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The treatment of corner and pole-type singularities in numerical conformal mapping techniques

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Abstract: This paper is a report of recent developments concerning the nature and the treatment of singularities that affect certain numerical conformal mapping techniques. The paper also includes some new results on the nature of singularities that the mapping function may have in the complement of the closure of the domain under consideration.

Keywords: Numerical conformal mapping, kernel function methods, Symm's integral equation, corner and pole-type singularities

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1. Introduction

This paper is a report of recent developments concerning the treatment of singularities in certain numerical methods for approximating the functions f_I, f_E and f_D which accomplish respectively the following three conformal maps:

CM1. The mapping of a domain interior to a closed Jordan curve onto the interior of the unit disc.

CM2. The mapping of a domain exterior to a closed Jordan curve onto the exterior of the unit disc.

CM3. The mapping of a doubly-connected domain, bounded by two closed Jordan curves, onto a circular annulus.

The main objectives of the paper are as follows:

(i) To present detailed information about the location and nature of the singularities that the three mappings may have on and near the boundary of the domain under consideration.

(ii) To indicate how the singularities of the conformal maps affect two different classes of numerical methods, viz. expansion and integral equation methods. (We do this by considering certain expansion methods which have been studied in [22,26–30], and an integral equation method which has received considerable attention recently [8–10,12–18,31,33–35,39,40].)

(iii) To present numerical examples illustrating certain important aspects concerning the treatment of singularities.

The paper is essentially a detailed survey of developments reported in [14,15,22,26–30]. However, in Section 5 we also present certain new results that provide additional information about the singular behaviour of the interior and exterior mapping functions f_I and f_E.

2. The conformal mapping problems

Let $\partial\Omega$ be a closed piecewise analytic Jordan curve in the complex z-plane, and assume that the origin 0 lies in $\mathrm{Int}(\partial\Omega)$. Then the two problems associated with the conformal maps CM1 and CM2 can be stated as follows:

Problem P1. To determine the function

$$w = f_I(z) \tag{2.1}$$

which maps $\Omega_I = \mathrm{Int}(\partial\Omega)$ one-to-one conformally onto the unit disc

$$D_I = \{w: |w| < 1\} \tag{2.2}$$

so that

$$f(0) = 0 \quad \text{and} \quad f'(0) > 0. \tag{2.3}$$

Problem P2. To determine the function

$$w = f_E(z) \tag{2.4}$$

which maps $\Omega_E = \mathrm{Ext}(\partial\Omega)$ one-to-one conformally onto the exterior of the unit disc

$$D_E = \{w: |w| > 1\} \tag{2.5}$$

so that

$$f_E(\infty) = \infty \quad \text{and} \quad \lim_{z\to\infty} f_E'(z) > 0. \tag{2.6}$$

The above two problems can be related to each other by means of the transformation

$$z \to z^{-1}. \tag{2.7}$$

This simple inversion transforms $\partial\Omega$ onto a piecewise analytic Jordan curve $\partial\hat{\Omega}$, and maps Ω_I onto $\hat{\Omega}_E = \mathrm{Ext}(\partial\hat{\Omega})$ and Ω_E onto $\hat{\Omega}_I = \mathrm{Int}(\partial\hat{\Omega})$. Therefore, if $\hat{f}_I$ and $\hat{f}_E$ are respectively the interior and exterior mapping functions associated with $\partial\hat{\Omega}$, then

$$f_E(z) = \{\hat{f}_I(z^{-1})\}^{-1} \quad \text{and} \quad f_I(z) = \{\hat{f}_E(z^{-1})\}^{-1}. \tag{2.8}$$

Thus, in theory at least, there is no need to consider the interior and exterior mapping problems as separate problems. Indeed, in the case of expansion methods it is generally computationally convenient to determine f_E by using (2.8) and the corresponding approximation to the interior mapping function $\hat{f}_I$; see e.g. [27]. In the case of integral equation methods, however, no numerical advantage can be gained by using the intermediate transformation (2.7), and it is generally preferable to treat the two mapping problems separately.

Let the parametric equation of $\partial\Omega$ be

$$z = \tau(s), \quad 0 \leqslant s \leqslant L, \tag{2.9}$$

where s is an appropriate real parameter, and assume that (2.9) defines a positive orientation of $\partial\Omega$ with respect to Ω_I. Then the interior and exterior boundary correspondence functions θ_I and θ_E associated with the Problems P1 and P2 are defined respectively by

$$f_I\{\tau(s)\} = \exp\{i\theta_I(s)\} \quad \text{and} \quad f_E\{\tau(s)\} = \exp\{i\theta_E(s)\}, \tag{2.10a}$$

i.e.

$$\theta_I(s) = \operatorname{Arg}\{f_I(\tau(s))\} \quad \text{and} \quad \theta_E(s) = \operatorname{Arg}\{f_E(\tau(s))\}, \tag{2.10b}$$

where $\operatorname{Arg}(\cdot)$ is a continuous argument as defined, for example, in [11, §4.6] and [18, §11.7]. As is shown in [8], the functions θ_I and θ_E play very important roles in both the theory and application of the integral method considered in the present paper.

Let now $\partial\Omega_1$ and $\partial\Omega_2$ be two closed piecewise analytic Jordan curves such that $\partial\Omega_1 \subset \operatorname{Int}(\partial\Omega_2)$ and $0 \in \operatorname{Int}(\partial\Omega_1)$, and denote by Ω_D the finite doubly-connected domain

$$\Omega_D = \operatorname{Ext}(\partial\Omega_1) \cap \operatorname{Int}(\partial\Omega_2). \tag{2.11}$$

Then the problem associated with the conformal map CM3 can be stated as follows:

Problem P3. To determine the function

$$w = f_D(z) \tag{2.12}$$

which maps Ω_D one-to-one conformally onto a circular annulus

$$A(r_1, r_2) = \{w : r_1 < |w| < r_2\} \tag{2.13}$$

so that

$$f_D(\zeta_1) = r_1, \tag{2.14}$$

where ζ_1 is some fixed point on $\partial\Omega_1$ and r_1 is a prescribed number.

The condition (2.14) uniquely determines the radius r_2 of the outer circle and ensures that $\partial\Omega_1$ and $\partial\Omega_2$ are mapped respectively onto the two circles $|w| = r_1$ and $|w| = r_2$. The ratio

$$M = r_2/r_1 \tag{2.15}$$

of the two radii of $A(r_1, r_2)$ is an important domain functional known as the conformal modulus of Ω_D.

Let the parametric equation of $\partial\Omega_D = \partial\Omega_1 \cup \partial\Omega_2$ be

$$z = \tau(s), \quad 0 \leqslant s \leqslant L, \tag{2.16a}$$

so that

$$\partial\Omega_1 = \{\tau(s) : 0 \leqslant s \leqslant L_1\}$$

and

$$\partial\Omega_2 = \{\tau(s) : L_1 < s \leqslant L\}, \tag{2.16b}$$

where, for notational simplicity, we take

$$\tau(L_1) = \tau(L_1 -) = \tau(0)$$

and

$$\tau(L) = \tau(L_1 +). \tag{2.16c}$$

Then, by analogy with the definitions (2.10) of θ_I and θ_E, we define the boundary correspondence function θ_D associated with the function f_D by

$$f_D\{\tau(s)\} = r(s)\exp\{i\theta_D(s)\}, \tag{2.17a}$$

where

$$r(s) = \begin{cases} r_1, & 0 \leqslant s \leqslant L_1, \\ r_2, & L_1 < s \leqslant L, \end{cases} \tag{2.17b}$$

i.e.

$$\theta_D(s) = \mathrm{Arg}\{f_D(\tau(s)\}. \tag{2.17c}$$

3. Numerical conformal mapping

3.1. Expansion methods

By an expansion method we mean a numerical method where the mapping function is approximated by an explicit formula involving a linear combination of a set of basis functions. The class of such methods includes the well-known kernel function methods described in [6, Chapter III], the variational method of [6, p. 249], and the numerical methods described in [4,5]. In the application of any of these methods, information about the dominant singularities of the mappings is needed for constructing the set of basis functions. This emerges from the observation that the computational efficiency of an expansion method improves considerably when the basis set contains functions that reflect the main singular behaviour of the mapping in the complement of the domain under consideration. In the present paper we illustrate the construction of such basis sets by considering the following typical expansion methods:

(i) The well-known Bergman kernel method (BKM) and the closely related Ritz variational method (RM) for determining approximations to the mapping functions f_I and f_E. The theory of both these methods is treated extensively in the literature; see e.g. [1,6,7,25,37].

(ii) The variational method (VM) of Gaier [6, p. 249] and the associated orthonormalization method (ONM), which emerges from the theory contained in [6, p. 249; 1, p. 102; 25, p. 373]; see also [28]. Both the VM and ONM are methods for approximating the mapping function f_D of Problem P3.

In both the BKM and RM the approximation to the interior mapping function f_I is determined after first approximating the derivative f_I' by an expansion of the form

$$f_{I,n}'(z) = \sum_{j=1}^{n} a_j \eta_j(z), \tag{3.1}$$

where the basis set $\{\eta_j\}$ is a complete set in the space $L_2(\Omega_I)$. (Here $L_2(\Omega_I)$ denotes the Hilbert space of all square-integrable analytic functions in Ω_{I}.) The choice of the basis set plays a very critical role in the application of the methods. That is, for the reasons explained in [22, Section 2] and [26, Section 4], the set $\{\eta_j\}$ must be chosen so that the resulting approximation series (3.1) converges rapidly. This can be achieved, as proposed in [22,26,29], by using an 'augmented basis' formed by introducing into the 'monomial set'

$$z^{j-1}, \quad j=1,2,3,\ldots \tag{3.2}$$

functions that reflect the dominant singularities of f_{I}' on $\partial\Omega$ and in $\mathrm{Ext}(\partial\Omega)$.

The same procedure for constructing the basis set is used in [27], where the BKM and RM are applied to the exterior mapping Problem P2. Here, however, the approximation to f_{E} is determined, by means of (2.8), from the corresponding approximation to the interior mapping function $\hat{f}_{\mathrm{I}}$. For this reason, in the case of Problem P2, the augmented basis is formed by introducing into the monomial set functions that reflect the singularities of $\hat{f}_{\mathrm{I}}'$ on $\partial\hat{\Omega}$ and in $\mathrm{Ext}(\partial\hat{\Omega})$.

In the case of Problem P3, both the VM and ONM approximation to the mapping function f_{D} are determined after first approximating the function

$$H(z)=f_{\mathrm{D}}'(z)/f_{\mathrm{D}}(z)-1/z \tag{3.3}$$

by an expansion of the form

$$H_n(z)=\sum_{j=1}^{n} a_j\eta_j(z). \tag{3.4}$$

Here, the set $\{\eta_j\}$ is a basis for the Hilbert space of all functions in $L_2(\Omega_{\mathrm{D}})$ which also possess single-valued indefinite integrals in Ω_{D}. In this case the augmented basis is formed by introducing into the 'monomial set'

$$z^{j-1}, \quad 1/z^{j+1}, \qquad j=1,2,\ldots \tag{3.5}$$

functions that reflect the singularities of H on $\partial\Omega_{\mathrm{D}}$ and in $\mathrm{compl}(\overline{\Omega}_{\mathrm{D}})=\mathrm{Int}(\partial\Omega_1)\cup\mathrm{Ext}(\partial\Omega_2)$; see [28,30] and [3].

3.2. An integral equation method

The integral equation method (IEM) considered in this section is based on certain formulations proposed originally by Symm [33–35], and for this reason the method is frequently referred to as 'Symm's method'.

In the IEM, the approximate conformal map is determined after first solving a weakly singular Fredholm integral equation of the first kind for an unknown density function ν. The three equations associated with the mapping Problems P1, P2 and P3 can be expressed in a unified manner by taking G to be the domain under consideration, letting

$$z=\tau(s), \quad 0\leqslant s\leqslant L \tag{3.6}$$

be the parametric equation of the boundary ∂G, and denoting by

$$w=F(z) \tag{3.7}$$

the corresponding mapping function. (That is, F denotes one of the functions f_{I}, f_{E} or f_{D}, depending on whether the domain G is interior, exterior or doubly-connected, i.e. depending on whether G is Ω_I, Ω_E or Ω_D.) With this notation, the integral equations for determining the density function ν can be expressed as

$$\int_0^L \nu(s) \log|\tau(\sigma)-\tau(s)|\mathrm{d}s=\delta(\sigma), \quad 0\leqslant\sigma\leqslant L, \tag{3.8a}$$

where

$$\delta(\sigma)=\begin{cases} -\log|\tau(\sigma)|, & G\equiv\Omega_I, \\ 1, & G\equiv\Omega_E, \\ (L_1-\sigma)_+^0, & G\equiv\Omega_D, \end{cases} \tag{3.8}$$

and where, with the usual notation,

$$x_+^0=\begin{cases} 1, & x\geqslant 0, \\ 0, & x<0. \end{cases}$$

The theory of the IEM is treated fully in [8,9], where in particular, the question of solvability of (3.8) is studied. It turns out that in the two cases $G\equiv\Omega_I$ and $G\equiv\Omega_E$, (3.8) has a unique solution provided that

$$\operatorname{cap}\partial\Omega\neq 1, \tag{3.9}$$

where, with the notation of problem P2,

$$\operatorname{cap}\partial\Omega=\lim_{z\to\infty}\{f_E'(z)\}^{-1} \tag{3.10}$$

is the capacity of the curve $\partial\Omega$. Similarly, when $G\equiv\Omega_D$, (3.8) has a unique solution provided that

$$\operatorname{cap}\partial\Omega_2\neq 1. \tag{3.11}$$

(In other words, a unique solution always exists subject only to a possible rescaling of G.) It is also shown in [8,9] that the density functions corresponding to the three mapping problems are related to the derivatives of the associated boundary correspondence functions as follows:

Problem P1:

$$2\pi\nu(s)=-\dot{\theta}_I(s). \tag{3.12}$$

Problem P2: Let $\gamma=\log\{\operatorname{cap}\partial\Omega\}$. Then

$$2\pi\gamma\nu(s)=\dot{\theta}_E(s). \tag{3.13}$$

Problem P3: Assume, without loss of generality, that the mapping is normalized so that $\Omega_D\to A(r_1, 1)$, and let $\gamma=\log M$, where $M=1/r_1$ is the modulus of Ω_D. Then

$$2\pi\gamma\nu(s)=\dot{\theta}_D(s). \tag{3.14}$$

In the two cases $G\equiv\Omega_E$ and $G\equiv\Omega_D$, the integral equations contained in (3.8) are due to Gaier [8,9] and differ somewhat from those used originally by Symm [34,35]. For the Problem P2 and P3, the formulations of [34] and [35] involve the determination of two density functions $\hat{\nu}_E$ and $\hat{\nu}_D$ which are related to the boundary correspondence functions θ_I, θ_E and θ_D as follows:

Problem P2:

$$2\pi\hat{\nu}(s)=\dot{\theta}_E-\dot{\theta}_I. \tag{3.15}$$

Problem P3: Let θ_{II} be the interior boundary correspondence function associated with the inner component $\partial\Omega_1$ of $\partial\Omega_{\text{D}}$. Then

$$2\pi\hat{\nu}_{\text{D}}(s)=\begin{cases}\dot{\theta}_{\text{II}}-\dot{\theta}_{\text{D}}, & 0\leqslant s\leqslant L_1,\\ \dot{\theta}_{\text{D}}, & L_1<s\leqslant L;\end{cases} \tag{3.16}$$

see [9] and [15].

As will become apparent in Section 4, if the domains under consideration involve corners, then the original formulations of Symm [34,35] are not as suitable as those based on the integral equation (3.8).

Regarding the treatment of singularities, in the IEM we are interested mainly in the singular behaviour of the unknown density function, rather than of F. For example, the asymptotic expansion of ν near a corner is used in the collocation method of [14,15] for approximating the solution of (3.8) by splines and singular functions. (A similar approach can of course be used in connection with the Galerkin method of Wendland [40]; see also [17] and [20].) Also, the so-called re-parametization method of Hoidn [13] requires knowledge of the singular behaviour of ν at a corner. In this method, the corner singularities associated with the solution of Problem P1 are treated by redefining the parametric equation of the boundary curve $\partial\Omega$. Finally, information about the location of the singularities of the mapping function F in $\text{compl}(G\cup\partial G)$ can be used, in collocation and Galerkin methods, for defining appropriate non-uniform distributions of the nodal points; see [14, Example 1, p. 142] and also Examples 1–4 of the present paper.

4. Corner singularities

Any boundary singularities of the mapping functions are corner singularities, similar to those that arise in the study of elliptic boundary value problems. The asymptotic form of these singularities can be determined from the results of Lehman [21], which generalize earlier work of Lichtenstein [24], Kellog [19], Warschawski [38] and Lewy [23].

With the unified notation introduced in section 3.2, assume that part of the boundary ∂G consists of two analytic arcs Γ_1 and Γ_2 which meet at a point z_0 and from there a corner of interior angle $\alpha\pi$, where $0<\alpha<2$. (By interior angle, we mean interior to the domain G under consideration.) Then, depending on whether α is rational or irrational, the results of [21] lead to the following two asymptotic expansions:

(i) If $\alpha=p/q$, with p and q relatively prime, then as $z\to z_0$,

$$F(z)-F(z_0)=\sum_{k,l,m}B_{k,l,m}(z-z_0)^{k+l/\alpha}(\log(z-z_0))^m, \tag{4.1a}$$

where k, l and m run over all integers $k\geqslant 0$, $1\leqslant l\leqslant p$, $0\leqslant m\leqslant k/q$, and where $B_{0,1,0}\neq 0$. Also, the terms in (4.1a) are ordered so that the term corresponding to $B_{k,l,m}$ precedes the term corresponding to $B_{k',l',m'}$ if either $k+l/\alpha<k'+l'/\alpha$ or $k+l/\alpha=k'+l'/\alpha$ and $m>m'$.

(ii) If α is irrational, then as $z\to z_0$,

$$F(z)-F(z_0)=\sum_{k,l}B_{k,l}(z-z_0)^{k+l/\alpha}, \tag{4.1b}$$

where now k and l run over all integers $k\geqslant 0$, $l\geqslant 1$ and where $B_{0,1}\neq 0$.

In the two cases $G \equiv \Omega_I$ and $G \equiv \Omega_E$, the expansions (4.1a) and (4.1b) simplify considerably when the two arms Γ_1, Γ_2 of the corner z_0 are both straight lines. Then, as $z \to z_0$,

$$F(z) - F(z_0) = \sum_{l=1}^{\infty} B_l (z - z_0)^{l/\alpha}, \quad B_1 \neq 0; \tag{4.1c}$$

see e.g. [25, pp. 189–194] and [2, p. 170]. Also, when $G \equiv \Omega_D$ and both Γ_1 and Γ_2 are straight lines the expansion (4.1b) holds for both rational and irrational α, and the same applies, in all three cases, $G \equiv \Omega_I$, Ω_E, Ω_D, when both Γ_1 and Γ_2 are circular arcs.

It follows from the above that the dominant term in the asymptotic expansion of F is always $(z - z_0)^{1/\alpha}$. This reflects the geometric property that, under the mapping F, the angle $\alpha\pi$ at $z_0 \in \partial G$ is transformed onto an angle π at the point $F(z_0)$. Therefore, when $1/\alpha$ is not an integer, a branch point singularity always occurs at the corner z_0. Furthermore, because of the logarithmic terms in (4.1a), a branch point singularity might occur even when $1/\alpha$ is an integer. This means, in particular, that the use of preliminary transformations, which is frequently proposed as a method for rectifying corners, does not necessarily completely remove corner singularities.

4.1. Singularities of the functions f_I and H

As we indicated in section 3.1, this information is needed for constructing appropriate 'augmented' basis sets for use with the four expansion methods which we denoted by BKM, RM, ONM and VM. The form of the 'singular' functions needed for augmenting the monomial sets (3.2) and (3.5) emerges from the asymptotic expansions (4.1). The details, for each of the three mapping problems, are as follows:

Problem P1. The BKM or RM basis set is constructed by introducing into the monomial set (3.2) the derivatives of the first few singular terms of the appropriate asymptotic series (4.1a), (4.1b) or (4.1c). That is, the singular basis functions for dealing with corner singularities are of the form

$$\eta(z) = \frac{\mathrm{d}}{\mathrm{d}z}\left\{(z - z_0)^r\right\}, \quad r = k + l/\alpha \quad \text{or} \quad r = l/\alpha \tag{4.2a}$$

and

$$\eta(z) = \frac{\mathrm{d}}{\mathrm{d}z}\left\{(z - z_0)^{k + l/\alpha}(\log(z - z_0))^m\right\}; \tag{4.2b}$$

see [22] and [26].

Problem P2. In this case, a corner of exterior angle $\alpha\pi$ at $z_0 \in \partial\Omega$ is transformed, under the inversion (2.7), into a corner of interior angle $\alpha\pi$ at the point $1/z_0 \in \partial\hat{\Omega}$. Therefore, since the BKM or RM approximation to the mapping function f_E is determined by means of (2.8) from the corresponding approximation to the interior mapping function $\hat{f}_I$, the details for constructing the augmented basis are the same as for Problem P1. However, it is important to observe that the inversion (2.7) transforms a straight line Γ into a straight line $\hat{\Gamma}$ only if Γ passes through the origin of the z-plane. This means that in the case of the function $\hat{f}_I$, the simple asymptotic expansion (4.1c) cannot be assumed, even when both the arms of the corner are straight lines; see [27].

Problem P3. The question regarding the choice of basis functions for dealing with the corner singularities at z_0 of the function H, defined by (3.3), can again be answered by using the asymptotic expansions (4.1). However, as was indicated in section 3.1, the ONM and VM basis functions must possess single-valued integrals in Ω_D. For this reason, the form of the singular functions used for augmenting the set (3.5) depends on whether the corner z_0 lies on the inner or outer component of $\partial\Omega_D$. That is, the singular functions are of the form (4.2) when z_0 is on the outer boundary $\partial\Omega_2$, and of the form

$$\eta(z)=\frac{\mathrm{d}}{\mathrm{d}z}\left\{\left(\frac{1}{z}-\frac{1}{z_0}\right)^r\right\}, \quad r=k+l/\alpha \quad \text{for} \quad r=l/\alpha \tag{4.3a}$$

and

$$\eta(z)=\frac{\mathrm{d}}{\mathrm{d}z}\left\{\left(\frac{1}{z}-\frac{1}{z_0}\right)\right\}^{k+l/\alpha}\left(\log\left(\frac{1}{z}-\frac{1}{z_0}\right)^m\right)\} \tag{4.3b}$$

when z_0 is on the inner boundary $\partial\Omega_1$; see [28] and [3].

4.2. Singularities of the source density function ν

As before, we use the unified notation of section 3.2 and assume that part of the boundary ∂G of the domain G under consideration consists of two analytic arcs which meet at a point z_0 and form there a corner of interior angle $\alpha\pi$, $0<\alpha<2$. We also take the parametric equation of ∂G to be

$$z=\tau(s), \quad 0\leqslant s\leqslant L, \tag{4.4}$$

and let

$$z_0=\tau(s_0). \tag{4.5}$$

Then in a neighbourhood of s_0, $\tau(s)$ has a series expansion of the form

$$\tau(s)=\tau(s_0)+\begin{cases}\sum\limits_{n=1}^{\infty}(s-s_0)^n\tau^{(n)}(s_0+)/n!, & s>s_0,\\ \sum\limits_{n=1}^{\infty}(s-s_0)^n\tau^{(n)}(s_0-)/n!, & s<s_0,\end{cases} \tag{4.6a}$$

where

$$\tau^{(n)}(s_0\pm)=\lim_{s\to s_0\pm}\{\mathrm{d}^n\tau/\mathrm{d}s^n\}. \tag{4.6b}$$

Let

$$\theta(s)=\operatorname{Arg}\{F(\tau(s))\} \tag{4.7}$$

denote the boundary correspondence function associated with the mapping F, i.e. θ is θ_I, θ_E or θ_D depending on whether G is Ω_I, Ω_E or Ω_D. Then,

$$\dot{\theta}(s)=-\mathrm{i}\dot{F}(\tau(s))\overline{F(\tau(s))}/|F(\tau(s))|^2 \tag{4.8}$$

and thus, from (3.12)–(3.14), the density function ν of (3.8) is related to F by means of

$$\nu(s)=\operatorname{Im}\{\dot{F}(\tau(s))\overline{F(\tau(s))}\}/2\pi\eta, \tag{4.9}$$

where $\eta = -1$ when $G \equiv \Omega_I$, $\eta = \log\{\text{cap } \partial\Omega\}$ when $G \equiv \Omega_E$, and

$$\eta = \begin{cases} r_1^2 \log M, & 0 \leqslant s \leqslant L_1, \\ \log M, & L_1 < s \leqslant L, \end{cases}$$

when $G \equiv \Omega_D$ and the mapping is $\Omega_D \to A(r_1, 1)$. Hence, by using (4.1), (4.6) and (4.9), we find that as $s \to s_0$,

$$\nu(s) = \begin{cases} \sum_{j=1}^{\infty} a_j^+ \phi_j(s - s_0), & s > s_0, \\ \sum_{j=1}^{\infty} a_j^- \phi_j(s_0 - s), & s < s_0, \end{cases} \tag{4.10}$$

where $a_1^\pm \neq 0$ and where the functions ϕ_j depend on the value of α and can be determined from the expansions (4.1). For example, when α is rational, then according to the ordering of (4.1a), the first four functions in (4.10) are defined respectively by

$$\phi(\sigma) = \sigma^{-1+1/\alpha}, \quad 0 < \alpha < 2, \tag{4.11a}$$

$$\phi_2(\sigma) = \begin{cases} \sigma^{1/\alpha}, & 0 < \alpha < 1, \\ \sigma \log \sigma, & \alpha = 1, \\ \sigma^{-1+2/\alpha}, & 1 < \alpha < 2, \end{cases} \tag{4.11b}$$

$$\phi_3(\sigma) = \begin{cases} \sigma^{1+1/\alpha}, & 0 < \alpha < \frac{1}{2}, \\ \sigma^3 \log \sigma, & \alpha = \frac{1}{2}, \\ \sigma^{-1+2/\alpha}, & \frac{1}{2} < \alpha < 1, \\ \sigma^{1/\alpha}, & 1 \leqslant \alpha < 2, \end{cases} \tag{4.11c}$$

$$\phi_4(\sigma) = \begin{cases} \sigma^{2+1/\alpha}, & 0 < \alpha < \frac{1}{3}, \\ \sigma^5 \log \sigma, & \alpha = \frac{1}{3}, \\ \sigma^{-1+2/\alpha}, & \frac{1}{3} < \alpha \leqslant \frac{1}{2}, \\ \sigma^{1+1/\alpha}, & \frac{1}{2} \leqslant \alpha < 1, \\ \sigma^2 (\log \sigma)^2, & \alpha = 1, \\ \sigma^{-1+3/\alpha}, & 1 < \alpha < 2. \end{cases} \tag{4.11d}$$

Regarding the coefficients $a_j^\pm$ in (4.10), it can be shown that, for certain values of j and α, a_j^+ and a_j^- are related. In particular, the following three relations hold:

$$a_1^- = \lambda^{1/\alpha} a_1^+, \quad 0 < \alpha < 2, \tag{4.12a}$$

$$a_2^- = -\lambda^{2/\alpha} a_2^+, \quad 1 \leqslant \alpha < 2, \tag{4.12b}$$

$$a_3^- = -\lambda^{2/\alpha} a_3^+, \quad \tfrac{1}{2} \leqslant \alpha < 1, \tag{4.12c}$$

where

$$\lambda = |\tau^{(1)}(s_0 -)/\tau^{(1)}(s_0 +)|; \tag{4.12d}$$

see [15] and [16].

Let $\nu^{(k)} = \mathrm{d}^k \nu / \mathrm{d}s^k$. Then, the following conclusions can be drawn from the above:

C1. If $1 < \alpha < 2$, i.e the corner is re-entrant, then the density function ν becomes unbounded at $s = s_0$.

C2. If $1/(1+q) < \alpha < 1/q$, where $q \geqslant 1$ is an integer, then $\nu^{(q)}$ becomes unbounded at $s = s_0$.

C3. If $\alpha = 1/q$, where $q \geqslant 1$ is an integer, then (4.10) does not involve fractional powers of $s - s_0$. In general, however, $a_1^+ \neq a_1^-$, and because of this, $\nu^{(q-1)}$ has a jump discontinuity at $s = s_0$. Also, for some $j > 1$, one of the functions ϕ_j in (4.10) is a logarithmic function of the form

$$\sigma^{2q-1} \log \sigma .$$

This means that in general, the left and right $(2q-1)$th derivatives of ν at $s = s_0$ become unbounded.

Consider now the two cases $G \equiv \Omega_{\mathrm{I}}$ and $G \equiv \Omega_{\mathrm{E}}$, and assume that the arms Γ_1, Γ_2 of the corner z_0 are both straight lines. Then the asymptotic expansion of F at z_0 is given by (4.1c), and we may take, without any loss of generality,

$$\tau(s) - \tau(s_0) = \begin{cases} s - s_0, & s \geqslant s_0, \\ (s_0 - s)\exp(\mathrm{i}\alpha\delta\pi), & s \leqslant s_0, \end{cases} \tag{4.13a}$$

where s denotes arc length and

$$\delta = \begin{cases} 1, & G \equiv \Omega_{\mathrm{I}}, \\ -1, & G \equiv \Omega_{\mathrm{E}}. \end{cases} \tag{4.13b}$$

The above two simplifications imply the following. If Γ_1, Γ_2 are both straight lines, then the asymptotic expansions of the density function corresponding to the interior and exterior mapping problems are given by (4.10), where the functions ϕ_j are defined, for any α, by

$$\phi_j(\sigma) = \sigma^{-1+j/\alpha}, \quad j = 1, 2, 3, \ldots, \tag{4.14a}$$

and the coefficients $a_j^{\pm}$ satisfy

$$a_j^+ = (-1)^{j+1} a_j^-, \quad j = 1, 2, 3, \ldots; \tag{4.14b}$$

see [14,16]. Regarding the nature of the singularity at z_0, the conclusions that emerge from the simpler expansion (4.10), (4.14) are similar to those stated above for the general case. More precisely, the conclusions C1 and C2 remain unaltered. However, when the simpler expansion holds, then the conclusions C3 simplifies to the following, rather surprising, result:

C3: If $\alpha = 1/q$, where $q \geqslant 1$ is an integer, then the functions (4.14a) do not involve any fractional powers, and, because of (4.14b):

(a) if q is odd, then there are no singularities in ν at $s = s_0$;

(b) if q is even, then in general $\nu^{(q-1)}$ has a finite jump discontinuity at $s_0 = s$.

We end this section by restating certain important observations made in [15] in connection with the density functions $\hat{\nu}_{\mathrm{E}}$ and $\hat{\nu}_{\mathrm{D}}$ corresponding to the original formulations of Symm [34,35] for the exterior and doubly-connected problems. In the case of the exterior problem, because of (3.15), the asymptotic expansion of $\hat{\nu}_{\mathrm{E}}$ at z_0 will involve terms of the form

$$(s - s_0)^{-1+1/\alpha} \quad \text{and} \quad (s - s_0)^{-1+1/(2-\alpha)}. \tag{4.15}$$

Similarly, for the doubly-connected problem, if $z_0 \in \partial\Omega_1$, then because of (3.16), the asymptotic expansion of $\hat{\nu}_{\mathrm{D}}$ will involve terms of the form (4.15). This means that for $G \equiv \Omega_{\mathrm{E}}$ and $G \equiv \Omega_{\mathrm{D}}$

with $z_0 \in \partial\Omega_1$, the densities $\hat{\nu}_E$ and $\hat{\nu}_D$ will become unbounded for any $\alpha \neq 1$. That is, if the original formulations of Symm [34,35] are used, a serious singularity might occur at $z = z_0$, even when the corner at z_0 is not re-entrant.

5. Pole and pole-type singularities

Apart from corner singularities, the three mapping functions f_I, f_E, f_D and the function H of (3.2) may also have serious singularities off the boundary, in the complement of the closure of the domain under consideration. The following two sections are concerned with the problem of determining the location and nature of such singularities.

5.1. Singularities associated with Problems P1 and P2

The main purpose of this section is to outline a procedure, which has been used recently in [29], for determining the dominant singularities of the function f_I in Ext($\partial\Omega$), i.e. the singularities of the analytic continuation of f_I which are 'closest' to $\partial\Omega$. Here, however, we extend somewhat the results of [29] by providing some additional information about the singularities of f_I, and by considering the singular behaviour of the exterior mapping function f_E in Int($\partial\Omega$).

With the notation of Problem P1, we let Γ be an analytic arc of $\partial\Omega$ with analytic parametric equation

$$z = \tau(s), \quad s_1 \leqslant s \leqslant s_2, \tag{5.1}$$

and assume that the function

$$z = \tau(\zeta), \tag{5.2}$$

of the complex variable $\zeta = s + \mathrm{i}t$, is one-to-one and analytic in some simply-connected domain Ω^* containing the straight line

$$L = \{\zeta : \zeta = s + \mathrm{i}t,\ s_1 < s < s_2,\ t = 0\}. \tag{5.3}$$

We also assume that Ω^* has a symmetric partition with respect to L, so that

$$\Omega^* = \Omega_1^* \cup L \cup \Omega_2^*, \tag{5.4}$$

where Ω_2^* is the mirror image of Ω_1^* in the straight line L, and where the image of Ω_1^* under the transformation (5.2) is contained within Ω_I. More precisely, we assume that (5.2) maps Ω^* conformally onto a domain $\Omega_1 \cup \Gamma \cup \Omega_2$ so that the straight line L and the domains Ω_i^*, $i = 1, 2$ are mapped respectively onto the arc Γ and the domains $\Omega_1 \subseteq \Omega_I$ and Ω_2. Then the function

$$\phi(z) = \begin{cases} f_I(z), & z \in \Omega_1 \cup \Gamma, \\ 1/\overline{f_I(I(z))}, & z \in \Omega_2, \end{cases} \tag{5.5a}$$

where

$$I(z) = \tau\{\overline{\tau^{[-1]}(z)}\}, \tag{5.5b}$$

is analytic in Ω_1, meromorphic in Ω_2, and defines the analytic continuation of f_I across Γ into Ω_2. This analytic extension of f_I is a particular case of the symmetry principle of analytic arcs, and the points z, $I(z)$ are called symmetric points with respect to the arc Γ; see e.g. [32, p. 102].

It follows from the above that the singularities of f_1 in $\overline{\Omega}_2$, i.e the singularities of the analytic extension ϕ, can be determined by examining the behaviour of the function (5.5). For example the results of the following two theorems can be established easily, by considering the behaviour of ϕ at the symmetric points of the origin 0 with respect to Γ; see [29, pp. 156–57].

Theorem 5.1. *If* $0 \in \Omega_1$, *then the equation*

$$\tau(\zeta) = 0 \tag{5.6}$$

has exactly one root ζ_0 *in* Ω_1^*, *and the function* ϕ *has a simple node at the symmetric point*

$$z_0 = \tau(\overline{\zeta_0}) = I(0) \tag{5.7}$$

of 0 *with respect to* Γ.

Theorem 5.2. *If* $0 \in \partial\Omega_1 \setminus \Gamma$, *then the equation* (5.6) *has at least one root on* $\partial\Omega_1^* \setminus L$. *Let* ζ_0 *be such a root and assume that* τ *is analytic at the points* ζ_0 *and* $\bar{\zeta}_0 \in \partial\Omega_2^* \setminus L$, *so that, for some integers* $m \geqslant 1$ *and* $n \geqslant 1$,

$$\tau(\zeta) = (\zeta - \zeta_0)^m \tau_1(\zeta) \tag{5.8a}$$

and

$$\tau(\zeta) - \tau(\bar{\zeta}_0) = (\zeta - \bar{\zeta}_0)^n \tau_2(\zeta), \tag{5.8b}$$

where τ_1 *and* τ_2 *are analytic and non-zero at* ζ_0 *and* $\bar{\zeta}_0$, *respectively. Then as* $z \to z_0 = \tau(\bar{\zeta}_0)$,

$$\phi(z) \sim (z - z_0)^{-m/n}. \tag{5.9}$$

The following three special cases of Theorem 5.2 occur frequently in applications:

(a) $m = n = 1$. In this case ϕ has a simple pole at z_0.

(b) $m = 2$, $n = 1$. In this case ϕ has a double pole at z_0.

(c) $m = 1$, $n = 2$. In this case ϕ has a branch point singularity of the form

$$(z - z_0)^{-1/2}. \tag{5.10}$$

The theorem stated below extends the results of [29]. and provides additional information about the singular behaviour of ϕ. The theorem emerges easily from the analysis contained in [29, p. 157], and for this reason, its proof is not presented here.

Theorem 5.3. *Let* $\zeta_0 \in \partial\Omega_1^* \setminus L$ *be such that*

$$\tau(\zeta_0) \neq 0 \quad \text{and} \quad \tau'(\bar{\zeta}_0) = 0, \tag{5.11}$$

and assume that τ *is analytic at* ζ_0, $\bar{\zeta}_0$, *so that for some integers* $m \geqslant 1$ *and* $n \geqslant 2$,

$$\tau(\zeta) - \tau(\zeta_0) = (\zeta - \zeta_0)^m \tau_1(\zeta) \tag{5.12a}$$

and

$$\tau(\zeta) - \tau(\bar{\zeta}_0) = (\zeta - \bar{\zeta}_0)^n \tau_2(\zeta), \tag{5.12b}$$

where τ_1, τ_2 *are analytic and non-zero at* ζ_0 *and* $\bar{\zeta}_0$ *respectively. Then as* $z \to z_0 = \tau(\bar{\zeta}_0)$,

$$\phi(z) - \phi(z_0) \sim (z - z_0)^{m/n}. \tag{5.13}$$

The theorem shows that if the values of m and n in (5.12) are such that m/n is not an integer, then the function ϕ has a branch point singularity at z_0. In particular, the case $m=1$, $n=2$, which leads to a singularity of the form

$$\phi(z)-\phi(z_0)\sim(z-z_0)^{1/2}, \tag{5.14}$$

occurs frequently in applications.

Before considering the singularities associated with the exterior mapping Problem P2, we make a number of general remarks, where for simplicity, we refer to the singularities of the analytic extension ϕ as 'pole-type singularities of the mapping function f_{I} with respect to the arc Γ'.

Remark 1. If $0 \notin \Omega_1 \cup (\partial\Omega_1/\Gamma)$ then f_{I} has no poles in Ω_2 and is finite in $\Omega_2 \cup (\partial\Omega_2 \setminus \Gamma)$. However, it is important to observe that f_{I} may have a branch point singularity of the type predicted by Theorem 5.3. More precisely, if $\tau'(\bar{\zeta}_0)=0$, where $\zeta_0 \in \partial\Omega_{\mathrm{I}}^* \setminus L$, and if in (5.12) m and n are such that m/n is not an integer, then f_{I} has a singularity of the form (5.13) at the point $z_0=\tau(\bar{\zeta}_0)$.

Remark 2. If Γ is a straight line segment or a circular arc, then we may take respectively

$$\tau(\zeta)=a+b\zeta \tag{5.15}$$

and

$$\tau(\zeta)=c+r\exp(\mathrm{i}\zeta), \tag{5.16}$$

where a, $b\neq 0$ and c are complex constants and $r \neq 0$ is real. Since the derivatives of (5.15) and (5.16) are never zero and since, in each case, we may take $\Omega_{\mathrm{I}}^* = \tau^{[-1]}(\Omega)$, it follows that only the conclusion of Theorem 5.1 applies. This conclusion leads to the results predicted by the well-known Schwarz reflection principle, i.e. if $0 \in \Omega_1 \cup \partial\Omega_1 \setminus \Gamma$, then f_{I} has a simple pole at the symmetric point $z_0 = I(0)$, where now z_0 coincides with the mirror image of 0 in the straight line or with the geometric inverse of 0 with respect to the circular arc. Therefore, the determination of the dominant pole-type singularities of f_{I} is particularly simple in the case where $\partial\Omega$ consists of straight lines and circular arcs. In fact, this is the only geometry for which Levin et al. [22] and Papamichael and Kokkinos [26] were able to determine the precise location and nature of the singularities of f_{I} in $\mathrm{Ext}(\partial\Omega)$. Examples dealing with singularities corresponding to more general geometries can be found in [29] and also in Section 6 of the present paper.

Remark 3. In the case of the BKM or RM, the procedure for treating pole-type singularities is exactly the same as that used in the case of singular corners. That is, the BKM or RM basis set is formed by introducing into the monomial set (3.2) singular functions that reflect the dominant singularities of f_{I} in $\mathrm{Ext}(\partial\Omega)$. For example, the singular functions for treating a simple pole and a branch point of the form (5.9), at $z_0 \in \mathrm{Ext}\,(\partial\Omega)$, are respectively

$$\eta(z)=\frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{z}{z-z_0}\right\} \tag{5.17}$$

and

$$\eta(z)=\frac{\mathrm{d}}{\mathrm{d}z}\left\{(z-z_0)^{-m/n}\right\}. \tag{5.18}$$

Remark 4. Pole-type singularities can also affect the accuracy of the IEM, but their damaging effect is not as serious as in expansion methods. Here, the cause of the difficulty is that if a boundary segment Γ: $z=\tau(s)$, $s_1<s<s_2$ lies close to a pole-type singularity, then for $s \in (s_1, s_2)$, the density function ν and its derivatives assume large magnitudes; see (4.8). In collocation and Galerkin methods this difficulty can be overcome, quite simply, by using an appropriate non-uniform distribution of boundary nodal points, involving a higher concentration of points on Γ. This means that, in the case of the IEM, we are interested mainly in the approximate location of the pole-type singularities of f_1, and not very much in their precise nature; see [14, Example 1], [16, §5.3, Example 3] and the examples in Section 6 of the present paper.

Remark 5. The form of a pole-type singularity depends on the position of 0 in Ω_1, and the type of singularity changes when 0 coincides with certain 'critical' points. (For example, when Γ is an arc of a conic, then the type of singularity changes when 0 coincides with a foci of the conic, see [29, Section 3].) Because of this, a difficulty arises, in connection with the construction of the BKM and RM basis sets, when 0 lies 'close' to but does not coincide with a critical point. However, as Example 1 of Section 6 illustrates, this difficulty can be overcome by introducing into the basis set a function that reflects the combined effect of the two types of singularities.

Remark 6. Another difficulty occurs, in connection with the BKM and RM, when the regions Ω_2 corresponding to two different analytic arcs overlap. Let Γ_1 and Γ_2 be two such arcs and denote by $\Omega_2^{(1)}$ and $\Omega_2^{(2)}$ the corresponding Ω_2 regions. Then, in general, the function f_1 has two different continuations in $\Omega_2^{(1)} \cap \Omega_2^{(2)}$, which may be regarded as the extensions of f_1 on two different sheets of a Riemann surface due to a branch point on $\partial\Omega$ or in $\text{Ext}(\partial\Omega)$. This situation arises frequently when Γ_1 and Γ_2 are the arms of a corner, where a serious branch point singularity occurs. In such cases, it is in general sufficient to reflect only the corner singularity, by introducing into the BKM or RM basis set functions of the form (4.2).

We consider next the exterior mapping problem P2 and recall that, for the application of the BKM or RM, we are interested in the singular behaviour of the function $\hat{f}_1$ associated with the interior domain $\hat{\Omega}_1$.

As before, we let Γ be an analytic arc of $\partial\Omega$ with analytic parametric equation (5.1). Then, under the inversion

$$\hat{z}=z^{-1}, \tag{5.19}$$

Γ is transformed into an analytic arc $\hat{\Gamma}$ with parametric equation

$$\hat{z}=\hat{\tau}(s), \quad s_1<s<s_2, \tag{5.20a}$$

where

$$\hat{\tau}(s)=1/\tau(s). \tag{5.20b}$$

Therefore, the pole-type singularities of $\hat{f}_1$ with respect to $\hat{\Gamma}$ can be determined by the procedure outlined above, with $\hat{\tau}$ replacing the function τ. Now, however, for many curves $\partial\Omega$ that occur in practice, the intermediate transformation (5.19) makes it less likely for Theorems 5.1 and 5.2 to predict singularities of the mapping function $\hat{f}_1$. This can be explained as follows.

With reference to (5.4), let

$$\hat{\Omega}^* = \hat{\Omega}_1^* \cup L \cup \hat{\Omega}_2^* \tag{5.21}$$

be the symmetric partition associated with the function

$$\hat{z} = \hat{\tau}(\zeta), \tag{5.22}$$

and observe that the singularities predicted by Theorems 5.1 and 5.2 occur at points given by

$$\hat{z}_0 = \hat{\tau}(\bar{\zeta}_0), \tag{5.23}$$

where $\zeta_0 \in \hat{\Omega}_1^* \cup \partial\hat{\Omega}_1^*/L$ is a root of the equation

$$\hat{\tau}(\zeta) = 0. \tag{5.24}$$

Also, observe that (5.24) can only have a root at a point where τ becomes unbounded. This means that if, as is frequently the case, τ is an entire function, and in addition, the largest admissible region $\hat{\Omega}_1^*$ is finite, then $\hat{f}_{\rm I}$ does not have simple poles or singularities of the form (5.9) in $\hat{\Omega}_2 \cup (\partial\hat{\Omega}_2 \backslash \Gamma)$, $\hat{\Omega}_2 = \hat{\tau}(\hat{\Omega}_2^*)$. However, since

$$\hat{\tau}'(\zeta) = -\tau'(\zeta)/\{\tau(\zeta)\}^2, \tag{5.25}$$

singularities of the type predicted by Theorem 5.3 can still occur. The above remarks are illustrated by the following three examples.

(i) If the original boundary $\partial\Omega$ is a polygon, then $\hat{f}_{\rm I}$ has no pole-type singularities.

(ii) If $\partial\Omega$ consists of straight line segments and circular arcs, then the only pole-type singularities of $\hat{f}_{\rm I}$ are due to the circular arcs. More precisely, a singularity occurs only if the centre of a circular arc is in $\operatorname{Int}(\partial\Omega)$ and does not coincide with the origin of the z-plane. If $z_0 \in \operatorname{Int}(\partial\Omega)$ is such a centre, then $\hat{f}_{\rm I}$ has a simple pole at the point $\hat{z}_0 = 1/z_0 \in \operatorname{Ext}(\partial\hat{\Omega})$.

(iii) If $\partial\Omega$ is the ellipse

$$x^2/a^2 + y^2/b^2 = 1, \quad 0 < b < a, \tag{5.26}$$

i.e. if

$$\tau(s) = ae\cos(s - \mathrm{i}\eta), \quad -\pi \leqslant s \leqslant \pi, \tag{5.27a}$$

where

$$e = \{1 - b^2/a^2\}^{1/2} \quad \text{and} \quad \cos h\eta = 1/e, \tag{5.27b}$$

then the only two pole-type singularities of $\hat{f}_{\rm I}$ are of the form $(\hat{z} - \hat{z}_0)^{1/2}$ and occur at the points $\hat{z}_0 = \pm 1/ae$.

The results (i) and (ii) can be established, as in [27, p. 193], directly from the Schwarz reflection principle. The result (iii) can be obtained at once from the known form of $f_{\rm E}$, which in the case of the ellipse (5.26) is

$$f_{\rm E}(z) = \{z + (z^2 - a^2e^2)^{1/2}\}/(a+b). \tag{5.28}$$

It is however instructive to also establish the result (iii) by considering the form of the function (5.27). This can be done as follows.

Since

$$\tau(\zeta) = ae\cos(\zeta - \mathrm{i}\eta) \tag{5.29}$$

is an entire function, and since the largest admissible symmetric domain $\hat{\Omega}^*$ is the rectangle

$$\hat{\Omega}^* = \{ z = s + \mathrm{i}t : -\pi < s < \pi, -\eta < t < \eta \}, \tag{5.30}$$

it follows that the function $\hat{f}_{\mathrm{I}}$ associated with the ellipse (5.26) does not have singularities of the form predicted by Theorems 5.1 and 5.2. However,

$$\hat{\tau}(\zeta) = \{\sec(\zeta - \mathrm{i}\eta)\}/ae, \tag{5.31}$$

and therefore, for any

$$\zeta_0 = k\pi - \mathrm{i}\eta, \quad k = 0, \pm 1, \pm 2, \ldots, \tag{5.32}$$

$$\hat{\tau}'(\bar{\zeta}_0) = 0, \quad \hat{\tau}''(\bar{\zeta}_0) \neq 0 \quad \text{and} \quad \hat{\tau}(\zeta_0) \neq 0. \tag{5.33}$$

The result (iii) then follows from Theorem 5.3 with $m = 1$ and $n = 2$, because

$$\hat{\tau}(k\pi + \mathrm{i}\eta) = \pm 1/ae, \quad k = 0, \pm 1, \pm 2, \ldots . \tag{5.34}$$

Finally, we note that the situation regarding the effect and treatment of singularities, in connection with the IEM solution of Problem P2, is exactly as described in Remark 5. To see this, let $\hat{z}_0 \in \hat{\Omega}_{\mathrm{E}}$ be a point where $\hat{f}_{\mathrm{I}}$ has a pole-type singularity, assume that $z_0 = 1/\hat{z}_0 \in \Omega_{\mathrm{I}}$ lies close to an arc $\Gamma: z = \tau(s)$, $s_1 < s < s_2$ of $\partial\Omega$, and recall that f_{E} is related to $\hat{f}_{\mathrm{I}}$ by (2.8). Thus, as in the case of Problem P1, (4.8) implies that the density function ν and its derivatives assume large magnitudes for $s \in (s_1, s_2)$.

5.2. Singularities associated with Problem P3

In the case of Problem P3, the situation regarding the singularities in compl$(\bar{\Omega}_{\mathrm{D}})$ of the mapping function f_{D} and of the function H, defined by (3.3), is much more involved. In fact, Papamichael and Kokkinos [28], who studied the application of the ONM and the VM, were unable to provide any information about the singularities of the analytic extensions of these two functions. However, the problem has also been studied recently in [30], where it is shown that, in many cases, f_{D} and H have singularities in compl$(\bar{\Omega}_{\mathrm{D}})$ at the so-called 'common symmetric points' with respect to the boundary components $\partial\Omega_1$ and $\partial\Omega_2$.

Let Γ_j, $j = 1, 2$ be analytic arcs of $\partial\Omega_j$, $j = 1, 2$ respectively. Also, let $I_j(z)$, $j = 1, 2$ be the two functions corresponding to (5.5b), which define respectively pairs of symmetric points $(z, I_j(z))$, $j = 1, 2$ with respect to the arcs Γ_j, $j = 1, 2$. Then, two points

$$\zeta_1 \in \mathrm{Int}(\partial\Omega_1) \quad \text{and} \quad \zeta_2 \in \mathrm{Ext}(\partial\Omega_2) \tag{5.35}$$

are said to be common symmetric points with respect to Γ_1 and Γ_2 if

$$\zeta_1 = I_j(\zeta_2) \quad \text{and} \quad \zeta_2 = I_j(\zeta_1), \quad j = 1, 2, \tag{5.36}$$

i.e. if ζ_1 and ζ_2 are both fixed points of the two composite functions

$$S_1 = I_1 \circ I_2 \quad \text{and} \quad S_2 = I_2 \circ I_1. \tag{5.37}$$

Although there are geometries for which no common symmetric points exist, in many cases the points ζ_1 and ζ_2 can be determined easily from the functions (5.37). In such cases, an analysis based essentially on the repeated application of the Schwarz reflection principle shows that, under certain conditions, the points ζ_1 and ζ_2 are singular points of the functions f_{D} and H. Full details of this analysis can be found in [30], where it is also shown that, for the purpose of the

ONM and VM, the singular behaviour of H may be reflected approximately by introducing into the monomial set (3.5) the two singular functions

$$\eta_1(z) = 1/(z - \zeta_1) - 1/z \tag{5.38a}$$

and

$$\eta_2(z) = 1/(z - \zeta_2). \tag{5.38b}$$

In the case of the IEM, the effect and treatment of the singularities of f_D at the points ζ_1 and ζ_2 is exactly the same as in the cases of Problems P1 and P2. In what follows we illustrate the above remarks by considering the case where Ω_D is a regular polygon with a circular hole. This special case is studied fully in [30, Section 2].

Let

$$\Omega_D = \mathrm{Ext}(\partial\Omega_1) \cap \mathrm{Int}(\partial\Omega_2), \tag{5.39a}$$

where the inner boundary $\partial\Omega_1$ is the circle

$$\partial\Omega_1 = \{ z : |z| = a,\ a < 1 \} \tag{5.39b}$$

and the outer boundary $\partial\Omega_2$ is a concentric N-sided regular polygon with

$$l = \{ z : z = 1 + \mathrm{i}y, |y| \leqslant \tan(\pi/N) \} \tag{5.39c}$$

as one of its sides. That is,

$$\partial\Omega_2 = \bigcup_{j=1}^{N} \gamma_j, \tag{5.39d}$$

where

$$\gamma_j = l\omega_N^{j-1}, \quad \omega_N = \exp\{2\pi\mathrm{i}/N\}, \qquad j = 1, 2, \ldots, N. \tag{5.39e}$$

Then, with

$$\Gamma_1 = \{ z : z = a\mathrm{e}^{\mathrm{i}\theta}, |\theta| \leqslant \pi/N \} \quad \text{and} \quad \Gamma_2 = l, \tag{5.40}$$

we have that

$$I_1(z) = a^2/\bar{z}, \quad I_2(z) = 2 - \bar{z}, \tag{5.41}$$

and hence

$$S_1(z) = a^2/(2 - z), \qquad S_2(z) = 2 - a^2/z. \tag{5.42}$$

Therefore, in this particular case, the common symmetric points with respect to Γ_1 and Γ_2 are

$$\zeta_1 = 1 - (1 - a^2)^{1/2} \quad \text{and} \quad \zeta_2 = 1 + (1 - a^2)^{1/2}. \tag{5.43}$$

More precisely, in this case, there are N pairs of common symmetric points associated with the circle $\partial\Omega_1$ and each of the N sides of the polygon $\partial\Omega_2$. These points are respectively

$$\zeta_1^{(j)} = \zeta_1\omega_N^{j-1} \quad \text{and} \quad \zeta_2^{(j)} = \zeta_2\omega_N^{j-1}, \quad j = 1, 2, \ldots, N, \tag{5.44}$$

where ω_N is as in (5.39e).

Let G denote the subdomain of Ω_D which is bounded by Γ_1, Γ_2 and the two rays $\theta = \pm\pi/N$. Also, let S_j, $j = 1, 2$ be the functions (5.42), and define recursively the point sequences $\{z_{k,1}\}$

and $\{z_{k,2}\}$ by means of

$$z_{k+1,j} = S_j(z_{k,j}), \quad k = 0, 1, 2, \ldots, \tag{5.45}$$

with $j = 1$ and $j = 2$ respectively. Then, the following results are established in [30, pp. 95–97]:

(i) For any $z_{0,j} \in \overline{G}$,

$$\lim_{k \to \infty} z_{k,j} = \zeta_j, \quad j = 1, 2, \tag{5.46}$$

and in each case the convergence is linear.

(ii) The mapping function f_{D} can be continued analytically across Γ_1 and Γ_2 into two regions which contain respectively the real intervals $\zeta_1 < x < a$ and $1 < x < \zeta_2$.

(iii) Let

$$\alpha_1 = -\alpha_2 = \log M / \log(\zeta_2/a), \tag{5.47}$$

where M is the conformal modulus of Ω_{D}. Then, for any $z_{0,j} \in \overline{G}$,

$$\lim_{k \to \infty} \left\{ (z_{k,j} - \zeta_j)^{-\alpha_j} f_{\mathrm{D}}(z_{k,j}) \right\} = \mu_j, \quad j = 1, 2, \tag{5.48}$$

where μ_1 and μ_2 are finite and non-zero numbers which depend respectively on the points $z_{0,1}$ and $z_{0,2}$.

(iv) For any $z_{0,j} \in \overline{G}$,

$$\lim_{k \to \infty} (z_{k,j} - \zeta_j) H(z_{k,j})\} = \lambda_j, \quad j = 1, 2, \tag{5.49}$$

where λ_1 and λ_2 are finite and, in general, non-zero numbers which depend respectively on the points $z_{0,1}$ and $z_{0,2}$.

The above results show that, in the case of the domain (5.39), the common symmetric points (5.43) are singular points of both the functions f_{D} and H. The results also justify the use of functions of the form (5.38) for approximately reflecting the singular behaviour of the function H. Similar results can be established for other more general geometries, and such examples can be found in [30, Section 3].

6. Numerical examples

Many numerical examples, illustrating the very considerable improvement in accuracy which is achieved by treating the singularities of the conformal maps in the manner described in earlier sections, can be found in references [3,14–16,22,26–30]. (Of these, [22,26,29] and [27] concern the use of the BKM and RM for the solution of Problems P1 and P2 respectively, [3,28,30] the use of the ONM and VM for the solution of Problem P3, and [14–16] the use of the IEM.) In this section we present five numerical examples whose purpose is to illustrate certain important aspects of the treatment of singularities which are not widely understood. More specifically, the purpose of the examples given below is to illustrate the following:

(i) In the application of expansion methods, the effect of pole-type singularities that lie close to the boundary can, in practice, be as damaging as that of serious corner singularities.

(ii) In expansion methods, the use of singular basis functions that reflect only approximately the pole-type singularities of the mapping often leads to some improvement in accuracy.

However, much better improvement is achieved when the exact location and nature of the dominant pole-type singularities are known, and the corresponding 'exact' singular basis functions are used.

(iii) Pole-type singularities that lie close to the boundary may also affect the IEM. As was previously remarked, in collocation and Galerkin methods this difficulty can be overcome, quite simply, by using an appropriate non-uniform distribution of the boundary nodal points.

(iv) The use of preliminary transformations does not necessarily completely remove the effect of corner singularities.

The expansion methods used in our examples are respectively the BKM for the three interior and one exterior domains of Examples 1, 2, 3, 5, and the ONM for the doubly-connected domain of Example 4. The computational details of the BKM and ONM procedures used are exactly as described in references [22,26–28]. Regarding the IEM, the method used in all examples is the collocation method of [15]. This method is based on approximating the density function ν by cubic splines and 'corner singular' functions, and it is described fully in [15,16].

In each example and for each method used, we give an estimate of the maximum error in the modulus of the corresponding approximate conformal map. In the cases of Problems P1 and P2, this error estimate is given respectively by

$$E_n = \max_j \left|1 - |f_{\mathrm{I},n}(z_j)|\right| \tag{6.1a}$$

and

$$E_n = \max_j \left|1 - |f_{\mathrm{E},n}(z_j)|\right|, \tag{6.1b}$$

where $f_{\mathrm{I},n}$ and $f_{\mathrm{E},n}$ denote the BKM or IEM approximations to f_I and f_E and where, in each case, $\{z_j\}$ is a set of 'boundary test points' on $\partial\Omega$. Similarly, in the case of Problem P3 the error estimate is given by

$$E_n = \max\left\{ \max_j \left|r_1 - |f_{\mathrm{D},n}(z_{1,j})|\right|, \max_j \left|r_1 M_n - |f_{\mathrm{D},n}(z_{2,j})|\right| \right\}, \tag{6.1c}$$

where $f_{\mathrm{D},n}$ and M_n denote respectively the ONM or IEM approximation to f_D and M, and $\{z_{1,j}\}$, $\{z_{2,j}\}$ are two sets of boundary test points on $\partial\Omega_1$ and $\partial\Omega_2$, respectively. In the cases of the BKM and ONM, the subscript n in (6.1) refers to the 'optimum' number $n = N_{\mathrm{opt}}$ of basis functions, which gives maximum accuracy in the sense described in [22, p. 178]. In the case of the IEM, n refers to the size of the linear system whose solution gives the collocation approximation to ν; see [15, p. 303].

In presenting the results, we use the abbreviations BKM/MB and BKM/AB to denote respectively the BKM with monomial basis (3.2) and with augmented basis. Similarly, we use ONM/MB and ONM/AB to denote the ONM with monomial basis (3.5) and with augmented basis.

The BKM and ONM results were computed on a CRAY I computer using programs written in single precision Fortran. Single length working on the CRAY I is between 14 and 15 significant figures. The IEM results were computed on a DEC 10 computer using programs written in double precision DEC Algol. Single length working on the DEC 10 is between 8 and 9 significant figures.

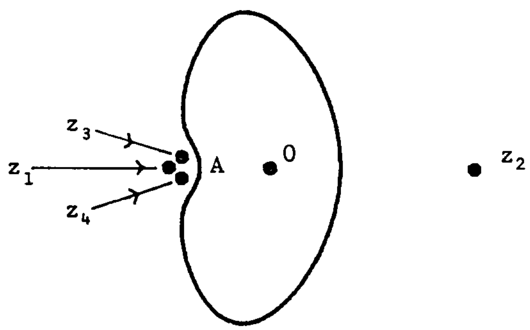

Fig. 1.

Example 1. Let Ω_I be the bean-shaped interior domain illustrated in Fig. 1. Its boundary $\partial\Omega$ is the analytic curve

$$z = \tau(s) = x(s) + \mathrm{i}\,y(s), \quad -\pi \leqslant s \leqslant \pi, \tag{6.2a}$$

where

$$x(s) = \tfrac{9}{4}\{0.2\cos(s) + 0.1\cos(2s) - 0.1\}$$

and (6.2b)

$$y(s) = \tfrac{9}{4}\{0.35\sin(s) + 0.1\sin(2s) - 0.02\sin(4s)\}.$$

The conformal mapping of the above domain is considered in Reichel [31, Example 2.3], where the problem of determining and treating the singularities of the function f_I is also discussed briefly. For the domain of Fig. 1, Reichel predicts, by arguments based on intuitive geometric considerations, that f_I has an 'approximate' simple pole at the point

$$\tilde{z}_1 = -0.61. \tag{6.3}$$

In what follows we show that f_I does in fact have a simple pole at a point reasonably close to $\tilde{z}_1$. However, we also show that this pole is not the 'dominant' singularity of f_I, i.e. there are other singularities at points that lie closer to $\partial\Omega$ than $\tilde{z}_1$. We do this, as outlined in section 5.1, by determining the zeros of the two functions $\tau(\zeta)$ and $\tau'(\zeta)$ in a neighbourhood of the straight line

$$L = \{\zeta\colon \zeta = s + \mathrm{i}t, -\pi \leqslant s \leqslant \pi,\ t = 0\}.$$

The details are as follows.

The function $\tau(\zeta)$ has a simple zero at each of the points

$$\zeta_1 = \mathrm{i}\ 0.660\,656\,454\,578$$

and

$$\zeta_2 = -\pi + \mathrm{i}\ 0.532\,733\,445\,375.$$

Therefore, f_I has a simple pole at each of the two points

$$z_1 = \tau(\bar{\zeta}_1) = -0.650\,225\,813\,375$$

and

$$z_2 = \tau(\bar{\zeta}_2) = 1.311\,282\,520\,094;$$

see Fig. 1.

The function $\tau'(\zeta)$ has a simple zero at each of the points

$$\zeta_3 = 0.376\,736\,147\,099 - \mathrm{i}\,0.492\,754\,434\,660$$

and

$$\zeta_4 = -\bar{\zeta}_3.$$

Therefore, since $\tau(\bar{\zeta}_j) \neq 0$, $j = 3, 4$, f_{I} has singularities of the form $(z - z_j)^{1/2}$, $j = 3, 4$, at the points

$$z_3 = \tau(\zeta_3) = -0.565\,672\,547\,402 + \mathrm{i}\,0.068\,412\,683\,544$$

and

$$z_4 = \tau(\zeta_4) = \bar{z}_3;$$

see Theorem 5.3.

BKM/AB: The points z_1, z_3 and z_4 lie close to each other. For this reason we construct the function

$$\mu(z) = \left\{ \frac{\left((z - z_4)^{1/2} - (z_3 - z_4)^{1/2}\right)^{1/2}}{(z - z_1)} \right\}'$$

and, because of the reflected symmetry of Ω, we take the augmented basis to be

$$\eta_1(z) = \{z/(z - z_1)\}', \quad \eta_2(z) = \mu(z) + \mu(\bar{z}), \quad \eta_3(z) = \mathrm{i}\left(\mu(z) - \overline{\mu(\bar{z})}\right),$$
$$\eta_4(z) = \{z/(z - z_2)\}', \quad \eta_{4+j}(z) = z^{j-1}, \qquad j = 1, 2, 3, \ldots.$$

IEM: We use a uniform mesh with respect to the parameter of s of (6.2). This gives rise to a non-uniform distribution of the nodal points with respect to arc length and, because $\tau'(\zeta)$ has zeros at the points ζ_3, ζ_4, this distribution involves a higher concentration of nodes near the point $A \equiv \tau(0)$. That is, in this example, a uniform mesh with respect to s defines a suitable non-uniform distribution of nodes for dealing with the dominant pole-type singularities at the points z_1, z_3, z_4; see Fig. 1.

Numerical results:
BKM/MB: $N_{\mathrm{opt}} = 30$, $E_{30} = 3.6 \times 10^{-2}$.
BKM/AB: $N_{\mathrm{opt}} = 20$, $E_{20} = 1.4 \times 10^{-5}$, $R_{20} = 0.570\,943\,922$.
IEM: $E_{67} = 3.2 \times 10^{-6}$, $R_{67} = 0.570\,943\,972$.

(In the above, the R_n denote approximations to the so-called conformal radius $R = 1/f_{\mathrm{I}}'(0)$ of Ω_{I} at 0.)

The use of an augmented basis involving only the singular function $z/(z - \bar{z}_1)\}'$, corresponding to the approximate simple pole (6.3) of Reichel [31], leads to the inferior BKM/AB results: $N_{\mathrm{opt}} = 10$, $E_{10} = 3.3 \times 10^{-3}$.

Example 2. Let Ω_{I} be the S-shaped interior domain illustrated in Fig. 2 whose boundary is the analytic curve

$$\begin{aligned} z &= \tau(s) \\ &= 2\cos(s) + \mathrm{i}\{\sin(s) + 2\cos^3(s)\}, \quad 0 \leqslant s \leqslant 2\pi. \end{aligned} \tag{6.4}$$

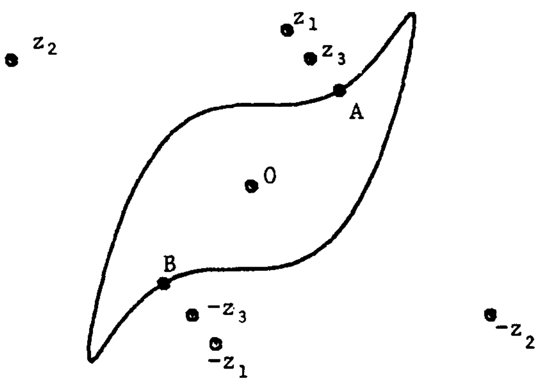

Fig. 2.

The mapping of this domain has been considered by Reichel [31, Example 1.1] and also by Ellacott [4, Example 3]. However, neither Reichel nor Ellacott provide any information about the pole singularities of the function f_1.

The following can be deduced by considering, as in Example 1, the zeros of the two functions $\tau(\zeta)$ and $\tau'(\zeta)$.

(i) The function f_1 has a simple pole at each of the four points $\pm z_1, \pm z_2$, where

$$z_1 = 0.454\,688\,019\,275 + \mathrm{i}\,1.902\,477\,887\,249$$

and

$$z_2 = -2.884\,939\,136\,035 + \mathrm{i}\,1.584\,060\,902\,263.$$

(ii) The function f_1 has singularities of the form $(z \pm z_3)^{1/2}$ at the points $\pm z_3$, respectively, where

$$z_3 = 0.731\,151\,125\,904 + \mathrm{i}\,546\,446\,051\,506.$$

BKM/AB: Because of the two-fold rotational symmetry about the origin, the monomial basis set is taken to be

$$z^{2j}, \quad j = 0, 1, 2, \ldots. \tag{6.5}$$

For the same reason, the augmented basis is constructed by introducing into the set (6.5) the three singular functions

$$\left\{z/\left(z^2 - z_j^2\right)\right\}, \quad j = 1, 2 \quad \text{and} \quad \left\{(z_3 - z)^{1/2} - (z_3 + z)^{1/2}\right\}'.$$

IEM: We use a uniform mesh with respect to the parameter s of (6.4). As in Example 1, because of the zeros of $\tau'(\zeta)$, the resulting distribution of boundary nodal points involves higher concentrations of nodes near the points $A = \tau(1)$ and $B = \tau(1 + \pi)$, which lie close to the singular points z_1, z_3 and $-z_1, -z_3$, respectively; see Fig. 2.

Numerical results:
BKM/MB: $N_{\text{opt}} = 17$, $E_{17} = 1.2 \times 10^{-3}$.
BKM/AB: $N_{\text{opt}} = 17$, $E_{17} = 1.1 \times 10^{-5}$, $R_{17} = 1.169\,091\,766$.
IEM: $E_{67} = 6.0 \times 10^{-6}$, $R_{67} = 1.169\,092\,036$.

(As in Example 1, the R_n denote approximations to the conformal radius of Ω_1 at 0.)

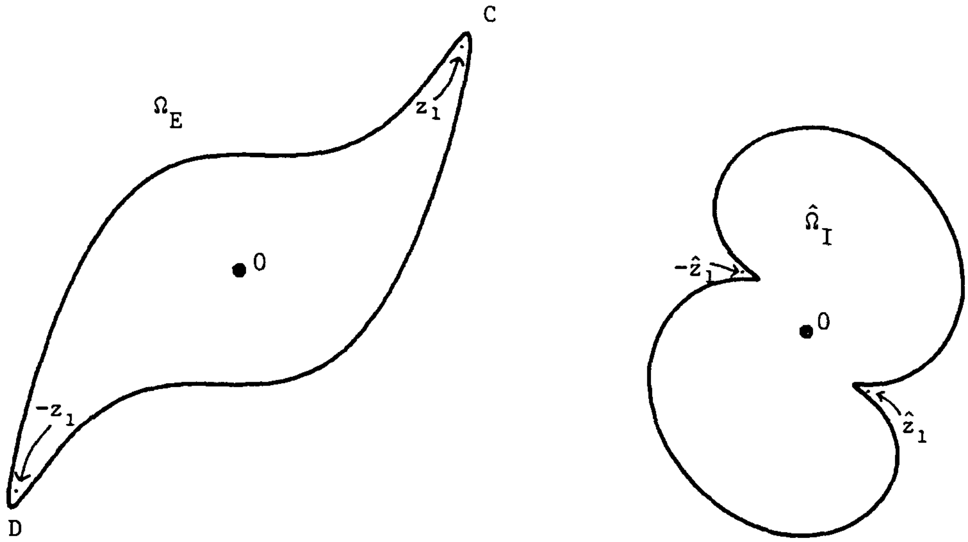

Fig. 3.

Example 3. Let Ω_E be the domain exterior to the S-shaped curve of Fig. 2, and recall the notation of Section 5.1. That is, let $\hat{\Omega}_I$ be the image of Ω_E under the inversion $\hat{z} = z^{-1}$, denote by f_I the mapping function associated with $\hat{\Omega}_I$, and let $\hat{\tau}(\zeta) = 1/\tau(\zeta)$, $\zeta = s + it$, where τ is defined by (6.4). Then the following can be deduced by considering the zeros of the function $\tau'(\zeta)$.

The function $\hat{\tau}'(\zeta)$ has a simple zero at each of the points

$$\zeta_1 = 0.150\,192\,355\,327 + \mathrm{i}\,0.052\,562\,788\,315$$

and

$$\zeta_2 = \pi + \zeta_1.$$

Also, $\hat{\tau}(\bar{\zeta}_j) \neq 0$, $j = 1, 2$, and $\hat{\tau}(\zeta_2) = -\hat{\tau}(\zeta_1)$. Therefore, by Theorem 5.3, the mapping function $\hat{f}_I$ has singularities of the form $(\hat{z} \pm \hat{z}_1)^{1/2}$ at the points $\pm \hat{z}_1 \in \text{Ext}(\partial\hat{\Omega})$, where

$$\hat{z}_1 = \hat{\tau}(\zeta_1) = 0.240\,671\,315\,273 - \mathrm{i}\,0.252\,916\,790\,376.$$

Of course, this also means that the function f_E has singularities of the form $(z \pm z_j)^{1/2}$ at the points $\mp z_1 \in \text{Int}(\partial\Omega)$, where $z_1 = 1/\hat{z}_1$; see Fig. 3.

BKM/AB: Because of the two-fold rotational symmetry of the domain Ω_I, the monomial basis set for determining the BKM/MB approximation to $\hat{f}_I$ is taken to be

$$\hat{z}^{2j}, \quad j = 0, 1, 2, \ldots. \tag{6.6}$$

In this example we consider the use of the following two augmented basis sets:

(i) *AB1:* This set is formed by introducing into (6.6) the singular function

$$\left\{(\hat{z}_1 - \hat{z})^{1/2} - (\hat{z}_1 + \hat{z})^{1/2}\right\}',$$

which, because of the rotational symmetry, corresponds to two singular functions of the form $(\hat{z} \pm \hat{z}_j)^{1/2}$.

(ii) *AB2:* This is the set $\{\eta_j(z)\}$ defined by

$$\left.\begin{array}{l}\eta_{2j-1}(z)=\left\{(\hat{z}_1-\hat{z})^{j-1/2}-(\hat{z}_1+\hat{z})^{j-1/2}\right\}',\\ \eta_{2j}(z)=\hat{z}^{2(j-1)},\end{array}\right\}\quad j=1,2,3,\ldots .$$

(That is, AB2 is constructed by assuming that at each of the points $\pm z_1$, $\hat{f}_1$ has an asymptotic expansion of the form (4.1b), with $\alpha = 2$.)

IEM: We use the same uniform mesh as in Example 2. (Because of the zeros of $\tau'(\zeta)$ at the points ζ_1, ζ_2, this mesh also involves higher concentrations of nodes near the points $C \equiv \tau(0)$ and $D = \tau(\pi)$, which lie close to the singular points $\pm z_1$; see Fig. 3).

Numerical results:
BKM/MB: $N_{\text{opt}} = 30$, $E_{30} = 4.3 \times 10^{-1}$.
BKM/AB1: $N_{\text{opt}} = 30$, $E_{30} = 2.6 \times 10^{-2}$.
BKM/AB2: $N_{\text{opt}} = 13$, $E_{13} = 1.6 \times 10^{-5}$, $c_{13} = 1.772\,414\,144$.
IEM: $E_{67} = 6.0 \times 10^{-8}$, $c_{67} = 1.772\,414\,138$.

(In the above the c_n denote approximations to the capacity of the curve $\partial\Omega$.)

The numerical results confirm our remark that in expansion methods, the effect of pole-type singularities can, in practice, be as damaging as that of serious corner singularities.

Example 4. Let Ω_D be a square with a 'large' circular hole. More specifically, let

$$\Omega_D = \text{Ext}(\partial\Omega_1) \cup \text{Int}(\partial\Omega_2), \tag{6.7a}$$

where

$$\partial\Omega_1 = \{z : |z| = 0.99\} \quad \text{and} \quad \partial\Omega_2 = \{z : z = 1 + iy, |y| \leqslant 1\}, \tag{6.7b}$$

i.e. (6.7) is the special case $N = 4$, $a = 0.99$ of the doubly-connected domain considered at the end of Section 5.2. Then in this particular case, the four pairs of common symmetric points, where the functions f_D and H have singularities of the form described by (5.48) and (5.49), are respectively

$$\zeta_{1,j} = (0.858\,932\,640)(\mathrm{i})^{j-1}, \quad \zeta_{2,j} = (1.141\,067\,360)(\mathrm{i})^{j-1}, \qquad j = 1, 2, 3, 4.$$

The mapping of the above domain has been considered recently in [30] and [16], and the ONM. IEM details given below are taken respectively from these two references.

ONM/AB: Because of the four-fold rotational symmetry, the monomial basis set is taken to be

$$z^{4j-1}, \quad j = \pm 1, \pm 2, \ldots . \tag{6.8}$$

For the same reason, the augmented basis constructed by introducing into the set (6.8) the two singular functions

$$4z^3/\left(z^4 - \zeta_{1,1}^4\right) - 4/z \quad \text{and} \quad 4z^3/\left(z^4 - \zeta_{2,1}^4\right);$$

see [30, p. 102].

IEM: In this example we perform the computations by using the following two distributions of nodes:

(i) *IEM1:* A uniform mesh, involving equally spaced nodes on each side of the square and on the circular inner boundary.

(ii) *IEM2:* A non-uniform mesh, such that the interval lengths between consecutive nodes decrease in arithmetic progression towards the points $\pm 0.99, \pm 0.99\mathrm{i}$ on $\partial\Omega_1$, and $\pm 1, \pm \mathrm{i}$ on $\partial\Omega_2$; see [16, p. 116] and [36, p. 119].

Numerical results:
ONM/MB: $N_{\mathrm{opt}} = 25$, $E_{25} = 1.9 \times 10^{-3}$.
ONM/AB: $N_{\mathrm{opt}} = 23$, $E_{23} = 1.8 \times 10^{-9}$, $M_{23} = 1.040\,412\,14$.
IEM1: $E_{71} = 1.9 \times 10^{-6}$.
IEM2: $E_{71} = 5.8 \times 10^{-8}$, $M_{71} = 1.040\,412\,13$.

(In the above, the M_n denote approximations to the conformal modulus M of Ω_{D}.)

Example 5. (i) Let Ω_{I} be the interior domain whose boundary consists of the straight line

$$\Gamma_1:\ z = 1 - 2s, \quad -1 \leqslant s \leqslant 0 \tag{6.9a}$$

and the two half ellipses

$$\Gamma_2:\ z = -1 + 2\cos(s) + \mathrm{i}\sin(s), \quad 0 \leqslant s \leqslant \pi \tag{6.9b}$$

and

$$\Gamma_3:\ z = 3\cos(s) + \mathrm{i}1.5\sin(s), \quad \pi \leqslant s \leqslant 2\pi; \tag{6.9c}$$

see Fig. 4(a).

The above domain has a re-entrant corner of interior angle $\frac{3}{2}\pi$ at the point $A \equiv (1, 0)$, and corners of angles π and $\frac{1}{2}\pi$ at the points $B \equiv (-3, 0)$ and $C \equiv (3, 0)$, respectively. Therefore, the mapping function f_{I} has a serious branch point singularity at A, a less serious one at B, and a 'weak' singularity at C; see (4.7a). The function f_{I} also has simple poles at the symmetric points

$$z_1 = \tfrac{2}{3} + \mathrm{i}\,1.885\,618\,083\,164 \quad \text{and} \quad z_2 = -\mathrm{i}\,3.464\,101\,615\,318 \tag{6.10}$$

of 0 with respect to the arcs Γ_2 and Γ_3, respectively; see Theorem 5.1 and [29, Section 3.1].

BKM/AB: The augmented basis is formed by introducing into the monomial set (3.2) the singular functions

$$\{z/(z - z_j)\}', \quad j = 1, 2, \qquad (z-1)^{(j-3)/3}, \quad j = 2, 4, 5, 7, 8,$$

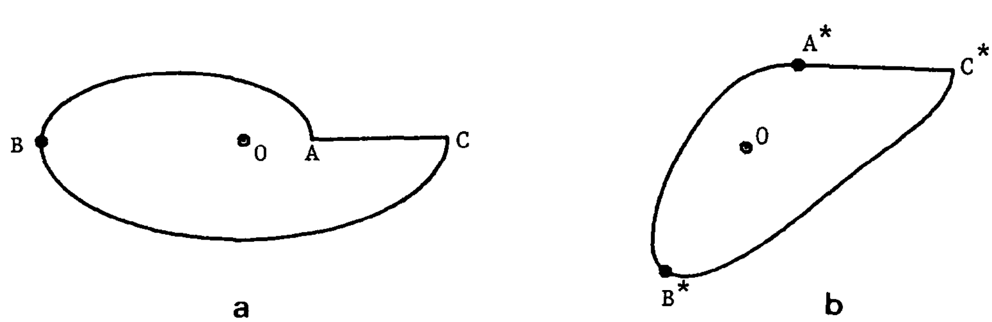

Fig. 4.

and

$$\{(z+3)^2\log(z+3)\}', \quad \{(z+3)^3(\log(z+3))^2\}',$$

which correspond respectively to the pole singularities at the points (6.10) and the corner singularities at the points A and B.

IEM: The procedure of [16] is designed to treat all corner singularities, i.e., the IEM uses singular functions for dealing with the singularities at each of the points A, B and C. In this example the pole singularities at the points (6.10) are not close to $\partial\Omega$, and we use a uniform mesh with respect to the parameter s of (6.9).

Numerical results:
BKM/MB: $N_{\text{opt}}=30$, $E_{30}=1.4\times10^{-1}$.
BKM/AB: $N_{\text{opt}}=29$, $E_{29}=2.3\times10^{-6}$, $R_{29}=1.219\ 403\ 701$.
IEM: $E_{79}=4.5\times10^{-4}$, $R_{79}=1.219\ 413\ 687$.

(As before, the R_n denote approximations to the conformal radius of Ω_{I} at 0.)

(ii) We now consider the possibility of treating the corner singularity at A by using the preliminary transformation

$$z\to(z-1)^{2/3}-(-1)^{2/3}. \tag{6.11}$$

This transformation maps Ω_{I}^* onto the domain Ω^* illustrated in Fig. 4(b), and transforms the corners A, B and C into the corners A^*, B^* and C^*, whose interior angles are respectively π, π and $\frac{1}{2}\pi$. That is, the singularities at B^* and C^* are as at B and C, but the transformation (6.11) reduces the severity of the singularity at A.

The results obtained by applying the BKM/MB to the domain Ω_{I}^* are as follows:
BKM/MB: $N_{\text{opt}}=22$, $E_{22}=3.8\times10^{-4}$, $R_{22}=1.219\ 404\ 136$.

Let z_1^* and z_2^* be the images of the points (6.10) under the transformation (6.11). Also, let $z_3^*\equiv A^*$ and $z_4^*\equiv B^*$. then the use of an augmented basis including the singular functions

$$\{z/(z-z_j^*)\}', \quad j=1,2$$

and

$$\{(z-z_j^*)^2\log(z-z_j^*)\}', \quad \{(z-z_j^*)^3\log(z-z_j^*)^2\}, \quad j=3,4$$

leads to the following results:
BKM/AB: $N_{\text{opt}}=27$, $E_{27}=1.3\times10^{-5}$, $R_{27}=1.219\ 403\ 703$.

The results of this example confirm our remark that the use of preliminary transformations does not necessarily completely remove the effect of corner singularities.

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