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Module categories of simple current extensions of vertex operator algebras

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Abstract

We study module categories of simple current extensions of rational C_2-cofinite vertex operator algebras of CFT-type and prove that they are semisimple. We also develop a method of induced modules for the simple current extensions.

MSC: 17B69

1. Introduction

Simple current extensions of vertex operator algebras (VOAs) seem to be the most realistic and effective construction of new VOAs, and have many applications (cf. [6] [22]). It is shown in [16] that the famous moonshine VOA $V^\natural$ is a $\mathbb{Z}_2$-graded simple current extension of the charge conjugate orbifold V_Λ^+ of the lattice VOA V_Λ associated to the Leech lattice Λ. On the other hand, it is initiated in [11] to study the moonshine VOA as a module for a sub VOA inside $V^\natural$. The merit of this method is that we can define some automorphisms using fusion rules for a sub VOA, see [27]. In [23] they constructed a series of $\mathbb{Z}_2$-graded simple current extensions of the unitary Virasoro VOAs and computed all the fusion rules for the extensions. A new kind of symmetry is obtained, and the symmetry is shown to be a 3A-triality of the Monster in [29]. In [24], they also use simple current extensions to study $V^\natural$. Thus, simple current extensions are useful to study the Monster through $V^\natural$. However, to define automorphisms by using a sub VOA, we have to establish the rationality of the sub

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VOA. In this paper, we establish the rationality of the simple current extensions of rational C_2-cofinite vertex operator algebras of CFT-type.

The main tool we use in this paper is the "associativity". By Huang's results, intertwining operators among simple current modules have a nice property so that in the representation theory of a simple current extension with an abelian symmetry we can find certain twisted algebras associated to pairs of the abelian group and its orbit spaces. The twisted algebras can be considered as a deformation or a generalization of group rings and play a powerful role in our theory. Using the twisted algebras, we can show that every module for a simple current extension of a rational C_2-cofinite VOA is completely reducible. Furthermore, we can parameterize irreducible modules for extensions by irreducible representations of the twisted algebras.

We also develop a method of induced modules. We show that every irreducible module of a rational C_2-cofinite vertex operator algebra of CFT-type can be lifted to a twisted module for a simple current extension with an abelian symmetry. This result concerns the following famous conjecture: "For a simple rational vertex operator algebra V and its finite automorphism group G, the G-invariants V^G, called the G-orbifold of V, is also a simple rational vertex operator algebra. Moreover, every irreducible module for the G-orbifold V^G is contained in a g-twisted V-module for some $g \in G$." Our result is the converse of this conjecture in a sense. Actually, we prove the following. Let V^0 be a simple rational VOA of CFT-type and D a finite abelian group. Then a D-graded simple current extension $V_D = \bigoplus_{\alpha \in D} V^\alpha$ is σ-regular for all $\sigma \in D^*$. Here the term "σ-regular" means that every weak σ-twisted module is completely reducible (cf. [30]). Moreover, for an irreducible V^0-module W, we can attach the group representation $\chi : D \to \mathbb{Z}/n\mathbb{Z}$ such that the powers of z in a V^0-intertwining operator of type $V^\alpha \times (V^\beta \boxtimes_{V^0} W) \to V^{\alpha+\beta} \boxtimes_{V^0} W$ are contained in $\chi(\alpha) + \mathbb{Z}$ for all $\alpha, \beta \in D$. Finally, we prove that W can be lifted to an irreducible $\hat{\chi}$-twisted V_D-module, where $\hat{\chi}$ is an element in D^* defined as $\hat{\chi}(\alpha) = e^{-2\pi\sqrt{-1}\chi(\alpha)}$ for $\alpha \in D$.

This paper is organized as follows. In Section 2, we mainly study "decompositions" of V_D-modules as V^0-modules. It is proved that V_D is σ-regular for all $\sigma \in D^*$. In Section 3, we study "induction" of V_D-modules from V^0-modules, which is a converse step of Section 2. As an application, we complete the classification of all $\mathbb{Z}_3$-twisted modules for the vertex operator algebra constructed in [29].

2. Representation theory of simple current extensions

Throughout this paper, V^0 denotes a simple rational C_2-cofinite vertex operator algebra (VOA) of CFT-type. In particular, V^0 is regular by Abe et al. [1]. In this paper, we assume that the module category of V^0 is well known in a sense that all irreducible modules and all fusion rules are classified.

We recall a definition of tensor product. A theory of tensor products of modules of a vertex operator algebra was first developed by Huang and Lepowsky [18–21], and Li considered it in a formal calculus approach in [25] (see also [5]). Huang and Lepowsky's approach contains not only intertwining operators, but also intertwining maps. However, since we will work over a simple situation, in this paper we use Li's definition which is enough for us.

Definition 2.1 (Li [25]). Let W^1, W^2 be V^0-modules. A *tensor product* (or a *fusion product*) for the ordered pair (W^1, W^2) is a pair $(W^1 \boxtimes_{V^0} W^2, F(\cdot,z))$ consisting of a V^0-module $W^1 \boxtimes_{V^0} W^2$ and a V^0-intertwining operator $F(\cdot,z)$ of type $W^1 \times W^2 \to W^1 \boxtimes_{V^0} W^2$ satisfying the following universal property: For any V^0-module U and any intertwining operator $I(\cdot,z)$ of type $W^1 \times W^2 \to U$, there exists a unique V^0-homomorphism ψ from $W^1 \boxtimes_{V^0} W^2$ to U such that $I(\cdot,z) = \psi \circ F(\cdot,z)$.

Remark 2.2. It is shown in [18–21] and [25] that if a VOA V is rational, then a tensor product for any two V-modules always exists. It follows from definition that if a tensor product exists, then it is unique up to isomorphism.

For simplicity, we will often choose a fixed fusion product $(W^1 \boxtimes_{V^0} W^2, F(\cdot,z))$, and write it as $W^1 \boxtimes_{V^0} W^2$. By abuse of notation, we will refer to it as *the* fusion product; all other fusion products $(W^1 \boxtimes_{V^0} W^2, F(\cdot,z))$ will be isomorphic to it.

Definition 2.3. An irreducible V^0-module U is called a *simple current* if it satisfies: for any irreducible V^0-module W, the fusion product $U \boxtimes_{V^0} W$ is also irreducible.

We study an extension of V^0 by simple current modules. Let D be a finite abelian group and assume that we have a set of irreducible simple current V^0-modules $\{V^\alpha \mid \alpha \in D\}$ indexed by D.

Definition 2.4. An extension $V_D = \bigoplus_{\alpha\in D} V^\alpha$ of V^0 is called a *D-graded simple current extension* [1] if V_D carries a structure of a simple VOA such that $Y(x^\alpha,z)x^\beta \in V^{\alpha+\beta}((z))$ for any $x^\alpha \in V^\alpha$ and $x^\beta \in V^\beta$.

Remark 2.5. It is shown in Proposition 5.3 of [9] that the VOA-structure of V_D over $\mathbb{C}$ is unique.

In the following, V_D always denotes a D-graded simple current extension of V^0. Let D^* be the dual group of D, that is, the group of characters of D. With the canonical action, D^* faithfully acts on V_D. So we may view D^* as a subgroup of $\mathrm{Aut}(V_D)$. Then it follows from [8] that V^α and V^β are inequivalent V^0-modules if $\alpha \neq \beta$.

Lemma 2.6. *V_D is C_2-cofinite.*

Proof. Since we have assumed that V^0 is a C_2-cofinite VOA of CFT-type, we can use a method of spanning sets for modules developed by several authors in [2,28,30], and we see that all V^α, $\alpha \in D$, are C_2-cofinite V^0-modules. Then the assertion immediately follows. $\square$

[1] We assume that the vacuum vector and the Virasoro vector of V_D is the same as those of V^0.

2.1. Twisted algebra $A_\lambda(D,\mathcal{S}_W)$

Let W be an irreducible V^0-module. In this subsection we describe a construction of the twisted algebra from W and V^α, $\alpha\in D$. Since all V^α, $\alpha\in D$, are simple current V^0-modules, all $V^\alpha\boxtimes_{V^0}W$, $\alpha\in D$, are also irreducible V^0-modules. By the results of Huang [14,15,17] fusion products among V^0-modules satisfy the associativity. Therefore $V^\alpha\boxtimes_{V^0}W\neq 0$ for all $\alpha\in D$ and $D_W:=\{\alpha\in D\mid V^\alpha\boxtimes_{V^0}W\simeq W\}$ forms a subgroup of D. Set $\mathcal{S}_W:=D/D_W$. Then $\mathcal{S}_W$ naturally admits an action of D. By definition, D_W acts on $\mathcal{S}_W$ trivially. Let $s\in\mathcal{S}_W$ and take a representative $\alpha\in D$ such that $s=\alpha+D_W$. We should note that irreducible V^0-modules $V^\alpha\boxtimes_{V^0}W$ and $V^\beta\boxtimes_{V^0}W$ are isomorphic if and only if $\alpha-\beta\in D_W$. Thus, the equivalent class of $V^\alpha\boxtimes_{V^0}W$ is independent of choice of a representative α in $\alpha+D_W$ and hence determined uniquely. So for each $s\in\mathcal{S}_W$, we define $W^s:=V^\alpha\boxtimes_{V^0}W$ after fixing a representative $\alpha\in D$ such that $s=\alpha+D_W$.

Let $\alpha,\beta,\gamma\in D$ and $s\in\mathcal{S}_W$. It follows from the associativity of fusion products that $V^\alpha\boxtimes_{V^0}W^s=W^{s+\alpha}$, where $s+\alpha$ denotes the action of $\alpha\in D$ to $s\in\mathcal{S}_W$. Take basis $I_s^\alpha(\cdot,z)$ of the one-dimensional spaces of V^0-intertwining operators of type $V^\alpha\times W^s\to W^{s+\alpha}$. By an associative property of V^0-intertwining operators (cf. [14,15,17]) there are (non-zero) scalars $\lambda_s(\alpha,\beta)\in\mathbb{C}$ such that the following equality holds:

$$\langle v, I_{s+\beta}^\alpha(x^\alpha,z_1)I_s^\beta(x^\beta,z_2)w^s\rangle=\lambda_s(\alpha,\beta)\langle v, I_s^{\alpha+\beta}(Y_{V_D}(x^\alpha,z_0)x^\beta,z_2)w^s\rangle|_{z_0=z_1-z_2},$$

where $x^\alpha\in V^\alpha$, $x^\beta\in V^\beta$, $w^s\in W^s$ and $v\in(W^{s+\alpha+\beta})^*$. We normalize intertwining operators $I_s^0(\cdot,z)$ to satisfy $I_s^0(\mathbf{1},z)=\mathrm{id}_{W^s}$. In other words, $I_s^0(\cdot,z)$ are vertex operators on V^0-modules W^s. By considering

$$\langle v', I_{s+\beta+\gamma}^\alpha(x^\alpha,z_1)I_{s+\gamma}^\beta(x^\beta,z_2)I_s^\gamma(x^\gamma,z_3)w^s\rangle,$$

we can deduce a relation

$$\lambda_{s+\gamma}(\alpha,\beta)\lambda_s(\alpha+\beta,\gamma)=\lambda_s(\alpha,\beta+\gamma)\lambda_s(\beta,\gamma).$$

By the normalization $I_s^0(\mathbf{1},z)=\mathrm{id}_{W^s}$, the $\lambda_s(\cdot,\cdot)$'s above also satisfy a condition $\lambda_s(0,\alpha)=\lambda_s(\alpha,0)=1$ for all $\alpha\in D$ and $s\in\mathcal{S}_W$. Using $\lambda_s(\alpha,\beta)$, we introduce the twisted algebra. Let $q(s)$, $s\in\mathcal{S}_W$, be formal symbols and $\mathbb{C}\mathcal{S}_W:=\bigoplus_{s\in\mathcal{S}_W}\mathbb{C}q(s)$ a linear space spanned by them. We define a multiplication on $\mathbb{C}\mathcal{S}_W$ by $q(s)\cdot q(t):=\delta_{s,t}q(s)$. Then $\mathbb{C}\mathcal{S}_W$ becomes a semisimple commutative associative algebra isomorphic to $\mathbb{C}^{\oplus|\mathcal{S}_W|}$. Let $U(\mathbb{C}\mathcal{S}_W):=\{\sum_{s\in\mathcal{S}_W}\mu_s q(s)\mid\mu_s\in\mathbb{C}^*\}$ be the set of units in $\mathbb{C}\mathcal{S}_W$. Then $U(\mathbb{C}\mathcal{S}_W)$ forms a multiplicative group in $\mathbb{C}\mathcal{S}_W$. Define an action of $\alpha\in D$ on $\mathbb{C}\mathcal{S}_W$ by $q(s)^\alpha:=q(s-\alpha)$. Then $U(\mathbb{C}\mathcal{S}_W)$ is a multiplicative right D-module. Set $\bar\lambda(\alpha,\beta)=\sum_{s\in\mathcal{S}_W}\lambda_s(\alpha,\beta)^{-1}q(s)\in U(\mathbb{C}\mathcal{S}_W)$. Then $\bar\lambda(\cdot,\cdot)$ defines a 2-cocycle $D\times D\to U(\mathbb{C}\mathcal{S}_W)$ because it satisfies a 2-cocycle condition

$$\bar\lambda(\alpha,\beta)^\gamma\cdot\bar\lambda(\alpha+\beta,\gamma)=\bar\lambda(\alpha,\beta+\gamma)\cdot\bar\lambda(\beta,\gamma).$$

Since the space of V^0-intertwining operators of type $V^\alpha\times W^s\to W^{s+\alpha}$ is one-dimensional, $\bar\lambda$ is unique upto 2-coboundaries. Namely, $\bar\lambda$ defines an element of the second cohomology group $H^2(D,U(\mathbb{C}\mathcal{S}_W))$. Let $\mathbb{C}[D]=\bigoplus_{\alpha\in D}\mathbb{C}e^\alpha$ be the group ring of

D and set

$$A_\lambda(D,\mathscr{S}_W) := \mathbb{C}[D]\underset{\mathbb{C}}{\otimes}\mathbb{C}\mathscr{S}_W = \left\{\sum \mu_{\alpha,s}e^\alpha \otimes q(s) \mid \alpha\in D, s\in\mathscr{S}_W, \mu_{\alpha,s}\in\mathbb{C}\right\}$$

and define a multiplication $*$ on $A_\lambda(D,\mathscr{S}_W)$ by

$$e^\alpha\otimes q(s) * e^\beta\otimes q(t) := \lambda_t(\alpha,\beta)^{-1}e^{\alpha+\beta}\otimes q(s)^\beta\cdot q(t).$$

Then $A_\lambda(D,\mathscr{S}_W)$ equipped with the product $*$ forms an associative algebra with the unit element $\sum_{s\in S_W} e^0\otimes q(s)$. We call $A_\lambda(D,\mathscr{S}_W)$ the *twisted algebra* associated to a pair $(D,\mathscr{S}_W)$.

Remark 2.7. The algebra $A_\lambda(D,\mathscr{S}_W)$ is called the *generalized twisted double* in [12]. It naturally appears in the orbifold theory, and has been studied in many papers. For reference, see [3,10,12,26].

Take an $s\in\mathscr{S}_W$ and set $\mathbb{C}[D]\otimes q(s) := \bigoplus_{\alpha\in D}\mathbb{C}e^\alpha\otimes q(s)$. Then $\mathbb{C}[D]\otimes q(s)$ is a subalgebra of $A_\lambda(D,\mathscr{S}_W)$. It has a subalgebra $\mathbb{C}[D_W]\otimes q(s) := \bigoplus_{\alpha\in D_W}\mathbb{C}e^\alpha\otimes q(s)$ which is isomorphic to the twisted group algebra $\mathbb{C}^{\lambda_s}[D_W]$ of D_W associated to a 2-cocycle $\lambda_s(\cdot,\cdot)^{-1}\in Z^2(D_W,\mathbb{C})$. There is a one-to-one correspondence between the category of $\mathbb{C}^{\lambda_s}[D_W]$-modules and the category of $A_\lambda(D,\mathscr{S}_W)$-modules given as below:

Theorem 2.8 (Mason [26], Dong and Yamskulna [2, Theorem 3.5]). *The functors*

$$\begin{aligned}
&\mathrm{Ind}_{\mathbb{C}^{\lambda_s}[D_W]}^{A_\lambda(D,\mathscr{S}_W)}: && M\in\mathbb{C}^{\lambda_s}[D_W]\text{-Mod} &&\mapsto && \mathbb{C}[D]\otimes q(s)\underset{\mathbb{C}[D_W]\otimes q(s)}{\otimes} M\in A_\lambda(D,\mathscr{S}_W)\text{-Mod},\\
&\mathrm{Red}_{\mathbb{C}^{\lambda_s}[D_W]}^{A_\lambda(D,\mathscr{S}_W)}: && N\in A_\lambda(D,\mathscr{S}_W)\text{-Mod} &&\mapsto && e^0\otimes q(s)N\in\mathbb{C}^{\lambda_s}[D_W]\text{-Mod}
\end{aligned}$$

define equivalences between the module categories $\mathbb{C}^{\lambda_s}[D_W]$-Mod *and* $A_\lambda(D,\mathscr{S}_W)$-Mod. *In particular*, $A_\lambda(D,\mathscr{S}_W)$ *is a semisimple algebra.*

2.2. Untwisted modules

Let M be an indecomposable weak V_D-module. Since V^0 is regular, we can find an irreducible V^0-submodule W of M. We use the same notation for D_W, $\mathscr{S}_W$, $A_\lambda(D,\mathscr{S}_W)$ and $\mathbb{C}^{\lambda_s}[D_W]$ as previously. We should note that the definition of D_W is independent of the choice of an irreducible component W. One can show the following.

Lemma 2.9 (Sakuma and Yamauchi [29, Lemma 3.6]). *All* $V^\alpha\cdot W = \{\sum a_n w \mid a\in V^\alpha, w\in W, n\in\mathbb{Z}\}$, $\alpha\in D$, *are non-trivial irreducible* V^0*-submodules of* M.

Let $D=\bigsqcup_{i=1}^n(t^i+D_W)$ be a coset decomposition of D with respect to D_W. Set $V_{D_W+t^i} := \bigoplus_{\alpha\in D_W}V^{\alpha+t^i}$. Then V_{D_W} forms a sub VOA of V_D, which is a D_W-graded simple current extension of V^0, and $V_D=\bigoplus_{i=1}^n V_{D_W+t^i}$ forms a D/D_W-graded simple current extension of V_{D_W}. As we have assumed that M is an indecomposable V_D-module, every irreducible V^0-submodule is isomorphic to one of $V^{t^i}\boxtimes_{V^0}W$, $i=1,\dots,n$.

Remark 2.10. Let $M_{D_W+t^i}$ be the sum of all irreducible V^0-submodules of M isomorphic to $V^{t^i}\boxtimes_{V^0} W$. Then we have the following decomposition of M into a direct sum of isotypical V^0-components with a D/D_W-grading:

$$M=\bigoplus_{t=1}^{n} M_{D_W+t^i},\qquad V_{D_W+t^i}\cdot M_{D_W+t^j}=M_{D_W+t^i+t^j}.$$

In particular, if M is irreducible under V_D, then each $M_{D_W+t^i}$ is an irreducible V_{D_W}-module. Thus, viewing V_D as a D/D_W-graded simple current extension of V_{D_W}, we can regard M as a D/D_W-stable V_D-module (for the D/D_W-stability, see Definition 2.15 below).

For each $s\in\mathscr{S}_W$, we set $W^s=V^s\boxtimes_{V^0} W$ by abuse of notation (because it is well-defined). Since all V^α, $\alpha\in D$, are simple current V^0-modules, there are unique V^0-intertwining operators $I_s^\alpha(\cdot,z)$ of type $V^\alpha\times W^s\to W^{s+\alpha}$ up to scalar multiples. We choose $I_s^0(\cdot,z)$ to satisfy the condition $I_s^0(\mathbf{1},z)=\mathrm{id}_{W^s}$, i.e., $I_s^0(\cdot,z)$ defines the vertex operator on a V^0-module W^s for each $s\in\mathscr{S}_W$. Then, by Huang [14,15,17], there exist scalars $\lambda_s(\alpha,\beta)\in\mathbb{C}$ such that

$$\langle v, I_{s+\beta}^{\alpha}(x^\alpha,z_1)I_s^\beta(x^\beta,z_2)w\rangle=\lambda_s(\alpha,\beta)\langle v, I_s^{\alpha+\beta}(Y_{V_D}(x^\alpha,z_0)x^\beta,z_2)w\rangle|_{z_0=z_1-z_2},\qquad(2.1)$$

where $x^\alpha\in V^\alpha$, $x^\beta\in V^\beta$, $w\in W^s$ and $v\in(W^{s+\alpha+\beta})^*$. Then by the same procedure as in the previous subsection, we can find the 2-cocycle $\bar{\lambda}(\cdot,\cdot)\in H^2(D,U(\mathbb{C}\mathscr{S}_W))$ and construct the twisted algebra $A_\lambda(D,\mathscr{S}_W)$. By assumption, M is a direct sum of some copies of W^s, $s\in\mathscr{S}_W$, as a V^0-modules so that we have $M\simeq\bigoplus_{s\in\mathscr{S}_W}W^s\otimes\mathrm{Hom}_{V^0}(W^s,M)$. Set $U^s:=\mathrm{Hom}_{V^0}(W^s,M)$. Clearly, all of U^s, $s\in\mathscr{S}_W$, are not zero because of Lemma 2.9. On $W^s\otimes U^s$, the vertex operator of $x^\alpha\in V^\alpha$ can be written as

$$Y_M(x^\alpha,z)|_{W^s\otimes U^s}=I_s^\alpha(x^\alpha,z)\otimes\phi_s(\alpha)\qquad(2.2)$$

with some $\phi_s(\alpha)\in\mathrm{Hom}_{\mathbb{C}}(U^s,U^{s+\alpha})$. Using (2.1) we can show that

$$\phi_{s+\beta}(\alpha)\phi_s(\beta)=\lambda_s(\alpha,\beta)^{-1}\phi_s(\alpha+\beta)$$

and hence we can define an action of $A_\lambda(D,\mathscr{S}_W)$ on $\bigoplus_{s\in\mathscr{S}_W}U^s$ by $e^\alpha\otimes q(s)\cdot\mu:=\delta_{s,t}\phi_s(\alpha)\mu$ for $\mu\in U^t$.

Lemma 2.11. *Under the action above, $\bigoplus_{s\in\mathscr{S}_W}U^s$ becomes an $A_\lambda(D,\mathscr{S}_W)$-module.*

Then we have

Proposition 2.12. *Suppose that M is irreducible under V_D. Then U^s is an irreducible $\mathbb{C}^{\lambda_s}[D_W]$-module for every $s\in\mathscr{S}_W$. Moreover, $\bigoplus_{s\in\mathscr{S}_W}U^s$ is an irreducible $A_\lambda(D,\mathscr{S}_W)$-module.*

Proof. Since $W^s\otimes U^s$ is an irreducible V_{D_W}-submodule of M, U^s is an irreducible $\mathbb{C}^{\lambda_s}[D_W]$-module. Moreover, there is a canonical $A_\lambda(D,\mathscr{S}_W)$-homomorphism from $P=A_\lambda(D,\mathscr{S}_W)\otimes_{\mathbb{C}[D_W]\otimes q(s)}U^s=\{\mathbb{C}[D]\otimes q(s)\}\otimes_{\mathbb{C}[D_W]\otimes q(s)}U^s$ to $\bigoplus_{s\in\mathscr{S}_W}U^s$. As

$$P=\bigoplus_{t\in\mathscr{S}_W}e^t\otimes q(s)\underset{\mathbb{C}[D_W]\otimes q(s)}{\otimes}U^s$$

and is irreducible under $A_\lambda(D,\mathscr{S}_W)$ by Theorem 2.8, we see that $U^t = e^t \otimes q(s) \otimes U^s$ and hence $P = \bigoplus_{s\in\mathscr{S}_W} U^s$. Consequently, $\bigoplus_{s\in\mathscr{S}_W} U^s$ is also irreducible under $A_\lambda(D,\mathscr{S}_W)$. $\square$

Corollary 2.13. *If M is an irreducible V_D-module, then irreducible V^0-components W^s and W^t have the same multiplicity in M for all $s,t\in\mathscr{S}_W$.*

Proof. Because U^s and U^t have the same dimension. $\square$

Theorem 2.14. *Let V^0 be a rational C_2-cofinite vertex operator algebra of CFT-type, and let $V_D = \bigoplus_{\alpha\in D} V^\alpha$ be a D-graded simple current extension of V^0. Then an indecomposable V_D-module M is completely reducible under V_D. Consequently, V_D is regular. As a V^0-module, an irreducible V_D-submodule of M has the shape $\bigoplus_{s\in\mathscr{S}_W} W^s\otimes U^s$ with each U^s an irreducible $\mathbb{C}[D_W]\otimes q(s) \simeq \mathbb{C}^{\lambda_s}[D_W]$-module for all $s\in\mathscr{S}_W$. Moreover, all U^t, $t\in\mathscr{S}_W$, are determined by one of them, say U^s, by the following rule*:

$$U^t \simeq \mathrm{Red}^{A_\lambda(D,\mathscr{S}_W)}_{\mathbb{C}^{\lambda_t}[D_W]}\,\mathrm{Ind}^{A_\lambda(D,\mathscr{S}_W)}_{\mathbb{C}^{\lambda_s}[D_W]} U^s.$$

Proof. Since $M = \bigoplus_{s\in\mathscr{S}_W} W^s\otimes \mathrm{Hom}_{V^0}(W^s,M)$ and the space $\bigoplus_{s\in\mathscr{S}_W}\mathrm{Hom}_{V^0}(W^s,M)$ carries a structure of a module for a semisimple algebra $A_\lambda(D,\mathscr{S}_W)$ by Proposition 2.12, M is also a completely reducible V_D-module because of (2.2). Since V^0 is regular, all weak V_D-modules are completely reducible. So V_D is also regular. Now assume that M is an irreducible V_D-module. The decomposition is already shown. It remains to show that U^t is determined by U^s by the rule as stated. It is shown in the proof of Proposition 2.12 that $U^t = e^t\otimes q(s)\otimes U^s$. It is easy to see that $e^t\otimes q(s)\otimes U^s = \mathrm{Red}^{A_\lambda(D,\mathscr{S}_W)}_{\mathbb{C}^{\lambda_t}[D_W]}\,\mathrm{Ind}^{A_\lambda(D,\mathscr{S}_W)}_{\mathbb{C}^{\lambda_s}[D_W]} U^s$. The proof is completed. $\square$

By the above theorem and Theorem 2.8, the number of inequivalent irreducible V_D-modules containing W as a V^0-submodule is $\dim_{\mathbb{C}} Z(\mathbb{C}^{\lambda_s}[D_W])$. In particular, if $D_W = 0$, then the structure of a V_D-module containing W is uniquely determined by its V^0-module structure. This fact is already observed in Proposition 3.8 of Sakuma and Yamauchi [29]. For convenience, we introduce the following notion.

Definition 2.15. An irreducible V_D-module N is said to be *D-stable* if $D_W = 0$ for some irreducible V^0-submodule W.

It is obvious that the definition of the D-stability is independent of the choice of an irreducible V^0-submodule W. Let N^i, $i = 1,2,3$, be irreducible D-stable V_D-modules, and let W^i be irreducible V^0-submodules of N^i for $i = 1,2,3$. Set $W^{i,\alpha} := V^\alpha \boxtimes_{V^0} W^i$. Then $W^{i,\alpha} \simeq W^{i,\beta}$ as V^0-modules if and only if $\alpha = \beta$, and N^i as a V^0-module is isomorphic to $\bigoplus_{\alpha\in D} W^{i,\alpha}$. The following lemma is a simple modification of Lemma 3.12 of Sakuma and Yamauchi [29]. We expect that this lemma would be used in our future study.

Lemma 2.16. *Let $N^i = \bigoplus_{\alpha\in D} W^{i,\alpha}$ be as above. Then we have the following isomorphism*:

$$\binom{N^3}{N^1 N^2}_{V_D} \simeq \bigoplus_{\alpha\in D} \binom{W^{3,\alpha}}{W^{1,0} W^{2,0}}_{V^0}.$$

In the above,

$$\binom{N^3}{N^1\ N^2}_{V_D}$$

denotes the space of V_D-intertwining operators of type $N^1 \times N^2 \to N^3$.

Proof. By the Huang's result in [17], we can replace the assumption on the Virasoro element in Lemma 3.12 of [29] by the C_2-cofinite condition. □

Remark 2.17. Let M be an irreducible V_D-module and W an irreducible V^0-submodule of M. Even if $D_W \neq 0$, we can apply the lemma above to M. We may consider V_D as a D/D_W-graded simple current extension of V_{D_W} as in Remark 2.10. Then we can view M as a D/D_W-stable V_D-module. So by replacing D by D/D_W, we can apply the above lemma to M.

2.3. Twisted modules

Let σ be an automorphism on V_D such that V^0 is contained in $V_D^{\langle\sigma\rangle}$, the space of σ-invariants of V_D. Let $V_D^{(r)}$ be a subspace $\{a \in V_D \mid \sigma a = e^{2\pi\sqrt{-1}r/|\sigma|}a\}$ for each $0 \leqslant r \leqslant |\sigma|-1$, where $|\sigma|$ is order of σ. Then $V_D^{(r)}$ are V^0-submodules of V_D so that there is a partition $D = \bigsqcup_{i=0}^{|\sigma|-1} D^{(i)}$ such that $V_D^{(i)} = \bigoplus_{\alpha\in D^{(i)}} V^\alpha$. One can easily verify that $D^{(0)}$ is a subgroup of D and each $D^{(i)}$ is a coset of D with respect to $D^{(0)}$. Namely, if $V^0 \subset V_D^{\langle\sigma\rangle}$, then σ is identified with an element of D^*, the dual group of D. Conversely, it is clear from definition that D^* is a subgroup of $\mathrm{Aut}(V_D)$. Thus, we have

Lemma 2.18. *An automorphism $\sigma \in \mathrm{Aut}(V)$ satisfies $V^0 \subset V_D^{\langle\sigma\rangle}$ if and only if $\sigma \in D^*$.*

The lemma above tells us that an automorphism σ is consistent with the D-grading of V_D if and only if σ belongs to D^*. We consider σ-twisted V_D-modules. Let M be an indecomposable admissible σ-twisted V_D-module. By definition, there is a decomposition

$$M = \bigoplus_{i=0}^{|\sigma|-1} M^{(i)}$$

such that $V_D^{(i)} \cdot M^{(j)} \subset M^{(i+j)}$. It is obvious that each $M^{(i)}$ is a V^0-module. Let W be an irreducible V^0-submodule of $M^{(0)}$, and let D_W, $\mathscr{S}_W$, $A_\lambda(D, \mathscr{S}_W)$ and $\mathbb{C}^{\lambda_s}[D_W]$ be as in Section 2.1. By replacing M by $\sum_{\alpha\in D} V^\alpha \cdot W$ if necessary, we may assume that all $V^{\alpha} \boxtimes_{V^0} W$, $\alpha \in D_W$, are contained in $M^{(0)}$ so that D_W is a subgroup of $D^{(0)}$. Since M is a completely reducible V^0-module, we have the following decomposition:

$$M = \bigoplus_{s \in \mathscr{S}_W} W^s \otimes \mathrm{Hom}_{V^0}(W^s, M),$$

where we set $W^s := V^s \boxtimes_{V^0} W$ for $s \in \mathscr{S}_W$ by abuse of notation. Set $U^s := \mathrm{Hom}_{V^0}(W^s, M)$ for $s \in \mathscr{S}_W$. As we did before, we can find a 2-cocycle $\bar{\lambda} \in H^2(D, U(\mathbb{C}\mathscr{S}_W))$ and a representation of $A_{\lambda}(D, \mathscr{S}_W)$ on the space $\bigoplus_{s \in \mathscr{S}_W} U^s$. Thus, by the same argument, we can show the following.

Theorem 2.19. *Let $\sigma \in D^*(\subset \mathrm{Aut}(V_D))$. Viewing as a V^0-module, an indecomposable admissible σ-twisted V_D-module M has the shape*

$$M = \bigoplus_{s \in \mathscr{S}_W} W^s \otimes U^s$$

such that the space $\bigoplus_{s \in \mathscr{S}_W} U^s$ carries a structure of an $A_{\lambda}(D, \mathscr{S}_W)$-module. In particular, M is a completely reducible V_D-module. If M is irreducible under V_D, then each U^s, $s \in \mathscr{S}_W$, is irreducible under $\mathbb{C}[D_W] \otimes q(s)$, and also $\bigoplus_{s \in \mathscr{S}_W} U^s$ is irreducible under $A_{\lambda}(D, \mathscr{S}_W)$. Moreover, for each pair s and $t \in \mathscr{S}_W$, U^s and U^t is determined by the following rule:

$$U^t \simeq \mathrm{Red}^{A_{\lambda}(D, \mathscr{S}_W)}_{\mathbb{C}^{\lambda_t}[D_W]} \mathrm{Ind}^{A_{\lambda}(D, \mathscr{S}_W)}_{\mathbb{C}^{\lambda_s}[D_W]} U^s.$$

Hence, all W^s, $s \in \mathscr{S}_W$, have the same multiplicity in M.

Remark 2.20. Since $D_W \subset D^{(0)}$, we note that the decomposition above is a refinement of the decomposition $M = \bigoplus_{i \in \mathbb{Z}/|\sigma|\mathbb{Z}} M^{(i)}$.

By the theorem above, V_D is σ-rational for all $\sigma \in D^*$. More precisely, we can prove that V_D is σ-regular, that is, every weak σ-twisted V_D-module is completely reducible (cf. [30]).

Corollary 2.21. *An extension V_D is σ-regular for all $\sigma \in D^*$.*

Proof. Let M be a weak σ-twisted V_D-module. Take an irreducible V^0-submodule W of M, which is possible because V^0 is regular. Then $\sum_{\alpha \in D} V^{\alpha} \cdot W$ is a σ-twisted admissible V_D-submodule. As we have shown that V_D is σ-rational, $\sum_{\alpha \in D} V^{\alpha} \cdot W$ is a completely reducible V_D-module. Thus, M is a sum of irreducible V_D-submodules and hence M is a direct sum of irreducible V_D-submodules. □

3. Module categories of simple current extensions

In the previous section, we studied a representation theory of a D-graded simple current extension V_D. In this section, we proceed a theory of induced modules, which is given as a converse step of the previous section. As a result, we can complete the classification of irreducible modules for V_D.

3.1. Induced modules

Let W be an irreducible V^0-module. We define the stabilizer D_W, the orbit space $\mathscr{S}_W$, intertwining operators $I_s^\alpha(\cdot,z)$, where $\alpha\in D$ and $s\in\mathscr{S}_W$, the twisted algebra $A_\lambda(D,\mathscr{S}_W)$ and the twisted group ring $\mathbb{C}^{\lambda_s}[D_W]$ as in Section 2.1. We set $W^s := V^s\boxtimes_{V^0} W$ for $s\in\mathscr{S}_W$, as we did previously. Let $h(s)$ be the top weight of a V^0-module W^s, which is a rational number by Theorem 11.3 of [7]. It follows from definition that the powers of z in an intertwining operator $I_s^\alpha(\cdot,z)$ are contained in $h(\alpha+s)-h(s)+\mathbb{Z}$. We set $\chi(\alpha,s):=h(\alpha+s)-h(s)\in\mathbb{Q}$. The following assertion is crucial for us:

Lemma 3.1. *The following hold for any* $\alpha,\beta\in D$ *and* $s\in\mathscr{S}_W$:
(i) $\chi(\alpha,\beta+s)-\chi(\alpha,s)\in\mathbb{Z}$, (ii) $\chi(\alpha,\beta+s)+\chi(\beta,s)-\chi(\alpha+\beta,s)\in\mathbb{Z}$.

Proof. Recall that the following results are established by Huang in [14,15,17]:

$$\langle v,I_{s+\beta}^\alpha(x^\alpha,z_1)I_s^\beta(x^\beta,z_2)w^s\rangle=\varepsilon_s(\alpha,\beta)\langle v,I_{s+\alpha}^\beta(x^\beta,z_2)I_s^\alpha(x^\alpha,z_1)w^s\rangle, \tag{3.1}$$

$$\langle v,I_{s+\beta}^\alpha(x^\alpha,z_1)I_s^\beta(x^\beta,z_2)w^s\rangle=\lambda_s(\alpha,\beta)\langle v,I_s^{\alpha+\beta}(Y_{V_D}(x^\alpha,z_0)x^\beta,z_2)w^s\rangle|_{z_0=z_1-z_2}, \tag{3.2}$$

where $x^\alpha\in V^\alpha$, $x^\beta\in V^\beta$, $w^s\in W^s$, $v\in(W^{s+\alpha+\beta})^*$, $\varepsilon_s(\alpha,\beta)$ is a suitable scalar in $\mathbb{C}$, and the equals above mean that the left-hand side and the right-hand side are analytic extensions of each other. Since all $I_s^\alpha(\cdot,z)$ are intertwining operators among modules involving simple currents, we note that by the convergence property in [14,15,17] the right-hand side of (3.2) has the form $z_2^{\chi(\alpha+\beta,s)+r}z_0^s f_1(z_0/z_2)|_{z_0=z_1-z_2}$ in the domain $|z_2|>|z_1-z_2|>0$, where r and s are some integers and $f_1(x)$ is an analytic function on $|x|<1$. Therefore, we have

$$\begin{aligned}&(z_1-z_2)^N\langle\mu,I_{s+\beta}^\alpha(x^\alpha,z_1)I_s^\beta(x^\beta,z_2)w^2\rangle\\&\quad=\varepsilon_s(\alpha,\beta)(z_1-z_2)^N\langle\mu,I_{s+\beta}^\alpha(x^\beta,z_2)I_s^\beta(x^\alpha,z_2)w^2\rangle\end{aligned} \tag{3.3}$$

in the domain $|z_1|>|z_1-z_2|\geqslant 0$, $|z_2|>|z_1-z_2|\geqslant 0$ for sufficiently large N. Since $z^{-\chi(\alpha,s)}I_s^\alpha(x^\alpha,z)w^s$ contains only integral powers of z, both

$$z_1^{-\chi(\alpha,\beta+s)}z_2^{-\chi(\beta,s)}\langle v,I_{s+\beta}^\alpha(x^\alpha,z_1)I_s^\beta(x^\beta,z_2)w^s\rangle$$

and

$$z_1^{-\chi(\alpha,s)}z_2^{-\chi(\beta,s+\alpha)}\langle v,I_{s+\alpha}^\alpha(x^\beta,z_2)I_{s+\alpha}^\beta(x^\beta,z_2)w^s\rangle$$

contain only integral powers of z_1 and z_2. Thus, by (3.1), (3.3) and the convergence property in [14,15,17], we obtain the following equality of the meromorphic functions:

$$\begin{aligned}&(z_1-z_2)^N\iota_{12}^{-1}z_1^{-\chi(\alpha,s+\beta)}z_2^{-\chi(\beta,s)}\langle v,I_{s+\beta}^\alpha(x^\alpha,z_1)I_s^\beta(x^\beta,z_2)w^s\rangle\\&\quad=(z_1-z_2)^N\varepsilon_s(\alpha,\beta)\iota_{21}^{-1}z_1^{-\chi(\alpha,s)}z_2^{-\chi(\beta,s+\alpha)}\cdot z_1^{\chi(\alpha,s)-\chi(\alpha,s+\beta)}z_2^{\chi(\beta,s+\alpha)-\chi(\beta,s)}\\&\qquad\times\langle v,I_{s+\alpha}^\beta(x^\beta,z_2)I_s^\alpha(x^\alpha,z_1)w^s\rangle.\end{aligned}$$

Since the equality above holds for any choices of $\log z_1$ and $\log z_2$ in the definitions of $z_1^r = e^{r\log z_1}$ and $z_2^r = e^{r\log z_2}$ (cf. [14,15,17]), we have $\chi(\alpha,s) - \chi(\alpha,s+\beta)\in\mathbb{Z}$ and $\chi(\beta,s+\alpha)-\chi(\beta,s)\in\mathbb{Z}$. This proves (i). The proof of (ii) is similar. By (3.2) and the convergence property in [14,15,17], we obtain the following equality of the meromorphic functions:

$$\lambda_s(\alpha,\beta)^{-1}\iota_{12}^{-1}z_1^{-\chi(\alpha,s+\beta)}z_2^{-\chi(\beta,s)}\langle v, I_{s+\beta}^{\alpha}(x^\alpha,z_1)I_s^{\beta}(x^\beta,z_2)w^s\rangle$$

$$= \iota_{20}^{-1}z_2^{-\chi(\alpha+\beta,s)}\langle v, I_s^{\alpha+\beta}(Y_{V_D}(x^\alpha,z_0)x^\beta,z_2)w^s\rangle$$

$$\cdot (z_2+z_0)^{-\chi(\alpha,s+\beta)}z_2^{\chi(\alpha+\beta,s)-\chi(\beta,s)}|_{z_0=z_1-z_2}.$$

Again the equality holds for any choices of $\log(z_1-z_2)$ and $\log z_2$ in the definitions of $(z_1-z_2)^r = e^{r\log(z_1-z_2)}$ and $z_2^r = e^{r\log z_2}$ (cf. [14,15,17]). Since $(z_2+z_0)^r = z_2^r(1+z_0/z_2)^r$ and $(1+x)^r = \sum_{i\geqslant 0}\binom{r}{i}x^i$ is analytic in the domain $|x|<1$, we see that $\chi(\alpha+\beta,s)-\chi(\alpha,s+\beta)-\chi(\beta,s)\in\mathbb{Z}$. This completes the proof of (ii). □

By the above lemma, we find that $\chi(\alpha,s)+\mathbb{Z}$ is independent of $s\in\mathscr{S}_W$. So we may set $\chi(\alpha) := \chi(\alpha,s)$ for $\alpha\in D$. Then by (ii) of Lemma 3.1 we find that $\chi(\cdot)$ satisfies the homomorphism condition $\chi(\alpha+\beta)+\mathbb{Z} = \chi(\alpha)+\chi(\beta)+\mathbb{Z}$. Since $\chi(\alpha)\in\mathbb{Q}$, there exists an $n\in\mathbb{N}$ such that χ defines a group homomorphism from D/D_W to $\mathbb{Z}/n\mathbb{Z} = \{j/n+\mathbb{Z}\mid 0\leqslant j\leqslant n-1\}$. It is clear that χ naturally defines an element $\hat{\chi}$ of D^* by $\hat{\chi}(\alpha) := e^{-2\pi\sqrt{-1}\chi(\alpha)}$. Thus χ gives rise to an element of $\mathrm{Aut}(V_D)$. In the following, we will construct irreducible $\hat{\chi}$-twisted V_D-modules which contain W as V^0-submodules.

Take an $s\in S_W$. Let φ be an irreducible representation of $\mathbb{C}^{\lambda_s}[D_W]$ on a space U. For each $t\in S_W$, set

$$U^t := \mathrm{Red}_{\mathbb{C}^{\lambda_t}[D_W]}^{A_\lambda(D,S_W)}\,\mathrm{Ind}_{\mathbb{C}^{\lambda_s}[D_W]}^{A_\lambda(D,S_W)}U.$$

Then each U^t, $t\in\mathscr{S}_W$, is a $\mathbb{C}^{\lambda_t}[D_W]$-module and a direct sum $\bigoplus_{s\in S_W}U^s$ naturally (and uniquely) carries a structure of an irreducible $A_\lambda(D,S_W)$-module. Set

$$\mathrm{Ind}_{V^0}^{V_D}(W,\varphi) := \bigoplus_{s\in S_W} W^s\otimes U^s$$

and define the vertex operator $\hat{Y}(\cdot,z)$ of V_D on $\mathrm{Ind}_{V^0}^{V_D}(W,\varphi)$ by

$$\hat{Y}(x^\alpha,z)w^t\otimes\mu^t := I_t^\alpha(x^\alpha,z)w^t\otimes\{e^\alpha\otimes q(t)\cdot\mu^t\}$$

for $x^\alpha\in V^\alpha$ and $w^t\otimes\mu^t\in W^t\otimes U^t$. We prove

Theorem 3.2. $(\mathrm{Ind}_{V^0}^{V_D}(W,\varphi),\hat{Y}(\cdot,z))$ *is an irreducible $\hat{\chi}$-twisted V_D-module.*

Proof. Since the powers of z in $\hat{Y}(x^\alpha,z)$ are contained in $\chi(\alpha)+\mathbb{Z}$, we only need to show the commutativity and the $\hat{\chi}$-twisted associativity of vertex operators. We use a technique of generalized rational functions developed in [4]. Let $x^\alpha\in V^\alpha$, $x^\beta\in V^\beta$, $w^s\otimes\mu^s\in W^s\otimes U^s$ and $v\in\mathrm{Ind}_{V^0}^{V_D}(W,\varphi)^*$. We note that $z^{-\chi(\alpha)}I_s^\alpha(x^\alpha,z)w^s\in W^{s+\alpha}((z))$.

For sufficiently large $N\in\mathbb{N}$, we have

$$
\begin{aligned}
&(z_1-z_2)^N\iota_{12}^{-1}z_1^{-\chi(\alpha)}z_2^{-\chi(\beta)}\langle v,\hat{Y}(x^\alpha,z_1)\hat{Y}(x^\beta,z_2)w^s\otimes\mu^s\rangle\\
&\quad=(z_1-z_2)^N\iota_{12}^{-1}z_1^{-\chi(\alpha)}z_2^{-\chi(\beta)}\langle v,I_{s+\beta}^\alpha(x^\alpha,z_1)I_s^\beta(x^\beta,z_2)w^s\\
&\qquad\otimes\{e^\alpha\otimes q(s+\beta)\cdot(e^\beta\otimes q(s)\cdot\mu^s)\}\rangle\\
&\quad=(z_1-z_2)^N\iota_{123}^{-1}z_1^{-\chi(\alpha)}z_2^{-\chi(\beta)}\langle v,I_{s+\beta}^\alpha(x^\alpha,z_1)I_s^\beta(x^\beta,z_2)I_s^0(\mathbf{1},z_3)w^s\\
&\qquad\otimes\{\lambda_s(\alpha,\beta)^{-1}e^{\alpha+\beta}\otimes q(s)\cdot\mu^s\}\rangle\\
&\quad=\iota_{345}^{-1}(z_3+z_4)^{-\chi(\alpha)}(z_3+z_5)^{-\chi(\beta)}\\
&\qquad\cdot\langle v,\lambda_s(\alpha,\beta)I_s^{\alpha+\beta}((z_4-z_5)^N Y_{V_D}(x^\alpha,z_4)Y_{V_D}(x^\beta,z_5)\mathbf{1},z_3)w^s\\
&\qquad\otimes\{\lambda_s(\alpha,\beta)^{-1}e^{\alpha+\beta}\otimes q(s)\cdot\mu^s\}\rangle|_{z_4=z_1-z_3,z_5=z_2-z_3}\\
&\quad=\iota_{354}^{-1}(z_3+z_4)^{-\chi(\alpha)}(z_3+z_5)^{-\chi(\beta)}\langle v,I_s^{\alpha+\beta}((z_4-z_5)^N\\
&\qquad\times Y_{V_D}(x^\beta,z_5)Y_{V_D}(x^\alpha,z_4)\mathbf{1},z_3)w^s\otimes\{e^{\alpha+\beta}\otimes q(s)\cdot\mu^s\}\rangle|_{z_4=z_1-z_3,z_5=z_2-z_3}\\
&\quad=(z_1-z_2)^N\iota_{123}^{-1}z_1^{-\chi(\alpha)}z_2^{-\chi(\beta)}\langle v,\lambda_s(\beta,\alpha)^{-1}I_{s+\alpha}^\beta(x^\beta,z_2)I_s^\alpha(x^\alpha,z_1)I_s^0(\mathbf{1},z_3)w^s\\
&\qquad\otimes\{(\lambda_s(\beta,\alpha)e^\beta\otimes q(s+\alpha)*e^\alpha\otimes q(s))\cdot\mu^s\}\rangle\\
&\quad=(z_1-z_2)^N\iota_{12}^{-1}z_1^{-\chi(\alpha)}z_2^{-\chi(\beta)}\langle v,\hat{Y}(x^\beta,z_2)\hat{Y}(x^\alpha,z_1)w^s\otimes\mu^s\rangle.
\end{aligned}
$$

Therefore, we get the commutativity. Similarly, we have

$$
\begin{aligned}
&\iota_{12}^{-1}z_1^{-\chi(\alpha)+N}z_2^{-\chi(\beta)}\langle v,\hat{Y}(x^\alpha,z_1)\hat{Y}(x^\beta,z_2)w^s\otimes\mu^s\rangle\\
&\quad=\iota_{12}^{-1}z_1^{-\chi(\alpha)+N}z_2^{-\chi(\beta)}\langle v,I_{s+\beta}^\alpha(x^\alpha,z_1)I_s^\beta(x^\beta,z_2)w^s\\
&\qquad\otimes\{e^\alpha\otimes q(s+\beta)\cdot(e^\beta\otimes q(s)\cdot\mu^s)\}\rangle\\
&\quad=\iota_{20}^{-1}\langle v,\lambda_s(\alpha,\beta)(z_2+z_0)^{-\chi(\alpha)+N}z_2^{-\chi(\beta)}I_s^{\alpha+\beta}(Y_{V_D}(x^\alpha,z_0)x^\beta,z_2)w^s\\
&\otimes\{\lambda_s(\alpha,\beta)^{-1}e^{\alpha+\beta}\otimes q(s)\cdot\mu^s\}\rangle|_{z_0=z_1-z_2}\\
&\quad=\iota_{20}^{-1}\langle v,(z_2+z_0)^{-\chi(\alpha)+N}z_2^{-\chi(\beta)}\hat{Y}(Y_{V_D}(x^\alpha,z_0)x^\beta,z_2)w^s\otimes\mu^s\rangle|_{z_0=z_1-z_2}.
\end{aligned}
$$

Hence, we obtain the associativity. □

Suppose that a simple VOA V and a finite group G acting on V is given. Then the G-invariants V^G of V, called the G-orbifold of V, is also a simple VOA by [8]. It is an important problem to classify the module category of V^G in the orbifold conformal field theory. It was conjectured in [3] that every irreducible V^G-module appears in a g-twisted V-module for some $g\in G$. In our case, V^0 is exactly the D^*-invariants of the extension V_D. By Theorem 3.2, we see that the conjecture is true for a pair (V_D,D^*).

Theorem 3.3. *Let V^0 be a rational, C_2-cofinite and CFT-type VOA, and D a finite abelian group. Let $V_D = \bigoplus_{\alpha \in D} V^\alpha$ be a D-graded simple current extension of V^0. Then every irreducible V^0-module W is contained in an irreducible σ-twisted V_D-module for some $\sigma \in D^*$. Moreover, σ is uniquely determined by W.*

3.2. Application to the 3A-algebra for the Monster

Here, we give an application of Theorem 3.2. We use the notation of [29] without any comments. A typical example of a vertex operator algebra generated by two conformal vectors is studied in [29]. We write it as U as in [29]. It has a shape $U = U^0 \oplus U^1 \oplus U^2$ with $U^0 = W(0) \otimes N(0)$, $U^1 = W(2/3)^+ \otimes N(4/3)^+$, $U^2 = W(2/3)^- \otimes N(4/3)^-$, and it is shown in [29] that U is a $\mathbb{Z}_3$-graded simple current extension of U^0. Define $\zeta \in \mathrm{Aut}(U)$ by $\zeta|_{U^i} = e^{2\pi\sqrt{-1}i/3} \cdot \mathrm{id}_{U^i}$. Then by Theorem 3.2 every irreducible U^0-module is uniquely induced to be a ζ^j-twisted U-module as follows:

(i) untwisted U-modules: $\mathrm{Ind}_{U^0}^U W(h) \otimes N(k)$, $h \in \{0, 2/5\}$, $k \in \{0, 1/7, 5/7\}$.
(ii) ζ-twisted U-modules: $\mathrm{Ind}_{U^0}^U W(h) \otimes N(k)^+$, $h \in \{0, 2/5\}$, $k \in \{4/3, 10/21, 1/21\}$.
(iii) ζ^2-twisted U-modules: $\mathrm{Ind}_{U^0}^U W(h) \otimes N(k)^-$, $h \in \{0, 2/5\}$, $k \in \{4/3, 10/21, 1/21\}$.

In the above, $\mathrm{Ind}_{U^0}^U X$ denotes $X \oplus (U^1 \boxtimes_{U^0} X) \oplus (U^2 \boxtimes_{U^0} X)$ for an irreducible U^0-module X (note that $U^i \boxtimes_{U^0} X \neq U^j \boxtimes_{U^0} X$ if $i \not\equiv j \bmod 3$). It is shown in [29] that $\mathrm{Aut}(U) = S_3$. Thus, we have a classification of $\mathbb{Z}_3$-twisted U-modules.

Theorem 3.4. *Every irreducible U^0-module is uniquely lifted to an irreducible ζ^i-twisted U-module for some $i = 0, 1, 2$.*

Remark 3.5. It is shown in [29] that U is contained in the famous moonshine VOA $V^\natural$ [13], and that ζ naturally induces a 3A element of Monster, the automorphism group of $V^\natural$.

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