



# New Ideas for Proving Convergence of Decomposition Methods

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**Abstract**—In this paper, we give new formulae which calculate easily the Adomian's polynomials used in decomposition methods. Then, the proof of convergence of the Adomian's technique becomes almost obvious by using a weak hypothesis on the nonlinear operator of the functional equation.

**Keywords**—Decomposition methods, Adomian's polynomials, Nonlinear functional equations, Series solutions.

## 1. INTRODUCTION

In the eighties, G. Adomian [1] proposed a new and fruitful method for solving exactly nonlinear functional equations of various kinds (algebraic, differential, partial differential, integral, ...). The technique uses a decomposition of the nonlinear operator as a series of functions. Each term of this series is a generalized polynomial called Adomian's polynomial. The Adomian technique is very simple in its principles. The difficulties consist in calculating the Adomian's polynomials and in proving the convergence of the introduced series. Some attempts to prove convergence have been given in [2–5]. None of these proofs tried to demonstrate directly the convergence of the series solution. In [5], a proof of convergence of the series solution is given but it uses a very strong hypothesis on the nonlinear operator. In the following, we shall prove convergence of the series solution owing to a new formula giving the Adomian's polynomials. This formula will express the series solution as a function of the first term of the series (this term is always known).

Let us first recall the basic ideas of the decomposition method of Adomian [1,3,4]. Consider the general functional equation

$$u = N(u) + f, \quad (1.1)$$

where  $N$  is a nonlinear operator from a Hilbert space  $H$  into  $H$ , and where  $f$  is a known function. We are looking for a solution  $u$  of (1.1) belonging to  $H$ . We shall suppose that (1.1) admits a unique solution. If (1.1) does not possess a unique solution, the decomposition method will give a solution among many (possible) other solutions. The decomposition method consists in looking for a solution having the series form

$$u = \sum_{i=0}^{\infty} u_i. \quad (1.2)$$

The nonlinear operator  $N$  is decomposed as

$$N(u) = \sum_{n=0}^{\infty} A_n, \tag{1.3}$$

where the  $A_n$  are functions called the Adomian's polynomials. In the first approach given by Adomian [1], the  $A_n$  were obtained by writing

$$v = \sum_{i=0}^{\infty} \lambda^i u_i, \quad N(v) = N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right) = \sum_{n=0}^{\infty} \lambda^n A_n. \tag{1.4}$$

We remark that the  $A_n$  are formally obtained from the relationship [1,3,4,6]

$$n! A_n = \frac{d^n}{d\lambda^n} \left[ N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots \tag{1.5}$$

These definitions are only formal, and nothing is proved or supposed about the convergence of the series  $\sum u_i$  and  $\sum A_n$ .

Putting (1.2) and (1.3) into (1.1) leads to the relationship

$$\sum_{i=0}^{\infty} u_i = \sum_{i=0}^{\infty} A_i + f, \tag{1.6}$$

and the Adomian's method consists in identifying the  $u_i$  by means of the formulae

$$\begin{aligned} u_0 &= f, \\ u_1 &= A_0, \\ u_2 &= A_1, \\ &\vdots \\ u_n &= A_{n-1}, \\ &\vdots \end{aligned} \tag{1.7}$$

In [2,4], it was proven that the  $A_i$  only depend on  $u_0, u_1, \dots, u_i$ , and thus the formulae (1.7) give all the terms  $u_i$  of the series in an explicit manner, if we are able to calculate the polynomials  $A_i$  for every  $i$ . But to prove convergence of the series  $\sum u_i$ , it is necessary (and sufficient) to prove the convergence of the series  $\sum A_n$ . Unfortunately, the convergence of  $\sum A_n$  is not a consequence of the previous definitions. In [3], the convergence is proved by using a new formulation of the method and by applying the fixed point theorem. In [4], a proof of convergence is obtained by means of properties of the substituted convergent series, and supposing that (1.1) admits a solution of the form  $\sum u_i$ , where the series is absolutely convergent. In the following, we shall prove convergence of the series  $\sum A_n$  owing to new formulae giving  $A_n$  in function of  $u_0$  only. This approach generalizes a result given in [5].

## 2. NEW FORMULAE FOR COMPUTING THE $A_n$

In [2], we have given a simpler formula than in [1,6] for calculating the  $A_i$ 's. Let us recall this formula:

$$\begin{aligned} A_n &= \sum_{\alpha_1 + \dots + \alpha_n = n} N^{(\alpha_1)}(u_0) \frac{u_1^{\alpha_1 - \alpha_2}}{(\alpha_1 - \alpha_2)!} \dots \frac{u_{n-1}^{\alpha_{n-1} - \alpha_n}}{(\alpha_{n-1} - \alpha_n)!} \frac{u_n^{\alpha_n}}{\alpha_n!}, \quad n \neq 0, \\ A_0 &= N(u_0), \end{aligned} \tag{2.1}$$

where  $(\alpha_i)_{i=1,\dots,n}$  is a decreasing sequence. The sum in (2.1) has to be done on all the solutions of the equation

$$\alpha_1 + \alpha_2 + \dots + \alpha_n = n, \quad \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n. \tag{2.2}$$

Let us call  $P(n)$  the number of solutions of (2.2). Using number theory [7] allows us to evaluate  $P(n)$ . Indeed, we have the following result.

**THEOREM 2.1.**

$$P(n) < \exp\left(\pi\sqrt{\frac{2}{3}n}\right), \quad \text{for } n \in N^*. \tag{2.3}$$

**PROOF.** Set

$$f(t) = \sum_{n=0}^{\infty} P(n) t^n = \prod_{k=1}^{\infty} (1 - t^k)^{-1}, \quad |t| < 1. \tag{2.4}$$

For  $0 < t < 1$ , we define  $g(t)$  as

$$g(t) = \log f(t). \tag{2.5}$$

Then, we have

$$g(t) = -\sum_{k=1}^{\infty} \log(1 - t^k) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{t^{kj}}{j} = \sum_{j=1}^{\infty} \left(\frac{t^j}{1 - t^j}\right) \frac{1}{j}. \tag{2.6}$$

If  $j \geq 1$  and  $0 < t < 1$ , we obtain

$$\frac{1 - t^j}{1 - t} = 1 + t + t^2 + \dots + t^{j-1}, \tag{2.7}$$

and then

$$j t^{j-1} < \frac{1 - t^j}{1 - t} < j, \quad 0 < t < 1, \quad j \geq 1. \tag{2.8}$$

Putting (2.8) into (2.6) implies

$$\sum_{j=1}^{\infty} \frac{t^{j-1}}{j^2} < \frac{1 - t}{t} g(t) < \sum_{j=1}^{\infty} \frac{1}{j^2}. \tag{2.9}$$

But we know that

$$\lim_{t \rightarrow 1} \frac{(1 - t)g(t)}{t} = \frac{\pi^2}{6}, \tag{2.10}$$

and, therefore,

$$P(n) t^n < f(t), \quad \text{if } 0 < t < 1, \quad \text{for } n \geq 0. \tag{2.11}$$

From these inequalities, we deduce that

$$\log P(n) < \frac{\pi^2}{6} \frac{t}{1 - t} + n \log\left(\frac{1}{t}\right). \tag{2.12}$$

Let us set  $1 + u = 1/t$ , then  $0 < u < \infty$ , for  $0 < t < 1$ , and we obtain

$$\log P(n) < \frac{\pi^2}{6} \frac{1}{u} + n u. \tag{2.13}$$

The function  $\pi^2/(6u) + nu$  has a minimum (with respect to  $u$ ) for  $u = \pi/\sqrt{6n}$ , and thus,

$$\log P(n) < \pi\sqrt{\frac{2}{3}n}. \tag{2.14}$$

This result will be used later for proving the convergence of the decomposition method. Let us give some properties of the  $A_n$  before proving the convergence of the series  $\sum A_n$ . In [2], we have redefined the Adomian's polynomials  $A_n$  by setting, for any sequence

$$u_n(\lambda) = \sum_{i=0}^n \lambda^i u_i, \tag{2.15}$$

$$N(u_n(\lambda)) = \sum_{i=0}^n \lambda^i A_i, \tag{2.16}$$

and thus we obtain the  $A_n$  by the relationships

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ N \left( \sum_{i=0}^n \lambda^i u_i \right) \right]_{\lambda=0}. \tag{2.17}$$

Then, we have the following results.

THEOREM 2.2.

$$(1) \quad \frac{\partial}{\partial u_0} A_{n-k} = \frac{\partial}{\partial u_k} A_n, \quad \forall n, k, \quad n \geq k. \tag{2.18}$$

In particular, for  $n = k$ , we obtain

$$\frac{\partial}{\partial u_0} A_0 = \frac{\partial}{\partial u_n} A_n, \quad \forall n, \tag{2.19}$$

$$(2) \quad A_{n+1} = \frac{1}{n+1} K A_n, \quad \text{where} \tag{2.20}$$

$$K = \sum_{k=0}^n (k+1) u_{k+1} \frac{\partial}{\partial u_k}. \tag{2.21}$$

PROOF. The first result is a direct consequence of the definition of  $A_n$ . For the second result, we have, by definition,

$$\begin{aligned} A_{n+1} &= \frac{1}{(n+1)!} \frac{d^{n+1}}{d\lambda^{n+1}} \left[ N \left( \sum_{i=0}^{n+1} \lambda^i u_i \right) \right]_{\lambda=0} \\ &= \frac{1}{(n+1)!} \sum_{k=0}^n c_n^k (k+1)! u_{k+1} (n-k)! \frac{\partial}{\partial u_0} A_{n-k} \\ &= \frac{1}{(n+1)!} \sum_{k=0}^n (k+1) u_{k+1} \frac{\partial}{\partial u_0} A_{n-k} = \frac{1}{n+1} \sum_{k=0}^n (k+1) u_{k+1} \frac{\partial}{\partial u_k} A_n. \quad \blacksquare \end{aligned}$$

Now, let us give a formula giving  $A_n$  as a function of  $u_0$ . We have the following result generalizing those of [5].

THEOREM 2.3.

$$\begin{aligned} A_n &= \sum_{\alpha_1 + \dots + \alpha_n = n} c_{\alpha_1, \dots, \alpha_n} (N(u_0))^{n+1-\alpha_1} (N'(u_0))^{\alpha_1-\alpha_2} \dots (N^{(n-1)}(u_0))^{\alpha_{n-1}-\alpha_n} \\ &\quad \times (N^{(n)}(u_0))^{\alpha_n}, \end{aligned} \tag{2.22}$$

where

$$c_{\alpha_1, \dots, \alpha_n} = \frac{n!}{(\alpha_1 - \alpha_2)! \cdot (\alpha_{n-1} - \alpha_n)! \alpha_n! (1!)^{\alpha_1 - \alpha_2} \dots (n-1)^{\alpha_{n-1} - \alpha_n} (n+1 - \alpha_1)!}. \tag{2.23}$$

PROOF. We first remark that

$$A_n = \frac{1}{(n+1)!} \frac{d^n}{d\lambda^n} [N(u_0)^{n+1}],$$

which can be proved easily by induction. Then, using a classical formula [8] for calculating the  $n^{\text{th}}$  derivative of a function, it leads to (2.22), with parameters  $c_{\alpha_1, \dots, \alpha_n}$  given by (2.23).

### 3. RESULTS OF CONVERGENCE

From Theorem 2.1, we deduce the following result.

**THEOREM 3.1.** *With the following hypotheses,*

- (a)  $N$  is  $C^{(\infty)}$  in a neighbourhood of  $u_0$  and  $\|N^{(n)}(u_0)\| \leq M'$ , for any  $n$  (the derivatives of  $N$  at  $u_0$  are bounded in norm);
- (b)  $\|u_i\| \leq M < 1$ ,  $i = 1, 2, \dots$ , where  $\|\cdot\|$  is the norm in the Hilbert space  $H$ ;

the series  $\sum_{n=0}^{\infty} A_n$  is absolutely convergent and, furthermore,

$$\|A_n\| \leq \left( \exp \left( \pi \sqrt{\frac{2}{3} n} \right) \right) M' M^n. \tag{3.1}$$

**PROOF.** We know that

$$A_n = \sum_{\alpha_1 + \dots + \alpha_n = n} N^{(\alpha_1)}(u_0) \frac{u_1^{\alpha_1 - \alpha_2}}{(\alpha_1 - \alpha_2)!} \dots \frac{u_{n-1}^{\alpha_{n-1} - \alpha_n}}{(\alpha_{n-1} - \alpha_n)!} \frac{u_n^{\alpha_n}}{\alpha_n!}.$$

Applying Theorem 2.1 implies

$$\|A_n\| \leq \left( \exp \left( \pi \sqrt{\frac{2}{3} n} \right) \right) M' M,$$

which is the general term of a convergent series. ■

But a difficulty remains. Indeed, the convergence of the series  $\sum A_n$  depends on the  $u_i$ 's by hypothesis (a) and the  $u_i$  are also obtained from the  $A_i$ 's. Practically, it will be difficult to prove that  $u_i$  is bounded in norm. The following result gives a more useful result proving directly that the series  $\sum A_n$  converges.

**THEOREM 3.2.** *If  $N$  is  $C^{(\infty)}$  and satisfies  $\|N^{(n)}(u_0)\| \leq M < 1$  for any  $n \in N$ , then the decompositional series  $\sum_{n=0}^{\infty} u_n$  is absolutely convergent and we have*

$$\|u_{n+1}\| = \|A_n\| \leq M^{n+1} n^{\sqrt{n}} \exp \left( \pi \sqrt{\frac{2}{3} n} \right). \tag{3.2}$$

**PROOF.** By using Theorems (2.1) and (2.3), one can prove that:

$$\sum_{\alpha_1 + \dots + \alpha_n = n} |c_{\alpha_1, \alpha_2, \dots, \alpha_n}| \leq n^{\sqrt{n}} \exp \left( \pi \sqrt{\frac{2}{3} n} \right), \quad \text{and therefore,}$$

$$\|A_n\| \leq M^{n+1} n^{\sqrt{n}} \exp \left( \pi \sqrt{\frac{2}{3} n} \right).$$

$\|A_n\|$  is majorized by the general term of a convergent series. ■

### 4. CONCLUSIONS

The decomposition method first developed by Adomian has now proved its power for solving all kinds of nonlinear functional equations. In [2,5] and in this paper, we have given practical formulae well adapted to numerical computations of the Adomian's polynomials  $A_n$ . The  $A_n$  can also be calculated by using software for formal calculus. Before these approaches, the computation of  $A_n$  was often a difficult problem when  $n \geq 3$ . Owing to this study, we are also able to prove the convergence of the Adomian's series without hypothesis on the  $u_i$ 's. It suffices to have some hypothesis on the nonlinear operator  $N$  at  $u_0$ . These hypotheses are not generally difficult to verify on concrete problems. Applications of decomposition methods are very important and numerous. Let us specially quote the applications to modelling where parameters have to be identified from experimental data. In that case, the decomposition method can be associated to an optimization technique. In the same way, the decomposition method may be applied to optimal control problems arising, for instance, in biomedicine. It allows us to transform an optimal control problem into a classical optimization problem.

## REFERENCES

1. G. Adomian, *Nonlinear Stochastic Systems Theory and Applications to Physics*, Kluwer, Dordrech, Holland, (1989).
2. K. Abbaoui and Y. Cherruault, Convergence of Adomian's method applied to differential equations, *Mathl. Comput. Modelling* **28** (5), 103–110.
3. Y. Cherruault, Convergence of Adomian's method, *Kybernetes* **18** (2), 31–38 (1989).
4. Y. Cherruault, G. Saccomandi and B. Somé, New results for convergence of Adomian's method applied to integral equations, *Mathl. Comput. Modelling* **16** (2), 85–93 (1992).
5. S. Guellal and Y. Cherruault, Practical formulae for calculation of Adomian's polynomials and application to the convergence of the decomposition method, *Int. J. of Biomedical Comp.* **36**, 223–228 (1994).
6. G. Adomian, A review of decomposition method and some recent results for nonlinear equations, *Mathl. Comput. Modelling* **13** (7), 17–43 (1990).
7. K. Chandrasekharan, *Arithmetical Functions*, Springer-Verlag, New York, (1970).
8. L. Schwartz, *Cours d'Analyse*, Hermann, Paris, France, (1981).