so by Zorn's lemma S_M has a minimal element, which is the required k-minimal model. Let N be the the following valuation: $N(p) = \min_{\leqslant_k}\{M_i(p) \mid M_i \in C\}$. N is defined since C is a chain, and *FOUR* has a finite number of elements. Obviously N bounds C. It remains to show that $N \in S_M$. Assume that $\psi \in \Gamma$ and let $\mathcal{A}(\psi) = \{p_1, \ldots, p_n\}$ (see Notation 5). Then: $N(p_1) = M_{i_1}(p_1), \ldots, N(p_n) = M_{i_n}(p_n)$. Since C is a chain we may assume, without a loss of generality, that $M_{i_1} \geqslant_k \cdots \geqslant_k M_{i_n}$, and so N is the same as M_{i_n} on every atom in $\mathcal{A}(\psi)$. Since M_{i_n} is a model of ψ, so is N. This is true for every $\psi \in \Gamma$ and so $N \in S_M$ as required.

[11] This lemma can be generalized for directed sets rather than sequences, but the above formulation is sufficient for our needs.

Now, back to the proof of the original proposition: The "only if" direction is trivial. For the other direction, suppose that $\Gamma \models_k^4 \Delta$, and let M be some model of Γ. By the previous lemma there must exist a k-minimal model N of Γ s.t. $M \geqslant_k N$. Thus there is a $\delta \in \Delta$ s.t. $N(\delta) \in \mathcal{D}$. Since all the operators that correspond to the connectives of Δ are monotone with respect to $\leqslant_k$, $M(\delta) \geqslant_k N(\delta)$. But $\mathcal{D}$ is upward-closed with respect to $\leqslant_k$, therefore $M(\delta) \in \mathcal{D}$ as well. $\square$

Corollary 32. *In the monotonic fragment of the language (i.e., without $\supset$), the logics $\models^4$ and $\models_k^4$ are identical.*

Proposition 31 shows that as long as we are interested in inferring formulae that do not include $\supset$, we can indeed limit ourselves to k-minimal models without any loss of generality. This in particular is the case when we are interested in inferring literals. Examples 29 and 30 show that this approach may lead to a considerable reduction in the number of models that should be checked.

The situation is completely different when we do allow the implication connective to appear on the right-hand side of $\models_k^4$:

Example 33 (*Tweety dilemma—continued*). For Γ of Example 3 we have

$$\Gamma \models_k^4 \neg \mathit{penguin}(\mathit{Tweety}) \supset f,$$

although

$$\Gamma \not\models^4 \neg \mathit{penguin}(\mathit{Tweety}) \supset f.$$ [12]

It follows that in the full language $\models_k^4 \neq \models^4$. This can be strengthened as follows:

Proposition 34. *$\models_k^4$ is nonmonotonic.*

Proof. $q \models_k^4 \neg q \supset p$, since $\{p : \bot,\ q : t\}$ is the only k-minimal model of q. On the other hand, $q, \neg q \not\models_k^4 \neg q \supset p$, since $\{p : \bot,\ q : \top\}$ is the only k-minimal model of $\{q, \neg q\}$. $\square$

Note. By Proposition 31, $\models_k^4$ *is* monotonic with respect to conclusions that do not contain $\supset$: If $\Gamma \models_k^4 \Delta$ then $\Gamma, \psi \models_k^4 \Delta$, provided that $\supset$ does not appear in the language of the formulae in Δ.

Using the example of the last proof, one can easily see that $q \models_k^4 \neg q \supset p$ and also $\neg q, \neg q \supset p \models_k^4 p$, but $\neg q, q \not\models_k^4 p$. It follows that $\models_k^4$ is not a consequence relation in the usual sense, since it is not closed under (multiplicative) cut. This is not surprising, since $\models_k^4$ is not monotonic, and it is usual to require a nonmonotonic relation to be closed only under *Cautious* Cut (see [30] and Section 4.5 below).

[12] The meaning of $\psi \supset f$ is that ψ *cannot* be true. This, of course, is stronger than saying that ψ is not a theorem, or even that $\neg\psi$ is a consequence of the assumptions.

Proposition 35. $\models^4_k$ *preserves Cautious Cut: If* $\Gamma, \psi_1, \ldots, \psi_n \models^4_k \Delta$ *and* $\Gamma \models^4_k \psi_i, \Delta$ *for* $i = 1, \ldots, n$, *then* $\Gamma \models^4_k \Delta$.

Proof. Suppose that M is a k-minimal model of Γ, but $M(\delta) \notin \mathcal{D}$ for every $\delta \in \Delta$. Since $\Gamma \models^4_k \psi_i, \Delta$, then $M(\psi_i) \in \mathcal{D}$ for $i = 1, \ldots, n$, and so M is a model of $\{\Gamma, \psi_1, \ldots, \psi_n\}$. Moreover, M must be a k-minimal model of $\{\Gamma, \psi_1, \ldots, \psi_n\}$, since any other model of this set which is strictly smaller than M with respect to $\leqslant_k$ must be a model of Γ, which is k-smaller than M. Now, $\Gamma, \psi_1, \ldots, \psi_n \models^4_k \Delta$, thus $M(\delta) \in \mathcal{D}$ for some $\delta \in \Delta$—a contradiction. □

Despite the nice properties of $\models^4_k$ (more of which will be shown in the sequel; see the note at the end of Section 4.5.2), we will see in what follows (see, e.g., Example 39 below) that this consequence relation appears to be "too conservative". In the following subsections we consider therefore more subtle consequence relations.

4.3. A consequence relation for preferring consistency

Recall that the basic idea in taking the k-minimal models was to avoid meaningless (or redundant) information. A "by-product" of this approach is a reduction in the level of inconsistency of our set of assumptions. When we assume less, the tendency of getting into conflicts decreases. In what follows we shall use a more direct approach of preserving consistency: Given a (possibly inconsistent) theory Γ, the idea is to give precedence to those models of Γ that minimize the amount of inconsistent belief in Γ.

Notation 36. Let ν be a four-valued valuation. Denote:
(a) $\mathcal{I}_1 = \{\top\}$.
(b) $I(\nu, \mathcal{I}_1) = \{p \mid p \text{ is atomic and } \nu(p) \in \mathcal{I}_1\}$.

Intuitively, $\mathcal{I}_1$ is the set of inconsistent values of $\langle FOUR\rangle$ (which in this case consists only of a single element), and $I(\nu, \mathcal{I}_1)$ corresponds to the inconsistent assignments of ν with respect to $\mathcal{I}_1$.

Definition 37. Let Γ be a set of formulae, and M, N models of Γ.
(a) M is *more consistent* than N with respect to $\mathcal{I}_1$ ($M >_{\mathcal{I}_1} N$) if $I(M, \mathcal{I}_1) \subset I(N, \mathcal{I}_1)$.
(b) M is a *most consistent* model of Γ with respect to $\mathcal{I}_1$ ($\mathcal{I}_1$-mcm, in short), if there is no other model of Γ which is more consistent than M with respect to $\mathcal{I}_1$. The set of all the $\mathcal{I}_1$-mcms of Γ is denoted $mcm(\Gamma, \mathcal{I}_1)$.

Definition 38. $\Gamma \models^4_{\mathcal{I}_1} \Delta$ if every $\mathcal{I}_1$-mcm of Γ is a model of some formula of Δ.

Example 39 (*Tweety dilemma—continued*). Consider again Examples 3, 21, 29, and 33. Denote by Γ' the knowledge base *before* Tweety is known to be a penguin, i.e.:

$bird(Tweety) \mapsto fly(Tweety)$
$penguin(Tweety) \supset bird(Tweety)$

penguin(Tweety) $\supset \neg$*fly(Tweety)*
bird(Tweety)

Γ' has 18 models altogether. They are listed in Fig. 4. Here $mcm(\Gamma', \mathcal{I}_1) = \{\text{M17}, \text{M18}\}$. Thus, using $\models^4_{\mathcal{I}_1}$ one can infer that *bird(Tweety)* (but $\neg$*bird(Tweety)* is not true), and *fly(Tweety)* (while $\neg$*fly(Tweety)* is not true). Also, nothing is yet known about Tweety being a penguin. Note that *fly(Tweety)* is *not* a consequence of $\models^4_k$ (and so not a consequence of $\models^4$ as well), although it seems to be an intuitive conclusion of Γ'. Therefore, as we have noted before, $\models^4_k$ might be considered as "overcautious".

Suppose now that a new datum arrives: *penguin(Tweety)*. The models of the modified knowledge base, Γ, are listed in Fig. 2. The mcms of Γ with respect to $\mathcal{I}_1$ are denoted there by M4 and M6. Therefore, according to the new information one should alter his belief and infer the intuitive conclusions, that *bird(Tweety)*, *penguin(Tweety)*, and $\neg$*fly(Tweety)*. The complements of these assertions cannot be inferred by $\models^4_{\mathcal{I}_1}$, as one expects.

Proposition 40. $\models^4_{\mathcal{I}_1}$ *is:* (a) *paraconsistent,* (b) *nonmonotonic.*

Proof.
(a) For example, $p, \neg p \not\models^4_{\mathcal{I}_1} q$. A countermodel assigns $\top$ to p and f to q.
(b) Consider, for instance, $\Gamma = \{p,\ \neg p \vee \neg q\}$. Then $\Gamma \models^4_{\mathcal{I}_1} \neg q$ but $\Gamma \cup \{q\} \not\models^4_{\mathcal{I}_1} \neg q$. □

Proposition 41.
(a) *If* $\Gamma \models^4 \Delta$ *then* $\Gamma \models^4_{\mathcal{I}_1} \Delta$.
(b) *If* $\Gamma \models^4_k \Delta$ *then* $\Gamma \models^4_{\mathcal{I}_1} \Delta$, *provided that the formulae of* Δ *do not contain* $\supset$.
(c) $\models^4_{\mathcal{I}_1} \neq \models^4$ *and* $\models^4_{\mathcal{I}_1} \neq \models^4_k$.

Proof.
(a) Immediate from the definition of $\models^4_{\mathcal{I}_1}$.
(b) Follows from part (a) and Proposition 31.
(c) Follows from Proposition 40(b) and its proof, since both $\models^4$ and $\models^4_k$ are monotonic with respect to the language of $\{\neg, \vee\}$. □

Model No.	*bird(Tweety)*	*fly(Tweety)*	*penguin(Tweety)*
M1–M8	$\top$	$\top, f$	$\top, t, f, \bot$
M9–M12	$\top$	$t, \bot$	$f, \bot$
M13–M16	t	$\top$	$\top, t, f, \bot$
M17–M18	t	t	$f, \bot$

Fig. 4. The models of Γ'.

Proposition 42. *If Γ, ψ are in the language of $\{\vee, \wedge, \neg, \supset, t, f\}$ and $\Gamma \models^4_{\mathcal{I}_1} \psi$, then ψ classically follows from Γ.*

Proof. Let M be a classical model of Γ. M is, of course, also a valuation in *FOUR*, and for formulae in the classical language ($\{\neg, \vee, \wedge, \supset, t, f\}$) there is really no difference between viewing M as a valuation in *FOUR* and viewing it as a valuation in $\{t, f\}$.[13] It follows that M is a model of Γ in *FOUR*, and since $I(M, \mathcal{I}_1) = \emptyset$, M must be an $\mathcal{I}_1$-mcm of Γ. Thus $M(\psi)$ is designated. But we also know that $M(\psi) \in \{t, f\}$, thus $M(\psi) = t$. It follows that M is a classical model of ψ, and so ψ classically follows from Γ. □

4.4. A consequence relation for preferring classical assignments

The approach presented in this subsection is similar to that of the previous one. The difference is that this time we prefer definite knowledge to an uncertain one. In particular, the approach taken here prefers classical inferences whenever their use is possible.

Notation 43. Let ν be a four-valued valuation. Denote:
(a) $\mathcal{I}_2 = \{\top, \bot\}$.
(b) $I(\nu, \mathcal{I}_2) = \{p \mid p \text{ is atomic and } \nu(p) \in \mathcal{I}_2\}$.

This time $\mathcal{I}_2$ is the set of the nonclassical values of *FOUR*, and $I(\nu, \mathcal{I}_2)$ corresponds to the nonclassical assignments of the valuation ν.

Definition 44. Let Γ be a set of formulae, and M, N models of Γ.
(a) M is *more consistent* than N with respect to $\mathcal{I}_2$ ($M >_{\mathcal{I}_2} N$) if $I(M, \mathcal{I}_2) \subset I(N, \mathcal{I}_2)$.
(b) M is a *most consistent* model of Γ with respect to $\mathcal{I}_2$ ($\mathcal{I}_2$-mcm, in short), if there is no other model of Γ which is more consistent than M with respect to $\mathcal{I}_2$. The $\mathcal{I}_2$-mcms of Γ are denoted by $mcm(\Gamma, \mathcal{I}_2)$.

Definition 45. $\Gamma \models^4_{\mathcal{I}_2} \Delta$ if every $\mathcal{I}_2$-mcm of Γ is a model of some formula of Δ.

Example 46 (*Tweety dilemma—continued*). Consider again Example 39 and Fig. 4. When taking $\mathcal{I}_2$ as the set of the "inconsistent" values, M17—the only classical model—is also the only $\mathcal{I}_2$-mcm of Γ'. It follows that according to $\models^4_{\mathcal{I}_2}$ one can infer that *bird(Tweety)*, *fly(Tweety)* (as in the case of $\models^4_{\mathcal{I}_1}$), and $\neg$*penguin(Tweety)* (which is not deducible when using $\models^4_{\mathcal{I}_1}$). The inverse assertions are not true, as expected.

Now, let $\Gamma = \Gamma' \cup \{penguin(Tweety)\}$. As in the case of $\models^4_{\mathcal{I}_1}$, $mcm(\Gamma, \mathcal{I}_2)$ consists of the valuations denoted M4 and M6 in Fig. 2. The new conclusions are, therefore, *bird(Tweety)*, *penguin(Tweety)*, and $\neg$*fly(Tweety)*. Again, the complements of these assertions cannot be inferred by $\models^4_{\mathcal{I}_2}$. These are the intuitive conclusions in this case as well.

The following propositions are analogous to Propositions 40, 41, and 42, respectively:

[13] This is so because $\{t, f\}$ is closed under the corresponding operators.

Proposition 47. $\models^4_{\mathcal{I}_2}$ *is*: (a) *paraconsistent,* (b) *nonmonotonic.*

Proof. The proof is the same as that of Proposition 40, using $\models^4_{\mathcal{I}_2}$ instead of $\models^4_{\mathcal{I}_1}$. $\square$

Proposition 48.

(a) *If* $\Gamma \models^4 \Delta$ *then* $\Gamma \models^4_{\mathcal{I}_2} \Delta$.

(b) *If* $\Gamma \models^4_k \Delta$ *then* $\Gamma \models^4_{\mathcal{I}_2} \Delta$, *provided that the formulae of* Δ *do not contain* $\supset$.

(c) $\models^4_{\mathcal{I}_2} \neq \models^4$ *and* $\models^4_{\mathcal{I}_2} \neq \models^4_k$.

Proof. The proof is the same as that of Proposition 41, using $\models^4_{\mathcal{I}_2}$ instead of $\models^4_{\mathcal{I}_1}$. $\square$

Proposition 49. *Suppose that* Γ, ψ *are in the language of* $\{\vee, \wedge, \neg, \supset, t, f\}$.

(a) *If* $\Gamma \models^4_{\mathcal{I}_2} \psi$, *then* ψ *classically follows from* Γ.

(b) *Suppose that* Γ *is classically consistent. Then* ψ *classically follows from* Γ *iff* $\Gamma \models^4_{\mathcal{I}_2} \psi$.

Proof. The proof of part (a) is the same as that Proposition 42. Part (b) follows from the fact that if Γ is classically consistent then the set of its classical models is the same of the set of the $\mathcal{I}_2$-mcms of Γ in *FOUR*. $\square$

It follows that $\models^4_{\mathcal{I}_2}$ is a nonmonotonic consequence relation that is equivalent to classical logic on consistent theories, and is nontrivial with respect to inconsistent theories.

4.5. *General properties of* $\models^4_{\mathcal{I}_1}$ *and* $\models^4_{\mathcal{I}_2}$

We begin with a comparison between $\models^4_{\mathcal{I}_1}$ and $\models^4_{\mathcal{I}_2}$. In general, neither of these consequence relations is stronger than the other. Consider, for instance, $\Gamma = \{p \supset \neg p, \neg p \supset p\}$. The only $\mathcal{I}_1$-mcm of Γ assigns $\bot$ to p, while this valuation as well as the one in which p is assigned $\top$ are the $\mathcal{I}_2$-mcms of Γ. Therefore, $\Gamma \models^4_{\mathcal{I}_1} p \supset q$ while $\Gamma \not\models^4_{\mathcal{I}_2} p \supset q$. On the other hand, $\models^4_{\mathcal{I}_2} p \vee \neg p$ but $\not\models^4_{\mathcal{I}_1} p \vee \neg p$.

Proposition 50. *Suppose that* $\mathcal{A}(\Gamma, \psi) = \{p_1, p_2, \ldots\}$. *Then* $\Gamma, p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots \models^4_{\mathcal{I}_1} \psi$ *iff* $\Gamma, p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots \models^4_{\mathcal{I}_2} \psi$.

Proof. Denote: $\Gamma' = \Gamma \cup \{p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots\}$. Then

$$mcm(\Gamma', \mathcal{I}_1) = mcm(\Gamma', \mathcal{I}_2),$$

since each model of Γ' assigns to the formulae in $\mathcal{A}(\Gamma, \psi)$ values from $\{t, f, \top\}$. $\square$

Next we consider some common properties of $\models^4_{\mathcal{I}_1}$ and $\models^4_{\mathcal{I}_2}$. In the rest of this section we shall write $\models^4_{\mathcal{I}}$ whenever the results apply to both these relations.

4.5.1. $\models^4_{\mathcal{I}}$ *and GBL-rules*

For future purposes we need the following obvious technical lemma:

Lemma 51. *Let Γ_1, Γ_2 be two sets of formulae s.t. $mod(\Gamma_1) \subseteq mod(\Gamma_2)$. Then every $\mathcal{I}$-mcm of Γ_2 which is also a model of Γ_1 must be an $\mathcal{I}$-mcm of Γ_1.*

Proposition 52 (Weak Soundness). *If $\Gamma \vdash_{GBL} \Delta$ then $\Gamma \models^4_{\mathcal{I}} \Delta$.*

Proof. Obvious from the fact that $\models^4$ is sound with respect to *GBL* and Propositions 41(a), 48(a). □

Note that what the previous proposition claims is that *GBL* is sound for $\models^4_{\mathcal{I}}$ in the *weak* sense; once we add another rule to *GBL* there is no guarantee that the extended system would be sound for $\models^4_{\mathcal{I}}$ anymore, even if the new rule itself is sound for $\models^4_{\mathcal{I}}$. Moreover, from Proposition 52 it does *not* follow that every single rule of *GBL* is sound for $\models^4_{\mathcal{I}}$. In fact, as part (b) of the following proposition shows, this is not the case.

Proposition 53.

(a) (Strong Soundness) *All the rules of GBL except* $[\supset\Rightarrow]$ *are valid for* $\models^4_{\mathcal{I}}$.

(b) $[\supset\Rightarrow]$ *is not valid for* $\models^4_{\mathcal{I}}$, *but its following weakened version* is *valid*:

$$[\supset\Rightarrow]_W \frac{\Gamma, \psi \supset \phi \Rightarrow \psi, \Delta \quad \Gamma, \psi \supset \phi, \phi \Rightarrow \Delta}{\Gamma, \psi \supset \phi \Rightarrow \Delta}.$$

Note. In every monotonic system with contraction, $[\supset\Rightarrow]_W$ is equivalent to $[\supset\Rightarrow]$: $[\supset\Rightarrow]_W$ follows from $[\supset\Rightarrow]$ by using contraction, and $[\supset\Rightarrow]$ is obtained from $[\supset\Rightarrow]_W$ by the addition of $\psi \supset \phi$ to the left-hand side of both premises. However, most of the consequence relations that we discuss are nonmonotonic, and so the nonweakened version of $[\supset\Rightarrow]$ will not be sound for them.

Proof of Proposition 53. The validity of Exchange and Contraction follows immediately from the definition of $\models^4_{\mathcal{I}}$. All the introduction rules on the right, except $[\Rightarrow\supset]$ (i.e., $[\Rightarrow\wedge], [\Rightarrow\neg\wedge], [\Rightarrow\vee], [\Rightarrow\neg\vee], [\Rightarrow\otimes], [\Rightarrow\neg\otimes], [\Rightarrow\oplus], [\Rightarrow\neg\oplus]$, $[\Rightarrow\neg\supset]$, and $[\Rightarrow\neg\neg]$) remain valid since the same formulae appear in them on the left-hand side of the premises and on the left-hand side of the conclusion, hence the same $\mathcal{I}$-mcms are involved, and the arguments in the case of $\models^4$ can be repeated. Similarly, the rules $[\wedge\Rightarrow], [\neg\vee\Rightarrow]$, $[\otimes\Rightarrow]$, $[\neg\otimes\Rightarrow], [\neg\oplus\Rightarrow], [\neg\supset\Rightarrow]$, and $[\neg\neg\Rightarrow]$ remain valid since the left-hand side of the premise and conclusion of each one of them have the same set of models. The validity of $[\neg\wedge\supset], [\vee\Rightarrow]$, and $[\oplus\Rightarrow]$ easily follows from Lemma 51. Finally, to show the validity of $[\Rightarrow\supset]$, suppose that $\Gamma \not\models^4_{\mathcal{I}} \psi \supset \phi, \Delta$. Then there is an $\mathcal{I}$-mcm M of Γ so that $M(\psi) \in \mathcal{D}$, $M(\phi) \notin \mathcal{D}$, and $M(\delta) \notin \mathcal{D}$ for every $\delta \in \Delta$. In particular M is a model of $\Gamma \cup \{\psi\}$. By Lemma 51, M is an $\mathcal{I}$-mcm of $\Gamma \cup \{\psi\}$. Therefore, $\Gamma, \psi \not\models^4_{\mathcal{I}} \phi, \Delta$—a contradiction.

(b) A counter-example: Let p, q be atomic formulae. Then $\models^4_{\mathcal{I}} (p \wedge \neg p) \supset f, q$ and $q \wedge \neg q \models^4_{\mathcal{I}} q$, but $((p \wedge \neg p) \supset f) \supset (q \wedge \neg q) \not\models^4_{\mathcal{I}} q$ (a counter $\mathcal{I}$-mcm assigns $\top$ to

p and f to q). For showing the validity of $[\supset\Rightarrow]_W$, suppose that $\Gamma, \psi \supset \phi \not\models^4_{\mathcal{I}} \Delta$. Then there is an $\mathcal{I}$-mcm M of $\Gamma \cup \{\psi \supset \phi\}$ such that $M(\delta) \notin \mathcal{D}$ for every $\delta \in \Delta$. Since $\Gamma, \psi \supset \phi \models^4_{\mathcal{I}} \psi, \Delta$, necessarily $M(\psi) \in \mathcal{D}$. But M is a model of $\psi \supset \phi$, so $M(\phi) \in \mathcal{D}$ and M is a model of $\Gamma \cup \{\psi \supset \phi, \phi\}$. Moreover, by Lemma 51 M must be an $\mathcal{I}$-mcm of $\Gamma \cup \{\psi \supset \phi, \phi\}$. Now, $\Gamma, \psi \supset \phi, \phi \models^4_{\mathcal{I}} \Delta$, hence there is a $\delta \in \Delta$ s.t. $M(\delta) \in \mathcal{D}$—a contradiction. □

Notes.

(1) Unlike the case of *GBL* and $\models^4$, not all the rules of *GBL* that are valid with respect to $\models^4_{\mathcal{I}}$ are also reversible. $[\Rightarrow\supset]$, for instance, is not (consider, e.g., $\Gamma = \{\neg p\}$, $\psi = p$, and $\phi = q$). This property for itself should not be considered as a drawback, and it is even desirable in nonmonotonic systems: Whenever $\Gamma, \phi \Rightarrow \psi \supset \phi$ holds (which is the case with $\models^4_{\mathcal{I}}$), then the assumption that $\Gamma \Rightarrow \phi$, together with (Cautious) Cut (which is also valid with respect to $\models^4_{\mathcal{I}}$; see below) yield $\Gamma \Rightarrow \psi \supset \phi$. This, and the inverse of $[\Rightarrow\supset]$, imply that $\Gamma, \psi \Rightarrow \phi$. Therefore, had $[\Rightarrow\supset]$ been reversible with respect to $\models^4_{\mathcal{I}}$, this consequence relation would have been monotonic.

(2) Proposition 53(a) implies that given some valid sequents, one can deduce others without checking all the models. Here is a simple example: Since for atomic formula p, q it holds that $\neg p, p \vee q \models^4_{\mathcal{I}} q$, then by $[\Rightarrow\supset]$ we have $p \vee q \models^4_{\mathcal{I}} \neg p \supset q$.

4.5.2. Comparison with general patterns of nonmonotonic reasoning

Being nonmonotonic, $\models^4_{\mathcal{I}_1}$ and $\models^4_{\mathcal{I}_2}$ do not respect weakening. Many rules for replacing weakening has been proposed in the study of general patterns of nonmonotonic reasoning (see, e.g., [23,24,29–31,33,34]). The logic proposed in most of these works is based on the two-valued propositional one. In particular, unlike in the present treatment, the consequence relations considered there are *not* paraconsistent.

In what follows we consider some of the proposals for what nonmonotonic systems should look like, and adapt them to the four-valued case. In this way we would be able to give them paraconsistent capabilities.

Definition 54 (Lehmann [30]). A *plausibility logic* in a language L is a relation $\Rightarrow$ between finite sets of formulae in L that satisfies the the following conditions:

Inclusion: $\Gamma, \psi \Rightarrow \psi$.
Right Monotonicity: If $\Gamma \Rightarrow \Delta$, then $\Gamma \Rightarrow \psi, \Delta$.
Cautious Left Monotonicity: If $\Gamma \Rightarrow \psi$ and $\Gamma \Rightarrow \Delta$, then $\Gamma, \psi \Rightarrow \Delta$. [14]
Cautious Cut: If $\Gamma, \psi_1, \ldots, \psi_n \Rightarrow \Delta$ and $\Gamma \Rightarrow \psi_i, \Delta$ for $i = 1, \ldots, n$, then $\Gamma \Rightarrow \Delta$.

Proposition 55. $\models^4_{\mathcal{I}}$ *is a plausibility logic.* [15]

Proof. Inclusion and Right Monotonicity follow immediately from the definition of $\models^4_{\mathcal{I}}$. Cautious Cut is shown as in Proposition 35. It is left to show Cautious Left Monotonicity:

[14] This rule was first proposed in [24].

[15] Recall that this means that the rules of Definition 54 are valid with respect to *both* $\models^4_{\mathcal{I}_1}$ and $\models^4_{\mathcal{I}_2}$.

Assume that $\Gamma \models^4_{\mathcal{I}} \psi$, and $\Gamma \models^4_{\mathcal{I}} \Delta$. Let M be an $\mathcal{I}$-mcm of $\Gamma \cup \{\psi\}$. In particular, M is a model of Γ. Moreover, it must be an $\mathcal{I}$-mcm of Γ as well, since otherwise there would be an $N \in mod(\Gamma)$, that is strictly more consistent than M. Since $\Gamma \models^4_{\mathcal{I}} \psi$, this N would have been an $\mathcal{I}$-mcm $\Gamma \cup \{\psi\}$ and therefore $N <_{\mathcal{I}} M$ with respect to $\Gamma \cup \{\psi\}$—a contradiction. Therefore, M is a $\mathcal{I}$-mcm of Γ. Now, since $\Gamma \models^4_{\mathcal{I}} \Delta$, M is a model of some $\delta \in \Delta$. Hence $\Gamma, \psi \models^4_{\mathcal{I}} \Delta$. □

The following definition is a generalization of the notion of *preferential logics*, which has been introduced in [29]:

Definition 56. Let $\models$ be a consequence relation (in the usual monotonic sense). Suppose that $\supset$ is a connective that is an internal implication with respect to $\models$ and $\leftrightarrow$ is a connective which is internal equivalence with respect to $\models$ (see Proposition 20). Then a $\models$*-preferential logic* is a relation $\Rightarrow$ that is closed under the following conditions:

Reflexivity: If $\Gamma \cap \Delta \neq \emptyset$, then $\Gamma \Rightarrow \Delta$.
Left Logical Equivalence: If $\Gamma \models \psi \leftrightarrow \phi$ and $\Gamma, \psi \Rightarrow \Delta$, then $\Gamma, \phi \Rightarrow \Delta$.
Right Weakening: If $\Gamma \models \psi \supset \phi, \Delta$ and $\Gamma \Rightarrow \psi, \Delta$, then $\Gamma \Rightarrow \phi, \Delta$.
Or: If $\Gamma, \psi \Rightarrow \Delta$ and $\Gamma, \phi \Rightarrow \Delta$, then $\Gamma, \psi \vee \phi \Rightarrow \Delta$. [16]
Cautious Left Monotonicity.
Cautious Cut.

Preferential logics form the central family of nonmonotonic logics among those considered in [29]. In their original definition [29] refer to the classical consequence relation together with the classical material implication and equivalence. Naturally, we prefer to use $\models^4$ instead:

Definition 57. A *four-valued preferential logic* is a $\models^4$-preferential logic, where $\supset$, $\leftrightarrow$ are the connectives defined in Definition 1 (see also Proposition 20).

Proposition 58. $\models^4_{\mathcal{I}}$ *is a four-valued preferential logic.*

Proof. By Proposition 20, $\supset$ is indeed an internal implication and $\leftrightarrow$ is an internal equivalence with respect to $\models^4$. It is left to show that the other conditions of Definition 56 are met. Reflexivity, Cautious Left Monotonicity, Cautious Cut, and $[\vee \Rightarrow]$ have already been proved in Propositions 53 and 55. It is left to show the validity of Left Logical Equivalence and Right Weakening.

Left Logical Equivalence: Let M be an $\mathcal{I}$-mcm of $\Gamma \cup \{\phi\}$, and suppose that $M(\delta) \notin \mathcal{D}$ for every $\delta \in \Delta$. M is in particular a model of Γ and thus it is a model of $\psi \leftrightarrow \phi$. By Proposition 2, $\Gamma \cup \{\psi\}$ and $\Gamma \cup \{\phi\}$ have the same models. Hence it is easily verified, using Lemma 51, that M is an $\mathcal{I}$-mcm of $\Gamma \cup \{\psi\}$. But this contradicts the assumption that $\Gamma, \psi \models^4_{\mathcal{I}} \Delta$.

[16] This rule was denoted by $[\Rightarrow \vee]$ in *GBL*.

Right Weakening: Suppose that M is an $\mathcal{I}$-mcm of Γ and $M(\phi), M(\delta) \notin \mathcal{D}$ for every $\delta \in \Delta$. Since $M \in mod(\Gamma)$ then by assumption, $M(\psi \supset \phi) \in \mathcal{D}$. But $M(\phi) \notin \mathcal{D}$, and so $M(\psi) \notin \mathcal{D}$ either—a contradiction to $\Gamma \models^4_{\mathcal{I}} \psi, \Delta$. □

Note. Similar proofs to those of Propositions 55 and 58 can be used for showing that $\models^4_k$ is also a plausibility logic as well as a four-valued preferential logic.

4.5.3. Reducing the amount of the preferred models

A we have already noted, one of the advantages of $\models^4_{\mathcal{I}_1}$ and $\models^4_{\mathcal{I}_2}$ with respect to $\models^4$ is that the set of models needed for drawing conclusions from the formers is never bigger than that of the latter. In this subsection we consider cases in which it is possible to reduce the amount of the relevant models even further, without changing the logic. The idea is to take the composition of $\leqslant_k$ and $\leqslant_{\mathcal{I}}$; Instead of considering every $\mathcal{I}_1$- [$\mathcal{I}_2$-] mcm of Γ, we use only the k-minimal models in this set. [17]

Proposition 59. *Suppose that the formulae of Δ are in the language without $\supset$. Then $\Gamma \models^4_{\mathcal{I}_1} \Delta$ iff every k-minimal element of $mcm(\Gamma, \mathcal{I}_1)$ is a model of some $\delta \in \Delta$.* [18]

Proof. If $\Gamma \models^4_{\mathcal{I}_1} \Delta$ then in particular every k-minimal element of $mcm(\Gamma, \mathcal{I}_1)$ is a model of some formula of Δ. For the converse, let M be an $\mathcal{I}_1$-mcm of Γ. By the lemma in the proof of Proposition 31, there exists a k-minimal model N of Γ s.t. $N \leqslant_k M$. It follows that for every atom p for which $N(p) = \top$, $M(p) = \top$ as well. Thus $I(N, \mathcal{I}_1) \subseteq I(M, \mathcal{I}_1)$. But M is an $\mathcal{I}_1$-mcm of Γ, so $I(N, \mathcal{I}_1) = I(M, \mathcal{I}_1)$, and N is also an $\mathcal{I}_1$-mcm of Γ. In particular, N is k-minimal among the $\mathcal{I}_1$-mcms of Γ, and so there is a $\delta \in \Delta$ s.t. $N(\delta) \in \mathcal{D}$. Since all the operators that correspond to the connectives of Δ are monotone with respect to $\leqslant_k$, $M(\delta) \geqslant_k N(\delta)$, and so $M(\delta) \in \mathcal{D}$ as well. Therefore $\Gamma \models^4_{\mathcal{I}_1} \Delta$. □

Note. Proposition 59 is no longer true when $\supset$ occurs in the conclusions. For a counter-example consider, e.g., $\Gamma = \{p, p \vee q\}$. The k-minimal element of $mcm(\Gamma, \mathcal{I}_1)$ assigns t to p and $\bot$ to q, therefore $q \supset \neg q$ is true in it. However, $p, p \vee q \not\models^4_{\mathcal{I}_1} q \supset \neg q$.

Proposition 60. *Proposition* 59 *is not true for $\models^4_{\mathcal{I}_2}$; It is* not sufficient *to consider only the k-minimal elements of $mcm(\Gamma, \mathcal{I}_2)$ for inferring $\Gamma \models^4_{\mathcal{I}_2} \Delta$, even if the formulae in Δ are all in the language without $\supset$.*

Proof. Consider the following infinite set:

$$\Gamma = \{p_i \vee \neg p_i \supset p_{i+1} \wedge \neg p_{i+1} \mid i \geqslant 1\}.$$

It is easy to verify that

$$mcm(\Gamma, \mathcal{I}_2) = \{M_1^t, M_1^f, M_2^t, M_2^f, \ldots\},$$

[17] See [5] for a practical usage of the k-minimal mcms of a theory.

[18] This result is a generalization of Theorem 4.3 of [5] to the case that Γ is infinite and may contain implications.

where, for every $j \geqslant 1$, M_j^t assigns $\bot$ to $\{p_1, \ldots, p_{j-1}\}$, t to p_j, and $\top$ to $\{p_{j+1}, p_{j+2}, \ldots\}$. M_j^f is the same valuation as M_j^t, except that p_j is assigned f instead of t. Therefore $\Gamma \not\models^4_{\mathcal{I}_2} p_1$. On the other hand, $mcm(\Gamma, \mathcal{I}_2)$ has no k-minimal element (since for every $j \geqslant 1$, $M_{j+1}^t <_k M_j^t$ and $M_{j+1}^f <_k M_j^f$), therefore everything would have followed from this set (in particular p_1), had we used only the k-minimal $\mathcal{I}_2$-mcms of Γ for drawing conclusions. □

Despite the previous proposition, we still have the following result:

Proposition 61. *Suppose that Γ is* finite, *and the formulae of Δ are in the language without $\supset$. Then $\Gamma \models^4_{\mathcal{I}_2} \Delta$ iff every k-minimal element of $mcm(\Gamma, \mathcal{I}_2)$ is a model of some $\delta \in \Delta$.*

Proof. Again, the "only if" direction is obvious. For the other direction, assume that the condition holds. Since Γ is finite, it has a finite number of (k-minimal models among the $\mathcal{I}_2$-most consistent) models. Therefore, for every $\mathcal{I}_2$-mcm M of Γ there is a model N which is k-minimal among the $\mathcal{I}_2$-mcms of Γ, and $N \leqslant_k M$. By our assumption, there is a $\delta \in \Delta$ s.t. $N(\delta) \in \mathcal{D}$. As in the proof of the Proposition 59, this implies that $M(\delta) \in \mathcal{D}$ as well, and so $\Gamma \models^4_{\mathcal{I}_2} \Delta$. □

Note. As in Proposition 59, the condition about Δ is necessary in Proposition 61 as well: For giving a counter-example in this case note that Γ must be inconsistent (otherwise the $\mathcal{I}_2$-mcms of Γ are its $\{t, f\}$-models, and so each $\mathcal{I}_2$-mcm is k-minimal). Consider, therefore, $\Gamma = \{p \supset \neg p, \neg p \supset p\}$. The k-minimal element of $mcm(\Gamma, \mathcal{I}_2)$ assigns $\bot$ to p, and so $p \supset f$ is true in it. On the other hand, $\Gamma \not\models^4_{\mathcal{I}_2} p \supset f$.

4.6. The monotonic classical fragment

We conclude this section with some results concerning the $\{\vee, \wedge, \neg, t, f\}$-fragment of the language. This fragment may be called the *monotonic classical language*. It is extensively discussed in the literature, and although it has relatively weak expressive power in the multi-valued setting, the corresponding fragments of our logics have many nice properties.

First, it is well known that with respect to the monotonic classical language, $\models^4$ is identical to the set of "first degree entailments" in relevance logic (see [1,14]). The exact connection is that $\psi_1, \ldots, \psi_n \models^4 \phi_1, \ldots, \phi_m$ iff $\psi_1 \wedge \cdots \wedge \psi_n \to \phi_1 \vee \cdots \vee \phi_m$ is a first degree entailment.

A second important observation is that relative to this language, $\models^4_{\mathcal{I}_2}$ is really a three-evalued logic:

Proposition 62. *Suppose that the formulae of Γ are in the language of $\{\vee, \wedge, \neg, t, f\}$ and that M is an $\mathcal{I}_2$-mcm of Γ. Then there is no formula ψ s.t. $M(\psi) = \bot$.*

Proof. Since $\{t, f, \top\}$ is closed under $\neg$, $\vee$ and $\wedge$, it is sufficient to show the proposition only for atomic formulae. Define a transformation $g : FOUR \to \{t, f, \top\}$ as follows:

$g(\bot) = t$, $g(b) = b$ otherwise. Obviously, for every atom p, $g \circ M(p) \geqslant_k M(p)$. Since every connective in the language of Γ is k-monotone, $\forall\gamma \in \Gamma$ $g \circ M(\gamma) \geqslant_k M(\gamma)$. Now, $\mathcal{D}$ is upward-closed with respect to $\leqslant_k$, and so $\forall\gamma \in \Gamma$ $g \circ M(\gamma) \in \mathcal{D}$. Thus $g \circ M$ is also a model of Γ. Since $g \circ M \geqslant_{\mathcal{I}_2} M$, necessarily $g \circ M = M$. □

Another important property of formulae in the monotonic classical language is that as in the classical case, every formula can be translated to an equivalent formula in standard conjunctive normal form (CNF) or standard disjunctive normal form (DNF):

Proposition 63. *Every formula ψ in the monotonic classical language can be translated to a CNF-formula ψ' and to a DNF-formula ψ'' s.t. for every valuation ν in FOUR, $\nu(\psi) = \nu(\psi') = \nu(\psi'')$.*

Proof. The proof is similar to that of the classical case, using the fact that De Morgan's laws, distributivity, commutativity, associativity, and the double negation rule ($\neg\neg\phi \equiv \phi$) remain valid in the four-valued case. □

Another connection with classical logic is the following:

Proposition 64. *Let Γ be a classically consistent set in the monotonic classical language, and suppose that ψ is a formula in CNF, none of its conjuncts is a tautology.*[19] *Then ψ classically follows from Γ iff $\Gamma \models^4_{\mathcal{I}_1} \psi$.*

Proof. ($\Rightarrow$) Assume first that ψ is a disjunction of literals, which is not a tautology. Suppose also that $\Gamma \not\models^4_{\mathcal{I}_1} \psi$. Let M be an $\mathcal{I}_1$-mcm of Γ s.t. $M(\psi) \notin \mathcal{D}$. Since Γ is classically consistent, it has a classical model, N. Since $I(N, \mathcal{I}_1) = \emptyset$, $I(M, \mathcal{I}_1) = \emptyset$ as well. Now, define:

$$M'(p) = \begin{cases} t & M(p) = t, \text{ or } (M(p) = \bot \text{ and } \neg p \in \mathcal{L}(\psi)), \\ f & \text{otherwise.} \end{cases}$$

All the connectives in Γ are k-monotonic. Therefore, since $M' \geqslant_k M$, and M is a model of Γ, M' is a (classical) model of Γ as well. It is easy to see that $M'(\psi) = f$, therefore ψ does not classically follow from Γ.

Suppose now that ψ is a formula in CNF, none of its conjuncts is a tautology, and $\Gamma \not\models^4_{\mathcal{I}_1} \psi$. Then it must have a conjunct ψ' s.t. $\Gamma \not\models^4_{\mathcal{I}_1} \psi'$. We have shown that ψ' cannot classically follow from Γ, therefore ψ also does not classically follow from Γ.

($\Leftarrow$) Follows from Proposition 42. □

The last two propositions together with Proposition 59 entail that for checking whether a formula classically follows from a consistent set Γ, it is sufficient to perform the following steps:

(1) convert the formula to a conjunctive normal form,
(2) drop all the conjuncts which are tautologies, and

[19] Classically, every formulae which is not a tautology is equivalent to some formula of this form.

(3) check the remaining formula only with respect to the k-minimal $\mathcal{I}_1$-mcms of Γ.[20]

The next proposition should be compared with Proposition 60:

Proposition 65. *Suppose that the formulae of Γ are in the monotonic classical language. Then $\Gamma \models^4_{\mathcal{I}_2} \Delta$ iff every k-minimal element of $mcm(\Gamma, \mathcal{I}_2)$ is a model of some $\delta \in \Delta$.*

Proof. By Proposition 62, in this case every $\mathcal{I}_2$-mcm of Γ is also k-minimal in $mcm(\Gamma, \mathcal{I}_2)$, and so the claim follows. □

Next we compare $\models^4_{\mathcal{I}_1}$ and $\models^4_{\mathcal{I}_2}$ in the monotonic classical language. At the beginning of Subsection 4.5 we have noted that in general, neither of these relations is stronger than the other. As Proposition 66 below shows, this is no longer true in the case of the $\{\vee, \wedge, \neg, t, f\}$-fragment:

Proposition 66. *Let Γ, Δ, ψ be in the monotonic classical language.*

(a) *If $\Gamma \models^4_{\mathcal{I}_1} \Delta$ then $\Gamma \models^4_{\mathcal{I}_2} \Delta$.*

(b) *If ψ is a CNF-formula, none of its conjuncts is a tautology, then $\Gamma \models^4_{\mathcal{I}_1} \psi$ iff $\Gamma \models^4_{\mathcal{I}_2} \psi$.*

Proof. (a) This follows from the fact that in the classical monotonic language every $\mathcal{I}_2$-mcm of Γ is also an $\mathcal{I}_1$-mcm of Γ. Indeed, let M be an $\mathcal{I}_2$-mcm of Γ, and suppose that N is another model of Γ s.t. $N >_{\mathcal{I}_1} M$. Define for every atom p a valuation M' as follows: $M'(p) = t$ if $N(p) = \bot$ and $M'(p) = N(p)$ otherwise. Since the language is k-monotonic and $M' \geqslant_k N$, $M' \in mod(\Gamma)$. Now,

$$I(M', \mathcal{I}_2) = I(M', \mathcal{I}_1) = I(N, \mathcal{I}_1) \subset I(M, \mathcal{I}_1).$$

Moreover, by Proposition 62, $I(M, \mathcal{I}_1) = I(M, \mathcal{I}_2)$, thus $I(M', \mathcal{I}_2) \subset I(M, \mathcal{I}_2)$, and so $M' >_{\mathcal{I}_2} M$—a contradiction.

(b) Obviously, it suffices to show the claim for a disjunction ψ of literals that does not contain an atomic formula and its negation. So assume that $\Gamma \not\models^4_{\mathcal{I}_1} \psi$. Then there is an $\mathcal{I}_1$-mcm M of Γ s.t. $M(\psi) \notin \mathcal{D}$. Consider the valuation M', defined as follows:

$$M'(p) = \begin{cases} t & \text{if } M(p) = \bot \text{ and } p \notin \mathcal{L}(\psi), \\ f & \text{if } M(p) = \bot \text{ and } p \in \mathcal{L}(\psi), \\ M(p) & \text{otherwise.} \end{cases}$$

(1) M' is a model of Γ, since $\forall \gamma \in \Gamma$ $M'(\gamma) \geqslant_k M(\gamma)$ and $\mathcal{D}$ is upward-closed with respect to $\leqslant_k$.

(2) M' is an $\mathcal{I}_2$-mcm of Γ, since if $\exists N \in mod(\Gamma)$ s.t. $N >_{\mathcal{I}_2} M'$ then

$$I(N, \mathcal{I}_1) \subseteq I(N, \mathcal{I}_2) \subset I(M', \mathcal{I}_2) = I(M', \mathcal{I}_1) = I(M, \mathcal{I}_1),$$

so $N >_{\mathcal{I}_1} M$—a contradiction.

[20] This process might be useful in case Γ is a *fixed* theory, but the check should be made for many different potential conclusions. Note that if Γ is consistent then the number of k-minimal $\mathcal{I}_1$-mcms is never greater than the number of classical models and is frequently smaller. We shall return to this point in Section 5.

(3) $M'(\psi) \notin \mathcal{D}$: This follows from the structure of ψ and from the fact that for every $l \in \mathcal{L}(\psi)$, $M'(l) \in \mathcal{D}$ iff $M(l) \in \mathcal{D}$.

By (1)–(3) it follows that $\Gamma \not\models^4_{\mathcal{I}_2} \psi$. □

Note. The converse of part (a) of Proposition 66 is not true in general. For instance, $\models^4_{\mathcal{I}_2} p \vee \neg p$ while $\not\models^4_{\mathcal{I}_1} p \vee \neg p$.

5. Four values are better than three

5.1. The three-valued logics in the context of FOUR

Three-valued logics might be roughly divided into two families according to the decision whether the middle element is taken to be designated or not. Logics of the first class are, in fact, logics that are based on the subset $\{t, f, \bot\}$ of *FOUR*, while logics of the other class are based on the subset $\{t, f, \top\}$. In both cases the languages of the corresponding standard logics are based on some fragment of the language of $\{\neg, \vee, \wedge, \oplus, \otimes, \supset, t, f, \top, \bot\}$ (see [6]). The interpretations of these connectives are the reductions of the corresponding operators of *FOUR* (provided that the three values are closed under the operations, which is the case for the classical connectives. Note that $\{t, f, \bot\}$ is closed under $\otimes$ while $\{t, f, \top\}$ is closed under $\oplus$). The functional completeness theorem concerning *FOUR* induces a corresponding theorem for the three-valued subsets:

Theorem 67.
(a) *The language of* $\{\neg, \wedge, \supset, \otimes, f\}$ *is functionally complete for* $\{t, f, \bot\}$.
(b) *The language of* $\{\neg, \wedge, \supset, \oplus, f\}$ *is functionally complete for* $\{t, f, \top\}$.

Proof. This easily follows from the fifth and the seventh items, respectively, of Theorem 14. □

Note. The connective $\supset$ of *FOUR* induces *two* different three-valued implications, depending on the interpretation of the third value as either $\bot$ or $\top$. Parts (a) and (b) of Theorem 67 refer, in fact, to these two different meanings of $\supset$. On the other hand, the three-valued truth tables of $\otimes$ in $\{t, f, \bot\}$ and of $\oplus$ in $\{t, f, \top\}$ are identical. The two parts of Theorem 67 do provide, therefore, two different functionally complete sets of three-valued connectives, but this is due to the different meanings of $\supset$.

5.2. Comparison with four-valued systems

The main advantage of using *FOUR* rather than three-valued systems is, of course, that it allows us to deal with *both* types of abnormal propositions in one system. This makes it possible to construct a system like $\models^4_{\mathcal{I}_1}$, which is quite strong on one hand, but allows also the use of constructive as well as relevant paradigms of reasonings on the other hand. In this section we show, moreover, that one can in any case do with *FOUR* everything one can do using only three values, sometimes even more efficiently. We start by showing that it is possible to simulate the basic three-valued logics in the context of *FOUR*. Denote by $\models^3_{\mathrm{Kl}}$ the consequence relation that corresponds to Kleene's logic (i.e., $\Gamma \models^3_{\mathrm{Kl}} \Delta$ iff every $\{t, f, \bot\}$-model of Γ is a $\{t, f, \bot\}$-model of some formula in Δ), and by $\models^3_{\mathrm{LP}}$ the consequence relation of the logic LP [21] (i.e., $\Gamma \models^3_{\mathrm{LP}} \Delta$ iff every $\{t, f, \top\}$-model of Γ is a $\{t, f, \top\}$-model of some formula in Δ). Then:

Proposition 68. *Let Γ, Δ be two sets of assertions with $\mathcal{A}(\Gamma, \Delta) = \{p_1, p_2, \ldots\}$.*
(a) $\Gamma \models^3_{\mathrm{Kl}} \Delta$ *iff* $\Gamma, p_1 \wedge \neg p_1 \supset f, p_2 \wedge \neg p_2 \supset f, \ldots \models^4 \Delta$.
(b) $\Gamma \models^3_{\mathrm{LP}} \Delta$ *iff* $\Gamma, p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots \models^4 \Delta$.

Proof. Part (a) follows from the fact that the $\{t, f, \bot\}$-models of Γ are the same as the four-valued models of $\Gamma \cup \{p_1 \wedge \neg p_1 \supset f, p_2 \wedge \neg p_2 \supset f, \ldots\}$. Similarly, in case (b) the $\{t, f, \top\}$-models of Γ are the same as the four-valued models of $\Gamma \cup \{p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots\}$. □

A basic drawback of standard three-valued logics in which the nonclassical value in not designated is that they are not paraconsistent [10]; $\{p, \neg p\}$ has in them no model, and so everything follows from this set. Since we consider paraconsistency as one of the major reasons for switching to multi-valued semantics, we shall concentrate in what follows on the other family of three-valued logics, in which the third value *is* designated.

We have already mentioned LP as the basic logic among the three-valued logics with middle element designated. It is well known that LP invalidates the Disjunctive Syllogism ($\psi, \neg\psi \vee \phi \not\models^3_{\mathrm{LP}} \phi$). Priest [37,38] argues that this is a drawback: a consistent theory should preserve classical conclusions. He suggests to resolve this drawback by considering as the relevant models of a set Γ only those that are *minimally inconsistent*. Such models assign $\top$ only to some minimal set of atomic formulae. The consequence relation $\models^3_{\mathrm{LPm}}$ of the resulting logic, LPm, is then defined as follows: $\Gamma \models^3_{\mathrm{LPm}} \psi$ iff every minimally inconsistent model of Γ is a model of ψ.

The original treatment of Priest defines LPm only for what we have called the monotonic classical language ($\{\vee, \wedge, \neg, t, f\}$). This idea, however, can easily be extended to richer languages, and that is what we just have done.

Like $\models^3_{\mathrm{LP}}$ and $\models^3_{\mathrm{Kl}}$, the logic of Priest can also easily be simulated in *FOUR*:

Proposition 69. *Suppose that $\mathcal{A}(\Gamma, \psi) = \{p_1, p_2, \ldots\}$. The following conditions are equivalent:*
(1) $\Gamma \models^3_{\mathrm{LPm}} \psi$.
(2) $\Gamma, p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots \models^4_{\mathcal{I}_1} \psi$.
(3) $\Gamma, p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots \models^4_{\mathcal{I}_2} \psi$.

Proof. The three-valued models of Γ are the same as the four-valued models of $\Gamma \cup \{p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots\}$. Since each one of them assigns to the atomic formulae in $\mathcal{A}(\Gamma, \psi)$

[21] Also known as J_3, RM_3, and PAC (see [6,13,39] and Chapter IX of [15]). Strictly speaking, J_3 and RM_3 are extensions of LP, since they have added conditional operators.

values from $\{t, f, \top\}$, the LPm models of Γ are the same as the $\mathcal{I}_1$-mcms and the $\mathcal{I}_2$-mcms of $\Gamma \cup \{p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots\}$. $\square$

Although the motivation for $\models^4_{\mathcal{I}_2}$ and especially for $\models^4_{\mathcal{I}_1}$ is similar to that of Priest's $\models^3_{\mathrm{LPm}}$ (all of them try to minimize the amount of inconsistency), they are *not* the same logic. For instance, $p \supset \neg p, \neg p \supset p \models^3_{\mathrm{LPm}} p$, while $p \supset \neg p, \neg p \supset p \not\models^4_{\mathcal{I}_j} p$ for $j = 1, 2$. On the other hand, the following proposition shows that in the monotonic classical language $\models^3_{\mathrm{LPm}}$ is identical to $\models^4_{\mathcal{I}_2}$, and has strong relations to $\models^4_{\mathcal{I}_1}$.

Proposition 70. *Let Γ, Δ be two sets of formulae and ψ a formula in the language of $\{\neg, \wedge, \vee, t, f\}$.*

(a) *$\Gamma \models^3_{\mathrm{LPm}} \Delta$ iff $\Gamma \models^4_{\mathcal{I}_2} \Delta$.*

(b) *Suppose that ψ is a formula in CNF, none of its conjuncts is a tautology. Then $\Gamma \models^3_{\mathrm{LPm}} \psi$ iff $\Gamma \models^4_{\mathcal{I}_1} \psi$.*

Proof. We leave the proof of part (a) to the reader. Part (b) immediately follows from part (a) and Proposition 66. $\square$

Proposition 70(b) together with Proposition 59 imply that a switch to four-valued semantics might improve the three-valued inference process of LPm: Let ψ be a formula in the monotonic classical language. For checking whether $\Gamma \models^3_{\mathrm{LPm}} \psi$, it is sufficient to convert ψ to a conjunctive normal form, remove every conjunct which contains some atomic formula together with its negation, and check the resulting formula only in the k-minimal $\mathcal{I}_1$-mcms of Γ. The number of such models is usually smaller (and never bigger!) than the number of the LPm-models. This is due to the fact that from every k-minimal $\mathcal{I}_1$-mcm one can obtain several LPm-models by changing every $\bot$-assignment to either t or f. Here is a very simple example: Let $\Gamma = \{\neg p \vee q, p \vee q\}$. q follows from Γ according to $\models^3_{\mathrm{LPm}}$ and so also according to $\models^4_{\mathcal{I}_1}$ (and classically as well, of course). Now, Γ has *two* LPm-models: $\{p : t, q : t\}$ and $\{p : f, q : t\}$ (these are also its classical models),

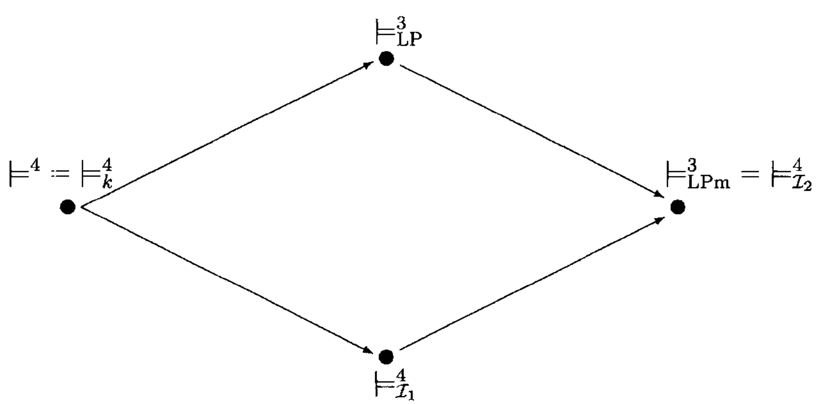

Fig. 5. Relationships among the three- and four-valued systems where $L = \{\neg, \wedge, \vee, t, f\}$.

but only *one* k-minimal $\mathcal{I}_1$-model: $\{p : \bot, q : t\}$. This single model suffices for inferring that q follows from Γ.

Fig. 5 summarizes the relationships among the three- and four-valued consequence relations with respect to the monotonic classical language. [22] One should remember, however, that important as it is, this language is quite limited.

6. More than four values are usually not necessary

In this section we consider a class of structures that naturally generalize $\langle FOUR \rangle$. We then generalize the above four-valued logics to those structures in an attempt to achieve more powerful inference mechanisms. The major result of this section is that this freedom to use more truth values does *not* add much; Each one of the multi-valued logics considered here can actually be characterized by one of our four-valued logics.

6.1. *Bilattices*

6.1.1. *Background and motivation*

Bilattices [26,27] are algebraic structures that naturally generalize Belnap's four-valued lattice, *FOUR*. The idea is to consider *arbitrary* number of truth values, and to arrange them (as in *FOUR*) in two closely related partial orders, each forming a lattice. As in the four-valued case, one intuitively understands one of the orderings as representing degrees of truth, and the other as representing degrees of knowledge.

The original motivation of Ginsberg for using bilattices was to provide a uniform approach for a diversity of applications in AI. In particular he treated first-order theories and their consequences, truth maintenance systems and formalisms for default reasoning. The algebraic structure of bilattices has been further investigated by Fitting and Avron [7, 18,21]. Fitting has also shown that bilattices are very useful tools for providing semantic for logic programs: He proposed an extension of Smullyan's tableaux-style proof method to bilattice-valued programs, and showed that this method is sound and complete with respect to a natural generalization of van Emden and Kowalski's operator (see [17,19]). Fitting also introduced a multi-valued fixedpoint operator (that generalizes the Gelfond–Lifschitz operator [25]) for providing bilattice-based stable models and well-founded semantics for logic programs (see [20]). A well-founded semantics for logic programs that is based on the bilattice *NINE* (Fig. 6) is considered also in [11]. Bilattices have also been found useful for nonmonotonic reasoning [3,4], temporal reasoning [22], model-based diagnostics [27], and reasoning with inconsistent knowledge bases [5,41].

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[22] The observation that $\models^3_{\mathrm{LP}}$ and $\models^4_{\mathcal{I}_1}$ are incomparable follows from the facts that excluded middle is valid with respect to $\models^3_{\mathrm{LP}}$ but not with respect to $\models^4_{\mathcal{I}_1}$, while the disjunctive syllogism (applied to atomic formulae) is valid in $\models^4_{\mathcal{I}_1}$ but not in $\models^3_{\mathrm{LP}}$.

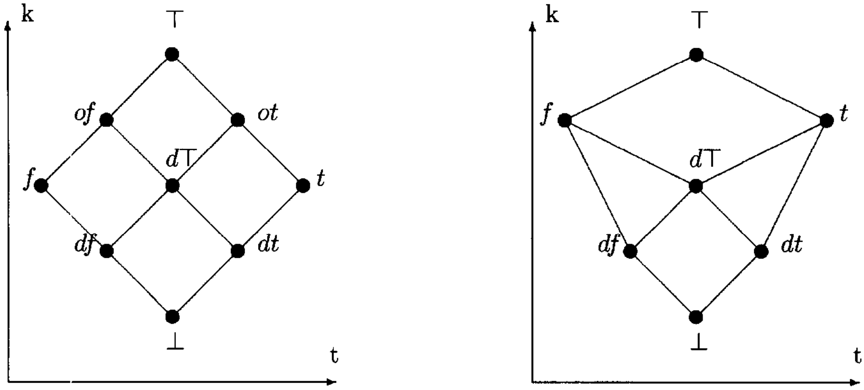

Fig. 6. *NINE*, and *DEFAULT*.

6.1.2. Preliminaries

Definition 71 (Ginsberg [27]). A *bilattice* is a structure $\mathcal{B} = (B, \leqslant_t, \leqslant_k, \neg)$ such that B is a nonempty set containing at least two elements; $(B, \leqslant_t)$, $(B, \leqslant_k)$ are complete lattices; and $\neg$ is a unary operation on B that has the following properties:

(a) if $a \leqslant_t b$, then $\neg a \geqslant_t \neg b$,
(b) if $a \leqslant_k b$, then $\neg a \leqslant_k \neg b$,
(c) $\neg\neg a = a$.[23]

In what follows we shall continue to use $\wedge$ and $\vee$ for the meet and join of $\leqslant_t$, and $\otimes$, $\oplus$ for the meet and join of $\leqslant_k$. Also, f and t still denote the respective least and greatest element with respect to $\leqslant_t$, while $\bot$ and $\top$—the least and the greatest element with respect to $\leqslant_k$. It is easy to see that t, f, $\top$, and $\bot$ are all distinct from each other.

Definition 72. A bilattice is called *distributive* [27] if all the twelve possible distributive laws concerning $\wedge$, $\vee$, $\otimes$, and $\oplus$ hold. It is called *interlaced* [17,19] if each one of $\wedge$, $\vee$, $\otimes$, and $\oplus$ is monotonic with respect to both $\leqslant_t$ and $\leqslant_k$.

The following subsets of the truth values in B are used for defining validity of formulae and the associated consequence relation. They provide a natural generalization of the set of the designated values $\{t, \top\}$ of $\langle FOUR \rangle$.

Definition 73 (Arieli and Avron [2,3]).

(a) A *bifilter* of a bilattice $\mathcal{B}$ is a nonempty set $\mathcal{F} \subset B$, $\mathcal{F} \neq B$ such that: $a \wedge b \in \mathcal{F}$ iff $a \in \mathcal{F}$ and $b \in \mathcal{F}$, $a \otimes b \in \mathcal{F}$ iff $a \in \mathcal{F}$ and $b \in \mathcal{F}$.
(b) A bifilter $\mathcal{F}$ is called *prime*, if it satisfies also: $a \vee b \in \mathcal{F}$ iff $a \in \mathcal{F}$ or $b \in \mathcal{F}$, $a \oplus b \in \mathcal{F}$ iff $a \in \mathcal{F}$ or $b \in \mathcal{F}$.

[23] Note that *FOUR* is the minimal nondegenerated bilattice.

Proposition 74 (Arieli and Avron [4]). *A subset $\mathcal{F}$ of an interlaced bilattice $\mathcal{B}$ is a (prime) bifilter iff it is a (prime) filter relative to $\leqslant_t$ and $\top \in \mathcal{F}$ (iff it is a (prime) filter relative to $\leqslant_k$ and $t \in \mathcal{F}$).*

From now on (unless otherwise stated) $\mathcal{F}$ will denote a prime bifilter. Obviously, if $a \in \mathcal{F}$ and $b \geqslant_t a$ or $b \geqslant_k a$, then $b \in \mathcal{F}$. It immediately follows that $t, \top \in \mathcal{F}$ while $f, \bot \notin \mathcal{F}$.

Example 75. Ginsberg's *DEFAULT* (Fig. 6, right) and Belnap's *FOUR* are bilattices that contain exactly one bifilter, $\{\top, t\}$, which is prime in both. *NINE* (Fig. 6, left), on the other hand, contains two bifilters: $\{b \mid b \geqslant_k t\}$ as well as $\{b \mid b \geqslant_k dt\}$; both are prime.

Definition 76 (Arieli and Avron [2,3]). A *logical bilattice* is a pair $(\mathcal{B}, \mathcal{F})$, where $\mathcal{B}$ is a bilattice, and $\mathcal{F}$ is a prime bifilter on $\mathcal{B}$.

Proposition 77 (Arieli and Avron [4]). *Every distributive bilattice can be turned into a logical bilattice.*

In [3] it is shown that if $\mathcal{B}$ is interlaced, then $\mathcal{D}(\mathcal{B}) = \{b \in \mathcal{B} \mid b \geqslant_t \top\}$ is always a bifilter, and even the smallest one.

Example 75—continued. $\langle FOUR \rangle = (FOUR, \{t, \top\})$, $(DEFAULT, \{t, \top\})$, $(NINE, \{b \mid b \geqslant_k t\})$, and $(NINE, \{b \mid b \geqslant_k dt\})$ are all logical bilattices.

The following definition of entailment is a natural generalization of Definition 1 for arbitrary logical bilattices.

Definition 78 (Arieli and Avron [2,6]). Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice $(\mathcal{B}, \mathcal{F})$. Define:

$$a \supset b = \begin{cases} b & \text{if } a \in \mathcal{F}, \\ t & \text{if } a \notin \mathcal{F}, \end{cases}$$

$$a \to b = (a \supset b) \wedge (\neg b \supset \neg a),$$

$$a \leftrightarrow b = (a \to b) \wedge (b \to a).$$

The following semantic notions are also obvious generalizations of the four-valued ones:

Definition 79.

(a) A *valuation* ν in B is a function that assigns a truth value from B to each atomic formula. Any valuation is extended to complex formulae in the standard way.
(b) Given $(\mathcal{B}, \mathcal{F})$, we will say that ν *satisfies* ψ ($\nu \models \psi$), iff $\nu(\psi) \in \mathcal{F}$.
(c) A valuation that satisfies every formula in a given set of formulae, Γ, is said to be a *model* of Γ. Given $(\mathcal{B}, \mathcal{F})$, the set of the models of Γ will be denoted $mod(\Gamma)$.

6.1.3. Types of truth values and valuations

We assign to every element of a bilattice $\mathcal{B}$ and to every valuation in $\mathcal{B}$ a specific type. This typing of the space of valuations on $\mathcal{B}$ will have a great significance in what follows.

Definition 80. Let $(\mathcal{B}_1, \mathcal{F}_1)$ and $(\mathcal{B}_2, \mathcal{F}_2)$ be two logical bilattices. Suppose that b_i is some element of B_i and that ν_i is a valuation on B_i for $i = 1, 2$.

(a) b_1 and b_2 are *of the same type* if: (i) $b_1 \in \mathcal{F}_1$ iff $b_2 \in \mathcal{F}_2$, and (ii) $\neg b_1 \in \mathcal{F}_1$ iff $\neg b_2 \in \mathcal{F}_2$.

(b) ν_1 and ν_2 are of the same type if for every atomic p, $\nu_1(p)$ and $\nu_2(p)$ are of the same type.

Note that the types depend on the identity of the bifilter, so two valuations might not be of the same type even in case they are identical and the underlying bilattice is the same. Consider, e.g., a valuation ν on *NINE* s.t. $\nu(p) = ot$ for some atom p. Then ν for $\mathcal{F} = \{b \mid b \geqslant_k t\}$ (is not of the same type as the same ν where the bifilter is $\mathcal{F} = \{b \mid b \geqslant_k dt\}$).

Proposition 81. *Let $(\mathcal{B}_1, \mathcal{F}_1)$ and $(\mathcal{B}_2, \mathcal{F}_2)$ be two logical bilattices and suppose that ν_1, ν_2 are two valuations on B_1, B_2 (respectively), which are of the same type. Then for every formula ψ, $\nu_1(\psi)$ and $\nu_2(\psi)$ are of the same type.*

Proof. By an induction on the structure of ψ (the fact that $\mathcal{F}$ is *prime* is crucial here!). □

Corollary 82. *Let ν_1, ν_2 be two valuations of the same type on a logical bilattice $(\mathcal{B}, \mathcal{F})$. Then for every formula ψ, $\nu_1(\psi)$ and $\nu_2(\psi)$ are of the same type.*

Theorem 83. *A model of Γ in $\langle FOUR \rangle$ is also a model of Γ in every logical bilattice $(\mathcal{B}, \mathcal{F})$.*

Proof. Let $M^{(4)}$ be a model of Γ in $\langle FOUR \rangle$, and suppose that $M^{(\mathcal{B},\mathcal{F})}$ is the same valuation defined on some logical bilattice $(\mathcal{B}, \mathcal{F})$. Since every bifilter $\mathcal{F}$ contains $t, \top$ and does not contain $f, \bot$, then $M^{(4)}$ and $M^{(\mathcal{B},\mathcal{F})}$ are of the same type. Hence, by Proposition 81, $M^{(4)}(\psi)$ and $M^{(\mathcal{B},\mathcal{F})}(\psi)$ are of the same type for every $\psi \in \Gamma$. In particular $M^{(\mathcal{B},\mathcal{F})}$ must be a model of Γ in $(\mathcal{B}, \mathcal{F})$ as well. [24] □

Lemma 84. *Let ν be a valuation in a logical bilattice $(\mathcal{B}, \mathcal{F})$. Then $\nu(\psi \leftrightarrow \phi) \in \mathcal{F}$ iff $\nu(\psi)$ and $\nu(\phi)$ are of the same type.*

Notation 85. Given a logical bilattice $(\mathcal{B}, \mathcal{F})$. Denote the four possible types of its elements by $\mathcal{T}_t^{\mathcal{B},\mathcal{F}}$, $\mathcal{T}_f^{\mathcal{B},\mathcal{F}}$, $\mathcal{T}_\top^{\mathcal{B},\mathcal{F}}$ and $\mathcal{T}_\bot^{\mathcal{B},\mathcal{F}}$, i.e.:

$$\mathcal{T}_t^{\mathcal{B},\mathcal{F}} = \{b \in B \mid b \in \mathcal{F}, \neg b \notin \mathcal{F}\}, \qquad \mathcal{T}_f^{\mathcal{B},\mathcal{F}} = \{b \in B \mid b \notin \mathcal{F}, \neg b \in \mathcal{F}\},$$

$$\mathcal{T}_\top^{\mathcal{B},\mathcal{F}} = \{b \in B \mid b \in \mathcal{F}, \neg b \in \mathcal{F}\}, \qquad \mathcal{T}_\bot^{\mathcal{B},\mathcal{F}} = \{b \in B \mid b \notin \mathcal{F}, \neg b \notin \mathcal{F}\}.$$

We shall usually omit the superscripts, and just write $\mathcal{T}_t, \mathcal{T}_f, \mathcal{T}_\top, \mathcal{T}_\bot$.

[24] In the specific case where $(\mathcal{B}, \mathcal{F})$ is interlaced, the last theorem immediately follows from Proposition 3.1 of [19], since it is shown there that *FOUR* is actually a sub-bilattice of every interlaced bilattice $\mathcal{B}$, so in this case $M^{(4)}(\psi)$ and $M^{(\mathcal{B},\mathcal{F})}(\psi)$ are not only of the same type, but are actually identical.

Definition 86. Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice. Define a function $h : \mathcal{B} \to FOUR$ as follows:

$$h(b) = \begin{cases} \top & \text{if } b \in \mathcal{T}_{\top}, \\ t & \text{if } b \in \mathcal{T}_t, \\ f & \text{if } b \in \mathcal{T}_f, \\ \bot & \text{if } b \in \mathcal{T}_{\bot}. \end{cases}$$

Proposition 87.
(a) *h is a homomorphism onto FOUR.*
(b) *M is a model in $(\mathcal{B}, \mathcal{F})$ of a set Γ of formulae iff the composition $h \circ M$ is a model of Γ in $\langle FOUR \rangle$.*

Proof. Left to the reader (see also [3, Theorems 2.17, 3.17]). □

6.2. Extending the four-valued logics to bilattice-based logics

In this section we introduce obvious generalizations of the logics of Section 4 to arbitrary logical bilattices. The main conclusion is that as in the case of the generalization of the classical two-valued logic to arbitrary Boolean algebra, no new logic is obtained.

6.2.1. The logics $\models^{\mathcal{B},\mathcal{F}}$ and $\models_k^{\mathcal{B},\mathcal{F}}$

Definition 88. Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice, and suppose that Γ, Δ are two sets of formulae.
(a) $\Gamma \models^{\mathcal{B},\mathcal{F}} \Delta$ if every model of Γ is a model of some formula in Δ.
(b) $\Gamma \models_k^{\mathcal{B},\mathcal{F}} \Delta$ if every k-minimal model of Γ is a model of some formula in Δ.

Note that $\models^4 = \models^{\langle FOUR \rangle}$ and $\models_k^4 = \models_k^{\langle FOUR \rangle}$. Therefore, in the particular case of $\langle FOUR \rangle$ we shall continue to use the abbreviations $\models^4$ and $\models_k^4$.

Theorem 89 (Arieli and Avron [3]). *$\Gamma \models^{\mathcal{B},\mathcal{F}} \Delta$ iff $\Gamma \models^4 \Delta$.*

Proof. One direction follows from Theorem 83. For the other, suppose that $\Gamma \not\models^{\mathcal{B},\mathcal{F}} \Delta$. Then there is a valuation M that is a model of Γ in $(\mathcal{B}, \mathcal{F})$ but $M(\delta) \notin \mathcal{F}$ for every $\delta \in \Delta$. Let $M' = h \circ M$. From Propositions 81 and 87 it follows that M' is a four-valued model of Γ s.t. $M'(\delta) \notin \{t, \top\}$ for every $\delta \in \Delta$. Therefore $\Gamma \not\models^4 \Delta$. □

Theorem 90. *Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice s.t.* $\inf_k \mathcal{F} \in \mathcal{F}$.[25] *Then*

$$\Gamma \models_k^{\mathcal{B},\mathcal{F}} \Delta \quad \textit{iff} \quad \Gamma \models_k^4 \Delta.$$

Proof. First, we prove some lemmas:

[25] This is clearly the case whenever $\mathcal{B}$ is finite. It can be shown also that if $\mathcal{B}$ is interlaced then $\inf_k \mathcal{F} \in \mathcal{F}$ iff $\inf_t \mathcal{F} \in \mathcal{F}$. Moreover, in this case $\inf_t \mathcal{F} = \inf_k \mathcal{F} \wedge \top$ while $\inf_k \mathcal{F} = \inf_t \mathcal{F} \otimes t$.

Lemma 90-A. *Suppose that $\emptyset \neq X \subseteq B$ and let $\neg X = \{\neg x \mid x \in X\}$. Then* $\inf_k \neg X = \neg \inf_k X$.

Proof.

$$x \in \neg X \Rightarrow \neg x \in X \Rightarrow \neg x \geqslant_k \inf_k X \Rightarrow x \geqslant_k \neg \inf_k X.$$

Thus: $\inf_k \neg X \geqslant_k \neg \inf_k X$. On the other hand, replacing X with $\neg X$ yields that $\inf_k \neg\neg X \geqslant_k \neg \inf_k \neg X$, i.e., $\inf_k X \geqslant_k \neg \inf_k \neg X$. Therefore $\neg \inf_k X \geqslant_k \inf_k \neg X$, and so $\neg \inf_k X = \inf_k \neg X$. □

Lemma 90-B. *For every $x \in \{t, f, \top, \bot\}$* $\inf_k \mathcal{T}_x \in \mathcal{T}_x$. *Moreover:* $\inf_k \mathcal{T}_\bot = \bot$, $\inf_k \mathcal{T}_t = \inf_k \mathcal{F} = \min_k \mathcal{F}$, $\inf_k \mathcal{T}_f = \neg \inf_k \mathcal{F} = \neg \min_k \mathcal{F}$, *and* $\inf_k \mathcal{T}_\top = \min_k \mathcal{F} \oplus \neg \min_k \mathcal{F}$.

Proof.

(i) The case $x = \bot$ is trivial, since $\bot \in \mathcal{T}_\bot$.

(ii) The case $x = t$: Let $a = \inf_k \mathcal{F}$. Since $\mathcal{T}_t \subseteq \mathcal{F}$, $\inf_k \mathcal{T}_t \geqslant_k a$. Now, $a \in \mathcal{F}$ (given). On the other hand, $t \in \mathcal{F}$. Hence $t \geqslant_k a$, and so $f \geqslant_k \neg a$. It follows that $\neg a \notin \mathcal{F}$ (otherwise $f \in \mathcal{F}$—a contradiction). Therefore $a \in \mathcal{T}_t$, and so $a = \min_k \mathcal{T}_t$.

(iii) The case $x = f$. Let again $a = \inf_k \mathcal{F}$. Since $\neg\mathcal{T}_f \subseteq \mathcal{F}$, by Lemma 90-A $\neg \inf_k \mathcal{T}_f \geqslant_k a$. Hence $\inf_k \mathcal{T}_f \geqslant_k \neg a$. On the other hand we just have shown that $\neg a \notin \mathcal{F}$, while $\neg\neg a = a \in \mathcal{F}$. It follows that $\neg a \in \mathcal{T}_f$, and so $\neg a = \min_k \mathcal{T}_f$.

(iv) The case $x = \top$: Since $\mathcal{T}_\top \subseteq \mathcal{F}$ and $\neg\mathcal{T}_\top \subseteq \mathcal{F}$, $\inf_k \mathcal{T}_\top \geqslant_k \inf_k \mathcal{F} \in \mathcal{F}$ and $\neg \inf_k \mathcal{T}_\top \geqslant_k \inf_k \mathcal{F} \in \mathcal{F}$. Hence $\inf \mathcal{T}_\top \in \mathcal{F}$ and $\inf \neg\mathcal{T}_\top \in \mathcal{F}$. By Lemma 90-A, then, $\inf \mathcal{T}_\top \in \mathcal{T}_\top$. For the other part note that $\min_k \mathcal{F} \oplus \neg \min_k \mathcal{F} \in \mathcal{F}$ and also

$$\neg(\min_k \mathcal{F} \oplus \neg \min_k \mathcal{F}) = \neg \min_k \mathcal{F} \oplus \min_k \mathcal{F} \in \mathcal{F}.$$

Thus $\min_k \mathcal{F} \oplus \neg \min_k \mathcal{F} \in \mathcal{T}_\top$, and so

$$\inf_k \mathcal{T}_\top \leqslant_k \min_k \mathcal{F} \oplus \neg \min_k \mathcal{F}.$$

On the other hand, $\forall b \in \mathcal{T}_\top$ $b \geqslant_k \min_k \mathcal{F}$ (by (ii)) and $\neg b \geqslant_k \neg \min_k \mathcal{F}$ (by (iii)). Hence

$$\forall b \in \mathcal{T}_\top \ b \geqslant_k \min_k \mathcal{F} \oplus \neg \min_k \mathcal{F}.$$

In particular,

$$\inf_k \mathcal{T}_\top \geqslant_k \min_k \mathcal{F} \oplus \neg \min_k \mathcal{F},$$

therefore

$$\inf_k \mathcal{T}_\top = \min_k \mathcal{F} \oplus \neg \min_k \mathcal{F}. \qquad \square$$

Lemma 90-C. *Suppose that M is a k-minimal model of Γ in $\langle \mathcal{B}, \mathcal{F} \rangle$, and let $h : \mathcal{B} \to FOUR$ be the homomorphism defined in Definition* 86. *Then $h \circ M$ is a k-minimal model of Γ in $\langle FOUR \rangle$.*

Proof. Suppose not. Then there is another model N of Γ, which is k-smaller than $h \circ M$ in $\langle FOUR \rangle$. By Theorem 83, N is also a model of Γ in $\langle \mathcal{B}, \mathcal{F} \rangle$. Define a valuation N' by

$N'(p) = \inf_k \mathcal{T}_{N(p)}$ (p atomic). By Corollary 82, N' is also a model of Γ in $(\mathcal{B}, \mathcal{F})$. Note that N and N' are of the same type, and so are M and $h \circ M$. Let p be an atomic formula.

Case A: If $N(p)$ and $(h \circ M)(p)$ are of the same type, then so are $N'(p)$ and $M(p)$. By the construction of N', $N'(p) \leqslant_k M(p)$.

Case B: If $N(p)$ and $(h \circ M)(p)$ are not of the same type, then since $N(p) \leqslant_k (h \circ M)(p)$, there are three possible cases: (i) $N(p) = \bot$ and $(h \circ M)(p) \in \{t, f, \top\}$, or (ii) $N(p) = t$ and $(h \circ M)(p) = \top$, or (iii) $N(p) = f$, and $(h \circ M)(p) = \top$. Let's consider each case:

Case B(i): In this case $N'(p) = \bot$ as well, while $M(p) \notin \mathcal{T}_\bot$, thus $M(p) \neq \bot$ and so $N'(p) <_k M(p)$.

Case B(ii): Since, by Lemma 90-B, $N'(p) = \min_k \mathcal{F}$ and $M(p) \in \mathcal{F}$, so $N'(p) \leqslant_k M(p)$. But $N'(p) \neq M(p)$ since $\neg M(p) \in \mathcal{F}$ while $\neg N'(p) \notin \mathcal{F}$. Therefore $N'(p) <_k M(p)$.

Case B(iii): Again, by Lemma 90-B, in this case $N'(p) = \min_k \neg\mathcal{F}$. But $\neg M(p) \in \mathcal{F}$, so $N'(p) <_k M(p)$ here as well.

Now, since N is a model of Γ in $\langle FOUR \rangle$, which is strictly k-smaller than $h \circ M$, there is at least one atom p_0 that falls under case B above. For this p_0, $N'(p_0) <_k M(p_0)$ while for any other atom p, $N'(p) \leqslant_k M(p)$. Hence N' is a model of Γ in $(\mathcal{B}, \mathcal{F})$ which is k-smaller than M—a contradiction. $\square$

The "if" direction of Theorem 90 now easily follows from Lemma 90-C: Suppose that for some logical bilattice $(\mathcal{B}, \mathcal{F})$, $\Gamma \not\models_k^{\mathcal{B},\mathcal{F}} \Delta$. Let M be a k-minimal model of Γ s.t. $M(\delta) \notin \mathcal{F}$ for every $\delta \in \Delta$. By Lemma 90-C, $h \circ M$ is a k-minimal model of Γ in $\langle FOUR \rangle$ of the same type as M. Therefore $(h \circ M)(\delta) \notin \{t, \top\}$ for every $\delta \in \Delta$, and so $\Gamma \not\models_k^4 \Delta$.

The other direction: Suppose that $\Gamma \not\models_k^4 \Delta$. Then there is a k-minimal model M of Γ in $\langle FOUR \rangle$ s.t. $M(\delta) \notin \{t, \top\}$ for every $\delta \in \Delta$. Define a valuation M' on B as follows: $M'(p) = \inf_k \mathcal{T}_{M(p)}$ (p atomic). By Corollary 82 and Lemma 90-B, $h \circ M' = M$. Hence (by Proposition 87) M' is a model of Γ, and $M'(\delta) \notin \mathcal{F}$ for every $\delta \in \Delta$. Moreover, M' is a k-minimal model of Γ, and so $\Gamma \not\models_k^{\mathcal{B},\mathcal{F}} \Delta$. Indeed, if N is another model of Γ s.t. $N <_k M'$, then $h \circ N \leqslant_k h \circ M' = M$. Also, there is p s.t. $N(p) <_k M'(p)$ and so $N(p) \notin \mathcal{T}_{M(p)}$. Hence $h(N(p)) \neq M(p)$, and so actually $h \circ N <_k M$. Since $h \circ N$ is a model of Γ in $\langle FOUR \rangle$ (because N is a model of Γ), M is not k-minimal—a contradiction. $\square$

6.2.2. *The logics of* $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$

Like $\models^4$ and $\models_k^4$, the logics $\models_{\mathcal{I}_1}^4$ and $\models_{\mathcal{I}_2}^4$ have also natural generalizations to bilattices.

Definition 91 (Arieli and Avron [2,3]). Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice, and b—an arbitrary element in B (the carrier of $\mathcal{B}$). A subset $\mathcal{I}$ of B is called an *inconsistency set* in $(\mathcal{B}, \mathcal{F})$, if it has the following properties: (a) $b \in \mathcal{I}$ iff $\neg b \in \mathcal{I}$, (b) $\mathcal{F} \cap \mathcal{I} = \mathcal{T}_\top$.

Lemma 92. *Suppose that $\mathcal{I}$ is an inconsistency set in $(\mathcal{B}, \mathcal{F})$. Then:*

(a) $\mathcal{T}_\top \subseteq \mathcal{I} \subseteq \mathcal{T}_\top \cup \mathcal{T}_\bot$.

(b) $\top \in \mathcal{I}$ *and* $t, f \notin \mathcal{I}$.

Proof. Immediate from Definition 91. $\square$

Example 93. $\mathcal{T}_\top$ and $\mathcal{T}_\top \cup \mathcal{T}_\bot$ are respectively the minimal and maximal inconsistency set in every logical bilattice. In $\langle FOUR \rangle$ the former set was denoted $\mathcal{I}_1$ (see Notation 36(a)) and the latter $\mathcal{I}_2$ (Notation 43(a)). These are the only inconsistency sets of $\langle FOUR \rangle$.

Notation 94. $I(\nu, \mathcal{I}) = \{p \mid p$ is atomic and $\nu(p) \in \mathcal{I}\}$. Intuitively, $I(\nu, \mathcal{I})$ is the set of the inconsistent assignments of a valuation ν with respect to an inconsistency set $\mathcal{I}$ (compare to Notations 36(b) and 43(b)).

The next two definitions are natural extensions of Definitions 37, 38, 44, and 45, to general logical bilattices:

Definition 95. Let Γ be a set of formulae, and M, N models of Γ.
(a) M is *more consistent* than N with respect to $\mathcal{I}$ ($M >_{\mathcal{I}} N$) if $I(M, \mathcal{I}) \subset I(N, \mathcal{I})$.
(b) M is a *most consistent* model of Γ with respect to $\mathcal{I}$ ($\mathcal{I}$-mcm, in short), if there is no other model of Γ which is more consistent than M. The set of all the $\mathcal{I}$-mcms of Γ is denoted $mcm(\Gamma, \mathcal{I})$.

Definition 96. $\Gamma \models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \Delta$ if every $\mathcal{I}$-mcm of Γ is a model of some formula of Δ. [26]

Note. Several relations similar to $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$ are considered in the literature. We have already mentioned, e.g., Priest's LPm [37,38]. In our terms, Priest considers the inconsistency set $\mathcal{I} = \mathcal{T}_\top$. In the three-valued case this is the only inconsistency set, and it consists only of $\top$. In the general (multi-valued) case there are many others.

Kifer and Lozinskii [28] also propose a similar relation (denoted there $\models\!\approx_\Delta$, where Δ stands for the values that are considered as representing inconsistent knowledge). This relation is considered in the framework of annotated logics [44,45]. See [3,5] for a discussion on the similarities and the differences between $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$ and $\models\!\approx_\Delta$.

We now show that again everything that one can infer by using $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$ may be inferred in $\langle FOUR \rangle$ together with either $\mathcal{I}_1$ or $\mathcal{I}_2$ as the inconsistency set:

Theorem 97. *For every logical bilattice $(\mathcal{B}, \mathcal{F})$ and an inconsistency set $\mathcal{I}$ there is a consistency set $\mathcal{J}$ in $\langle FOUR \rangle$ s.t. $\Gamma \models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \Delta$ iff $\Gamma \models_{\mathcal{J}}^{4} \Delta$.*

Proof. In the course of this proof we shall use the following convention: whenever ν is a function from the atomic formulae to $\{t, f, \top, \bot\}$, ν^4 denotes its expansion to complex

[26] There is a slight (but significant) change between the relation $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$ defined here and the relation $\models_{con(\mathcal{I})}^{\mathcal{B},\mathcal{F}}$ (abbreviation: $\models_{con}$), considered in [2,3]. The difference is that instead of considering the inconsistent assignments of ν on *every* atomic formulae as we do here, in [2,3] only the assignments on the atomic formulae that appear in the language of the set of assumptions, Γ, are considered. In other words, the relevant set of assignments there is $I(\nu, \Gamma, \mathcal{I}) = \{p \in \mathcal{A}(\Gamma) \mid \nu(p) \in \mathcal{I}\}$ (cf. Definition 94). Our new definition has certain advantages over the original one. Thus, Proposition 49(b) fails for $\models_{con(\{\top,\bot\})}^{4}$ and Proposition 53(a) fails for both $\models_{con(\{\top\})}^{4}$ and $\models_{con(\{\top,\bot\})}^{4}$.

formulae in *FOUR*, and ν^B denotes the corresponding valuation on B.[27]

Let $(\mathcal{B},\mathcal{F})$ be a logical bilattice, and let $h:(\mathcal{B},\mathcal{F})\to FOUR$ be the homomorphism onto *FOUR*, defined in 86.

Lemma 97-A. $\nu^4 = h\circ\nu^B$.

Proof. We show by induction on the structure of a formula ψ that $\nu^4(\psi)=h\circ\nu^B(\psi)$. For atomic formulae this follows from the fact that on $\{t,f,\top,\bot\}$, h is the identity function. For more complicated formulae we use the fact that h is an homomorphism. □

Lemma 97-B. *ν^B is a model of Γ in $(\mathcal{B},\mathcal{F})$ iff ν^4 is a model of Γ in $\langle FOUR\rangle$.*

Proof. Immediate from Lemma 97-A and the fact that $\nu^B(\psi)\in\mathcal{F}$ iff $\nu^4(\psi)=h\circ\nu^B(\psi)\in\{t,\top\}$. □

The rest of the proof is divided into two cases that correspond to the two possibilities of defining an inconsistency set in $\langle FOUR\rangle$:

- *Case* A: $\mathcal{T}_\bot\subseteq\mathcal{I}$.
- *Case* B: $\mathcal{T}_\bot\setminus\mathcal{I}\neq\emptyset$.

For each case define a corresponding inconsistency set in $\langle FOUR\rangle$. In Case A let $\mathcal{J}=\mathcal{I}_2=\{\top,\bot\}$, and in Case B let $\mathcal{J}=\mathcal{I}_1=\{\top\}$.

Lemma 97-C. *In Case A, M is an $\mathcal{I}$-mcm of Γ in $(\mathcal{B},\mathcal{F})$ iff $h\circ M$ is an $\mathcal{I}_2$-mcm of Γ in $\langle FOUR\rangle$.*

Proof. By Lemma 92(a) in Case A, $\mathcal{I}=\mathcal{T}_\top\cup\mathcal{T}_\bot$ and so $b\in\mathcal{I}$ iff $h(b)\in\mathcal{I}_2$. Therefore, for every two valuations M_1 and M_2 in B,

$$\begin{aligned}
&M_1 >^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} M_2\\
&\quad\Leftrightarrow \{p\mid M_1(p)\in\mathcal{I}\}\subset\{p\mid M_2(p)\in\mathcal{I}\}\\
&\quad\Leftrightarrow \{p\mid (h\circ M_1)(p)\in\mathcal{I}_2\}\subset\{p\mid (h\circ M_2)(p)\in\mathcal{I}_2\}\\
&\quad\Leftrightarrow h\circ M_1 >^4_{\mathcal{I}_2} h\circ M_2.
\end{aligned}$$

It immediately follows that if $h\circ M$ is an $\mathcal{I}_2$-mcm of Γ in $\langle FOUR\rangle$ then M is an $\mathcal{I}$-mcm of Γ in $(\mathcal{B},\mathcal{F})$. For the converse, assume that $h\circ M$ is not an $\mathcal{I}_2$-mcm of Γ in $\langle FOUR\rangle$. Let ν be an assignment in *FOUR* s.t. ν^4 is a model of Γ in $\langle FOUR\rangle$ and $\nu^4 >^4_{\mathcal{I}_2} h\circ M$. By Lemma 97-A, $\nu^4=h\circ\nu^B$. Thus $h\circ\nu^B >^4_{\mathcal{I}_2} h\circ M$, and so $\nu^B >^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} M$. Moreover, by 97-B ν^B is a model of Γ in B. Hence M is not an $\mathcal{I}$-mcm of Γ in $(\mathcal{B},\mathcal{F})$. □

Corollary 97-D. *In Case A, $\Gamma\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}\Delta$ iff $\Gamma\models^4_{\mathcal{I}_2}\Delta$.*

[27] Note that although $\nu^4(p)=\nu^B(p)$ when p is atomic, this might not be the case in general, unless $\mathcal{B}$ is interlaced.

Proof. Suppose that $\Gamma \not\models^4_{\mathcal{I}_2} \Delta$. Then there is an assignment ν in *FOUR* s.t. ν^4 is an $\mathcal{I}_2$-mcm of Γ in $\langle FOUR \rangle$ that is not a model of any $\delta \in \Delta$. By Lemma 97-A, $\nu^4 = h \circ \nu^B$ and by Lemmas 97-B and 97-C, ν^B is an $\mathcal{I}$-mcm of Γ in $(\mathcal{B}, \mathcal{F})$ s.t. $\nu^B(\delta) \notin \mathcal{F}$ for every $\delta \in \Delta$. Hence $\Gamma \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \Delta$. For the converse, assume that M is an $\mathcal{I}$-mcm of Γ in $(\mathcal{B}, \mathcal{F})$ which is not a model of any formula in Δ. Then, by Lemmas 97-B and 97-C, $h \circ M$ is an $\mathcal{I}_2$-mcm of Γ in $\langle FOUR \rangle$, and $h \circ M(\delta) \in \{f, \bot\}$ for every $\delta \in \Delta$. Therefore $\Gamma \not\models^4_{\mathcal{I}_2} \Delta$. $\square$

Let us turn now to Case B, in which there is an $\alpha \in \mathcal{T}_\bot \setminus \mathcal{I}$. Suppose that M is a model of Γ in $(\mathcal{B}, \mathcal{F})$. Consider the valuation M_α, defined for every atomic formula p as follows:

$$M_\alpha(p) = \begin{cases} \alpha & \text{if } M(p) \in \mathcal{T}_\bot \cap \mathcal{I}, \\ M(p) & \text{otherwise.} \end{cases}$$

Since obviously $h \circ M = h \circ M_\alpha$, then in particular:

$$I(h \circ M, \mathcal{I}_1) = I(h \circ M_\alpha, \mathcal{I}_1). \tag{1}$$

Lemma 97-E. *For every $\psi \in \Gamma$, $M(\psi) \in \mathcal{F}$ iff $M_\alpha(\psi) \in \mathcal{F}$.*

Proof. Immediate from Proposition 81. $\square$

Corollary 97-F. *If M is an $\mathcal{I}$-mcm of Γ then $M = M_\alpha$.*

Proof. In other words, we have to show that there is no atom p such that $M(p) \in \mathcal{T}_\bot \cap \mathcal{I}$. Assume otherwise. Then $M_\alpha >^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} M$. Since by Lemma 97-E M_α is also a model of Γ, this implies that M is not an $\mathcal{I}$-mcm of Γ. $\square$

Lemma 97-G. *If $M = M_\alpha$ then*:

$$I(M, \mathcal{I}) = I(h \circ M, \mathcal{I}_1). \tag{2}$$

Proof. If $M = M_\alpha$, there is no atom p such that $M(p) \in \mathcal{T}_\bot \cap \mathcal{I}$. Hence, by Lemma 92,

$$M(p) \in \mathcal{I} \Leftrightarrow M(p) \in \mathcal{T}_\top \Leftrightarrow (h \circ M)(p) \in \mathcal{I}_1,$$

and so $I(M, \mathcal{I}) = I(h \circ M, \mathcal{I}_1)$. $\square$

Lemma 97-H. *In Case B, If M is an $\mathcal{I}$-mcm of Γ in $(\mathcal{B}, \mathcal{F})$ then $h \circ M$ is an $\mathcal{I}_1$-mcm of Γ in $\langle FOUR \rangle$.*

Proof. Suppose that M is an $\mathcal{I}$-mcm of Γ in $(\mathcal{B}, \mathcal{F})$. Assume that ν is a valuation in *FOUR* s.t. ν^4 is a model of Γ in $\langle FOUR \rangle$ and $\nu^4 >^4_{\mathcal{I}_1} h \circ M$. By Lemma 97-B, ν^B is a model of Γ in $(\mathcal{B}, \mathcal{F})$. Now, since obviously $(\nu^B_\alpha)_\alpha = \nu^B_\alpha$, we have:

$$\begin{aligned}
I(\nu^B_\alpha, \mathcal{I}) &= I(h \circ \nu^B_\alpha, \mathcal{I}_1) && \text{by Lemma 97-G} \\
&= I(h \circ \nu^B, \mathcal{I}_1) && \text{by Eq. (1)} \\
&= I(\nu^4, \mathcal{I}_1) && \text{by Lemma 97-A} \\
&\subset I(h \circ M, \mathcal{I}_1) && \text{by the assumption} \\
&= I(M, \mathcal{I}) && \text{by Corollary 97-F and Lemma 97-G.}
\end{aligned}$$

Hence $\nu_\alpha^B >_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} M$, and so M is not an $\mathcal{I}$-mcm of Γ in $(\mathcal{B},\mathcal{F})$, a contradiction. □

Corollary 97-I. *In Case B, $\Gamma \models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \Delta$ iff $\Gamma \models_{\mathcal{I}_1}^{4} \Delta$.*

Proof. If $\Gamma \not\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \Delta$ then there exists an $\mathcal{I}$-mcm M of Γ s.t. $M(\delta) \notin \mathcal{F}$ for every $\delta \in \Delta$. By Lemma 97-H, $h \circ M$ is an $\mathcal{I}_1$-mcm of Γ in $\langle FOUR \rangle$ and $(h \circ M)(\delta) \notin \{t, \top\}$ for every $\delta \in \Delta$. Therefore $\Gamma \not\models_{\mathcal{I}_1}^{4} \Delta$. For the converse, assume that $\Gamma \not\models_{\mathcal{I}_1}^{4} \Delta$. Suppose that ν is an assignment in *FOUR* s.t. ν^4 is an $\mathcal{I}_1$-mcm of Γ in $\langle FOUR \rangle$ and $\nu^4(\delta) \notin \{t, \top\}$ for every $\delta \in \Delta$. By Lemma 97-A $\nu^4 = h \circ \nu^B$. By Lemma 97-B and its proof, ν^B is a model of Γ in $(\mathcal{B},\mathcal{F})$ s.t. $\nu^B(\delta) \notin \mathcal{F}$ for every $\delta \in \Delta$. By Lemma 97-E the same is true for ν_α^B. It is left to show, then, that ν_α^B is an $\mathcal{I}$-mcm of Γ in $(\mathcal{B},\mathcal{F})$. Suppose otherwise. Then there is an $\mathcal{I}$-mcm M of Γ, s.t. $M >_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \nu_\alpha^B$. Since $(\nu_\alpha^B)_\alpha = \nu_\alpha^B$ and (by Corollary 97-F) $M = M_\alpha$, we have:

$$\begin{aligned} I(h \circ M, \mathcal{I}_1) &= I(M, \mathcal{I}) && \text{by Lemma 97-G} \\ &\subset I(\nu_\alpha^B, \mathcal{I}) && \text{by the assumption} \\ &= I(h \circ \nu_\alpha^B, \mathcal{I}_1) && \text{by Lemma 97-G} \\ &= I(h \circ \nu^B, \mathcal{I}_1) && \text{by Eq. (1).} \end{aligned}$$

Therefore $(h \circ M) >_{\mathcal{I}_1}^{4} (h \circ \nu^B) = \nu^4$. Since $h \circ M$ is a model of Γ (because M is), this is a contradiction. This concludes the proof of Corollary 97-I and Theorem 97. □

The following conclusion easily follows from the proof of Theorem 97:

Corollary 98. *Let $(\mathcal{B},\mathcal{F})$ and $\mathcal{I}$ be some logical bilattice and an inconsistency set in it. Then:*

(a) *If $\mathcal{T}_\perp^{\mathcal{B},\mathcal{F}} \not\subset \mathcal{I}$ then $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \equiv \models_{\mathcal{I}_1}^{4}$.*
(b) *If $\mathcal{T}_\perp^{\mathcal{B},\mathcal{F}} \subset \mathcal{I}$ then $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \equiv \models_{\mathcal{I}_2}^{4}$.*

Note. The relation $\models_{con}^{\mathcal{B},\mathcal{F}}$ of [2,3] (see footnote after Definition 96) can also be characterized by $\langle FOUR \rangle$; $\Gamma \models_{con(\mathcal{I})}^{\mathcal{B},\mathcal{F}} \Delta$ iff there is an inconsistency set $\mathcal{J}$ in *FOUR* s.t. $\Gamma \models_{con(\mathcal{J})}^{4} \Delta$. The proof is similar to that of Theorem 97; We omit the details.

7. Summary and conclusion

Bilattices are algebraic structures that have been shown useful in several areas of computer science. The smallest nondegenerated bilattice, $\langle FOUR \rangle$, consists of four elements, and it is usually associated with Belnap four-valued logic. The goal of this work has been to show that the logical role of $\langle FOUR \rangle$ among (logical) bilattices is similar to that the two-valued (classical) lattice has among Boolean algebras. As such, $\langle FOUR \rangle$ provides a useful framework for capturing classical reasoning (in cases its use is appropriate) as well as some standard non-monotonic methods and paraconsistent techniques.

We began this work by providing appropriate interpretations of the classical connectives in terms of $\langle FOUR\rangle$, and adding to them connectives that correspond to the basic bilattice operations. We have examined the expressive power of the various fragments of the resulting language, and showed that (a fragment of) our language is functionally complete for *FOUR*.

With this syntactical tool in our disposal, we turned to considering the use of $\langle FOUR\rangle$ as our main semantical tool. The existence of elements like $\top$ and $\bot$, as well as the idea of ordering data according to degrees of knowledge, suggest that this structure should be particularly suitable for reasoning with uncertainty.

During the discussion on the importance of $\langle FOUR\rangle$ we have considered several inference relations that allow plausible reasoning mechanisms:

- $\models^4$: This is a consequence relation in the standard sense of Tarski and Scott. It was called here "the basic consequence relation". We have shown that this relation is sound and complete with respect to the cut-free Gentzen type system *GBL*, monotonic, compact, and paraconsistent. Its main drawbacks are that it is strictly weaker than classical logic even for consistent theories, and that it *always* invalidates some intuitively justified inference rules, like the Disjunctive Syllogism.
- $\models^4_k$: This relation considers only the k-minimal models for making inferences. The idea behind its definition is that we should not assume anything that is not really known. We have shown that as long as we are interested in inferring formulae that do not include our nonmonotonic $\supset$, $\models^4_k$ is equivalent to $\models^4$. Therefore, in such cases we can indeed limit ourselves to the k-minimal models without any loss of generality, and so reduce the amount of models required for making inferences.
- $\models^4_{\mathcal{I}_1}$: The idea here is to give precedence to the models that minimize the amount of inconsistent belief. This approach reflects the intuition that contradictory data corresponds to inadequate information about the real world, and therefore should be minimized. This relation is a plausibility logic, paraconsistent, nonmonotonic, and preferential. In the monotonic classical fragment of the language this relation can be used for efficiently checking which element of a given set of formulae classically follows from a given consistent theory.
- $\models^4_{\mathcal{I}_2}$: This relation prefers definite knowledge to an uncertain one. Thus, the approach taken here is to prefer classical inferences whenever possible. Indeed, for consistent theories in the classical fragment this inference relation is identical to the classical one. In general, however, $\models^4_{\mathcal{I}_2}$ is different than classical logic, since it is paraconsistent and nonmonotonic.

All these consequence relations can be generalized in a natural way to arbitrary logical bilattices. A natural question that arises at this point is whether by this generalization one obtains something that is not already available in $\langle FOUR\rangle$. Alternatively, one may wonder whether only three values suffice. Our answer to both questions is basically negative. We have shown that everything that can be done using three values is also possible in the four-valued setting, and even more efficiently, while the converse is not true. On the other hand, we gave a sequence of theorems that show that it is possible to characterize in $\langle FOUR\rangle$ any bilattice-valued version of the consequence relations mentioned above. The outcome is, as the title of this paper implies, a strong evidence for the fundamental logical role and usefulness of the four-valued framework.

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