Diophantine Approximation and Continued Fraction
Expansions of Algebraic Power Series in
Positive Characteristic

Alain Lasjaunias*

Université de Bordeaux I, Mathématiques Pures,
351, Cours de la Liberation, F-33405 Talence Cedex, France

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In a recent paper M. Buck and D. Robbins have given the continued fraction
expansion of an algebraic power series when the base field is \( \mathbb{F}_3 \). We study its
rational approximation property in relation with Roth’s theorem, and we show
that this element has an analog for each power of an odd prime number. At last
we give the explicit continued fraction expansion of another classical example.

1. INTRODUCTION

Let \( K \) be a field. We denote \( K((T^{-1})) \) the set of formal Laurent series
with coefficients in \( K \). If \( \pi = \sum_{k \leq \delta} a_k T^k \) is an element of \( K((T^{-1})) \), with
\( a_{\delta_k} \neq 0 \), we introduce the absolute value \( |\pi| = |T|^k \) and \( |0| = 0 \), with \( |T| > 1 \).
It is well known that Roth’s theorem (if \( \pi \) is an element of \( K((T^{-1})) \), irrational
algebraic over \( K(T) \), then for all real \( \varepsilon > 0 \) we have \( |\pi - P/Q| > |Q|^{-(2+\varepsilon)} \) for all \( P/Q \in K(T) \) with \( |Q| \) large enough) fails if \( K \) has a positive
characteristic \( p \). In this case, which is the one we consider here, Liouville’s
theorem (there is a real positive constant \( C \) such that \( |\pi - P/Q| \geq C |Q|^{-n} \) for all \( P/Q \in K(T) \), where \( n \) is the degree of \( \pi \) over \( K(T) \)) holds and is
optimal.

Many examples can be studied. A special case is the one where \( \pi \) satisfies
an equation of the form \( \pi = (Ax^s + B)/(Cx^r + D) \), where \( A, B, C, D \) belong
to \( K[T] \), with \( AD - BC \neq 0 \), and \( s \) is a positive integer. Those elements
have been studied by Baum and Sweet, Mills and Robbins, Voloch, de
Mathan [1, 5–7]. To simplify we will say that such an irrational algebraic
element is an element of class I. It is also possible to study some particular

*e-mail: lasjauni@math.u-bordeaux.fr.
rational functions, with coefficients in $K[T]$, of an element of class I (this was done by Voloch in [8]). For such simple examples, if $d$ is a real number such that, for every $\epsilon > 0$, we have $|x - P/Q| > |Q|^{-d+\epsilon}$ for $|Q|$ large enough, then there is a real positive constant $C$ such that $|x - P/Q| \geq C |Q|^{-d}$, for all $P/Q$. But all these examples seem to be exceptions. It seems that, except for “particular” elements, Roth’s theorem holds, and for an irrational algebraic element, for every $\epsilon > 0$, we have $|x - P/Q| \geq C |Q|^{-2}$ for all $P/Q$. Nevertheless, no algebraic element $x$, for which this result could be established, was known. It has only been proved that if $x$ is an algebraic element of degree $n$, not of class I, then Thue’s theorem holds, i.e., $|x - P/Q| > |Q|^{-\lfloor(n/2)\rfloor+\epsilon}$ for $|Q|$ large enough ([3]).

Buck and Robbins [2] have given the continued fraction expansion of a particular algebraic element of $F_3((T^{-1}))$. What is very curious in this example is that it does not belong to the set of exceptions already known. Indeed this element satisfies, for $|Q|$ large enough, $|x - P/Q| > |Q|^{-2+\epsilon}$ but not $|x - P/Q| \geq C |Q|^{-2}$, for all $P/Q$. Actually there are two real positive constants $\lambda_1$ and $\lambda_2$ such that, for some rationals $P/Q$ with $|Q|$ arbitrary large, we have $|x - P/Q| \leq |Q|^{-2+\lambda_1\sqrt{|Q|}},$ and for all rationals $P/Q$ with $|Q| > 1$, we have $|x - P/Q| \geq |Q|^{-2+\lambda_2\sqrt{|Q|}}$.

We have observed that $\alpha(T) = \beta(T)$, where $\beta$ satisfies $\beta = 1/(T + \beta^2)$; that is to say, $\beta^2$ is a rational function of an element of class I, but not such that it can be studied by the method mentioned above. This new approach allows us to give another proof of the result due to Buck and Robbins [2]. Let $\alpha$ be an irrational element of $K((T^{-1}))$. Then it may be expanded uniquely as a continued fraction. We write this continued fraction expansion as $\alpha = [a_0, a_1, a_2, ..., a_n, ...]$, where $a_k \in K[T]$ for $k \geq 0$ and $\deg a_k > 0$ for $k > 0$. With these notations, we will prove that, in $F_3((T^{-1}))$, we have

$$[T, T^2, T^3, ..., T^n, ...]^2 = [\lim_{n} \Omega_n],$$

where $(\Omega_n)_{n \geq 0}$ is a sequence of elements of $F_3[T]$, defined inductively by

$$\Omega_0 = \emptyset, \quad \Omega_1 = T^2, \quad \Omega_2 = \Omega_{n-1}, 2T^2, \Omega_{n-2}^2, 2T^2, \Omega_{n-1} \text{ for } n \geq 2$$

and $\lim_{n} \Omega_n$ denotes the sequence beginning with $\Omega_n$ for all $n \geq 0$. This has been obtained by studying a general case. Let $q$ be a power of an odd prime number $p$; then we have considered, in $F_p((T^{-1}))$, the continued fraction expansion of $[T, T^2, ..., T^n, ...]_{(q-1)/2}$. We have not been able to describe it entirely for $q > 3$, but we show that it has an interesting structure which implies the above result for $q = 3$. The possibility of describing completely the general case, or even of improving the description given in this paper, is an open question.
At last we give the continued fraction expansion of a classical example of algebraic element, first introduced by Mahler [4].

2. A BADLY APPROXIMABLE ELEMENT

In [2], Buck and Robbins have given the continued fraction expansion of an element of $\mathbb{F}_3((T^{-1}))$. If $K = \mathbb{F}_3$, they show that the algebraic equation

$$x^4 + x^2 - Tx + 1 = 0$$  \hspace{1cm} (1)

has a unique solution in $K((T^{-1}))$, the continued fraction expansion of which can be totally described. Indeed, they define recursively the following polynomial sequences:

$$\Omega_0 = \emptyset, \quad \Omega_1 = T, \quad \Omega_n = \Omega_{n-1} - T, \quad \Omega_n^{(3)} - T, \quad \Omega_{n-1} \quad \text{for} \quad n \geq 2$$  \hspace{1cm} (2)

(here $\Omega_n^{(3)}$ denotes the sequence obtained by cubing each element of $\Omega_n$ and commas indicate juxtaposition of sequences); then they prove that $[0, \Omega_n]$ is the beginning for all $n > 0$ of the continued fraction expansion of this solution. Using this result we can prove:

**Theorem A.** Let $\alpha$ be the unique root of (1) in $\mathbb{F}_3((T^{-1}))$. Then there exist explicit positive real constants $\lambda_1$ and $\lambda_2$ such that for some rationals $P/Q$ with $|Q|$ arbitrarily large, we have

$$|\alpha - P/Q| \leq |Q|^{-(2 + \lambda_1/\deg Q)}$$  \hspace{1cm} (3)

and, for all rationals $P/Q$ with $|Q|$ sufficiently large, we have

$$|\alpha - P/Q| \leq |Q|^{-(2 + 2\lambda_2/\deg Q)}$$  \hspace{1cm} (4)

(We can take $\lambda_1 = 2/\sqrt{3}$ and $\lambda_2 > 2/\sqrt{3}$.)

**Proof.** We write $\alpha = [a_0, a_1, a_2, \ldots]$. For $k > 0$, we put $d_k = \deg a_k$ and $P_k/Q_k = [a_0, \ldots, a_k]$. It results, from the inductive definition (2), that all partial quotients are monomials, and all have a power of 3 as the degree. For $t \geq 1$, we define $k_t = \inf\{k \geq 1/d_k = 3^t\}$. If $k_t \leq k < k_{t+1}$, we have $d_k \leq d_{k_t} = 3^t$. For each $n \geq 0$, let us define the sequence $\Omega^*_n$ of the degrees of the elements of $\Omega_n$. We get

$$\Omega^*_0 = \emptyset, \quad \Omega^*_1 = 1, \quad \Omega^*_2 = 1111, \quad \Omega^*_3 = 1111111111.$$
From the recursive definition \(2\), we see, by induction on \(k\), that
\[
\sup_{n \geq k} \Omega^*_{n} = \sup_{n \geq k} \Omega^*_{n+1} = 3^k \quad \text{for} \quad k \geq 0;
\]
therefore, for \(k \geq 0\), \(2k + 1\) is the smallest integer \(n\) such that \(3^n\) belongs to \(\Omega^*_n\). Again, from \(2\) and by induction on \(k\), we see that \(\Omega^*_{n+1}\) has an odd number of terms, has \(3^n\) as the central term, and is reversible. All of this leads to
\[
\sum_{d_k} d_k = 3^t + 2 \sum_{k < k_j} d_k. \tag{5}
\]
Now we put \(\omega_n = \sum_{d_k} \omega_k d_k\). From \(2\), we obtain
\[
\omega_0 = 0, \quad \omega_1 = 1, \quad \omega_n = 2\omega_{n-1} + 3\omega_{n-2} + 2 \quad \text{for} \quad n \geq 2. \tag{6}
\]
It is easy to check that the sequence \(((3^n - 1)/2)^{n \geq 0}\) is the one satisfying \(6\). Hence by \(5\), we have
\[
\deg Q_{k-1} = \sum_{k < k_i} d_k = (\omega_{2i+1} - 3)/2 = (3^{2i+1} - 2.3^t - 1)/4. \tag{7}
\]
Thus \(3^t \geq (2/\sqrt{3}) \sqrt{\deg Q_{k-1}}, \) which gives \(|T|^{-3^t} \leq |Q_{k-1}|^{-2\sqrt{\deg Q_{k-1}}}.\) Also we have, for \(i \geq 1\)
\[
|a - P_{k-1}/Q_{k-1}| = |T|^{-3^t} |Q_{k-1}|^{-2}. \tag{8}
\]
This shows that \(3\) holds for \(P/Q = P_{k-1}/Q_{k-1}\) and for \(i \geq 1\), with \(\lambda_1 = 2/\sqrt{3}.\)

On the other hand, we see that \(\deg Q_{k-1} < \deg Q_k \leq \deg Q_{k+1} - 1\) implies \(|x - P_k/Q_k| = |T|^{k+1} |Q_k|^{-2} \geq |T|^{-3^t} |Q_{k-1}|^{-2}.\) As, by \(7\), the sequence \((3^t/\sqrt{\deg Q_{k-1}})_{k \geq 1}\) converges to \(2/\sqrt{3}\), then, if \(\lambda_2 > 2/\sqrt{3}\), we can write \(3^t \leq \lambda_2 \sqrt{\deg Q_{k-1}} \leq \lambda_2 \sqrt{\deg Q_k}\) for \(i\) large enough. It follows that \(4\) holds for \(P_k/Q_k\) with \(k\) large enough. Since the convergents are the best rational approximations, this is also true for all \(P/Q\) with \(|Q|\) large enough. So the theorem is proved.

Remark. The fact that for this element and for all \(x > 0\), we have \(|x - P/Q| > |Q|^{-12 + \varepsilon}\) for \(|Q|\) large enough, but not \(|x - P/Q| > C |Q|^{-2}\) for all \(P/Q\), implies that it is not of class I, according to the theorem proved in [5] or [7]. In the same paper [2], the authors have considered the unique solution, in \(K(T^{-1})\), of the algebraic equation \(1\), when the base field is \(K = F_{13}\). In that situation the solution is actually of class I. After some calculation, it can be seen that \(1\) implies \(x = (Ax^{13} + B)/(C^{13} + D)\) with \(A = T^2 + 1, B = T^5 + 2T^2 + 2T, C = 9T, \) and \(D = T^8 + T^4 + 11T^2 + 1.\) (We can observe that the conjecture made by the authors,
3. A POWER OF A SIMPLE ELEMENT OF CLASS I

Here we come back to the element of $\mathbb{F}_3((T^{-1}))$, mentioned above, first introduced by Mills and Robbins in [6], satisfying

$$x^4 + x^2 - Tx + 1 = 0$$ (1)

Let $p$ be an odd prime number, let $q$ be a power of $p$, and let $K = \mathbb{F}_p$. We consider the elements $x_q$ of $K((T^{-1}))$ defined by its continued fraction expansion:

$$x_q = [0, T, T^q, ..., T^{q^n}, ...].$$ (2)

This element is of class I, being the unique root, in $K((T^{-1}))$, of the algebraic equation

$$x^{q+1} + Tx - 1 = 0.$$ (3)

We put $r = (q + 1)/2$ and we consider the element $\theta_q$, of $K((T^{-1}))$, defined by $\theta_q = x_q'$. We observe that (3) implies $x_q = (1/T)(1 - x_q^r)$ which leads to $\theta_q = (1/T')(1 - x_q^r)$. So $\theta_q$ is a solution of the algebraic equation

$$x = (1/T')(1 - x^r).$$ (4)

If $x$ is a solution of (4), in $K((T^{-1}))$, we must have $|x| \leq 1$. Since otherwise $|x| > 1$ gives $|(1 - x^r)| = |x|^b$, and by (4), $|T'| = |x|^q$ which is impossible. We consider the set $E = \{x \in K((T^{-1}))/|x| \leq 1\}$, and the map $f$ of $E$ into itself defined by $f(x) = (1/T')(1 - x^r)$. Then we can see that $f$ is a contraction mapping, $E$ is complete, and therefore $f(x) = x$ has a unique solution in $E$. So $\theta_q$ is the unique root of (4) in $K((T^{-1}))$. Also, the coefficients of this equation are elements of $K(T)$; thus its solution $\theta_q$ is an element of $K((T^{-1}))$. Then we can introduce the element $\theta_q^*$ of $K((T^{-1}))$, defined by $\theta_q(T) = \theta_q^*(T')$. So $\theta_q^*$ is the unique solution, in $K((T^{-1}))$, of the algebraic equation

$$x = (1/T)(1 - x^r).$$ (5)
Now we see that, if $q=3$, we have $\theta_3^* = (1/T)(1 - (\theta_3^*)^2)^2 = (1/T)(1 + (\theta_3^*)^2 + (\theta_3^*)^4)$, so that $\theta_3^*$ is the root of $(1)$ in $\mathbb{F}_3(T^{-1})$.

Here we shall see that the link between $\theta_q$ and $\varepsilon_q$ is simple enough to give a partial description of the continued fraction expansion of this element, this description being complete for $q=3$. We start from the continued fraction expansion of $\varepsilon_q$. Let us consider the usual two sequences of polynomials of $K[T]$, defined inductively by

$$
P_0 = 0, \quad P_1 = 1, \quad Q_0 = 1, \quad Q_1 = T,
$$

$$
P_n = T^{\varepsilon_q^{-1}}P_{n-1} + P_{n-2}, \quad Q_n = T^{\varepsilon_q^{-1}}Q_{n-1} + Q_{n-2}
$$

for $n \geq 2$. So $(P_n/Q_n)_{n \geq 0}$ is the sequence of the convergents to $\varepsilon_q$. By $(2)$, for $n \geq 1$, we have

$$
P_n/Q_n = [0, T, T^{\varepsilon_q}, ..., T^{\varepsilon_q^{-1}}] = 1/(T + [0, T, T^{\varepsilon_q}, ..., T^{\varepsilon_q^{-1}}]^\varepsilon) = 1/(T + (P_{n-1}/Q_{n-1})^\varepsilon).
$$

Since $P_n$ and $Q_n$ are coprime and both unitary, we obtain

$$
P_0 = 0, \quad P_n = Q_n, \quad Q_0 = 1, \quad Q_n = TQ_{n-1} + P_{n-1} \quad \text{for} \quad n \geq 1.
$$

Now let us consider the continued fraction expansion of $\theta_q$. We set $\theta_q = [a_0, a_1, ..., a_n, ...]$. We observe that $a_0 = 0$ from the definition of $\theta_q$ since $|\varepsilon_q| < 1$. Then we introduce the usual two sequences of polynomials of $K[T]$, defined inductively by

$$
U_0 = 0, \quad U_1 = 1, \quad V_0 = 1, \quad V_1 = a_1,
$$

$$
U_n = a_n U_{n-1} + U_{n-2}, \quad V_n = a_n V_{n-1} + V_{n-2}
$$

for $n \geq 2$. So $(U_n/V_n)_{n \geq 0}$ is the sequence of the convergents to $\theta_q$.

First we are going to give some special subsequences of convergents to $\theta_q$. We use the following auxiliary results:

**Lemma 1.** For $n \geq 0$, the polynomial $a_n$ is an odd polynomial in the indeterminate $T$ and the rational $(P_n/Q_n)^\varepsilon$ is a convergent to $\theta_q$.

**Proof.** We know that Eq. $(5)$ has $\theta_3^*$ as unique solution in $K(T^{-1})$. From $(5)$ we see that

$$
\theta_3^*(-T) = (-1/T)(1 - (\theta_3^*(-T))^2)^\varepsilon.
$$
thus
\[ -\theta_q^*(-T) = (1/T)(1 - (-\theta_q^*(-T))^2). \]

Therefore \(-\theta_q^*(-T)\) is also a solution of (5), and we have \(-\theta_q^*(-T) = \theta_q^*(T)\). That is to say, \(\theta_q^*\) is an odd element of \(K(T^{-1})\) and by induction we see that the partial quotients of the continued fraction expansion of \(\theta_q^*\) are odd polynomials of \(K[T]\). If we write \(\theta_q^* = [a_0^*(T), a_1^*(T), \ldots, a_s^*(T), \ldots]\), then, because of the identity \(\theta_q^*(T') = \theta_q(T)\), we have \(a_i(T) = a_i^*(T')\).

Now we show that \((P_n/Q_n)^r\) is a convergent to \(\theta_q\). Indeed, for \(n \geq 0\)
\[
|x_q - (P_n/Q_n)^r| = |x_q - P_n/Q_n| \sum_{0 \leq i < r} x_q^i (P_n/Q_n)^{r-i-1}.
\]

Since \(|x_q| = |P_n/Q_n| = |T|^{-1}\), we have \(r\) terms in the sum, each with absolute value \(|T|^{-r+1}\) and dominant coefficient 1. Therefore, as \(r\) and \(p\) are coprime, this becomes
\[
|x_q - (P_n/Q_n)^r| = |x_q - P_n/Q_n| |T|^{-r+1} = |Q_n Q_{n+1}|^{-1} |T|^{-r+1}.
\]

From (6) we get \(|Q_{n+1}| = |Q_n| |T|^{-1}\), which gives
\[
|\theta_q - (P_n/Q_n)^r| = |Q_n|^{-2} |T|^{-r}. \quad (7)
\]
This shows that \((P_n/Q_n)^r\) is a convergent to \(\theta_q\), and the lemma is proved.

**Lemma 2.** Let \(P\) and \(Q\) be two polynomials of \(K[T]\), with \(Q \neq 0\), and \(n\) a positive integer. If
\[
|Q| < |Q_n|^{r} \quad \text{and} \quad |PQ_n^r - QP_n^r| < |Q_n|^{r}/|Q| \quad (8)
\]
then \(P/Q\) is a convergent to \(\theta_q\). Moreover, if \(P\) and \(Q\) are coprime and the convergent \(P/Q\) is \(U_k/V_k\), then we have
\[
|a_{k+1}| = |PQ_n^r - QP_n^r|^{-1} |Q|^{-1} |Q_n|^r. \quad (9)
\]

**Proof.** By (7) and (8), we have
\[
|\theta_q - (P_n/Q_n)^r| = \frac{1}{|Q_n|^{r-1} |T|^{-r}} < \frac{1}{|Q_n|^{r-1} |T|} \leq \frac{|PQ_n^r - QP_n^r|}{|Q_n|^{r-1} |Q|}. \quad (10)
\]

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since \(|Q| < |Q_n|\)' and \((P_n, Q_n) = 1\) implies \(PQ_n - QP_n \neq 0\). Hence,

\[ |\theta_q - (P_n/Q_n)| < |P/Q - (P_n/Q_n)|.\]

Therefore,

\[ |\theta_q - P/Q| = |\theta_q - (P_n/Q_n) + (P_n/Q_n) - P/Q| = |P/Q - (P_n/Q_n)|\]

and by (8)

\[ |\theta_q - P/Q| < |Q|^{-2}.\]

This shows that \(P/Q\) is a convergent to \(\theta_q\). Now if \(P\) and \(Q\) are coprime and \(P/Q = U_k/V_k\), we have \(|Q| = |V_k|\). Besides, we know that

\[ |\theta_q - U_k/V_k| = |V_k|^{-2} |a_{k+1}|^{-1}.\]

Since

\[ |\theta_q - U_k/V_k| = |P/Q - (P_n/Q_n)|,\]

it is clear that (9) holds. So Lemma 2 is proved.

**Lemma 3.** Let us consider the elements of \(K(T)\), defined by

\[ \Theta_q(T) = \frac{T^q}{(T^2 + 1)^q} \quad \text{and} \quad \Theta'_q(T) = \frac{T^q}{(T^2 - 1)^q}.\]

Then we have the continued fraction expansions in \(K(T)\):

\[ \Theta_q(T) = [0, T, 2T, 2T, \ldots, 2T, T] \]

\( (2T \text{ is repeated } q-1 \text{ times}) \) \hspace{1cm} (10)

\[ \Theta'_q(T) = [0, T, -2T, 2T, \ldots, -2T, 2T, -T] \]

\( (-2T, 2T \text{ is repeated } q-1 \text{ times}) \). \hspace{1cm} (11)

**Proof.** Let \((R_k)_{0 \leq k \leq q+1}\) be the sequence of elements of \(K(T)\), defined inductively by

\[ R_0 = 0, \quad R_1 = 1, \quad R_k = 2TR_{k-1} + R_{k-2} \]

for \(2 \leq k \leq q\), \(R_{q+1} = TR_q + R_{q-1}\). \hspace{1cm} (12)
Then, by the usual property of a linear recurrent sequence, we have
\[ R_k = \frac{1}{2\sqrt{T^2 + 1}}((T + \sqrt{T^2 + 1})^k - (T - \sqrt{T^2 + 1})^k) \quad \text{for} \quad 1 \leq k \leq q. \quad (12') \]

Now we introduce the sequence \((S_k)_{0 \leq k \leq q + 1}\) of elements of \(K[T]\), defined inductively by
\[ S_0 = 1, \quad S_1 = T, \quad S_k = 2TS_{k-1} + S_{k-2} \]
for \(2 \leq k \leq q, \quad S_{q+1} = TS_q + S_{q-1}. \quad (13) \]

So \((R_k/S_k)_{0 \leq k \leq q + 1}\) are the convergents to \([0, T, 2T, \ldots, 2T, T]\), and (10) will be proved if we show that:
\[ R_{q+1} = T^q \quad \text{and} \quad S_{q+1} = (T^2 + 1)^r \quad (14) \]

First we prove that
\[ S_k = TR_k + R_{k-1} \quad (13') \]
holds for \(1 \leq k \leq q\). By induction, since \(S_k\) and \(R_k\) satisfy the same recursive relation, it suffices to see that \((13')\) is satisfied for \(k = 1\) and \(k = 2\).

Now we prove that:
\[ R_q = (T^2 + 1)^{r-1} \quad \text{and} \quad S_q = T^q \quad (15) \]

Indeed, by \((12')\), we have
\[ R_q = \frac{1}{2\sqrt{T^2 + 1}}((T^q + (\sqrt{T^2 + 1})^q) - (T^q - (\sqrt{T^2 + 1})^q)) = (T^2 + 1)^{r-1} \]
\[ R_{q-1} = \frac{1}{2\sqrt{T^2 + 1}} \left( \frac{T^q + (\sqrt{T^2 + 1})^q}{T + \sqrt{T^2 + 1}} - \frac{T^q - (\sqrt{T^2 + 1})^q}{T - \sqrt{T^2 + 1}} \right) \]
\[ = T^q - T(T^2 + 1)^{r-1}. \]

Then, by \((13')\), we get \(S_q = TR_q + R_{q-1} = T^q\). By \((12)\), we also get \(R_{q+1} = TR_q + R_{q-1} = T^q\). Now we compute \(S_{q+1}\). From the classical identity \(R_{q+1}S_q - S_{q+1}R_q = 1\), we obtain, with \((14)\) and \((15)\), \(S_{q+1}R_q = T^{2q} + 1 = (T^2 + 1)^q\); hence \(S_{q+1} = (T^2 + 1)^r\). So (10) is proved.
Now we show that (11) is a consequence of (10). Let $u$ be a square root of $-1$ eventually in an extension of $K$. We have

$$u \Theta_q(uT) = \frac{u^{2qT^q}}{(-T^2 + 1)^q} = \frac{T^q}{(T^2 - 1)^q} = \Theta_q'(T).$$

From this identity and (10), it follows that

$$\Theta_q'(T) = u[0, uT, 2uT, ..., 2uT, uT].$$

Using the property of the multiplication of a continued fraction expansion by a scalar, we have

$$\Theta_q'(T) = [0, T, 2u^2T, 2T, ..., 2T, u^2T].$$

So (11) is proved.

We observe, from (12') and (13'), that the polynomial $R_i$ has the opposite parity to the integer $i$ and the polynomial $S_i$ has the same parity as the integer $i$. For $0 \leq i \leq q + 1$, we introduce the elements of $K[T]$, defined by

$$R_i = R_i(uT), \quad S_i(T) = -uS_i(uT) \quad \text{for } i \text{ odd}$$

$$R_i' = uR_i(uT), \quad S_i'(T) = S_i(uT) \quad \text{for } i \text{ even}. \quad (16)$$

Since we have $u(R_i/S_i)(uT) = (R_i'/S_i')(T)$, it is clear, by the same argument as above, that $R_i'/S_i'$ are the convergents to $\Theta_q'(T)$.

**Lemma 4.** For $1 \leq i \leq q$, let $R_i$, $S_i$, $R_i'$, and $S_i'$ be the elements of $K[T]$ introduced in Lemma 2. Notations being as above, for $n \geq 0$, we put

$$R_{i, n} = P_{i, n}R_i(Q_{i, n}), \quad S_{i, n} = S_i(Q_{i, n}) \quad \text{for } n \text{ odd}$$

$$R_{i, n} = P_{i, n}R_i'(Q_{i, n}), \quad S_{i, n} = S_i(Q_{i, n}) \quad \text{for } n \text{ even.} \quad (16)$$

Then, for $n \geq 0$, $R_{i, n}/S_{i, n}$ is a convergent to $0_q$. Further $R_{i, n}$ and $S_{i, n}$ are coprime, and if $m(i, n)$ is the integer such that $U_{m(i, n)}/V_{m(i, n)} = R_{i, n}/S_{i, n}$, then $d_{m(i, n) + 1} = \lambda_{i, n}T^r$, where $\lambda_{i, n}$ is a nonzero element of $K$. Moreover, for $n \geq 0$, we have

$$R_{i, n}/S_{i, n} = P_{i, n}/Q_{i, n}, \quad R_{q, n}/S_{q, n} = Q_{q, n - 1}P_{q, n - 1}/P_{q, n} \quad (17)$$

and the convergent preceding $R_{i, n}/S_{i, n}$ is $R_{q, n - 1}/S_{q, n - 1}$; i.e.,

$$R_{q, n - 1}/S_{q, n - 1} = U_{m(1, n - 1)}/V_{m(1, n - 1)} \quad \text{for all } n \geq 1. \quad (18)$$
Proof. Let \( n \) and \( i \) be integers such that \( n \geq 0 \) and \( 1 \leq i \leq q \). We shall apply Lemma 2 with \( P = R_{i,n} \) and \( Q = S_{i,n} \). First, by (13) and (16), we have \( |S_{i}| = |S'_{i}| = |T| \); hence we have \( |S_{i,n}| = |Q_{i}|^q \). Then by (6), \( |Q_{i}|^q \leq |Q_{i+1}|^q < |Q_{n+1}|^q \). Thus we have \( |S_{i,n}| < |Q_{n+1}|^q \), which is the first part of condition (8). We put \( \delta_{i,n} = R_{i,n} Q_{n+1}^n - S_{i,n} P_{n+1}^n \). For \( n \) odd, we have
\[
\delta_{i,n} = P_{n}^n R_{i,n} (Q_{n}^n) Q_{n+1}^n - S_{i,n} P_{n+1}^n.
\]
By (6), (14), and since we have \( P_{n+1} Q_{n} - P_{n} Q_{n+1} = -1 \), we get
\[
\delta_{i,n} = (Q_{n}^n + 1)^r (Q_{n}^n) - S_{i,n} P_{n+1}^n.
\]
In the same way, for \( n \) even, by (6), (14), and since we have \( P_{n+1} Q_{n} - P_{n} Q_{n+1} = 1 \), we get
\[
\delta_{i,n} = (Q_{n}^n - 1)^r (Q_{n}^n) - S_{i,n} P_{n+1}^n.
\]
We observe, from (14) and (16), that \( R_{q+1} = (-1)^r T^q \) and \( S_{q+1} = (-T^2 + 1)^r \), so we obtain
\[
\delta_{i,n} = (1)^r A' (Q_{n}^n) \quad \text{with} \quad A' = S_{q+1} R_{q+1} - S_{q} R_{q+1}.
\]
Also we have \( |R_{q+1}/S_{q+1} - R_{i}/S_{i}| = |S_{q+1}/S_{i}| \) and, therefore,
\[
|A'| = |S_{q+1}/S_{i}| |R_{q+1}/S_{q+1} - R_{i}/S_{i}| = |S_{q+1}/S_{i}|.
\]
By (12) and (13), we see that \( |S_{i}| = |T| \) and \( |R_{i}| = |T|^{-1} \), then we get \( |A'| = |T|^{q-r} \). The same way, by (16), \( |S_{i}^q| = |S'_{i}| \), \( |R_{i}^q| = |R'_{i}| \), so we obtain \( |A'_{i}| = |T|^{q-r} \). Thus, as \( |S_{i,n}| = |Q_{i}|^{q-r} \), and by (6) \( |Q_{n+1}|^q = |Q_{n}|^q \), we get
\[
|\delta_{i,n}| = |Q_{i}^n|^{q-r} < |Q_{n+1}|^q < |Q_{n}|^q < |Q_{i,n}|^q,
\]
which is the second part of condition (8). So by Lemma 2, \( R_{i,n}/S_{i,n} \) is a convergent to \( \theta_{i} \) for \( n \geq 0 \) and \( 1 \leq i \leq q \).

Now we prove that \( R_{i,n} \) and \( S_{i,n} \) are coprime. First we show that \( A_{i} \) and \( S_{i} \) are coprime (the same for \( A'_{i} \) and \( S'_{i} \)). We have \( A_{i} + S_{i} T^q = (T^2 + 1)^r R_{i} \) (or \( A'_{i} + (-1)^r S'_{i} T^q = (T^2 + 1)^r R_{i} \)). Hence, since \( R_{i} \) and \( S_{i} \) are coprime (or \( R_{i}^q \) and \( S_{i}^q \) are coprime), we see that if \( A_{i} \) is a prime common divisor of \( A_{i} \) and \( S_{i} \) (or of \( A'_{i} \) and \( S'_{i} \)), then it divides \( T^2 + 1 \) (or \( T^2 - 1 \)). Now if \( S_{i} \)
has such a divisor then we have \( S_i(u) = 0 \) or \( S_i(-u) = 0 \), where \( u \) is a square root of \(-1\). From (13) we deduce
\[
S_0(u) = 1, \quad S_1(u) = u, \quad S_i(u) = 2uS_{i-1}(u) + S_{i-2}(u) \quad \text{for} \quad 1 \leq i \leq q,
\]
and this implies \( S_i(u) = u^i \) for \( 1 \leq i \leq q \). As \( S_i \) is alternatively an odd or even polynomial, we also have \( S_i(-u) = (-1)^i S_i(u) \). Therefore \( S_i(\pm u) \neq 0 \), and consequently \( A_i \) and \( S_i \) are coprime. For \( A_i \) and \( S_i \), the same proof holds.

Here we have to prove that \( S_i(\pm 1) \neq 0 \), and this is derived from (16) and the fact that \( S_i(\pm u) \neq 0 \). Hence there are polynomials \( E \) and \( F \) of \( K[T] \) such that
\[
E A_i + F S_i = 1
\]
where from \( E(Q^r_i) A_i(Q^r_i) + F(Q^r_i) S_i(Q^r_i) = 1 \).

Thus \( A_i(Q^r_i) \) and \( S_i(Q^r_i) \) are coprime (the same for \( A_i(Q^r_i) \) and \( S_i(Q^r_i) \)). Now we return to \( R_{i n} \) and \( S_{i n} \). If \( B \) is a common divisor of both of them, then \( B \) divides \( R_{i n} Q_{n+1} - S_{i n} Q_{n+1} = A_i(Q^r_i) \) and \( S_{i n} = S_i(Q^r_i) \) (or \((-1)^i A_i(Q^r_i) \) and \( S_i(Q^r_i) \)), and therefore divides 1. So we have the desired result.

Then Lemma 2 applies. By (9), we obtain
\[
| a_{n i, n} | = | \delta_{n i, n} |^{-1} | S_{i n} |^{-1} | Q_{n+1} | = | Q_n |^{-r} | Q_n |^{-1} | Q_{n+1} | = | T |,
\]
so by Lemma 1, \( a_{n i, n} = l_{i, n} T \), where \( l_{i, n} \) is a nonzero element of \( K \).

Now we compute \( R_{1 n} / S_{1 n} \) and \( R_{q n} / S_{q n} \). Since \( R_1 = R_1 = 1 \) and \( S_1 = S_1 = T \), the definition gives immediately the first part of (17). By (15), we have \( (R_{i n} / S_{i n})(T) = (T^2 + 1)^{-1} | T^r | \). By (15) and (16), we obtain \( (R_{i n} / S_{i n})(T) = (T^2 + 1)^{-1} | T^r | \). Therefore \( R_{q n} / S_{q n} = P_{(Q^r_n)}(Q_n^r) \) \((-1)^{r-1} | T^r | \). Moreover, by (6), we have \( Q_n^r = P_{n+1} \) and then \( Q_n^r = P_{n+1} \). So \( R_{q n} / S_{q n} = P_{(Q_n^r)}(Q_n^r) \). Finally, we have
\[
| S_{q n} | = | Q_n r^r | = | (Q_n + 1) | / | T | = | S_{1 n} | / | T |.
\]
Since the denominators of the convergents are polynomials of \( K[T^r] \), \( R_{q n} / S_{q n} \) must be the convergent preceding \( R_{1 n+1} / S_{1 n+1} \). This is (18), and so Lemma 4 is proved.

Now we can describe partially the continued fraction expansion of \( \theta_q \).

With the notations of Lemma 4, we can write \( R_{i n} / S_{i n} = [a_1, a_2, \ldots, a_{n i, n}] \), for \( n \geq 0 \) and for \( 1 \leq i \leq q \). We put \( \Omega_{1 n} = a_1, a_2, \ldots, a_{n i, n} \), for all \( n \geq 1 \). We can write explicitly \( \Omega_{1 n} \) and \( \Omega_{1} \). By (17), we have \( R_{i n} / S_{i n} = [0, \Omega_{1, n}] = (P_{n} / Q_{n}) \). By (6), we get \( R_{1 n} / S_{1 n} = (P_{1} / Q_{1}) \). So \( \Omega_{1} = a_1 = T \).

Further, by (6) and with the notations of Lemma 3, we have
\[
R_{1, 2} / S_{1, 2} = (P_{2} / Q_{2}) = T^r / (T^r + 1)^{r} = \Theta_{r}(T^r).
\]
Therefore, by (10), we get
\[ \Omega_{1,2} = T'; 2T', 2T', ..., 2T', T' \quad (q + 1 \text{ terms}). \] (19)

We observe that, for \( n \geq 1 \), we have \( m(1, n) < m(2, n) < \cdots < m(q, n) \). Indeed \( |S_{1,n}| > |S_{2,n}| \), since \( |S_{1,n}| = |Q_n|' \) and \( |Q_n| > 1 \), for \( n \geq 1 \). Then we put \( \Omega_{i,n} = a_{m(i-1,n)+1}, \cdots, a_{m(i,n)} \), for \( n \geq 1 \) and \( 2 \leq i \leq q \). We define also \( \Omega_{i,n} \) by \( \Omega_{i,n} = a_{m(i-1,n)+1}, \Omega_{i,n} \) and \( \Omega'_{i,n} \) by \( \Omega'_{i,n} = T'; \Omega'_{i,n} \).

If \( \Omega = x_1, x_2, \ldots, x_k \) is a sequence of polynomials, we denote \( \bar{\Omega} \) the sequence obtained by reversing the terms of \( \Omega \), i.e., \( \bar{\Omega} = x_k, x_{k-1}, \ldots, x_1 \).

Also if \( \varepsilon \) is a nonzero element of \( K \) we write \( \varepsilon[\Omega] \) for \( \varepsilon x_1, \varepsilon^{-1}, x_2, \ldots, \varepsilon^{(-1)^{k-1}}x_k \). Notice that if \( [\Omega] \) denotes the element of \( K(T) \) which has \( \Omega \) as a continued fraction expansion, we have \( \varepsilon[\Omega] = [\varepsilon\Omega] \). Now we can prove the following result.

**Lemma 5.** There exists a sequence \((\varepsilon_n)_{n \geq 1}\) of nonzero elements of \( K \), such that
\[ a_{m(1,n)-k} = \varepsilon_{n}^{(-1)^{k}} a_{k+1} \quad \text{for each } (k, n) \text{ with } 0 \leq k < m(1, n) - 1; n \geq 1. \] (20)

Further we have for \( n \geq 2 \),
\[ \Omega_{q,n} = \varepsilon_{n+\frac{1}{2}}^{\frac{-1}{2}} \bar{\Omega}_{q,n}, \quad \Omega_{q-i,n} = \varepsilon_{n+i}^{\frac{-1}{2}} \bar{\Omega}_{i,n} \quad \text{for } 1 \leq i \leq r - 2 \]
\[ \lambda_{q,n} = \varepsilon_{n+\frac{1}{2}}^{\frac{-1}{2}} \lambda_{q,n}, \quad \lambda_{q-i,n} = \varepsilon_{n+i}^{\frac{-1}{2}} \lambda_{i,n} \quad \text{for } 1 \leq i \leq r - 1. \] (21)

**Proof.** By (17) and (18), we can write
\[ U_{m(1,n)} = \varepsilon_{n}\xi_{n}, \quad V_{m(1,n)} = \varepsilon_{n}^{\frac{-1}{2}} \bar{\xi}_{n} \]
and
\[ U_{m(1,n)-1} = \varepsilon_{n}^{\frac{-1}{2}} \xi_{n-\frac{1}{2}} \bar{\xi}_{n-1} \quad \text{for } n \geq 2. \]
(22)

where \( \varepsilon_n \) and \( \xi_n \) are nonzero elements of \( K \). We write \( \varepsilon_n = \varepsilon_n'/\varepsilon_n'' \).

From the definition of \( V_k \), for each \( k \geq 1 \), we have \( V_{k}/V_{k-1} = [a_k, a_k^{-1}, \ldots, a_1] \), so we can write \( V_{m(1,n)} / V_{m(1,n)-1} = [a_{m(1,n)}, a_{m(1,n)-1}, \ldots, a_1] \).

On the other hand, by (22) and (23), we have
\[ \frac{V_{m(1,n)}}{V_{m(1,n)-1}} = \varepsilon_{n} \]
and
\[ \frac{V_{m(1,n)-1}}{U_{m(1,n)}} = [0, a_1, \ldots, a_{m(1,n)}] = \varepsilon_{n} [a_1, \ldots, a_{m(1,n)}]; \]
Using Lemma 5, (24) becomes

This implies (20) and can be written

By Lemma 4 and (18), we have

and (20) becomes

and also

We put

then by (20) we have

being the sum of the degrees of its terms. We have

For each finite sequence of nonzero polynomials, we define its degree as

In this case the continued fraction expansion of \( \theta \) will be given explicitly below. We prove the result already obtained by Buck and Robbins in [2].
Here \( \Omega_{1,n-1}^{(3)} \) denotes the sequence obtained by cubing each element of \( \Omega_{1,n-1} \).

**Proof.** Let \( n \) be an integer with \( n \geq 2 \). First we are going to explicit \( \Omega_{2,n} \). We have \( U_{m(1,n)}/V_{m(1,n)} = [0, \Omega_{1,n}], \) \( U_{m(1,n)+1}/V_{m(1,n)+1} = [0, \Omega_{1,n}, \Omega_{1,n}^2] \) and \( U_{m(2,n)}/V_{m(2,n)} = [0, \Omega_{1,n}, \Omega_{1,n}^2, \Omega_{2,n}] \). If we denote \( x_{2,n} \), the element of \( K(T) \) defined by \( [\Omega_{2,n}] \), then it is a classical fact that we have

\[
\frac{U_{m(2,n)}}{V_{m(2,n)}} = \frac{x_{2,n}U_{m(1,n)+1} + U_{m(1,n)}}{x_{2,n}V_{m(1,n)+1} + V_{m(1,n)}}.
\]

(26)

We know that \( U_{m(2,n)}/V_{m(2,n)} = R_{2,n}/S_{2,n} \). We have \( R_1(T) = 2T, S_1(T) = 2T^2 + 1 \), and also \( R'_1(T) = uR_1(uT) = -2T, S'_1(T) = S_1(uT) = -2T^2 + 1 \). It follows that \( R_{2,n}/S_{2,n} = P_n^2 Q_n^2(Q_n^2 + (-1)^n) \). We put

\[
P' = P_n^2 Q_n^2 \quad \text{and} \quad Q' = Q_n^4 + (-1)^n.
\]

(27)

Then formula (26) can be solved for \( x_{2,n} \), and by (22), we obtain

\[
x_{2,n} &= x_n' \frac{P_n^2 Q' - Q_n^2 P'}{Q' - U_{m(1,n) + 1} Q_n^2}.
\]

(26')

We have to determine \( U_{m(1,n) + 1}/V_{m(1,n) + 1} \). We use Lemma 2 and the fact that \( R_{1,n-1}/S_{1,n-1} \) and \( R_{1,n}/S_{1,n} \) are, by Lemma 4, the two convergents preceding it.

So, we consider the polynomials \( P \) and \( Q \) of \( K(T) \), defined by

\[
P = 2T^2 P_n^2 + P_{n-1}^3 Q_n, \quad Q = 2T^2 Q_n^2 + P_n^2.
\]

(28)

We apply Lemma 2, to show that \( P/Q \) is a convergent to \( \theta_1 \). First we have deg \( Q = 2 \deg Q_n + 2 \) and thus \( Q \neq 0 \). By (28) and (6), we have

\[
P Q_n^2 = Q^3_n - P_{n-1} Q_n^2 = P_n^3 - P_n^3 - P_n^2 = ( -1)^n,
\]

so that \( (P, Q) = 1 \). Since \( 2 \deg Q_n + 2 < 2 \deg Q_{n+1} \) for \( n \geq 2 \), the first part of condition (8), i.e. \( |Q| < |Q_{n+1}|^2 \), is satisfied. Let us show that \( |P Q_{n+1}^2 - Q_{n+1}^3| < |Q_{n+1}|^2/|Q| \) is also satisfied. We put

\[
X_1 = Q_{n+1}^2 + P_n^2 Q_{n+1}^2 - Q_n^2 P_{n+1}^2, \quad X_2 = P_{n-1} Q_n Q_{n+1}^2 - P_n^2 P_{n+1}.
\]

By (28), we observe that \( P Q_{n+1}^2 - Q_{n+1}^3 = 2T^2 X_1 + X_2 \). As \( P_{n+1} Q_n - Q_{n+1} P_n = (-1)^n \), and using (6), we have

\[
X_1 = (-1)^{n+1} (2Q_n P_{n+1} + (-1)^{n+1})
\]

\[
= (-1)^{n+1} (2Q_n^3 + (-1)^{n+1}) = (-1)^n Q_n^4 + 1;
\]

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then
\[ X_2 = Q_{n+1}^2 + P_{n+1}^2, \]
\[ X_3 = (Q_{n+1}/Q_n)^2 - (1)^n + (P_n/Q_n)^2, \]
\[ X_4 = ((Q_{n+1}/Q_n)^2 + (P_n/Q_n)^2)(-1)^n + (P_n/Q_n)^3. \]

We put \( X = PQ_{n+1}^2 - Q_{n+1}^2. \) As \( X = 2T^2X_1 + X_2, \) we have
\[ X = 2T^2 + (-1)^n (2T^2 Q_n^4 + (Q_{n+1}/Q_n)^2 + (P_n/Q_n)^2) \]
\[ X = 2T^2 + (-1)^n (2T^2 Q_n^4 + (TQ_n^2 + P_n/Q_n)^2 + (P_n/Q_n)^2) \]
\[ = (-1)^{n+1} P_n^2 Q_n^3 - (P_n/Q_n) \]
\[ = (-1)^{n+1} P_n^2 Q_n^3 - (P_n/Q_n), \]

As \( Q_n^4 - TP_n Q_n^3 - P_n^4 = P_{n+1} Q_n - Q_{n+1} P_n = (-1)^n, \) we get
\[ X = 2T^2 + (-1)^n (2TQ_n^3 + P_n^2 Q_n^3 + (P_n/Q_n)^2) \]
\[ X = 2T^2 = (-1)^n (TP_n Q_n^3 + 2P_n^2 Q_n^3) \]
\[ X = 2T^2 = (-1)^n (TQ_n Q_n^3 + 2P_n^2 Q_n^3) \]
\[ = (-1)^n P_n^2 Q_n^3 - (P_n/Q_n) \]
\[ = (-1)^n P_n^2 Q_n^3 - (P_n/Q_n), \]

Since, for \( n \geq 2, \) \(|P_{n-1}^3| \leq |Q_n| \) and \(|P_n| \leq |Q_n|, \) this equality implies
\[ |X| < |Q_n|^4 = \frac{|Q_{n+1}|^2}{|Q|}, \]

so (8) is satisfied. Hence, \( P/Q \) is a convergent to \( \theta \) and, since \( \deg Q = \deg V_{m(1, n) + 2} \) and \( \theta \in \mathbb{F}((T^{-2})) \), it is the one following \( U_{m(1, n)}/V_{m(1, n)}. \)

Therefore we can write
\[ U_{m(1, n) + 1} = \eta_n P, \]
\[ V_{m(1, n) + 1} = \eta_n Q, \]

where \( \eta_n \) is an invertible element of \( \mathbb{F}_3 \). By (22), (23), (28), and \( \varepsilon^{-1} = \varepsilon \) for \( \varepsilon \in \mathbb{F}_3^*, \) the first equality of (29) can be written
\[ a_{m(1, n) + 1} U_{m(1, n) + 1} = a_{m(1, n) + 1} = \eta_n e_n 2T^2 U_{m(1, n) + 1} + \eta_n e_n U_{m(1, n) + 1}. \]

Since we have \( \deg U_{m(1, n) + 1} > \deg U_{m(1, n) + 1} \), it follows that \( a_{m(1, n) + 1} = \eta_n e_n 2T^2 \), and \( \eta_n e_n = 1, \) i.e. \( \eta_n = e_n. \) Thus, since \( e_n^2 e_n = e_n \), we obtain
\[ a_{m(1, n) + 1} = e_n 2T^2. \]

Now we come back to (26). By (29), as \( \eta_n = e_n \) and \( \varepsilon_n = e_n \), (26) implies
\[ X_{2, n} = e_n \frac{P_n^2 Q_n^2 - Q_n^2 P_n}{Q_n P_n}. \]
So we can compute $x_{2,n}$. By (27) and (6),

$$
P_n^2 Q' - Q_n^2 P' = P_n^2 (Q_n^4 + (-1)^n) - Q_n^2 P_n^2 Q_n^2 = (-1)^n P_n^2 = (-1)^n Q_n^{2n-1}
$$

$$
P_n^2 Q' - Q_n^2 P' = (-1)^n P_n^2 = (-1)^n Q_n^{2n-1}.
$$

By (27), (28), and (6),

$$
Q'P - PQ = P_n^4 Q_n^2 (2T^2 Q_n^2 + P_n^2) - (Q_n^4 + (-1)^n (2T^2 P_n^2 + P_{n-1} Q_n))
$$

$$
Q'P - PQ = P_n^4 Q_n^2 - Q_n^2 P_n^4 - (-1)^n (2T^2 P_n^2 + P_{n-1} Q_n)
$$

$$
Q'P - PQ = (Q_n^2 P_n Q_n - Q_n P_n)^3 + (-1)^n (T^2 P_n^2 - Q_n^2 + T Q_n P_n)
$$

$$
Q'P - PQ = (1)^n (T^2 P_n^2 + Q_n^2 + T Q_n P_n)
$$

$$
Q' - PQ = (1)^n (Q_n - T P_n)^2 = (-1)^n P_{n-1}^n.
$$

Hence, by (31), we obtain

$$
x_{2,n} = e_n (Q_{n-1}/P_{n-1})^6.
$$

Now we observe that

$$\begin{align*}
[a_1, \ldots, a_{m(1,n-1)}] &= 1/([0, a_1, \ldots, a_{m(1,n-1)}] \\
&= 1/(P_{n-1}/Q_{n-1})^2 = (Q_{n-1}/P_{n-1})^2
\end{align*}
$$

and, since $K = \mathbb{F}_3$, we have

$$
e_n (Q_{n-1}/P_{n-1})^6 = [e_n a_1^3, \ldots, e_n a_{m(1,n-1)}^3].
$$

So, by (32) and $x_{2,n} = [\Omega_{2,n}]$, we obtain

$$\Omega_{2,n} = e_n a_1^3, \ldots, e_n a_{m(1,n-1)}^3.
$$

According to (30) and (33), we can write (24) as

$$\Omega_{1,n+1} = \Omega_{1,n}, e_n 2T^2; e_n a_1^3, \ldots, e_n a_{m(1,n-1)}^3; e_{n+1} 2T^2, e_{n+1} \tilde{Q}_{1,n}.
$$

So by Lemma 5 and (20), we have simultaneously $e_n a_{m(1,n-1)}^3 = e_{n+1} e_n a_1^3$, which implies $a_{m(1,n-1)} = e_{n+1} a_1$ and $a_{m(1,n-1)} = e_{n-1} a_1$. Therefore $e_{n+1} = e_{n-1}$ for all $n \geq 2$. Since $e_2 = e_1 = 1$, it follows that $e_n = 1$ for all $n \geq 1$.

Finally, by (20), the sequence $\Omega_{1,n}$ is reversible for all $n \geq 1$, and so $\Omega_{1,n} = \Omega_{1,n}$. So (34) becomes (25) for $n \geq 2$, and the theorem is proved.
Remark. We have observed the beginning of the continued fraction expansion of $\theta_q$ by computer, for $q \leq 27$. In all cases and for the values of $n$ that we could reach, we had

$$e_n = 1, \quad \lambda_{q,n} = 1, \quad \lambda_{q,n} = 2 \quad \text{for} \quad 1 \leq i < q, \quad \Omega_{r,n} = \Omega_{1,n}^{(q)},$$

as it is for $q = 3$. So, for $q > 3$, we can conjecture that (24) becomes

$$\Omega_{1,n+1} = \Omega_{1,n}^{(q)} 2T', \Omega_{2,n}^{(q)}, \ldots, \Omega_{r-1,n}^{(q)} 2T', \Omega_{1,n-1}^{(q)} 2T', \Omega_{r-1,n}^{(q)} = \Omega_{1,n}^{(q)}.$$

For $n \geq 2$, we denote $J_n(q) = 2T', \Omega_{2,n}^{(q)}, \ldots, \Omega_{r-1,n}^{(q)} 2T'$, and $j_n(q)$ the degree of $J_n(q)$, we have $j_n(q) = (r - 2) \omega_n + (r - 1) n = (r - 2) q + (r - 1) r$. We denote $j_n(q)$ the highest degree in $T'$ of the terms in $J_n(q)$; then we have $j_n(q) \leq \omega_n \leq q \deg Q_{n-1}$. Now we observe that if $j_n(q)$ were not too large, then the number of terms in $J_n(q)$ would increase with $n$, because $j_n(q)$ does so. In that direction, we have observed the data about $J_n(q)$ found in Table I.

Of course we expect the element $\theta_q$ to satisfy Roth's theorem for all power $q$ of an odd prime number $p$, as it does for $q = 3$. Using the same arguments as the one developed in Section 2, this would result from the conjecture $\Omega_{r,n} = \Omega_{1,n}^{(q)}$ and $j_{2n+1}(q) < q^r, j_{2n+2}(q) < q^r$ (cf. Table I).

If we replace the element $\theta_q$ by the element $\theta_k$ for $1 \leq k \leq r$, we can see, as we did for $\theta_q$, that $(P_n/Q_n)^k$ is a convergent to $\theta_k$, as soon as $k$ and $p$ are coprime. Therefore, in that situation, the approximation exponent of $\theta_k$ is at least $(q + 1)/k - 1$. We may suppose that this approximation exponent is indeed equal to $(q + 1)/k - 1$ (i.e., there are no essentially better approximations to $\theta_k$ than $(P_n/Q_n)^k$; consequently $\theta_q$ satisfies Roth's theorem). This is proved in [8] for $(q + 1)/k$ sufficiently large. If it were true for all $k$, with $(k, p) = 1$, we wonder whether it could be established without the help of the continued fraction expansion of $\theta_q$.

### Table I

The Number of Terms of $J_n(q)$ and (between Brackets) the Higher Degree (in $T'$) of Those Terms

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4. THE CONTINUED FRACTION EXPANSION OF A CLASSICAL EXAMPLE

In this last section we would like to give a result which is indirectly connected with the subject presented above. When we started our investigation from Buck and Robbins paper \([2]\), we studied the method they have used to be able to describe the continued fraction expansion of \(\theta_s^p\). Their idea is to start from an algebraic element, to observe the beginning of its continued fraction expansion by computer, to guess its pattern, and then to show that the element defined by this expansion satisfies the desired equation. We have tried to apply this approach to the celebrated example given by Mahler in \([4]\), so we have succeeded in describing entirely the continued fraction expansion of this element. Curiously this result does not seem to be known, so we give it here. We will only give a brief survey of the proof.

We have the following result:

**Theorem C.** Let \(p\) be a prime number, \(q = p^s\) for \(s \in \mathbb{N} - \{0\}\), \(q > 2\), and \(K = \mathbb{F}_p\). Let \(\alpha\) be the element of \(K((T^{-1}))\), defined by

\[
\alpha = 1/T + \alpha^q \quad \text{and} \quad |\alpha| = |T|^{-1}.
\]

Let us define the sequence \((\Omega_n)_{n > 0}\) of finite sequences of elements of \(K[T]\) recursively by

\[
\Omega_1 = T, \quad \Omega_n = \Omega_{n-1} - T^{(q-2)q^{n-2}} - \bar{\Omega}_{n-1} \quad \text{for} \quad n \geq 2,
\]

where \(\bar{\Omega} = a_m, a_{m-1}, ..., a_1\) and \(-\bar{\Omega} = -a_1, -a_2, ..., -a_m\), if \(\bar{\Omega} = a_1, a_2, ..., a_m\).

Let \(\Omega_n\) be the infinite sequence beginning by \(\Omega_n\) for all \(n \geq 1\). Then the continued fraction expansion of \(\alpha\) is \([0, \Omega_\infty]\).

To prove this, we start from the element \(\alpha = [0, \Omega_\infty]\). For \(n \geq 1\), we put

\[
\Omega_n = a_1, a_2, ..., a_{m(n)}, \quad r_n/s_n = [0, a_1, a_2, ..., a_{m(n)-1}],
\]

\[
t_n/u_n = [0, a_1, a_2, ..., a_{m(n)}].
\]

Then we show, from the relation (R), that, for \(n \geq 1\), we have

\[
r_n = -u_n z_n^2, \quad s_n = 1 - u_n z_n, \quad t_n = 1 + u_n z_n, \quad u_n = T^{q^{n-1}},
\]

where \(z_n = \sum_{a \leq k \leq n-2} T^{-q^k}\) for \(n \geq 2\) and \(z_1 = 0\). Now we define \(\delta_n = r_n/s_n - (t_n/u_n)^{q^{n-1}} (t_n/u_n) - T^{-1}\). It is clear that \(\delta_n\) tends to \(\alpha - \alpha^q - T^{-1}\). At last we show that \(\lim_n \delta_n = 0\), and so the proof is complete.
ACKNOWLEDGMENTS

In conclusion, the author expresses his indebtedness to Professor B. de Mathan for his valuable suggestions during the preparation of this paper.

REFERENCES