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General correspondence for continued fractions

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Abstract: We introduce a new concept of correspondence for continued fractions, based on modified approximants.

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1. Introduction

A continued fraction

$$\mathrm{K}\frac{a_n}{b_n} = \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \frac{a_3|}{|b_3} + \cdots, \quad a_n \neq 0$$

with a_n, $b_n \in \mathscr{M}$— the field of functions meromorphic at 0, can define a function $f \in \mathscr{M}$ in several ways. For instance

(i) $\mathrm{K}(a_n(z)/b_n(z))$ converges pointwise to $f(z)$,
(ii) $\mathrm{K}(a_n(z)/b_n(z))$ corresponds to $f(z)$,
(iii) $\mathrm{K}(a_n/b_n)$ converges to f in some metric on $\mathscr{M}$.

Correspondence, (ii), can be seen as a special case of (iii). (For definition and further information, see Section 3.) These types of convergence are based on ordinary approximants of $\mathrm{K}(a_n/b_n)$.

Recently the use of *modified approximants*

$$S_n(w_n) = \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_n|}{|b_n + w_n}, \quad w_n \in \mathscr{M}$$

has been preferred in several connections, ([2, 6, 7, 8] and more). As a consequence of this, an alternative concept of convergence of $\mathrm{K}(a_n/b_n)$ based on modified approximants has been introduced, [3]. This so-called general convergence is described in Section 2. In Section 3 we apply it to define general correspondence.

2. General convergence

We shall first consider continued fractions $\mathrm{K}(a_n/b_n)$ with complex elements. Let $d(x, y)$ denote the chordal metric on the Riemann sphere $\hat{\mathbb{C}}$. (See for instance [1, p. 20].) That is,

$$d(x, y) = \begin{cases} \dfrac{2|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} & \text{if } x, y \in \mathbb{C}, \\ \dfrac{2}{\sqrt{1+|x|^2}} & \text{if } y = \infty. \end{cases}$$

Definition 2.1 [3]. We say that $\mathrm{K}(a_n/b_n)$ *converges generally* to the value $f \in \hat{\mathbb{C}}$, if there exist two sequences $\{v_n\}$ and $\{w_n\}$ of elements from $\hat{\mathbb{C}}$ such that

(i) $\liminf_{n\to\infty} d(v_n, w_n) > 0$

and

(ii) $\lim_{n\to\infty} S_n(v_n) = \lim_{n\to\infty} S_n(w_n) = f.$

The importance of this concept lies in its nice properties. Among other things we have:

(P1) The limit f is unique in the sense that if $\mathrm{K}(a_n/b_n)$ converges generally to f and to g, then $f = g$.

(P2) If $\mathrm{K}(a_n/b_n)$ converges to f, then $\mathrm{K}(a_n/b_n)$ converges generally to f.

(P3) Let $\mathrm{K}(a_n/b_n)$ converge generally to f, and let $\{g^{(n)}\}$ be a sequence of g-wrong tails; that is, $f \neq g^{(0)} = S_n(g^{(n)})$ for all n. Then

$$\lim S_n(w_n) = f \quad \text{for } \forall \{w_n\} \text{ s.t. } \liminf d(w_n, g^{(n)}) > 0.$$

(P4) Let $\mathrm{K}(a_n/b_n)$ converge generally, and let $\{g^{(n)}\}$ and $\{p^{(n)}\}$ be two sequences of g-wrong tails. Then $d(g^{(n)}, p^{(n)}) \to 0$.

(P5) Let $\{f^{(n)}\}$, $\{g^{(n)}\}$ and $\{p^{(n)}\}$ be three distinct sequences of tails for $\mathrm{K}(a_n/b_n)$ such that

$$d(g^{(n)}, p^{(n)}) \to 0 \quad \text{and } \liminf d(g^{(n)}, f^{(n)}) > 0.$$

Then $\mathrm{K}(a_n/b_n)$ converges generally to $f^{(0)}$.

General convergence can thus replace the use of (ordinary) convergence, $\lim S_n(0) = f$, if we work with modified approximants. It is even better than ordinary convergence in the following ways:

(1) If is often easier to prove general convergence.

(2) Truncation error estimates are often better and easier to obtain for certain modified approximants. Uniform convergence is therefore often easier to prove for these approximants.

(3) From (P3) follows that the continued fractions which converge generally but not in the ordinary sense, are continued fractions where $\liminf |g^{(n)}| = 0$. That is, the 'converging character' of $\mathrm{K}(a_n/b_n)$ is destroyed by the restriction $w_n = 0$ for all n. Examples of such continued fractions are the periodic ones that diverge by Thiele oscillation.

The proofs of (P1)–(P5) are given in [3]. For (P2) it is based on the observation

$$S_n(\infty) = S_{n-1}(0) \quad \text{for all } n. \tag{2.1}$$

For all the other properties, the proofs rely only on the invariance of the cross ratio under linear fractional transformations. Indeed, the proofs still hold if $\mathbb{C}$ is replaced by any field $\mathbb{F}$ and d is replaced by any bounded metric in $\hat{\mathbb{F}} = \mathbb{F} \cup \{\infty\}$ with this invariance property. That is, for continued fractions $\mathrm{K}(a_n/b_n)$ with $a_n, b_n \in \mathbb{F}$, $a_n \neq 0$ we have:

Proposition 2.2. *Let d be a bounded metric in the extended field $\hat{\mathbb{F}} = \mathbb{F} \cup \{\infty\}$, such that*

$$\frac{d(u_1, u_3)}{d(u_1, u_4)} \bigg/ \frac{d(u_2, u_3)}{d(u_2, u_4)} = \frac{d(T(u_1), T(u_3))}{d(T(u_1), T(u_4))} \bigg/ \frac{d(T(u_2), T(u_3))}{d(T(u_2), T(u_4))} \tag{2.2}$$

for every set u_1, u_2, u_3, u_4 of distinct elements from $\hat{\mathbb{F}}$ and every non-singular linear fractional transformation T in $\mathbb{F}$. Then general convergence in $\hat{\mathbb{F}}$ with respect to d has the properties (P1)–(P5).

Proof. (P1) The proof of Theorem 3.3 in [3] can be copied line for line, except that the use of [3, Lemma 3.4] is replaced by the use of the invariance of the cross ratio (2.2).

(P2) Follows by (2.1) since $d(0, \infty) > 0$.

(P3) Assume that for some $N \in \mathbb{N}$ and $d > 0$ we have

$$f = \lim S_n(u_n) = \lim S_n(v_n), \quad d(u_n, v_n) \geqslant d \quad \text{for } n \geqslant N.$$

Let $f \neq \infty$. It then suffices to prove that $d(S_n(w_n), S_n(v_n)) \to 0$ if $\liminf d(w_n, -h_n) \geqslant d_1 > 0$, where $h_n = -S_n^{-1}(\infty)$ for all n. This clearly holds if

$$d(S_n(w_n), S_n(u_n))/d(S_n(u_n), S_n(v_n)) \leqslant M \quad \text{from some } n \text{ on},$$

or, since $f = \lim S_n(u_n) \neq \lim S_n(-h_n) = \infty$, if

$$Q_n = \frac{d(S_n(w_n), S_n(v_n))}{d(S_n(u_n), S_n(v_n))} \bigg/ \frac{d(S_n(w_n), S_n(-h_n))}{d(S_n(u_n), S_n(-h_n))} \leqslant M_1.$$

By (2.2) follows that

$$Q_n = \frac{d(w_n, v_n)}{d(u_n, v_n)} \bigg/ \frac{d(w_n, -h_n)}{d(u_n, -h_n)} \leqslant \frac{d(w_n, v_n) \cdot d(u_n, -h_n)}{d \cdot d_1/2} \leqslant \frac{8}{d \cdot d_1}$$

from some n on.

If $f = \infty$ we repeat the argument for the continued fraction $\mathrm{K}(a_{n+1}/b_{n+1})$.

(P4) Let

$$f = \lim S_n(u_n) = \lim S_n(v_n), \quad d(u_n, v_n) \geqslant d > 0 \quad \text{for } n \geqslant N.$$

Without loss of generality we assume that $f \neq \infty$. Let $\{g^{(n)}\}$ be a sequence of g-wrong tails with $g^{(0)} \neq \infty$. Then $S_n(g^{(n)}) = g^{(0)} \neq f$, and we have

$$d(S_n(g^{(n)}), S_n(v_n))/d(S_n(u_n), S_n(v_n)) \to \infty.$$

Hence Q_n as defined above, with $w_n = g^{(n)}$ becomes

$$Q_n = \frac{d(g^{(n)}, v_n) \cdot d(u_n, -h_n)}{d(u_n, v_n) \cdot d(g^{(n)}, -h_n)} \to \infty.$$

That is, $d(g^{(n)}, -h_n) \to 0$.

(P5) Without loss of generality we assume that $f^{(0)} \neq \infty$, $g^{(0)} \neq \infty$. Let $\{w_n\}$ be an arbitrary sequence from $\hat{\mathbb{F}}$ such that

$$d(w_n, -h_n) \geqslant \epsilon, \quad d(w_n, f^{(n)}) \geqslant \epsilon \quad \text{for all } n$$

for an $\epsilon > 0$. Since $S_n(f^{(n)}) = f^{(0)} \to f^{(0)}$, it suffices to prove that $S_n(w_n) \to f^{(0)}$, that is, that

$$\frac{d(S_n(w_n), f^{(0)})}{d(f^{(0)}, g^{(0)})} = \frac{d(S_n(w_n), S_n(f^{(n)}))}{d(S_n(f^{(n)}), S_n(g^{(n)}))} \to 0,$$

or, since $d(S_n(-h_n), S_n(g^{(n)})) = d(\infty, g^{(0)}) \neq 0$, that

$$Q_n = \frac{d(S_n(w_n), S_n(f^{(n)}))}{d(S_n(f^{(n)}), S_n(g^{(n)}))} \Big/ \frac{d(S_n(w_n), S_n(-h_n))}{d(S_n(-h_n), S_n(g^{(n)}))}$$

$$= \frac{d(w_n, f^{(n)}) \cdot d(-h_n, g^{(n)})}{d(g^{(n)}, f^{(n)}) \cdot d(-h_n, w_n)} \to 0.$$

If $p^{(0)} = \infty$ then $p^{(n)} = -h_n$ and the result follows immediately.

If $p^{(0)} \neq \infty$ then by use of (2.2)

$$0 < \frac{d(g^{(0)}, p^{(0)})}{d(f^{(0)}, p^{(0)})} \Big/ \frac{d(g^{(0)}, \infty)}{d(f^{(0)}, \infty)} = P = \frac{d(g^{(n)}, p^{(n)})}{d(f^{(n)}, p^{(n)})} \Big/ \frac{d(g^{(n)}, -h_n)}{d(f^{(n)}, -h_n)}.$$

That is, $d(g^{(n)}, -h_n) \to 0$ and the result still holds. □

This result is of interest in view of (iii) in Section 1, and, as we shall see in the next section, in view of (ii) in Section 1.

3. General correspondence

Let $\mathrm{K}(a_n/b_n)$ be a continued fraction with elements in $\mathscr{M}$. Let $L(f)$ denote the Laurent expansion of $f \in \mathscr{M}$ at 0. To each $f \in \mathscr{M}$ there exists a unique $L(f) \in \mathscr{L}$—the field of all formal Laurent series. That is, $\mathscr{M}$ is imbedded in $\mathscr{L}$.

Definition 3.1 [5, p. 149]. We say that $\mathrm{K}(a_n/b_n)$ corresponds to $L \in \mathscr{L}$ if $S_n(0) \in \mathscr{M}$ for all n and $L(S_n(0))$ coincides with L up to the term $c_{k_n} z^{k_n}$, where $k_n \to \infty$ as $n \to \infty$.

If in particular $L = L(f)$ for some $f \in \mathscr{M}$, we say that $\mathrm{K}(a_n/b_n)$ corresponds to f, and we write $f \sim \mathrm{K}(a_n/b_n)$. The connection between f and $\mathrm{K}(a_n/b_n)$, or rather its approximants $S_n(0)$, is then of the same nature as the connection between f and its Padé approximants.

We shall first see that correspondence is just convergence in $\mathscr{M}$ or $\mathscr{L}$ in a certain metric. Let λ: $\mathscr{L} \to \hat{\mathbb{R}}$ be given by

$$\lambda(L) = \begin{cases} \infty & \text{if } L = 0, \\ m & \text{if } L = \sum_{k=m}^{\infty} c_k z^k, \quad c_m \neq 0, \end{cases}$$

and $\| \ \|: \mathscr{L} \to \mathbb{R}$ be the norm

$$\| L \| = \begin{cases} 0 & \text{if } L = 0, \\ 2^{-\lambda(L)} & \text{if } L \neq 0. \end{cases}$$

Then $\mathrm{K}(a_n/b_n)$ corresponds to L iff $\lambda(L - L(S_n(0))) \to \infty$; that is, iff $\| L - L(S_n(0)) \| \to 0$. This proves the assertion. (This result is given in [5, §5.1].)

Working with modified approximants, it is unnatural to have the correspondence concept based exclusively on (ordinary) approximants $S_n(0)$. We shall use the results from Section 2 to define general correspondence. Let $\hat{\mathscr{L}} = \mathscr{L} \cup \{ l_\infty \}$, where l_∞ is the set of all doubly infinite series $\sum_{-\infty}^{\infty} c_k z^k$ with $c_k \neq 0$ for arbitrary large $-k$, regarded as one single element. With $l_0 =$ the 0-series we define (in a natural way) $l / l_\infty = l_0$ for all $l \neq l_\infty$ and $l / l_0 = l_\infty$ for all $l \neq l_0$. Moreover, $l \cdot l_\infty = l_\infty$ for $l \neq l_0$ and $l_\infty + l = l_\infty$ for $l \neq l_\infty$. Similarly, we define $\hat{\mathscr{M}} = \mathscr{M} \cup \{ f_\infty \}$, where f_∞ is the function $f_\infty(z) \equiv \infty$, and the arithmetic operations with f_∞ in the similar natural way. Finally we define $L(f_\infty) = l_\infty$, $\lambda(l_\infty) = -\infty$ and thus $\| l_\infty \| = \infty$.

In analogy with the chordal metric in $\hat{\mathbb{C}}$, we define the metric $d^*: \hat{\mathscr{L}} \times \hat{\mathscr{L}} \to [0, 2]$ by

$$d^*(L_1, L_2) = \begin{cases} \dfrac{2 \| L_1 - L_2 \|}{\sqrt{1 + \| L_1 \|^2} \sqrt{1 + \| L_2 \|^2}} & \text{if } L_1, L_2 \in \mathscr{L}, \\ 2 / \sqrt{1 + \| L_2 \|^2} & \text{if } L_1 = l_\infty, \ L_2 \in \mathscr{L}, \\ 0 & \text{if } L_1 = L_2 = l_\infty. \end{cases}$$

Definition 3.2. We say that $\mathrm{K}(a_n/b_n)$ *corresponds generally* to $L \in \hat{\mathscr{L}}$ if there exist two sequences $\{ v_n \}$, $\{ w_n \}$ of functions from $\hat{\mathscr{M}}$ such that

(i) $\liminf_{n \to \infty} d^*(L(v_n), L(w_n)) > 0$

and

(ii) $\lim_{n \to \infty} d^*(L, L(S_n(v_n))) = \lim_{n \to \infty} d^*(L, L(S_n(w_n))) = 0.$

In most applications correspondence to $L = l_\infty$ is not interesting. Restricting ourselves to $L \in \mathscr{L}$, we have the following (equivalent) definition.

Definition 3.3. We say that $\mathrm{K}(a_n/b_n)$ *corresponds generally* to $L \in \mathscr{L}$ if there exist two distinct sequences $\{ v_n \}$, $\{ w_n \}$ of functions from $\hat{\mathscr{M}}$ such that

(i) $\limsup_{n \to \infty} \lambda(L(v_n) - L(w_n)) < \infty,$

(ii) $\lim_{n \to \infty} \lambda(L - L(S_n(v_n))) = \lim_{n \to \infty} \lambda(L - L(S_n(w_n))) = \infty.$

To prove that general correspondence has the nice properties (P1)–(P5) too, we shall prove that the cross ratio in the embedding of $\hat{\mathscr{M}}$ in $\hat{\mathscr{L}}$ with the metric d^* is invariant under linear fractional transformations (Proposition 2.2.).

Lemma 3.4. *Let v_1, v_2, v_3 and v_4 be distinct elements $\in \hat{\mathscr{M}}$, and let T be a non-singular linear fractional transformation in $\mathscr{M}$. Then*

$$\frac{d^*(L(T(v_1)), L(T(v_3)))}{d^*(L(T(v_1)), L(T(v_4)))} \Bigg/ \frac{d^*(L(T(v_2)), L(T(v_3)))}{d^*(L(T(v_2)), L(T(v_4)))} = \frac{d^*(L(v_1), L(v_3))}{d^*(L(v_1), L(v_4))} \Bigg/ \frac{d^*(L(v_2), L(v_3))}{d^*(L(v_2), L(v_4))}.$$

Proof. We introduce the notation $T_k = L(T(v_k))$ and $L_k = L(v_k)$ for $k = 1, 2, 3, 4$. Assume first that all T_k and L_k are $\in \mathscr{L}$. Then

$$\frac{d^*(T_1, T_3)}{d^*(T_1, T_4)} \Bigg/ \frac{d^*(T_2, T_3)}{d^*(T_2, T_4)} = \frac{\| T_1 - T_3 \| \cdot \| T_2 - T_4 \|}{\| T_1 - T_4 \| \cdot \| T_2 - T_3 \|} = \frac{2^{-\lambda(T_1 - T_3)} \cdot 2^{-\lambda(T_2 - T_4)}}{2^{-\lambda(T_1 - T_4)} \cdot 2^{-\lambda(T_2 - T_3)}}$$
$$= 2^{-\lambda(T_1 - T_3) - \lambda(T_2 - T_4) + \lambda(T_1 - T_4) + \lambda(T_2 - T_3)} = 2^{-\lambda(((T_1 - T_3)/(T_1 - T_4))/((T_2 - T_3)/(T_2 - T_4)))},$$

and the result follows since

$$\frac{T(v_1) - T(v_3)}{T(v_1) - T(v_4)} \Bigg/ \frac{T(v_2) - T(v_3)}{T(v_2) - T(v_4)} = \frac{v_1 - v_3}{v_1 - v_4} \Bigg/ \frac{v_2 - v_3}{v_2 - v_4},$$

and thus by the embedding of $\mathscr{M}$ in $\mathscr{L}$,

$$\frac{T_1 - T_3}{T_1 - T_4} \Bigg/ \frac{T_2 - T_3}{T_2 - T_4} = \frac{L_1 - L_3}{L_1 - L_4} \Bigg/ \frac{L_2 - L_3}{L_2 - L_4}.$$

If $T_k = l_\infty$ for some k (only one since v_1, v_2, v_3 and v_4 are distinct) or/and $L_k = l_\infty$ for some k, then the result still holds. This can be seen by similar observations. □

This means in particular that if $\mathrm{K}(a_n/b_n)$ corresponds generally to L, then L is unique, and if $\mathrm{K}(a_n/b_n)$ corresponds to L in the classical sense, then it corresponds generally to L.

As for general convergence, there exist continued fractions $\mathrm{K}(a_n/b_n)$ which correspond generally to a series $L \in \mathscr{L}$, but not in the classical sense. Since $-h_n = S_n^{-1}(f_\infty)$ then is a sequence of g-wrong tails, we then must have $\lim\sup \lambda(L(h_n)) = \infty$. This excludes for instance C-fractions and general T-fractions. For these we therefore have general correspondence to an $L \in \mathscr{L}$ if and only if we have classical correspondence to L. What we gain by introducing general correspondence is for instance:

(1) Working with modified approximants we do not have to check the correspondence for the ordinary approximants $S_n(0)$.

(2) In convergence theorems like for instance [5, Theorem 5.11] we can replace ordinary convergence and correspondence by general convergence and correspondence and uniform convergence of certain modified approximants.

4. Final remarks

A generalization of general convergence is given in [4]:

Definition 4.1. We say that a sequence $\{T_n\}$ of non-singular linear fractional transformations is restrained if there exist two sequences $\{u_n\}$, $\{v_n\}$ in $\hat{\mathbb{C}}$ such that

(i) $\liminf_{n\to\infty} d(u_n, v_n) > 0$ ($d=$ the chordal metric), and

(ii) $\{T_n(u_n)\}$ and $\{T_n(v_n)\}$ have the same limiting structure; i.e. $\lim_{n\to\infty} d(T_n(u_n), T_n(v_n)) = 0$.

Clearly, if $T_n(w) = S_n(w)$ for a continued fraction $\mathrm{K}(a_n/b_n)$, and $T_n(u_n) = S_n(u_n) \to f \in \hat{\mathbb{C}}$, then $\{T_n\}$ is restrained if and only if $\mathrm{K}(a_n/b_n)$ converges generally.

It turns out that the limiting structure of $\{T_n(u_n)\}$, $\{T_n(v_n)\}$ then is unique (P1) and that properties similar to (P3)–(P5) hold for restrained sequences [4]. These results are also proved by means of the invariance of the cross ratio under linear fractional transformations. It is straightforward to check that Proposition 2.2 still holds if we replace 'general convergence' by 'restrained' (and exclude (P2)).

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