On the varieties of special divisors

by Marc Coppens and Gerriet Martens

Katholieke Industriële Hogeschool der Kempen, Campus H.I. Kempen, Kleinhoefstraat 4,
B-2440 Geel, Belgium
marc.coppens@khk.be
Mathematisches Institut, Universität Erlangen-Nürnberg, Bismarckstrasse 1 1/2
D-91054 Erlangen, Germany
e-mail: martens@mi.uni-erlangen.de

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ABSTRACT

Based on a relation between the varieties $W_d^r(C)$ of special divisors on a curve $C$ and subloci of effective divisors on $C$ imposing a suitable number of conditions on a certain linear series we develop a tool for the construction of irreducible components of $W_d^r(C)$. Using this we discover new irreducible components of $W_d^r(C)$, for a general $k$-gonal curve $C$ of genus $g$, and in some cases we can identify the duals of these components in $K_C - W_d^r(C) = W_d^{r'}(C)(d' = 2g - 2 - d, r' = g - 1 - d + r)$.

1. INTRODUCTION

$C$ always denotes a smooth irreducible projective curve of genus $g > 0$ over the complex numbers, and $W_d^r$ is its variety of complete linear series of degree $d$ and dimension at least $r$ on $C$. For non-negative integers $e$ and $f$ we let

$$V_{e}^{e-f}(g_d^r) := \{ E \in C^{(e)} | \dim(g_d^r(-E)) \geq r - e + f \},$$

the sublocus of effective divisors of degree $e$ on $C$ imposing at most $e - f$ linear conditions on a fixed linear series $g_d^r$ on $C$; it has a natural scheme structure.

In particular, fixing a non-negative integer $n$ and a pencil $g_k^r$ on $C$ we will consider $V_{e}^{e-f}(|K_C - ng_k^r|)$. Then $\dim |ng_k^1 + E| \geq \dim |ng_k^1| + f \geq n + f$ for $E \in V_{e}^{e-f}(|K_C - ng_k^r|)$, and so we have a natural map

$$\sigma: V_{e}^{e-f}(|K_C - ng_k^r|) \to W_{nk+e}^{n+f}, \ E \mapsto |ng_k^1 + E|.$$
For \( n = 0 \) it displays \( V^e_{ \text{e} - f}(\{K_C\}) \) as a \( \mathbb{P}^f \)-bundle over \( W^f_{ \text{e} - f} \). For \( n > 0 \), however, it usually does not map irreducible components of \( V^e_{ \text{e} - f}(\{K_C - ng_k\}) \) onto irreducible components of \( W^{n + f}_{nk+e} \) (even in the case \( \dim |ng_k| = n \); e.g., cf. [CM2], 4.2.2).

It is the purpose of this paper to show how to construct this way (i.e. via \( \nu \)) irreducible components of \( W^{n + f}_{nk+e} \) and to show that the method can be applied in some interesting situations. Our tool for the construction is Theorem 1 in Section 1. In Section 2 we use it to exhibit irreducible components of \( W''_d \), on a general \( k \)-gonal curve \( C \). Specifically, we prove

\[
\text{Let } C \text{ be a general } k \text{-gonal curve of genus } g \text{ and } d, r \text{ and } \alpha \text{ be positive integers. Assume that } \alpha < k \text{ and that } \alpha \text{ divides } r \text{ or } r + 1. \text{ Then } W'_d \text{ has an irreducible component } V_{\alpha} \text{ of dimension } \dim V_{\alpha} = \rho_g(d, r) + (r + 1 - \alpha)(g - d + r - k + \alpha) \text{ provided that this number is non-negative and not less than } \rho_g(d, r), \text{ the Brill-Noether number} (\text{note these are natural obstructions}).
\]

Contrary to the components of \( W'_d \) constructed in [CM2], Section 2 these components \( V_{\alpha} \), for \( \alpha \neq r + 1 \), are contained in \( W'^1_1 + W_{d-k} \). The \( V_{\alpha} \) have a certain description, and it would be interesting to have a description of their duals \( K_C - V_{\alpha} \), too. For example, it is easy to see that the dual of \( V_{k-1} \subseteq W'_d \) is \( V_1 \subseteq W''_d \) \((d' = 2g - 2 - d, r' = g - 1 - d + r)\), and in Section 3 we prove that the dual of \( V_{k-2} \subseteq W'_d \) is \( V_2 \subseteq W''_d \). Using this description we show that an ‘unexpected’ linear series \( g^{k-3}_d \) on a general \( k \)-gonal curve \( C \) (i.e. a \( g^{k-3}_d \) such that \( \rho_g(d, k - 3) < 0 \)) always contains the unique pencil \( g^1_k \) of \( C \); this answers, for series of dimension \( k - 3 \), a question asked in [CM2].

**Notations.** We use the same notations and conventions as in [CM2] (which mostly agree with those in [ACGHI]). In particular, \( C^{(d)} \) is the set of effective divisors of degree \( d \) on \( C \). Dealing with \( W'_d \) we make tacitly use of the Abel-Jacobi map \( \iota : C^{(d)} \to \text{Jac}(C) \) into the Jacobian variety \( \text{Jac}(C) \) of \( C \). However, in order to make the notation not too cumbersome we do not denote a complete linear series \( g^{(d)}_d \) on \( C \) and its image \( \iota(g^{(d)}_d) \) in \( W'_d \) by different symbols, and a canonical divisor \( K_C \) on \( C \) sometimes is identified with the point \( \iota(K_C) \) on \( \text{Jac}(C) \).

\[
\rho_g(d, r) := g - (r + 1)(g - d + r) \text{ is the Brill-Noether number. For fixed } d \text{ and } r \text{ we put } r' := g - 1 - d + r \text{ (i.e. } r' + 1 \text{ is the index of speciality } h^1(g^{(d)}_d) \text{ of a complete linear series } g^{(d)}_d \text{ on } C); \text{ so } \rho_g(d, r) = g - (r + 1)(r' + 1). \text{ Fixing a complete and base point free pencil } g^1_k \text{ on } C \text{ we call a complete } g^1_k \text{ on } C \text{ a series of type 2 if its dual is compounded of the } g^1_k, \text{ i.e. if } |K_C - g^1_k| \text{ away from the base locus is the multiple } r'g^1_k \text{ of the } g^1_k, \text{ and } r' \geq 1.
\]

**2. THE TOOL.**

We begin with a simple but useful application of the base point free pencil trick. (In this Section \( C \) is any smooth curve of genus \( g \).)
Lemma 1. Let $L, M$ be (maybe, incomplete) linear series on $C$. Assume that $M$ is base point free and $\dim(L - M) \leq \dim(L) - \gamma$. Then, for any integer $m \geq 0$, $\dim(L + mM) \geq \dim(L) + m\gamma$.

Proof. We may assume that $L \neq \emptyset$. Obviously,

$$\dim(L + mM) = \dim(L - M) + \sum_{n=0}^{m} (\dim(L + nM) - \dim(L + (n - 1)M)),$$

and by the base point free pencil trick (cf. [ACGH], III, ex. B-4), for $n \geq 1$,

$$\dim(L + nM) - \dim(L + (n - 1)M) \geq \dim(L + (n - 1)M) - \dim(L + (n - 2)M).$$

Thus $\dim(L + mM) \geq \dim(L - M) + (m + 1)(\dim(L) - \dim(L - M)) = \dim(L) + m(\dim(L) - \dim(L - M)) \geq \dim(L) + m\gamma$. □

Corollary. For a base point free pencil $g^1_k$ on $C$ and non-negative integers $e, f, n$ let $E \in V^e_{g^1_k}([K_C - ng^1_k])$ (so $E \in C^{(e)}$ with $\dim|ng^1_k + E| \geq n + f$). Assume that $\dim|(n - 1)g^1_k + E| = n - 1$. Then

$$\dim|(n + \lambda)g^1_k + E| \geq n + f + \lambda(f + 1) \text{ for any integer } \lambda \geq 0.$$

In view of this corollary, condition (C3) in the next theorem (the construction tool) expresses a behaviour which one likes to expect for a general choice of $E$ in $V^e_{g^1_k}([K_C - ng^1_k])$.

Theorem 1. Let $g^1_k$ be a base point free pencil on $C$, and $e, f, n$ be non-negative integers such that $f \leq k - 2$. By $Z$ we denote an irreducible component of $V^{e - f}_{g^1_k}([K_C - ng^1_k]) \neq \emptyset$, and $\sigma : Z \to W^{n + f}_{nk + e}$ is the natural map defined by $F \mapsto |ng^1_k + F|$. For a general $E \in Z$ we assume that the following conditions are satisfied:

(C1) $\dim|(n - 1)g^1_k + E| = n - 1$

(C2) If $n \geq 1$, $|(n - 1)g^1_k + E|$ is not a specialization of a linear series $g^{n-1}_{(n-1)k + e}$ satisfying $|g^{n-1}_{(n-1)k + e} - (n - 1)g^1_k| = \emptyset$

(C3) $\dim|(n + \lambda)g^1_k + E| = \max((n + \lambda)k + e - g, n + f + \lambda(f + 1)) \text{ for all } 0 \leq \lambda \leq n.$

Then, for any integer $t \geq 0$ such that $\dim[K_C - (n + t)g^1_k - E] \geq k - 2 - f$ the set $\sigma(Z) + tg^1_k = \{|(n + t)g^1_k + F| : F \in Z\}$ is an irreducible component of $W^{n + f + (f + 1)}_{(n + t)k + e}$.

Before we prove the theorem we want to show its usefulness by a simple example:
Corollary. Let \( g_k (k \geq 2) \) be a base point free pencil on \( C \) such that \( |\lambda g_k^1| = \lambda g_k^1 \) (i.e. \( \dim |\lambda g_k^1| = \lambda \)) for any \( 0 \leq \lambda \in \mathbb{Z} \) unless \( |\lambda g_k^1| \) is non-special. Let \( e \) and \( r \) be non-negative integers such that \( \rho_g (rk + e, r) \leq e \). Then \( rg_k + W_e \) is an irreducible component of \( W_k + e \).

Proof of the corollary. For \( r = 0 \) the result is trivial. So let \( r \geq 1 \). In the Theorem take \( f = n = 0 \). Of course, \( \mathcal{V}_e(V|K_C|) = C(e) \neq \emptyset \) is irreducible for \( e > 0 \). Choose \( E \in \mathbb{Z} := C(e) \) general. Then \( |E - g_k^1| = \emptyset \) (which is (C1)) since \( \rho_g (rk + e, r) \leq e \) implies \( e - k < g \). (C2) is void, and (C3) holds by our assumption on the multiples of the pencil \( g_k^1 \) since \( E \in C(e) \) is general. Thus \( rg_k + W_e \) is an irreducible component of \( W_k^{r + e} \) as long as \( \dim |K_C - rg_k^1 - E| \geq k - 2 \). But this latter inequality is equivalent to \( \rho_g (rk + e, r) \leq e \) since \( r \geq 1 \).

We add some remarks on the Corollary:

At first, by dualization we obtain further components made up by series of type 2; this generalizes [CM1], 3.1. Also, the Corollary is best possible because for \( rg_k + W_e \) to be an irreducible component of \( W_k^{r + e} \) it's necessary that \( e = \dim (rg_k + W_e) \geq \rho_g (rk + e, r) \). The hypothesis on the multiples of the \( g_k^1 \) is satisfied by a general \( k \)-gonal curve \( C \) ([B]); but there are also ‘special’ \( k \)-gonal curves satisfying it (e.g., smooth curves in \( \mathbb{P}^1 \times \mathbb{P}^1 \) of bi-degree \( (h, k) \) with \( h \geq k \)).

Proof of the theorem. (i) First we need some preparation.

We may assume that \( |ng_k^1 + E| \) is special because otherwise there is nothing to prove. Then we may write \( \dim |K_C - ng_k^1 - E| = t_0(k - 1 + f) + \epsilon \) with \( 0 \leq t_0 \in \mathbb{Z} \) and \( 0 \leq \epsilon \leq k - 2 - f \), i.e. we let

\[
t_0 := \left\lceil \frac{\dim |K_C - ng_k^1 - E|}{k - 1 - f} \right\rceil.
\]

By (C3) we know that \( \dim |ng_k^1 + E| = n + f \).

Claim 1. For \( 0 \leq \lambda \in \mathbb{Z} \), \( |(n + \lambda)g_k^1 + E| \) is special if and only if \( \lambda \leq t_0 \), and then we have

\[
\dim |(n + \lambda)g_k^1 + E| = n + f + \lambda(f + 1).
\]

Proof of the claim. By (C3), \( |(n + \lambda)g_k^1 + E| \) is special if and only if \( \dim |(n + \lambda)g_k + E| = n + f + \lambda(f + 1) \geq (n + \lambda)k + \epsilon - g \), i.e. \( t_0(k - 1 - f) + \epsilon = \dim |K_C - ng_k^1 - E| = g - 1 - nk - e + n + f > \lambda(k - 1 - f) - 1 \), or, equivalently,

\[
\frac{\epsilon + 1}{k - 1 - f} > \lambda - t_0
\]

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(recall that \( f \leq k - 2 \)). Since
\[
0 < \frac{\epsilon + 1}{k - 1 - f} \leq 1
\]
the claim follows.

Note that, by (C1), (1) also holds for \( \lambda = -1 \). Clearly, then, by the Riemann-Roth theorem,
\[
\dim |K_C - (n + \lambda)g^1_k - E| = \dim |K_C - ng^1_k - E| - \lambda(k - 1 - f)
= (t_0 - \lambda)(k - 1 - f) + \epsilon \quad \text{for} \quad -1 \leq \lambda \leq t_0.
\]

In particular, \( L_0 := |K_C - (n + t_0)g^1_k - E| \) is a series of dimension \( \epsilon \). Since, by the claim, \(|(n + \lambda)g^1_k + E|\) is non-special for \( \lambda > t_0 \) we know that \(|L_0 - g^1_k| = \phi\), and from (2) it follows that
\[
\dim |L_0 + mg^1_k| = m(k - 1 - f) + \epsilon
\]
for any integer \( m \) with \( 0 \leq m \leq t_0 + 1 \).

Thus \( \dim |L_0 + g^1_k| = k - 1 - f + \epsilon = 2\epsilon + 1 + \tau \) with \( 0 \leq \tau := k - 2 - f - \epsilon \leq k - 2 - f \). However, if \( L \) is some complete linear series with the same degree and dimension as \( L_0 \) and such that \(|L - g^1_k| = \phi\) one only knows, by the base point free pencil trick, that \( \dim |L + g^1_k| \geq 2\dim(L) - \dim |L - g^1_k| = 2\epsilon + 1 \); this indicates trouble in the case \( \tau > 0 \).

We now introduce the following notation. Let \( Z_\tau \subseteq W^\tau_{\deg(L_0)} \) be the closure of the series \( L \) as above (recall \(|L - g^1_k| = \phi\)) with the additional property \( \dim |L + g^1_k| = 2\epsilon + 1 + \tau \). We just observed that \( K_C - t_0g^1_k - \sigma(Z) \subseteq Z_\tau \). Since \( Z \) is irreducible there is an irreducible component, \( Z_{\tau,0} \) say, of \( Z_\tau \) containing \( K_C - t_0g^1_k - \sigma(Z) \).

Claim 2. \( Z_{\tau,0} = K_C - t_0g^1_k - \sigma(Z) \).

**Proof of the claim.** Let \( L \in Z_{\tau,0} \) be general.

First we show that
\[
\dim |L + mg^1_k| = m(k - 1 - f) + \epsilon \quad \text{for any integer} \quad 0 \leq m \leq t_0 + 1.
\]

In fact, we know that \( \dim |L + g^1_k| = 2\epsilon + 1 + \tau = k - 1 - f + \epsilon \), by definition of \( Z_\tau \). Applying Lemma 1 to the series \(|L + g^1_k|\) and \( g^1_k \) (i.e. using \( \dim L = \dim |L + g^1_k| - (k - 1 - f) \)) we obtain, for \( m \geq 1 \),
\[
\dim |L + mg^1_k| = \dim |(L + g^1_k) + (m - 1)g^1_k|
\geq \dim |L + g^1_k| + (m - 1)(k - 1 - f) = m(k - 1 - f) + \epsilon
\]
(and, obviously, this also holds for $m = 0$). But since $|L_0 + mg_k^1|$ is a specialization of $|L + mg_k^1|$ we conclude from (3), by semi-continuity, that we have equality in the latter inequality, for $m = t_0 + 1$. This proves (4).

Applying (4) for $m = t_0$ and $m = t_0 + 1$ we see, by Riemann-Roch, that dim $|K_C - (L + t_0g_k^1)| = n + f$ and dim $|K_C - (L + (t_0 + 1)g_k^1)| = n - 1$. (In fact, this is clear without carrying out computations since it is, by (1), true for $I_0$ instead of I, and (3) and (4) have the same right hand side.)

Let $n = 0$. Then $|K_C - (L + t_0g_k^1)|$ specializes to $|K_C - (L_0 + t_0g_k^1)| = |E| = \sigma(E)$ where $E \in Z$ is general. But $\sigma(Z)$ is, for $n = 0$, an irreducible component of $W_f^\epsilon$. This implies that $K_C - Z_{r,0} - t_0g_k^1 \subseteq \sigma(Z)$.

Let $n \geq 1$. Then $|K_C - (L + (t_0 + 1)g_k^1)|$ specializes to $|K_C - (L_0 + (t_0 + 1)g_k^1)| = |(n-1)g_k^1 + E|$. Therefore (C2) implies that $|K_C - (L + (t_0 + 1)g_k^1)| = |(n-1)g_k^1 + F|$ for some $F \in C^\epsilon$. Clearly, $F \in V_\epsilon \setminus \{ |K_C - ng_k^1| \}$, and $F$ specializes to $F$. Since $E \in Z$ is general we must have $F \in Z$. Hence $K_C - Z_{r,0} - t_0g_k^1 \subseteq \sigma(Z)$, and the claim is proved.

(ii) Now we are in a position to prove the Theorem. We keep the notation of (i).

We have to show that $\sigma(Z) + tg_k^1 (0 \leq t \in \mathbb{Z})$ is an irreducible component of $W_{(n+1) + \epsilon}^\epsilon(n+1)$ as long as dim $|K_C - (n + \epsilon)g_k^1 - E| \geq k - 2 - f$, i.e. dim $|L_0 + (t_0 + \epsilon)g_k^1| \geq k - 2 - f \geq 0$. By (3), this condition means: $t \leq t_0$, and in case $\epsilon < k - 2 - f$ we even must have $t < t_0$. Thus, putting $\lambda := t_0 - t$ we suppose that $\lambda \geq 0$, and that $\lambda = 0$ implies $\epsilon = k - 2 - f$. Since, by claim 2, $\sigma(Z) + tg_k^1 = K_C - Z_{r,0} - \lambda g_k^1$ we must show (up to dualization) that $Z_{r,0} + \lambda g_k^1$ is an irreducible component of $W_{\deg L_0 + \lambda k}^{\lambda(k-1-\epsilon)}$. Recall that, by claim 2, $L_0$ is general in $Z_{r,0}$.

First, let $\lambda = 0$ (i.e. $t = t_0$). Then we have $\epsilon = k - 2 - f$, i.e. $t = k - 2 - f - \epsilon = 0$. But for $t = 0$ we have seen in (i) that dim $|L_0 + g_k^1| = 2\epsilon + 1$, the minimum value. This implies that (by its definition) $Z_{0,0}$ is an irreducible component of $W_{\deg L_0}^\epsilon$, and so we are done in this case.

Next, assume $\lambda = 1$. Let $Y$ be an irreducible component of $W_{\deg L_0 + k}^{\epsilon + (k-1)}$ containing the set $Z_{r,0} + g_k^1$, and let $M \in Y$ be general. Then $|M - g_k^1|$ specializes to $L_0$, and it follows, by semi-continuity, that dim $|M - g_k^1| \leq \epsilon$. If we have dim $|M - g_k^1| = \epsilon$ we have $|M - g_k^1| \in Z_{r,0}$, and since $Z_{r,0}$ is an irreducible component of $Z_r$ it follows $|M - g_k^1| \in Z_{r,0}$, i.e. $Y \subseteq Z_{r,0} + g_k^1$, as wanted. So assume that dim $|M - g_k^1| < \epsilon$. By the base point free pencil trick, dim $|M + g_k^1| \geq 2 \dim |M| - \dim |M - g_k^1| \geq 2(\epsilon + k - 1 - f) - (\epsilon - 1) > \epsilon + 2(k - 1 - f)$. But $|M + g_k^1|$ specializes to $|L_0 + 2g_k^1| \in Z_{r,0} + 2g_k^1$, and we know that dim $|L_0 + 2g_k^1| = 2(k - 1 - f) + \epsilon$, by (3) (note that for $t_0 = 1$ we use (3) here at its limit validity $t_0 + 1 = 2$). This contradiction to semi-continuity settles the case $\lambda = 1$.

Finally, let $\lambda \geq 2$. We proceed by induction thus assuming that $Z_{r,0} + (\lambda - 1)g_k^1$ is already known to be as irreducible component of $W_{\deg L_0 + (\lambda - 1)k}^{\epsilon + (\lambda - 1)(k-1-\epsilon)}$. Let $Y$ be an irreducible component of $W_{\deg L_0 + (\lambda - 1)k}^{\epsilon + (\lambda - 1)(k-1-\epsilon)}$ containing $Z_{r,0} + \lambda g_k^1$, and let $M \in Y$ be general. Then $|M - g_k^1|$ specializes to a general element of $Z_{r,0} + (\lambda - 1)g_k^1$ whence dim $|M - g_k^1| \leq (\lambda - 1)$.
\[(k - 1 - f) + \epsilon.\] If we have equality here then \[|M - g_k^1| \in Z_{r,0} + (\lambda - 1)g_k^1\] since the latter set, by induction hypothesis, is an irreducible component of \[W^r_{\deg(l_0)+\lambda g_k^1} \subset Z_{r,0} + (\lambda - 1)g_k^1\], and so we obtain \[Y \subseteq Z_{r,0} + \lambda g_k^1.\] The possibility \(<\) in the last inequality is ruled out in the same way as for \(\lambda = 1\), using the base point free pencil trick for \(\dim |M + g_k^1|\) and using (3).

The Theorem is thereby proved. \(\Box\)

In the closing lines of this Section we want to compute the dimension of the set \(\sigma(Z) + t g_k^1\) considered in the Theorem. Assuming, of course, that \(g_k^1\) is base point free and \(Z\) is an irreducible component of \(V_{e-f}(K_C - n g_k^1) \neq \emptyset\) we use the following abbreviations \((0 \leq t \in \mathbb{Z}) : d := (n + t)k + e, r := n + f + t(f + 1), r' := g - 1 - d + r\) and \(\kappa := \dim Z - (e - f (g - n(k - 1) - e + f)).\) Obviously, \(r \geq f,\) and it is known that \(\kappa \geq 0\) \((\kappa = 0\) if and only if \(Z\) has its ’expected’ dimension).

**Lemma 2.** Assume (Cl) holds for \(Z\). Then, using the above notation, \(\sigma(Z) + t g_k^1 \subseteq W^r_d,\) and its dimension is

\[
\rho(g(d,r) + (r-f)(r' - (k-2-f)) + \kappa \quad \text{ if } n = 0
g(d,r) + (r-f)(r' - (k-2-f)) + \kappa - (n-1)f \quad \text{ if } n > 0.
\]

**Proof.** By (Cl), for \(n > 0\) we have \(h^0(E) = 1\) for a general \(E \in Z\). Hence the natural map \(\sigma : Z \to \sigma(Z) \subseteq W_{nk+e}^{n+f}\) has generic fibre dimension \(f\) resp. \(0\) if \(n = 0\) resp. \(n > 0\). So

\[
\dim(\sigma(Z) + t g_k^1) = \dim(\sigma(Z)) =
\dim(Z) - f = e - f (g - e + f) + \kappa - f = \rho(g,e,f) + \kappa
\]

for \(n = 0\) and \(\dim(\sigma(Z) + t g_k^1) = \dim Z = e - f (g - n(k - 1) - e + f) + \kappa\) for \(n > 0\). Some tedious calculation gives the above dimension formula. Finally, \(\sigma(Z) + t g_k^1 \subseteq W^r_d\) follows from (Cl) and Lemma 1. \(\Box\)

**Corollary.** Assume (Cl) holds for \(Z\).

(i) If \(r > f\) (i.e. \((n, t) \neq (0, 0))\) and \(\kappa = 0\) then \(\sigma(Z) + t g_k^1\) is not an irreducible component of \(W^r_d\) unless we choose \(t\) as in the theorem (i.e. such that \(r' \geq k - 2 - f\)).

(ii) \(\dim(\sigma(Z) + t g_k^1) \geq \rho(g,d,r)\) if \(r' \geq k - 2 - f\) and \(n \leq 1\).

(iii) \(\dim(\sigma(Z) + t g_k^1) \geq \rho(g,d,r)\) if \(r' \geq k - 2\).

Note that in Lemma 2 it is not excluded that \(\sigma(Z) + t g_k^1 \subseteq W^r_d + 1\) in which case it clearly cannot be an irreducible component of \(W^r_d\). This explains (C3) in the Theorem. As for (C2) cf. the next section. In view of [CM2], 1.1.1, (Cl) is a very natural condition for \(Z\).
3. NEW COMPONENTS OF $W_d$

In this Section $C$ denotes a general $k$-gonal curve of genus $g$ and $g^1_k$ a fixed (complete and base point free) pencil of degree $k$ on $C$. Then $g > 2(k-2)$. If $g > 2(k-1)$ the $g^1_k$ on $C$ is unique ([AC]).

**Proposition 1.** Let $e, f$ and $n \geq 0$ be integers such that $0 < f \leq k-2$ and $g - n(k-1) - e + f > 0$. Assume that

$$e - f(g - n(k-1) - e + f) \geq \begin{cases} f & \text{for } n = 0 \\ 0 & \text{for } n > 0 \end{cases}.$$

Then there exists an irreducible component $Z$ of $V^{e-f}_e(|K_C - ng^1_k|)$ of the expected dimension $e - f(g - n(k-1) - e + f)$ which satisfies the conditions (C1) and (C3) of Theorem 1.

**Proof.** The existence of $Z$ and property (C1) for it are settled by the Main Theorem of [CM2]. So we need to show that (C3) holds for this component $Z$, more precisely, for a general element of $Z$ which we denote by $A$ here. To do so we adapt the notation and procedure of the Proof of Theorem 4.2.1 in [CM2]. Specifically, let $C'$ be a general $(f+2)$-gonal curve of genus $g$ and $P_1, \ldots, P_{k-2-f}$ be general points on $C'$ such that $(C;g^1_k)$ specializes to $(C';g^1_{f+2} + P_1 + \ldots + P_{k-2-f})$. Under this specialization $Z$ specializes to a subset $Z' \subseteq C'(e)$ (called $W$ in loc. cit.) which has the following properties: For the complete linear series

$$L := |K_{C'} - (g - 1 - n(k-1) - e + f + n)g^1_{f+2} - n(P_1 + \ldots + P_{k-2-f})|$$

on $C'$ one computes (loc. cit.)

$$m := \dim L = e - f(g - n(k-1) - e + f)$$

and

$$\deg L = e + m + (n-1)(k-2) + (k-2-f).$$

By our assumptions, $\deg L \geq e$ if $n > 0$. For $n = 0$ we even assume $m \geq f$ whence $\deg L = e + m - f \geq e$, again. As in loc. cit. we see that $Z'$ is an irreducible component of the set $\{ E \in C'(e) : |L - E| \neq \emptyset \}$. The claim in loc. cit. shows that $Z'$ in fact is an irreducible component of $V^{e-f}_e(|K_{C'} - n(g^1_{f+2} + P_1 + \ldots + P_{k-2-f})|)$; more precisely the Proof of that claim showed that, for general $E \in Z'$,

$$|K_{C'} - n(g^1_{f+2} + P_1 + \ldots + P_{k-2-f}) - E| =$$

$$|(g - 1 - n(k-1) - e + f)g^1_{f+2} + E'|$$

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with some fixed effective divisor $E'$ of $C'$ (so $E' \in |L - E|$). For $0 \leq \lambda \in \mathbb{Z}$ let

$$M := |(n + \lambda)(g^1_{g+2} + P_1 + \ldots + P_{k-2-f}) + E|.$$ 

Then one computes:

$$\dim |K_{C'} - M| = \dim |K_{C'} - n(g^1_{g+2} + P_1 + \ldots + P_{k-2-f}) - E| - \lambda(g^1_{g+2} + P_1 + \ldots + P_{k-2-f})| = \dim |(g-1-n(k-1)-e+f-\lambda)g^1_{g+2} + E'| - \lambda(P_1 + \ldots + P_{k-2-f})| = \dim |(g-1-n(k-1)-e+f-\lambda)g^1_{g+2} - \lambda(k-2-f)|$$

(note that none of the points $P_i, i = 1, \ldots, k-2-f$, is contained in $E'$; in fact, since they are general w.r.t. $|K_{C'} - (g-1-n(k-1)-e+f+n)g^1_{g+2}|$ none of them is a base point of $L$, and since $E$ is general in $Z'$ so is $E + E'$ in $L$)

$$= (g-1-n(k-1)-e+f-\lambda) - \lambda(k-2-f),$$

as long as $\lambda(k-2-f) \leq g-n(k-1)-e+f-\lambda$.

If $\lambda(k-2-f) > g-n(k-1)-e+f-\lambda$ then $|K_{C'} - M| = \phi$. Hence, by Riemann-Roch,

$$\dim M = \max((n + \lambda)k + e - g, n + f + \lambda(f + 1)).$$

Since the series $M$ on $C'$ is a specialization of the series $|(n + \lambda)g^1_k + A|$ on $C$ we have, by semi-continuity,

$$\dim |(n + \lambda)g^1_k + A| \leq \max((n + \lambda)k + e - g, n + f + \lambda(f + 1)).$$

Lemma 1 implies that we have equality. \(\square\)

For $n \leq 1$ condition (C2) in Theorem 1 is void. Therefore, Theorem 1, Lemma 2 and Proposition 1 imply the

**Theorem 2.** Let $d, f, n \leq 1$ and $t$ be non-negative integers such that $0 < f \leq k-2$. Put $r := n + f + t(f + 1)$ and $r' := g-1-d+r$. Then $W'_d$ contains an irreducible component $V$ of dimension $\dim V = \rho_q(d, r) + (r-f)$ $(r' - (k-2-f))$ as long as $\dim V \geq \max(0, \rho_q(d, r))$. The general series in $V$ is base point free (and simple if $r \geq 2$).

**Proof.** Clearly, $r \geq f$. If $r = f$, i.e. $n = t = 0$, the Theorem reduces to [CM 2], 2.3.2. So assume $r > f$. Then the condition $\dim V \geq \rho_q(d, r)$ means $r' \geq k-2-f$. We want to choose $V = \sigma(Z) + tg^1_k$ with $Z$ as in the Proposition; then the general series in $V$ is base point free since this is so for the general series in $\sigma(Z)$.
([CM2]), and simpleness follows from [CM1], 1.6 if \( r \geq 2 \). There is only some
calculation left. Let \( e := d - (n + t)k \). Then

\[
k - 2 - f \leq r' = g - 1 - d + r = (g - n(k - 1) - e + f) - t(k - 1 - f) - 1
\]

implies that

\[
g - n(k - 1) - e + f \geq (t + 1)(k - 1 - f) > 0.
\]

Calculating \( t \) from \( r = n + f + t(f + 1) \) we obtain \( (f + 1)d = (f + 1)e + (r + nf - f)k \); so the hypothesis on \( e - f(g - n(k - 1) - e + f) \) in Proposition 1
turns out to be equivalent with \( (f + 1)d \geq kr + f(g + 1 + f - k) \) (both for \( n = 0 \)
and \( n = 1 \)) which in turn is equivalent with \( \dim V \geq 0 \). (This is also clear from
the proof of Lemma 2.)

Finally, to apply Theorem 1 we need to know that \( \dim [K \cap (n + t)g_k] - E \) \begin{align*}
&\geq k - 2 - f \\
&\text{for } E \in Z \text{ general. But, using (Cl) and Lemma 1, we get}
\end{align*}

\[
\begin{align*}
\dim [K \cap (n + t)g_k] - E &= g - 1 - d + \\
\dim [(n + t)g_k + E] &\geq g - 1 - d + r = r' \geq k - 2 - f.
\end{align*}
\]

(Of course, since \( f \leq k - 2 \) we have that \( (n + t)g_k^1 + E \) is special and so
\( \dim [(n + t)g_k^1 + E] = r \).)

Note that we have \( |L - g_k| \neq \phi \) for any series \( L \) in the component \( V \) of Theorem
2 if and only if \( r > f \). It is an interesting question how the dual \( K - V \) of \( V \)
may look like; we turn to this question in Section 3.

**Examples.** Let \( V \) be as in Theorem 2.

(i) In the special case \( n = f = 1, t = 0 \) (then \( r = 2 \)) \( V \) is an irreducible component
of \( W'_{d} \) made up by Segre's nets (cf. [S]; [CM1]), for \( k \geq 4 \).

(ii) It follows from [CM2], 1.1.1 that for maximal \( f \), i.e. \( f = k - 2 \), \( V \) consists
of series of type 2 if \( r' \geq 1 \). The existence of this component was already ob-
served in Section 1.

**Corollary.** Let \( d, r \) and \( \alpha \) be positive integers such that \( \alpha < k \) and \( \alpha \) divides \( r \) or
\( r + 1 \). Then \( W'_d \) has an irreducible component \( V_\alpha \) of dimension
\( \dim V_\alpha = \rho_\alpha(d, \alpha - 1) - (r + 1 - \alpha)k = d - rk + (\alpha - 1)(d - g + k - \alpha) = \rho_\alpha(d, r) + (r + 1 - \alpha)(g - d + r - k + \alpha) \) provided that this number is non-negative and not less
than \( \rho_\alpha(d, r) \).

**Proof.** If \( \alpha = 1 \) choose the irreducible component \( r g_k^1 \) of \( W'_d \) found in
the Corollary in Section 1. So let \( 2 \leq \alpha < k \).

If \( \alpha \) divides \( r \) take \( f := \alpha - 1, n = 1 \) and \( t := \frac{\alpha - 1}{\alpha} \) in Theorem 2.
If $\alpha$ divides $r + 1$ take $f := \alpha - 1, n = 0$ and

$$t := \frac{r + 1}{\alpha} - 1$$

in Theorem 2.

Then choose $V_{\alpha} := V$. □

We have $V_{\alpha} \subseteq W_k^1 + W_{d-k}$ unless $\alpha = r + 1$. The components $V_1, V_2$ and, if $r \leq k - 2$, also the components $V_r, V_{r+1}$ are always present as long as the related dimensions $\dim V_1 = d - rk$, $\dim V_2 = \rho_g(d, 1) - (r - 1)k, \dim V_r = \rho_g(d, r-1) - k$ and $\dim V_{r+1} = \rho_g(d, r)$ are non-negative and $\geq \rho_g(d, r)$. For fixed $d < g$ and $r$, $\dim V_{\alpha}$ is a decreasing function in

$$\alpha \geq \frac{k - 1}{2}.$$

Remark. Let $k \geq 6, k - 3 \leq r \not\equiv 1 \mod 3$ and $d = g - 1$. Since 2 and 3 divide $r$ or $r + 1$ we can apply the corollary for $\alpha = 2$ and $\alpha = 3$. The irreducible component $V_3$ of $W_{d-1}^r$ has bigger dimension than $V_2$ thus showing that the ‘optimistic guess’ on $\dim W_d^r$ made in the closing lines of [CM2], Section 4 has to be refined, for $k \geq 6$. (For $k \leq 5$ the guess is true; [CM2], [P].) Of course, it is not hard to write down this refinement if one only considers the components $V_{\alpha}$ of $W_d^r$; but there might be others. □

In order to get rid of condition (C2) we applied Theorem 1 only for $n \leq 1$. The following examples show that we get troubles with (C2) for $n \geq 2$; so in that case the discovery of irreducible components (if any) seems to be more delicate.

Examples. (i) Take $Z$ as in Proposition 1, such that $g - n(k - 1) - e + f = k - 1 - f$. (Hence $\dim |K_C - ng_k^1 - E| = k - 2 - f$ for a general $E \in Z$.) Assume that $n \geq 2$. Then, by Lemma 2, $\dim \sigma(Z) = \rho_g(d, r) - (n - 1)f < \rho_g(d, r)$, and so $\sigma(Z)$ cannot be an irreducible component of $W_{d-1}^r (d = nk + e, r = n + f)$. By Proposition 1 and Theorem 1 the condition (C2) must be violated.

(ii) In Proposition 1, let $k = 4, f = 1, n = 2, e = g - 8$. Then $\sigma(Z) \subseteq W_3^1$ has dimension $g - 11 > g - 12 = \rho_g(g, 3)$. But $W_3^3 = K_C - W_3^{g-2}$ and, by [CM1], 3.2, $W_3^{g-2}$ has irreducible components of dimension $g - 12$ and $g - 10$ only. Thus $\sigma(Z)$ is not an irreducible component of $W_3^g$, and so (C2) is false again.

4. DUALIZATION

In this Section (which generalizes results of [P]) $C$ denotes a general $k$-gonal curve of genus $g$ and $g_k^i$ a fixed pencil of degree $k$ on $C$, for $k \geq 4$. Recall that $\dim |\lambda g_k^1| = \lambda$ for any $0 \leq \lambda \in \mathbb{Z}$ if $|\lambda g_k^1|$ is special ([B]).

We want to discuss the dual $K_C - (\sigma(Z) + tg_k^1)$ of the set $\sigma(Z) + tg_k^1$ ($0 \leq t \in \mathbb{Z}$) where $Z$ is an irreducible component of $V_{\alpha}^{\text{gen}} (|K_C - ng_k^1|) \neq \phi$ satisfying
(C1). By an example in Section 2 we already know what happens in the case of maximal \( f \), i.e. \( f = k - 2 \). Here we restrict to the case \( f = k - 3 \).

First we recall from [CM1], [CM2] the

**Definition.** A linear series \( g'_d \) on \( C \) is called a series of type 3 if it is complete and base point free with \( r \geq 1 \) and \( r' = r - 1 + d + r \geq 1 \) and if \( g'_d = \lfloor (r-1)g^1_k + E \rfloor \) for some effective divisor \( E \) on \( C \) such that \( \dim |g'_d - g^1_k| = r - 2 \).

(Note that \( E \neq 0 \), by [B], and \( h^0(E) = 1 \) in case \( r \geq 2 \).)

The crucial fact on series \( g'_d \) of type 3 is that we know the dimension of the set they constitute in \( W'_{g'} \) ([CM2], 4.3.1). To apply this we need the following technical result.

**Lemma 3.** For positive integers \( d, r \) let \( g'_d \) be a complete linear series on \( C \) such that \( \dim |g'_d - g^1_k| = r - 2 \). Then

(i) \( g'_d = \lfloor (r-1)g^1_k + F \rfloor \) for some effective divisor \( F \) with \( |F - g^1_k| = \phi \).

(Here \( F = 0 \) if and only if \( \dim |(r-1)g^1_k| = r \), and then \( g'_d \) is non-special.)

Or:

(ii) Away from its base locus \( g'_d \) has the form \( t g^1_k + g^0_x \) where \( g^x_x \) is a series of type 3, and \( s + 2t = r \geq s \).

Note that for \( t = 0 \) case (ii) is contained in (i), and for \( r' \geq 1 \) case (i) is contained in (ii) with \( t = 0 \).

**Proof.** Since \( \dim |g'_d - g^1_k| = r - 2 \) we have \( r \geq 1 \), and if \( r = 1 \) we obviously are in case (i). So let \( r \geq 2 \). Write \( |g'_d - g^1_k| = g'^{r-2} + F \) where \( F \) is the base locus of \( |g'_d - g^1_k| \). (In particular, \( |F - g^1_k| = \phi \).) Clearly, \( \dim |g'^{r-2} + g^1_k| \geq r - 1 \). If we have equality then, by [CM1], 1.8, \( g'^{r-2} - (r-2)g^1_k \), and so \( g'_d - \lfloor (r-1)g^1_k + F \rfloor \) with \( \dim |(r-1)g^1_k| = r - 1 \), and we are in case (i) with \( F \neq 0 \). So let \( \dim |g'^{r-2} + g^1_k| = r \). Then \( r \geq 3 \), \( F \) is the base locus of \( g'_d \), and the base point free part \( g'_d - F \) of \( g'_d \) is the sum of \( g^1_k \) and the base point free \( g'^{r-2} \). By the base point free pencil trick,

\[
    r - 3 \geq \dim |g'^{r-2} - g^1_k| \geq 2(r - 2) - \dim |g'^{r-2} + g^1_k| = r - 4.
\]

Assume \( \dim |g'^{r-2} - g^1_k| = r - 3 \). Then \( \dim |g'^{r-2} = \dim |g'^{r-2} - g^1_k| + \dim g^1_k \) which implies, by [CM1], 1.8, that \( g'^{r-2} = (r-2)g^1_k \), and we are in case (i) again.

But now we have \( \dim |(r-1)g^1_k| = \dim |g'_d - F| = r \); hence \( g'_d - F \) (and, in particular, \( g'_d \)) is non-special ([B]), and so \( d \geq r \) by \( \dim |g'_d - F| = d \) \deg \ F \ g \), i.e. \( F = 0 \). (Conversely, if \( F = 0 \) in (i) then \( \dim |(r-1)g^1_k| = r \), and \( g'_d \) is non-special.)

Now, let \( \dim |g'^{r-2} - g^1_k| = r - 4 = (r - 2) - 2 \). Then \( h^1(g'^{r-2}) \geq 2 \): In fact, if \( h^1(g'^{r-2}) \leq 1 \) we obtain, by Riemann-Roch,
whence \( k + \deg F \leq 3 \) contradicting \( k \geq 4 \). By the above reasoning, applied to the base point free series \( g^e_{e-2} \) (instead of \( g^f_d \)), we conclude that \( g^e_{e-2} \) is of type 3, or is the sum of \( g^i_k \) and a base point free \( g^{e-2}_{e-k} \) \((r \geq 5)\) with \( h^1(g^{e-2}_{e-k}) \geq 2 \) such that \( \dim |g^{e-2}_{e-k} - g^i_k| = (r - 4) - 2 \). Proceeding inductively we see that the base point free part \( g^e_d - F \) of \( g^e_d \) is of form

\[
|g^i_k + \text{series of type 3 of dimension } s |
\]

such that \( s + 2t = r \), and we are in case (ii), with \( t \geq 1 \).  

Returning to our sets \( \sigma(Z) + tg^i_k \), for \( f = k - 3 > 0 \), and assuming (Cl) for \( Z \) we know that \( \sigma(Z) + tg^i_k \subseteq W^r_d \) for \( d := (n + t)k + e \) and \( r := n + f + t(f + 1) = (t + 1)(k - 2) + n - 1 \). (Except for the closing lines, \( d \) and \( r \) always have this meaning, and we let \( r' := g - 1 - d + r \).) It is not clear yet if \( \sigma(Z) + tg^i_k \subseteq W^r_{d+1} \). However we have the

**Corollary.** Let \( Z \) be an irreducible component of \( V_e^{e-k}(K_c - ng^i_k) \) for which (Cl) holds \((0 \leq n \in Z)\) and let \( r' \geq 1 \). Assume that the general element \( L \) of \( \sigma(Z) + tg^i_k \) is not of type 2 (w.r.t. our chosen \( g^i_k \)). Then (C3) holds for \( Z \), and the dual \(|K_c - L|\) of \( L \) has a description as in part (ii) of Lemma 3.

**Proof.** By Lemma 1, \( \dim L \geq (t + 1)(k - 2) + n - 1 = r \). Hence \( \dim |K_c - L| \geq r' \geq 1 \). To show (C3) for \( Z \) we have to show that \( \dim L = r \). To see this we use the

**Claim.** \( \dim |L - \lambda g^i_k| \geq \dim L - \lambda(k - 2) \) for any \( 0 \leq \lambda \in Z \).

By applying the claim with \( \lambda = t + 1 \) we obtain, by (Cl),

\[
n - 1 = \dim |L - (t + 1)g^i_k| \geq \dim L - (t + 1)(k - 2),
\]

i.e. \( \dim L \leq (t + 1)(k - 2) + n - 1 = r \).

To prove the claim (which turns out to be true for any complete series \( L \) with \( \dim |K_c - L| \geq 1 \) which is not of type 2) we first observe that it is valid for \( \lambda = 0 \). Assume it is true for \( \lambda = \lambda_0 \geq 0 \) but violated for \( \lambda = \lambda_0 + 1 \). Then

\[
\dim |K_c - (L - \lambda_0 g^i_k)| + \dim g^i_k \leq \dim |(K_c - (L - \lambda_0 g^i_k)) + g^i_k| = \\
= \dim |K_c - (L - (\lambda_0 + 1)g^i_k)| = g - 1 - (d - (\lambda_0 + 1)k) + \dim |L - (\lambda_0 + 1)g^i_k| \leq \\
\leq g - 1 - (d - (\lambda_0 + 1)k) + \dim L - (\lambda_0 + 1)(k - 2) - 1 = \\
= g - (d - \lambda_0 k) + \dim L - \lambda_0(k - 2) \leq g - (d - \lambda_0 k) + \dim |L - \lambda_0 g^i_k| = \\
= \dim |K_c - (L - \lambda_0 g^i_k)| + \dim g^i_k,
\]
i.e. we have \( \dim |(K_C - (L - \lambda_0 g^1_k)) + g^1_k| = \dim |K_C - (L - \lambda_0 g^1_k)| + \dim g^1_k \). By [CM1], I.8 this implies that \( |L - \lambda_0 g^1_k| \) is of type 2. But \( |L - \lambda_0 g^1_k| \) is not of type 2 since \( L \) is not. This contradiction proves the claim.

It remains to show the assertion on \( L' := [K_C - L] \). Since \( |L - g^1_k| \) is a general element of \( \sigma(Z) + (t - 1)g^1_k \) and is not of type 2 (since \( L \) is not) note that we have \( \dim |L - g^1_k| = t(k - 2) + n - 1 = r - k + 2 \) (also for \( t = 0 \)). Hence \( \dim L' = r' \) and \( \dim |L' + g^1_k| = \dim |K_C - (L - g^1_k)| = g - 1 - (d - k) + r - k + 2 = r' + 2 \), and applying Lemma 3 to \( |L' + g^1_k| \) we obtain that \( |L' + g^1_k| \) is as in (i) or (ii) of that lemma. In case (i) we see that the base point free part of \( L' \) is \( r'g^1_k \) \((r' \geq 1)\) whence \( L \) is of type 2, a contradiction. Hence \( |L' + g^1_k| \) is as in part (ii) of Lemma 3 (with \( t \geq 1 \) there), and so also \( L' \) is as in part (ii) of Lemma 3, i.e. we have

\[
L' - \text{base locus} = |r'g^1_k + \text{series of type 3 of dimension } s'|
\]

for some integers \( s' \geq 1, \tau \geq 0 \) such that \( s' + 2\tau = r' \).

If we now restrict, in the corollary, to \( n = 0 \) or \( n = 1 \) we obtain a result which goes far beyond Theorem 2 in the case \( f' = k - 3 \): 

\textbf{Theorem 3.} Let \( Z \) be an irreducible component of \( V_{e^{-(k-3)}}((K_C - ng^1_k)) \) for \( n = 0 \) or \( n = 1 \), and assume that (CI) holds for \( Z \). Let \( 0 \leq t \in \mathbb{Z}, d = (n + t)k + e, r = (t + 1)(k - 2) + n - 1, \) and assume that \( r' = g - 1 - d + r \geq 1 \). Then \( \sigma(Z) + tg^1_k \) is an irreducible component of \( W' \) of the (expected) dimension \( \rho_{\tau}(d, r) + (r - (k - 3))(r' - 1) \), and \( K_C - (\sigma(Z) + tg^1_k) = \tau g^1_k + X_0 \) where \( X_0 \) is an irreducible component of \( W'_{d', r} \) resp. \( W'_{d' - r} \) \((d' := 2g - 2 - d)\) made up by pencils resp. nets of type 3 if \( r' = 2\tau + 1 \) is odd resp. \( r' = 2\tau + 2 \) is even.

\textbf{Proof.} Let us abbreviate \( \delta := d - tk = nk + e, \ s := n + k - 3 \) and \( s' := g - 1 - \delta + s \). Then we have \( s' = r' + 2\tau \geq 1 \) and \( \sigma(Z) \subseteq W'_{\delta} \).

First we claim that \( \sigma(Z) \) is not contained in the irreducible subset \( W \) of \( W'_{\delta} \) made up by the linear series of type 2 (w.r.t. our chosen \( g^1_k \)) which have the same degree \( \delta \) and the same dimension \( \geq s \) as the general element of \( \sigma(Z) \). Assume \( \sigma(Z) \subseteq W \). Since \( \dim W \leq \rho_{\delta}(\delta, s) + (s - (k - 2))s' \) (cf. [CM2], 2.3.5; the \( \leq \) is due to the fact that the series in \( W \) are of dimension \( \geq s \)) and, by Lemma 2, \( \dim \sigma(Z) \geq \rho_{\delta}(\delta, s) + (s - (k - 3))(s' - 1) \) we see that this is impossible for \( s = k - 3 \). If \( s = k - 2 \) (i.e. \( n = 1 \)) we obtain \( s' = 1 \) and \( \dim \sigma(Z) = \dim W \) whence the general series of type 2 is contained in \( \sigma(Z) \). But this is impossible by the Proposition in [CM2], Section 1.

From the Corollary of Lemma 3 we conclude that a general series in \( \sigma(Z) \) is a complete \( g^1_\delta \) such that

\[
|K_C - g^1_\delta| - \text{base locus} = |r'g^1_k + \text{series of type 3 of dimension } s'|
\]
where \( s^* + 2t^* = s' = r' + 2t \) for some integers \( s^* \geq 1, t^* \geq 0 \). Let \( \epsilon \) denote the degree of this base locus and \( \delta' := \deg|K_C - g_k^1| = 2g - 2 - \delta \). Then the involved series of type 3 and dimension \( s^* \) has degree \( \beta := \delta' - \epsilon - t^*k \), and according to [CM2], 4.3.1 the closure \( X \) of the set of series of type 3 in \( W^r_d \) is equidimensional of dimension \( \rho_g(\beta, 1) \) if \( s^* = 1 \) resp. \( \rho_g(\beta, 1) + 1 - (k + 1)(s^* - 1) \) if \( s^* \geq 2 \). One computes that

\[
\rho_g(\beta, 1) = \rho_g(\delta', 1) - 2\epsilon - 2t^*k = \rho_g(\delta', 1) - 2\epsilon - (s' - s^*)k = \rho_g(\delta', 1) - 2\epsilon - (s' - 1)k + (s^* - 1)k.
\]

Hence it follows that

\[
dim X = \rho_g(\delta', 1) - 2\epsilon - (s' - 1)k \quad \text{for} \ s^* = 1, \quad \text{and} \quad \dim X = \rho_g(\delta', 1) - 2\epsilon - (s' - 1)k - (s^* - 2) \quad \text{for} \ s^* \geq 2.
\]

Since \( K_C - \sigma(Z) \subseteq t^*g_k^1 + X + W \), we obtain

\[
\epsilon + \dim X \geq \dim(K_C - \sigma(Z)) = \dim \sigma(Z) \geq \rho_g(\delta, s) + (s - (k - 3))(s' - 1) = \rho_g(\delta', 1) - (s' - 1)k,
\]

and so \( \epsilon = 0 \) (i.e. \( |K_C - g_k^1| \) is base point free), \( 1 \leq s' \leq 2 \) and \( K_C - \sigma(Z) = t^*g_k^1 + X_0 \) for an irreducible component \( X_0 \) of \( X \).

It follows that, for a general series \( L \in o(Z) + tg_k^1 \), we have

\[
|K_C - L| = |(t^* - t)g_k^1 + \text{series of type 3 of dimension } s^*|
\]

with \( 1 \leq s^* \leq 2 \) and \( s^* + 2(t^* - t) = s' - 2t = r' \), and

\[
K_C - (\sigma(Z) + tg_k^1) = \tau g_k^1 + X_0
\]

with \( \tau := t^* - t = \frac{1}{2}(r' - s^*) \geq 0 \) (recall \( r' \geq 1, s^* \leq 2 \)).

We have already observed that \( \dim(\sigma(Z) + tg_k^1) - \rho_g(\delta, s) + (s - (k - 3))(s' - 1) \), and some easy calculation shows that the latter number equals \( \rho_g(d, r) + (r - (k - 3))(r' - 1) \). Finally, to see that \( \sigma(Z) + tg_k^1 \) is an irreducible component of \( W^r_d \) we have, by Theorem 1 and the Corollary of Lemma 3, only to check that its general element \( L \) is not of type 2. But by the above \( |K_C - L| \) is a base point free; if it were compounded of the \( g_k^1 \), i.e. if \( |K_C - L| = \lambda g_k^1 (r' \leq \lambda \in \mathbb{Z}) \), we would obtain that \( |(\lambda - \tau)g_k^1| \) were of type 3 (note that \( \lambda \geq r' > \tau \) which is impossible. \( \square \)

We noticed in Section 2 that the irreducible component \( V_{k-1} \) of \( W^r_d \) (made up by series of type 2 if \( r' \geq 1 \)) is coupled, via dualization, with \( V_1 = r'g_k^1 + W_{d'-r_k} \) in \( W^r_{d'} (d' = 2g - 2 - d, r' = g - 1 - d + r) \). According to Theorem 3, the irreducible component \( V_{k-2} \) of \( W^r_d \) is coupled with \( V_2 \) in \( W^r_{d'} \) (pro-
vided that \( k - 2 \) divides \( r \) or \( r + 1 \). One may wonder if such a correspondence continues to hold. Nevertheless, if \( V_{\alpha'} \) denotes the irreducible component of \( W'_{d_4} \) found in the Corollary of Theorem 2, the variety \( W'_{d_4} \) contains the irreducible components \( V_{\alpha} \) for positive divisors \( \alpha < k \) of \( r \) or \( r + 1 \) and the irreducible components \( K_C - V_{\alpha'} \) for positive divisors \( \alpha' < k \) of \( r' \) or \( r' + 1 \), and \( \dim(K_C - V_{\alpha'}) = \dim V_{\alpha} \) if \( \alpha' = k - \alpha \). The main question is if there still are components of \( W'_{j} \) of other dimensions, for \( k \geq 6 \). If \( r = k - 3 \) we have, by Theorem 3, the following partial answer likewise answering, in this special case, a question asked in [CM2], Section 2.

**Corollary.** \( W'_{d_4}^{k-3} \) has the expected dimension \( \rho_g(d,k-3) \) away from \( W_{k_1}^{1} + W_{d-k}^{k} \).

**Proof.** Let \( V \) be an irreducible component of \( W'_{d_4}^{k-3} \) not contained in \( W_{k_1}^{1} + W_{d-k}^{k} \). Then there is an irreducible component \( Z \) of \( V_{d}^{d-(k-3)}([K_C]) \) satisfying (C1) such that \( \sigma(Z) = V \). Taking \( n = t = 0 \) in the Theorem 3 we obtain \( \dim V = \rho_g(d,k-3) \) (also for \( r' < 1 \)).

Assume that the general \( k \)-gonal curve \( C \) admits 'unexpected' linear series, i.e. \( g'_d \) such that \( \rho_g(d,r) < 0 \). Then \( \rho_g(k,1) < 0 \), and \( C \) has only one pencil \( g^1_k \) of degree \( k \) ([AC]). One may wonder ([CM2]) if this unique pencil \( g^1_k \) occurs in any unexpected series on \( C \). (This question asks for a more general version of the Conjecture 4.1 in [CKM] which has been proved almost completely in [K].) Since any series \( g'_d \) of dimension \( r \geq k \) contains the \( g^1_k \) this question is of interest only for \( r < k \), and the answer is known to be affirmative for \( r \geq k - 2 \) ([CM2]) and \( r = 1 \) ([AC]). The Corollary of Theorem 3 implies that it is affirmative for \( r = k - 3 \), too.

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