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The maximum dimension of a subspace of nilpotent matrices of index 2

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A matrix M is nilpotent of index 2 if $M^2 = 0$. Let V be a space of nilpotent $n \times n$ matrices of index 2 over a field $\boldsymbol{k}$ where $card\,\boldsymbol{k} > n$ and suppose that r is the maximum rank of any matrix in V. The object of this paper is to give an elementary proof of the fact that $dim\,V \leqslant r(n-r)$. We show that the inequality is sharp and construct all such subspaces of maximum dimension. We use the result to find the maximum dimension of spaces of anti-commuting matrices and zero subalgebras of special Jordan Algebras.

1. Introduction

An $n \times n$ matrix M is nilpotent if $M^t = 0$ for some $t > 0$. We are concerned with linear spaces of nilpotent matrices over a field $\boldsymbol{k}$. As far back as 1959, Gerstenhaber [4] showed that the maximum dimension of a space of nilpotent matrices was $\frac{n(n-1)}{2}$. In this paper we are interested in matrices nilpotent of index 2. Naturally such a space will have smaller dimension. We are able to show that the maximum dimension of such a space depends on the maximum r of the ranks of matrices in the space: $r(n-r)$. This bound is sharp and we characterize those spaces attaining this maximum dimension. While this might seem to be a very specialized result, it has some important consequences. It gives an immediate proof that $r(n-r)$ is the maximum possible dimension of a space of anti-commuting matrices over any field of $card\,\boldsymbol{k} > n/2$ (and $char\,\boldsymbol{k} \neq 2$). It also shows that $r(n-r)$ is the maximum

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dimension of a zero subalgebra of a special Jordan Algebra. All of the proofs involve only elementary linear algebra.

Related work has been done by Brualdi and Chavey [2]. They have investigated the more general problem of finding the maximal dimension of a space of nilpotent matrices of *bounded* index k. Their arguments are combinatorial in nature and do not imply our result. Atkinson and Lloyd [1] and others have also studied spaces of matrices of bounded rank, but their results do not overlap ours.

2. Preliminary theorems

Theorem 1. *Let V be a space of $n \times n$ matrices over a field $\boldsymbol{k}$ where card $\boldsymbol{k} \geqslant n$. Let $A \in V$ have the property that $r =$ rank $A \geqslant$ rank X for every $X \in V$. If $a \in$ ker A then $Ba \in$ Im A for all $B \in V$.*

Proof. The result is obvious when $r = n$ so assume $r < n$.

Let $S = \{a_1, a_2, \ldots, a_k\}$ be a basis of *ker A* and extend S to a basis $B_1 = \{a_1, a_2, \ldots, a_k, a_{k+1}, \ldots, a_n\}$ of $\boldsymbol{k}^n$. Then $T = \{Aa_{k+1}, Aa_{k+2}, \ldots, Aa_n\}$ is a basis of *Im A* and we extend T to a basis $B_2 = \{c_1, c_2, \ldots, c_k, Aa_{k+1}, \ldots, Aa_n\}$ of $\boldsymbol{k}^n$.

Now let the vectors in B_1 form the columns of a matrix Q and the vectors in B_2 form the columns of a matrix P. Then

$$P^{-1}AQ = \begin{pmatrix} 0 & 0 \\ 0 & I_r \end{pmatrix},$$

where I_r is an $r \times r$ identity matrix. Let B be any matrix in V and assume

$$P^{-1}BQ = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix},$$

where B_4 is an $r \times r$ matrix. Then for any $x \in \boldsymbol{k}$ we have

$$P^{-1}(B + xA)Q = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 + xI_r \end{pmatrix}.$$

Let S be any $(r+1) \times (r+1)$ submatrix of $P^{-1}(B + xA)Q$ containing $B_4 + xI_r$. Then *det* $S = 0$. Since *card* $\boldsymbol{k} \geqslant n > r$, each term of this polynomial must be identically 0. The fact that the coefficient of x^n is 0 implies that each element of B_1 must be 0. So

$$P^{-1}BQ = \begin{pmatrix} 0 & B_2 \\ B_3 & B_4 \end{pmatrix}.$$

Now suppose $a_0 \in$ *ker A*. Then

$$a_0 = \sum_{i=1}^{k} x_i\, a_i = Q \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So we have

$$Ba_0 = BQ\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} = P\begin{pmatrix} 0 & B_2 \\ B_3 & B_4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} = P\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ y_{k+1} \\ \vdots \\ y_n \end{pmatrix}$$

$$= \sum_{j=1}^{n-k} y_{k+j}\, A\, a_{k+j} \in Im\, A. \quad \square$$

We note here that the proof actually only required the cardinality of the field $\boldsymbol{k}$ to be more than r, the maximum rank of a matrix in V.

Lemma 1. *Let V be a space of $m \times n$ matrices over any field $\boldsymbol{k}$ and partition the elements of V as $V = \left\{\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}\right\}$. Let $W = \{A_1\}$ and $U = \left\{\begin{pmatrix} 0 & A_2 \\ A_3 & A_4 \end{pmatrix}\right\} \cap V$. Then $dim\, W + dim\, U = dim\, V$.*

Proof. Let $dim\, W = s$, $dim\, U = t$, and $dim\, V = k$. There must exist independent matrices $B_1, B_2, \ldots, B_s \in V$ so that if $B_i = \left\{\begin{pmatrix} B_{i,1} & B_{i,2} \\ B_{i,3} & B_{i,4} \end{pmatrix}\right\}$, then $\{B_{1,1}, B_{2,1}, \ldots, B_{s,1}\}$ is a basis of W. Extend $B_1, B_2, \ldots, B_s$ to a basis

$$B = \{B_1, B_2, \ldots, B_s, \ldots, B_k\}$$

of V.

For $1 \leqslant j \leqslant k - s$ let

$$B_{s+j} = \begin{pmatrix} B_{s+j,1} & B_{s+j,2} \\ B_{s+j,3} & B_{s+j,4} \end{pmatrix}.$$

Then

$$B_{s+j,1} = c_{s+j,1}\, B_{1,1} + c_{s+j,2}\, B_{2,1} + \cdots + c_{s+j,s}\, B_{s,1}$$

for suitable scalars $c_{s+j,1}, c_{s+j,2}, \ldots, c_{s+j,s}$. Replace B_{s+j} by

$$B'_{s+j} = B_{s+j} - (c_{s+j,1}\, B_1 + c_{s+j,2}\, B_2 + \cdots + c_{s+j,s}\, B_s)$$

and let $B' = \left\{B_1, B_2, \ldots, B_s, B'_{s+1}, B'_{s+2}, \ldots, B'_k\right\}$. It is easy to show that B' is a basis of V and

$$U = span\,\left\{B'_{s+1}, B'_{s+2}, \ldots, B'_k\right\}. \quad \square$$

Note: Lemma 1 is more significant than it first appears. For our proof we chose A_1 as the space W, but in principle there is nothing special about that choice. In fact a result similar to the statement of the Lemma holds if A_1 is replaced by any set of r fixed positions in the matrices found in V, so long as U is chosen as the set of complementary positions. We will use this principle repeatedly in Section 3.

We need the following lemma in Section 3. It is equivalent to a known result, but we include a short and simple proof.

Lemma 2. *Let V be a subspace of $M_{m,n}(\boldsymbol{k})$ where $\boldsymbol{k}$ is any field and let $V^R = \left\{A \in M_{n,m}(\boldsymbol{k}) \mid XA = 0 \text{ for all } X \in V\right\}$. Then $dim\, V + dim\, V^R \leqslant mn$.*

Proof. Let V_i be the space spanned by the ith rows of the elements of V, and let V_i^R be the space spanned by the ith columns of the elements of V^R. Then $V_i^R \subseteq$ nullspace of A_i where A_i is a matrix whose rows form a basis of V_i. This implies that

$$dim\, V_i + dim\, V_i^R \leqslant n.$$

But now

$$\begin{aligned} dim\, V + dim\, V^R &\leqslant \sum_{i=1}^{m} dim\, V_i + \sum_{i=1}^{m} dim\, V_i^R \\ &= \sum_{i=1}^{m} (dim\, V_i + dim\, V_i^R) \\ &\leqslant mn. \qquad \square \end{aligned}$$

We note that Lemmas 1 and 2 hold for any scalar field $\boldsymbol{k}$.

The following known result is needed in a later section. It was first proved by Flanders [3]. The first step of our proof is similar to that of Flanders but then our argument is considerably shorter because of Theorem 1 and Lemmas 1 and 2. We also note that the restriction on the size of the scalar field $\boldsymbol{k}$ has been removed by Meshulam [5].

Theorem 2. *Let V be a space of $n \times n$ matrices over a field $\boldsymbol{k}$ of card $\boldsymbol{k} \geqslant n$. If rank $A \leqslant r$ for all $A \in V$ then $dim\, V \leqslant nr$.*

Proof. As in the proof of Theorem 1 we can assume that each $B \in V$ is of the form

$$\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}, \tag{1}$$

where B_4 is $r \times r$, and we showed there that $B_1 = 0$. There we considered the determinant of any $(r+1) \times (r+1)$ submatrix of $P^{-1}(B + xA)Q$ containing $B_4 + xI_r$. The fact that the coefficient of x^{n-1} in this polynomial must be 0 implies that each row of B_2 is orthogonal (in the usual sense) to each column of B_3. Hence $B_2B_3 = 0$.

Let $W = \{(B_4)\}$ and $U = \left\{\begin{pmatrix} 0 & B_2 \\ B_3 & 0 \end{pmatrix}\right\} \cap V$.

In U, let $W_1 = \{(B_3)\}$ and $U_1 = \left\{\begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix}\right\} \cap U$.

If $B = \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \in U_1$ and $C = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \in U$, then $B + C \in U$, which implies that $B_2Y = 0$, and so $W_1 \subseteq \{(B_2)\}^R$. Now by Lemma 2

$$dim\, W_1 + dim\, \{(B_2)\} = dim\, W_1 + dim\, U_1 \leqslant (n-r)r.$$

Then by Lemma 1

$$dim\, U = dim\, W_1 + dim\, U_1 \leqslant (n-r)r.$$

and

$$dim\, W + dim\, U = dim\, V \leqslant r^2 + (n-r)r = nr. \qquad \square$$

Using the techniques of the previous lemmas and theorems, we show how to construct (up to equivalence) all spaces of $n \times n$ matrices of bounded rank r and dimension nr. The derivation is somewhat tedious, but we include it here because we will use the same method in Section 4 to characterize the spaces of matrices of nilindex 2 having maximum dimension.

Let V be such a space. Then if $A \in V$ we may assume that

$$A = \begin{pmatrix} 0 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

where $A_2A_3 = 0$, $dim\ W = dim\ \{(A_4)\} = r^2$ and

$$dim\ U = dim\ \left\{\begin{pmatrix} 0 & A_2 \\ A_3 & 0 \end{pmatrix}\right\} \cap V = r(n-r).$$

Suppose there exists a $B \in U$ of the form

$$B = \begin{pmatrix} 0 & B_2 \\ B_3 & 0 \end{pmatrix}$$

such that there is a non-zero entry $b = b_{i,j}$ in the B_2 corner of B. Let $c = b_{k,l}$ be any entry in the B_3 corner of B.

Since $dim\ W = r^2$ there must exist a matrix D in V such that

$$D = \begin{pmatrix} 0 & D_2 \\ D_3 & D_4 \end{pmatrix}$$

in which the jth column of D_4 and the kth row of D_4 are filled with 0s and the remaining submatrix of D_4 consists of the identity I_{r-1}. Now let S be the $(r+1) \times (r+1)$ submatrix of $B + xD$ containing b, c and D_4. Then $det\ S = 0$ and the coefficient of x^{r-1} of the polynomial is $\pm bc$ and so $c = 0$. Hence

$$B = \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix}.$$

But if $X \in U$ then $X + xB \in U$ and it follows that in fact

$$U = \left\{\begin{pmatrix} 0 & A_2 \\ 0 & 0 \end{pmatrix}\right\}.$$

Finally let E be any matrix in V where

$$E = \left\{\begin{pmatrix} 0 & E_2 \\ E_3 & E_4 \end{pmatrix}\right\}$$

and let X be any matrix in U. Then $E + xX \in V$ and hence $E_3 \subseteq \{(A_2)\}^R$. But

$$dim\ \{(A_2)\} = dim\ U = r(n-r)$$

and so Lemma 2 implies that $E_3 = 0$; therefore

$$V = \left\{\begin{pmatrix} 0 & A_2 \\ 0 & A_4 \end{pmatrix}\right\},$$

where A_2 and A_3 are arbitrary. A similar argument shows that if no $B \in U$ has a suitable $b_{i,j} \neq 0$ in the B_2 corner, then

$$V = \left\{\begin{pmatrix} 0 & 0 \\ A_3 & A_4 \end{pmatrix}\right\}.$$

We summarize this discussion in the following

Theorem 3. *Let V be a space of $n \times n$ matrices of rank at most r over a field $\boldsymbol{k}$ with card $\boldsymbol{k} \geqslant n$. If $dim\ V = nr$ then up to equivalence V is of the form*

$$\left\{\begin{pmatrix} 0 & A_2 \\ 0 & A_4 \end{pmatrix}\right\} \quad \text{or} \quad \left\{\begin{pmatrix} 0 & 0 \\ A_3 & A_4 \end{pmatrix}\right\}, \tag{2}$$

where A_4 is $r \times r$.

3. The main result

Theorem 4. *Let V be a space of $n \times n$ nilpotent matrices of index 2 over a field $\boldsymbol{k}$ where card $\boldsymbol{k} > n/2$. Suppose rank $X \leqslant r$ for all $X \in V$. Then dim $V \leqslant r(n-r)$.*

Proof. Let $Q \in V$ such that *rank* $Q = r$. By rearranging a Jordan basis it is easy to show that Q is similar to

$$\begin{pmatrix} 0 & I_r \\ 0 & 0 \end{pmatrix},$$

where I_r is an $r \times r$ identity matrix. Without loss of generality we replace Q by the above matrix. Let $A \in V$. In the proof of Theorem 1 we actually require that *card* $\boldsymbol{k} > r$ but we know that $r \leqslant n/2$ so we can apply Theorem 1 to assume that

$$A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 0 & 0 & A_4 \\ 0 & 0 & A_5 \end{pmatrix},$$

where A_1, A_3, and A_5 are $r \times r$ matrices. But $(Q + A)^2 = QA + AQ = 0$ and this shows that $A_5 = -A_1$. We proceed in 3 cases depending on whether any of A_1 or A_2 and A_4 are 0.

Case 1. Suppose that $A_1 = 0$ for every $A \in V$ so that each $A \in V$ is of the form

$$\begin{pmatrix} 0 & A_2 & A_3 \\ 0 & 0 & A_4 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now let $W = \{ (A_3) \}$ and $U = \left\{ \begin{pmatrix} 0 & A_2 & 0 \\ 0 & 0 & A_4 \\ 0 & 0 & 0 \end{pmatrix} \right\} \cap V$. Clearly *dim* $W \leqslant r^2$.

In U, let $W_1 = \{ (A_2) \}$ and $U_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A_4 \\ 0 & 0 & 0 \end{pmatrix} \right\} \cap U$.

Suppose $B = \begin{pmatrix} 0 & B_2 & 0 \\ 0 & 0 & B_4 \\ 0 & 0 & 0 \end{pmatrix} \in U$ and $C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & C_4 \\ 0 & 0 & 0 \end{pmatrix} \in U_1$.

Then $(B + C)^2 = 0$ implies that $B_2C_4 = 0$. So in U_1, if $T = \{ (A_4) \}$ then $T \subseteq W_1^R$ and by Lemma 2,

$$dim\, W_1 + dim\, T = dim\, W_1 + dim\, U_1 \leqslant (n - 2r)r.$$

So by Lemma 1

$$dim\, U = dim\, W_1 + dim\, U_1 \leqslant (n - 2r)r \tag{3}$$

and

$$dim\, V = dim\, W + dim\, U \leqslant r^2 + (n - 2r)r = nr - r^2.$$

Case 2. Suppose A_2 and A_4 do not exist. In this case $r = n/2$ and each $A \in V$ is of the form $A = \begin{pmatrix} A_1 & A_2 \\ 0 & -A_1 \end{pmatrix}$.

If each $A_1 = 0$ then *dim* $V \leqslant r^2 = nr - r^2$ and so we assume there exists an $A_1 \neq 0$. Let $W = \{ (A_1) \}$. Let r_1 be the largest rank of any matrix in W. Then W is a space of nilpotent matrices of index 2 and bounded rank r_1 so by induction we may assume *dim* $W \leqslant r_1(r - r_1)$. As above, there exists a matrix in V which is similar to

$$Q_1 = \begin{pmatrix} 0 & I_{r_1} & A_2' & A_2'' \\ 0 & 0 & A_2''' & A_2'''' \\ 0 & 0 & 0 & -I_{r_1} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where A_2''' is an $r_1 \times (r - r_1)$ matrix.

Also $Q_1^2 = 0$ implies that $A_2^{'''} = 0$. Now let

$$U = \left\{ \begin{pmatrix} 0 & 0 & B_1 & B_2 \\ 0 & 0 & B_3 & B_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \cap V.$$

If $B \in U$ then $(Q_1 + B)^2 = 0$ implies that $B_3 = 0$ and so $dim\, U \leqslant r^2 - r_1(r - r_1)$. Hence by Lemma 1

$$dim\, V = dim\, W + dim\, U \leqslant r\, r_1 - r_1^2 + r^2 - r_1(r - r_1) = r^2 = nr - r^2.$$

Case 3. Suppose there exists an $A_1 \neq 0$ and A_2 and A_4 do exist. Note that $r < n/2$. Each $A \in V$ is of the form

$$A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 0 & 0 & A_4 \\ 0 & 0 & -A_1 \end{pmatrix}.$$

Let $W = \{ (A_1) \}$. Then as above, by induction we may assume $dim\, W \leqslant r\, r_1 - r_1^2$ where r_1 is the largest rank of any matrix in W. Also we may assume there is a matrix in V which is similar to

$$Q_1 = \begin{pmatrix} 0 & I_{r_1} & A_2' & A_3' & A_3^{''} \\ 0 & 0 & A_2^{''} & A_3^{'''} & A_3^{''''} \\ 0 & 0 & 0 & A_4^{''} & A_4' \\ 0 & 0 & 0 & 0 & -I_{r_1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where A_2' is $(r - r_1) \times (n - 2r)$, $A_2^{''}$ is $r_1 \times (n - 2r)$, $A_4^{''}$ is $(n - 2r) \times r_1$ and A_4' is $(n - 2r) \times (r - r_1)$. Then $Q_1^2 = 0$ implies that $A_2^{''} = A_4^{''} = 0$. Let

$$U = \left\{ \begin{pmatrix} 0 & 0 & A_2' & A_3' & A_3^{''} \\ 0 & 0 & A_2^{''} & A_3^{'''} & A_3^{''''} \\ 0 & 0 & 0 & A_4^{''} & A_4' \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\} \cap V.$$

As above, if $B \in U$ then $(Q_1 + B)^2 = 0$ implies that

$$B = \begin{pmatrix} 0 & 0 & B_2' & B_3' & B_3^{''} \\ 0 & 0 & 0 & B_3^{'''} & B_3^{''''} \\ 0 & 0 & 0 & 0 & B_4' \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $B_3^{'''}$ is $(r - r_1) \times (r - r_1)$. Also if $x \in \boldsymbol{k}$ then $rank\,(Q_1 + xB) \leqslant r$ and this implies that the $rank\, B_3^{'''} \leqslant (r - 2r_1)$. Hence if in U, $S = \left\{ (B_3^{'''}) \right\}$ then $dim\, S \leqslant (r - r_1)(r - 2r_1)$ by Theorem 2. Now in U let

$$W_1 = \left\{ \begin{pmatrix} B_3 & B_3^{''} \\ B_3^{'''} & B_3^{''''} \end{pmatrix} \right\} \quad \text{and} \quad T = \left\{ \begin{pmatrix} B_3 & B_3^{''} \\ 0 & B_3^{''''} \end{pmatrix} \right\}. \tag{4}$$

Then using Lemma 1 again:

$$\begin{aligned} dim\, W_1 &= dim\, S + dim\, T \\ &\leqslant (r - r_1)(r - 2r_1) + 2rr_1 - r_1^2 \\ &= r^2 - rr_1 + r_1^2. \end{aligned}$$

Finally let

$$U_1 = \left\{ \begin{pmatrix} 0 & 0 & B_2' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_4' \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\} \cap U.$$

Using Lemma 2 and the argument as found in Case 1 it can be shown that $dim\, U_1 \leqslant (n-2r)(r-r_1)$. Now using Lemma 1

$$\begin{aligned} dim\, V &= dim\, W + dim\, U \\ &= dim\, W + dim\, W_1 + dim\, U_1 \\ &\leqslant rr_1 - r_1^2 + r^2 - rr_1 + r_1^2 + (n-2r)(r-r_1). \end{aligned}$$

Simplifying:

$$dim\, V \leqslant nr - r^2 + 2rr_1 - nr_1.$$

But $r < n/2$ so $2rr_1 < nr_1$ and hence

$$dim\, V < r(n-r). \quad \square$$

There are some important consequences that follow immediately from this theorem. Indeed, it was questions like these that originally interested us in spaces of nilpotent matrices.

Corollary 1. *Let V be a space of anticommuting $n \times n$ matrices over a field $\boldsymbol{k}$ where card $\boldsymbol{k} > n/2$ and char $\boldsymbol{k} \neq 2$ If rank $A \leqslant r$ for all $A \in \boldsymbol{k}$ then dim $V \leqslant r(n-r)$.*

Proof. Since *char* $\boldsymbol{k} \neq 2$ the matrices in V must be nilpotent of index 2.

Let A be the algebra of all $n \times n$ matrices over a field $\boldsymbol{k}$ where *char* $\boldsymbol{k} \neq 2$. Define a new multiplication $\circ$ as

$$X \circ Y = \frac{1}{2}(XY + YX).$$

Then A with its new operation $\circ$ is a Jordan Algebra. It is often called a special Jordan Algebra. $\square$

Corollary 2. *Let A be a special Jordan Algebra constructed from $n \times n$ matrices over a field $\boldsymbol{k}$ where card $\boldsymbol{k} > n/2$ and char $\boldsymbol{k} \neq 2$. If A_1 is a zero subalgebra of A then dim $A_1 \leqslant r(n-r)$ where r is the maximum rank of any matrix in A_1.*

Proof. In a zero subalgebra $X \circ Y = 0$ and the result follows directly from Corollary 1. $\square$

4. The spaces of maximum dimension

We now show that the inequality in our main result is sharp by constructing spaces which have the maximum dimension $r(n-r)$. In addition, the spaces constructed below are, up to similarity, the only ones reaching the maximum dimension. Again we consider the three cases.

Case 1. Let $V_1 = \left\{ \begin{pmatrix} 0 & A_2 & A_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$ where A_2 is any $r \times (n-2r)$ matrix and A_3 is any $r \times r$ matrix. Clearly V_1 is a space of nilpotent matrices of index 2 and bounded rank r and $dim\, V_1 = nr - r^2$. Similarly

$V_2 = \left\{ \begin{pmatrix} 0 & 0 & A_3 \\ 0 & 0 & A_4 \\ 0 & 0 & 0 \end{pmatrix} \right\}$ where A_4 is any $(n-2r) \times r$ matrix and A_3 is any $r \times r$ matrix is also a space of nilpotent matrices of index 2 and bounded rank r and $dim\, V_2 = nr - r^2$.

If a space is of this type has maximum dimension $r(n-r)$, we can show that these are the only such subspaces. The argument is very similar to that in the derivation of Theorem 3, so we omit the details.

Case 2. Let $V_3 = \left\{ \begin{pmatrix} 0 & A_2 \\ 0 & 0 \end{pmatrix} \right\}$ where A_2 is any $n/2 \times n/2$ matrix (n is even). Then V_3 is a space of nilpotent matrices of index 2 and bounded rank $r = n/2$ and $dim\, V_3 = r(n-r) = n^2/4$.

Again we can show that these are the only subspaces of this type that achieve maximum dimension. The argument is similar to that of Theorem 3 and we omit it.

Case 3. Note that in this case we showed that $dim\, V < nr - r^2$ and so no subspaces of maximum dimension of this type exist.

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