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Summation of series defined by counting blocks of digits

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Abstract

We discuss the summation of certain series defined by counting blocks of digits in the B-ary expansion of an integer. For example, if $s_2(n)$ denotes the sum of the base-2 digits of n, we show that $\sum_{n\geqslant 1} s_2(n)/(2n(2n+1)) = (\gamma + \log\frac{4}{\pi})/2$. We recover this previous result of Sondow and provide several generalizations.

MSC: 11A63; 11Y60

1. Introduction

A classical series with rational terms, known as Vacca's series [17] or in an equivalent integral form as Catalan's integral [7] (see also [6,16]), evaluates to Euler's constant γ:

$$\gamma = \sum_{n\geqslant 1} \frac{(-1)^n}{n} \left\lfloor \frac{\log n}{\log 2} \right\rfloor = \int_0^1 \frac{1}{1+x} \sum_{n\geqslant 1} x^{2^n-1}\, dx.$$

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In a recent paper [15] Sondow gave the following two formulas:

$$\gamma^{\pm} = \sum_{n \geqslant 1} \frac{N_1(n) \pm N_0(n)}{2n(2n+1)}$$

where $\gamma^+ = \gamma$ is the Euler constant, $\gamma^- = \log \frac{4}{\pi}$ is the "alternating Euler constant" [14], and $N_1(n)$ (respectively $N_0(n)$) is the number of 1's (respectively 0's) in the binary expansion of the integer n. The series for $\gamma^+ = \gamma$ is equivalent to Vacca's. The formulas for $\gamma^{\pm}$ show in particular that

$$\sum_{n \geqslant 1} \frac{s_2(n)}{2n(2n+1)} = \frac{\gamma + \log \frac{4}{\pi}}{2}$$

where $s_2(n)$ is the sum of the binary digits of the integer n.

This last formula reminds us of one of the problems posed at the 1981 Putnam competition [9]: Determine whether or not

$$\exp\left(\sum_{n \geqslant 1} \frac{s_2(n)}{n(n+1)}\right)$$

is a rational number. In fact, $\sum \frac{s_2(n)}{n(n+1)} = 2 \log 2$. A generalization was proven by Shallit [13], where the base 2 is replaced by any integer base $B \geqslant 2$. A more general result, where the sum of digits is replaced by the function $N_{w,B}(n)$, which counts the number of occurrences of the block w in the B-ary expansion of the integer n, was given by Allouche and Shallit [2].

The purpose of the present paper is to show that the result of [15] cited above can be deduced from a general lemma in [2]. Furthermore, we sum the series

$$\sum_{n \geqslant 1} \frac{N_{w,2}(n)}{2n(2n+1)} \quad \text{and} \quad \sum_{n \geqslant 1} \frac{N_{w,2}(n)}{2n(2n+1)(2n+2)},$$

thus generalizing Corollary 1 in [15] and a series for Euler's constant in [1,5,11] (dated February 1967, August 1967, February 1968), respectively. Finally, we indicate some generalizations of our results, including an extension to base $B > 2$, and a method for giving alternate proofs without using the general lemma from [2].

2. A general lemma

The first lemma in this section is taken from [2]; for completeness we recall the proof. We also give two classical results presented as lemmas, together with a new result (Lemma 4).

We start with some definitions. Let $B \geqslant 2$ be an integer. Let w be a word on the alphabet of digits $\{0, 1, \ldots, B-1\}$ (that is, w is a finite block of digits). We denote by $N_{w,B}(n)$ the number of (possibly overlapping) occurrences of w in the B-ary expansion of an integer $n > 0$, and we set $N_{w,B}(0) = 0$.

Given w as above, we denote by $|w|$ the length of the word w (i.e., if $w = d_1 d_2 \cdots d_k$, then $|w| = k$). Denote by w^j the concatenation of j copies of the word w.

Given w and B as above, we denote by $v_B(w)$ the value of w when w is interpreted as the base B-expansion (possibly with leading 0's) of an integer.

Remark 1. The occurrences of a given word in the B-ary expansion of the integer n may overlap. For example, $N_{11,2}(7) = 2$.

If the word w begins with 0, but $v_B(w) \neq 0$, then in computing $N_{w,B}(n)$ we assume that the B-ary expansion of n starts with an arbitrarily long prefix of 0's. If $v_B(w) = 0$ we use the usual B-ary expansion of n without leading zeros. For example, $N_{011,2}(3) = 1$ (write 3 in base 2 as $0\cdots011$) and $N_{0,2}(2) = 1$.

Lemma 1. [2] *Fix an integer $B \geqslant 2$, and let w be a non-empty word on the alphabet $\{0, 1, \dots, B-1\}$. If $f : \mathbb{N} \to \mathbb{C}$ is a function with the property that $\sum_{n \geqslant 1} |f(n)| \log n < \infty$, then*

$$\sum_{n \geqslant 1} N_{w,B}(n)\biggl(f(n) - \sum_{0 \leqslant j < B} f(Bn + j)\biggr) = \sum f\bigl(B^{|w|}n + v_B(w)\bigr),$$

where the last summation is over $n \geqslant 1$ if $w = 0^j$ for some $j \geqslant 1$, and over $n \geqslant 0$ otherwise.

Proof. (See [2].) As $N_{w,B}(n) \leqslant \lfloor \frac{\log n}{\log B} \rfloor + 1$, all series $\sum N_{w,B}(un + v) f(un + v)$, where u and v are nonnegative integers, are absolutely convergent. Let ℓ be the last digit of w, and let $g := B^{|w|-1}$. Then

$$\sum_{n \geqslant 0} N_{w,B}(n) f(Bn + \ell) = \sum_{0 \leqslant k < g} \sum_{n \geqslant 0} N_{w,B}(gn + k) f(Bgn + Bk + \ell)$$

and

$$\sum_{n \geqslant 0} N_{w,B}(Bn + \ell) f(Bn + \ell) = \sum_{0 \leqslant k < g} \sum_{n \geqslant 0} N_{w,B}(Bgn + Bk + \ell) f(Bgn + Bk + \ell).$$

Now, if either $n \neq 0$ or $v_B(w) \neq 0$, then for $k = 0, 1, \dots, g-1$ we have

$$N_{w,B}(Bgn + Bk + \ell) - N_{w,B}(gn + k) = \begin{cases} 1, & \text{if } k = \lfloor \frac{v_B(w)}{B} \rfloor; \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, if $n = 0$ and $v_B(w) = 0$ (hence $\ell = 0$), then the difference equals 0 for every $k \in \{0, 1, \dots, g-1\}$. Hence

$$\sum_{n \geqslant 0} N_{w,B}(Bn + \ell) f(Bn + \ell) - \sum_{n \geqslant 0} N_{w,B}(n) f(Bn + \ell) = \sum f\biggl(Bgn + B\biggl\lfloor \frac{v_B(w)}{B} \biggr\rfloor + \ell\biggr)$$
$$= \sum f\bigl(B^{|w|}n + v_B(w)\bigr), \qquad (*)$$

the last two summations being over $n \geqslant 0$ if w is not of the form 0^j, and over $n \geqslant 1$ if $w = 0^j$ for some $j \geqslant 1$. We then write

$$\sum_{n\geqslant 0} N_{w,B}(n)f(n) = \sum_{0\leqslant j<B}\sum_{n\geqslant 0} N_{w,B}(Bn+j)f(Bn+j)$$
$$= \sum_{j\in[0,B)\setminus\{\ell\}}\sum_{n\geqslant 0} N_{w,B}(Bn+j)f(Bn+j) + \sum_{n\geqslant 0} N_{w,B}(Bn+\ell)f(Bn+\ell)$$

which together with (∗) gives

$$\sum_{n\geqslant 0} N_{w,B}(n)\biggl(f(n) - \sum_{0\leqslant j<B} f(Bn+j)\biggr) = \sum f\bigl(B^{|w|}n + v_B(w)\bigr).$$

Since $N_{w,B}(0) = 0$, the proof is complete. □

Now let Γ be the usual gamma function, let $\Psi := \Gamma'/\Gamma$ be the logarithmic derivative of the gamma function, let $\zeta(s)$ be the Riemann zeta function, let $\zeta(s,x) := \sum_{n\geqslant 0}(n+x)^{-s}$ be the Hurwitz zeta function, and let γ denote Euler's constant.

Lemma 2. *If a and b are positive real numbers, then*

$$\sum_{n\geqslant 1}\left(\frac{1}{an} - \frac{1}{an+b}\right) = \frac{1}{b} + \frac{\gamma + \Psi(b/a)}{a}.$$

Proof. We write

$$\sum_{n\geqslant 1}\left(\frac{1}{an} - \frac{1}{an+b}\right) = \lim_{s\to 1_+}\sum_{n\geqslant 1}\left(\frac{1}{(an)^s} - \frac{1}{(an+b)^s}\right) = \frac{1}{a}\lim_{s\to 1_+}\sum_{n\geqslant 1}\left(\frac{1}{n^s} - \frac{1}{(n+\frac{b}{a})^s}\right)$$
$$= \frac{1}{a}\lim_{s\to 1_+}\left(\zeta(s) - \zeta\left(s, \frac{b}{a}\right) + \left(\frac{a}{b}\right)^s\right)$$
$$= \frac{1}{b} + \frac{1}{a}\lim_{s\to 1_+}\left(\left(\zeta(s) - \frac{1}{s-1}\right) - \left(\zeta\left(s, \frac{b}{a}\right) - \frac{1}{s-1}\right)\right)$$
$$= \frac{1}{b} + \frac{1}{a}\left(\gamma + \frac{\Gamma'(b/a)}{\Gamma(b/a)}\right) = \frac{1}{b} + \frac{\gamma + \Psi(b/a)}{a}$$

(see, for example, [18, p. 271]). □

Lemma 3. *For $x > 0$ we have*

$$\sum_{r\geqslant 1}\left(\frac{x}{r} - \log\left(1 + \frac{x}{r}\right)\right) = \log x + \gamma x + \log\Gamma(x).$$

Proof. Take the logarithm of the Weierstraß product for $1/\Gamma(x)$ (see, for example, [8, Section 1.1] or [18, Section 12.1]). □

The next lemma in this section is the last step before proving our theorems.

Lemma 4. *Let a and b be positive real numbers. Then*

$$\sum_{n\geqslant 1}\left(\frac{1}{an}-\log\frac{an+1}{an}\right)=\log\Gamma\left(\frac{1}{a}\right)+\frac{\gamma}{a}-\log a$$

and

$$\sum_{n\geqslant 0}\left(\frac{1}{an+b}-\log\frac{an+b+1}{an+b}\right)=\log\Gamma\left(\frac{b+1}{a}\right)-\log\Gamma\left(\frac{b}{a}\right)-\frac{\Psi(b/a)}{a}.$$

Proof. The proof is straightforward. The first formula follows directly from Lemma 3. To prove the second, write the nth term of the series for $n\geqslant 1$ as the following sum of nth terms of three absolutely convergent series:

$$\frac{1}{an+b}-\frac{1}{an}-\frac{b}{an}+\log\left(1+\frac{b}{an}\right)+\frac{b+1}{an}-\log\left(1+\frac{b+1}{an}\right);$$

then use Lemmas 2 and 3. □

3. Two theorems

In this section we give two theorems that are consequences of Lemma 1, and that generalize results in [15] and [1,5,11].

Theorem 1. *Let w be a non-empty word on the alphabet $\{0,1\}$, and let Ψ denote the logarithmic derivative of the gamma function.*

(a) *If $v_2(w)=0$, then*

$$\sum_{n\geqslant 1}\frac{N_{w,2}(n)}{2n(2n+1)}=\log\Gamma\left(\frac{1}{2^{|w|}}\right)+\frac{\gamma}{2^{|w|}}-|w|\log 2.$$

(b) *If $v_2(w)\neq 0$, then*

$$\sum_{n\geqslant 1}\frac{N_{w,2}(n)}{2n(2n+1)}=\log\Gamma\left(\frac{v_2(w)+1}{2^{|w|}}\right)-\log\Gamma\left(\frac{v_2(w)}{2^{|w|}}\right)-\frac{1}{2^{|w|}}\Psi\left(\frac{v_2(w)}{2^{|w|}}\right).$$

Proof. Let

$$A_n:=\frac{1}{n}-\log\frac{n+1}{n}$$

for $n\geqslant 1$. Noting that $A_n-A_{2n}-A_{2n+1}=\frac{1}{2n(2n+1)}$, the theorem follows from Lemma 1 with $B=2$, and $f(n):=A_n$ for $n\geqslant 1$, together with Lemma 4. □

Example 1. Taking $w = 0$ and $w = 1$, and recalling that $\Gamma(1/2) = \sqrt{\pi}$ and $\Psi(1/2) = -\gamma - 2\log 2$ by Gauß's theorem (see, for example, [8, p. 19] or [10, p. 94]), we get

$$\sum_{n\geqslant 1} \frac{N_{0,2}(n)}{2n(2n+1)} = \frac{1}{2}\log\pi + \frac{\gamma}{2} - \log 2$$

and

$$\sum_{n\geqslant 1} \frac{s_2(n)}{2n(2n+1)} = \sum_{n\geqslant 1} \frac{N_{1,2}(n)}{2n(2n+1)} = -\frac{1}{2}\log\pi + \frac{\gamma}{2} + \log 2.$$

These equalities imply the formulas in the introduction:

$$\sum_{n\geqslant 1} \frac{N_{1,2}(n) \pm N_{0,2}(n)}{2n(2n+1)} = \gamma^{\pm}$$

where (following the notations of [15]) $\gamma^+ := \gamma$ and $\gamma^- := \log\frac{4}{\pi}$, which is Corollary 1 of [15].

Remark 2. The formulas in Theorem 1 are analogous to those in [2, p. 25]. The analogy becomes more striking if one uses Gauß's theorem to write all expressions of the form $\Psi(x)$, with x a rational number in (0, 1], using only trigonometric functions, logarithms, and Euler's constant.

Theorem 2. *Let w be a non-empty word on the alphabet* {0, 1}.

(a) *If $v_2(w) = 0$, then*

$$\sum_{n\geqslant 1} \frac{N_{w,2}(n)}{2n(2n+1)(2n+2)} = \log\Gamma\left(\frac{1}{2^{|w|}}\right) + \frac{\gamma}{2^{|w|+1}} - |w|\log 2 - \frac{1}{2^{|w|+1}}\Psi\left(\frac{1}{2^{|w|}}\right) - \frac{1}{2}.$$

(b) *If $v_2(w) \neq 0$, then*

$$\sum_{n\geqslant 1} \frac{N_{w,2}(n)}{2n(2n+1)(2n+2)} = \log\Gamma\left(\frac{v_2(w)+1}{2^{|w|}}\right) - \log\Gamma\left(\frac{v_2(w)}{2^{|w|}}\right) - \frac{1}{2^{|w|+1}}\left(\Psi\left(\frac{v_2(w)}{2^{|w|}}\right) + \Psi\left(\frac{v_2(w)+1}{2^{|w|}}\right)\right).$$

Proof. Noting that $\frac{1}{2n(2n+1)} - \frac{1}{4}\cdot\frac{1}{n(n+1)} = \frac{1}{2n(2n+1)(2n+2)}$, it suffices to use Theorem 1 and the following result, deduced from [2, top of p. 26] in the case $B = 2$.

(a) If $v_2(w) = 0$, then

$$\sum_{n\geqslant 1} \frac{N_{w,2}(n)}{n(n+1)} = \frac{1}{2^{|w|-1}}\left(\Psi\left(\frac{1}{2^{|w|}}\right) + \gamma + 2^{|w|}\right).$$

(b) If $v_2(w) \neq 0$, then

$$\sum_{n \geqslant 1} \frac{N_{w,2}(n)}{n(n+1)} = \frac{1}{2^{|w|-1}} \left(\Psi\left(\frac{v_2(w)+1}{2^{|w|}}\right) - \Psi\left(\frac{v_2(w)}{2^{|w|}}\right)\right). \qquad \square$$

Example 2. Taking $w = 0$ and $w = 1$, we get

$$\sum_{n \geqslant 1} \frac{N_{0,2}(n)}{2n(2n+1)(2n+2)} = \frac{1}{2}\log \pi + \frac{\gamma}{2} - \frac{1}{2}\log 2 - \frac{1}{2}$$

and

$$\sum_{n \geqslant 1} \frac{s_2(n)}{2n(2n+1)(2n+2)} = \sum_{n \geqslant 1} \frac{N_{1,2}(n)}{2n(2n+1)(2n+2)} = -\frac{1}{2}\log \pi + \frac{\gamma}{2} + \frac{1}{2}\log 2.$$

Hence

$$\sum_{n \geqslant 1} \frac{N_{1,2}(n) \pm N_{0,2}(n)}{2n(2n+1)(2n+2)} = \delta^{\pm}$$

where $\delta^{+} := \gamma - \frac{1}{2}$ and $\delta^{-} := \frac{1}{2} - \log\frac{\pi}{2}$, which are respectively a formula given in [1,5,11] and a seemingly new companion formula.

Remark 3. As mentioned, all expressions of the form $\Psi(x)$, with x a rational number in (0, 1], can be written using only trigonometric functions, logarithms, and Euler's constant.

4. Generalizations

Several extensions or generalizations of our results are possible. We give some of them in this section.

4.1. Variation on A_n

Instead of applying Lemma 1 with $f(n) = A_n = \frac{1}{n} - \log\frac{n+1}{n}$ for $n \geqslant 1$, we could replace A_n with

$$A_n^{(k)} := \frac{1}{n+k} - \log\frac{n+1}{n}$$

for $n \geqslant 1$, where k is a nonnegative integer. Defining the (rational) function $Q^{(k)}$ by

$$Q^{(k)}(n) := A_n^{(k)} - A_{2n}^{(k)} - A_{2n+1}^{(k)}$$

and noting that summing $\sum_{n \geqslant 1} A_{an+b}^{(k)}$ boils down to summing $\sum_{n \geqslant 1}(\frac{1}{an+b+k} - \frac{1}{an+b})$, which as in the proof of Lemma 2 involves the Hurwitz zeta function, we obtain explicit formulas for the sum of the series $\sum_{n \geqslant 1} N_{w,2}(n) Q^{(k)}(n)$.

4.2. Extension to base $B > 2$

Lemma 1 has been used above only for base $B = 2$. There are applications to other bases in [2]. We also note that the relation among the A_n's,

$$A_n = \frac{1}{2n(2n+1)} + A_{2n} + A_{2n+1}$$

for $n \geqslant 1$, can be generalized to base B. Namely,

$$A_n = Q(n, B) + R(n, B)$$

where

$$Q(n, B) := \frac{1}{Bn(Bn+1)} + \frac{2}{Bn(Bn+2)} + \cdots + \frac{B-1}{Bn(Bn+B-1)}$$

and

$$R(n, B) := A_{Bn} + A_{Bn+1} + \cdots + A_{Bn+B-1}.$$

This allows us to use Lemmas 1 and 4 to sum, for example, the series

$$\sum_{n \geqslant 1} N_{w,3}(n) \frac{9n+4}{3n(3n+1)(3n+2)},$$

since

$$Q(n, 3) = A_n - A_{3n} - A_{3n+1} - A_{3n+2} = \frac{9n+4}{3n(3n+1)(3n+2)}.$$

4.3. Weighted A_n's

In this section we consider a weighted form of the A_n's. First we need to study a relation between sequences of real numbers.

Lemma 5. *Let $(r_n)_{n \geqslant 1}$ and $(R_i)_{i \geqslant 1}$ be sequences of real numbers. Set $r_0 := 0$ and $R_0 := 0$. Then the following two properties are equivalent*:

(1) *for $i \geqslant 1$*

$$R_i = \sum_{k \geqslant 0} r_{\lfloor \frac{i}{2^k} \rfloor} = r_i + r_{\lfloor \frac{i}{2} \rfloor} + r_{\lfloor \frac{i}{4} \rfloor} + \cdots$$

(*note that this is actually a finite sum*);

(2) *for $n \geqslant 1$*

$$r_n = R_n - R_{\lfloor \frac{n}{2} \rfloor}.$$

Proof. The implication (1) ⇒ (2) is easily seen by considering the cases n even and n odd. Likewise, for (2) ⇒ (1) take i even and i odd. □

Remark 4. See [4, Theorem 9] for more about this relation.

Theorem 3. *Assume that $r_1, r_2, \ldots$ and $R_1, R_2, \ldots$ are real numbers related as in Lemma* 5. *Then the series $\sum |r_n| n^{-2}$ converges if and only if the series $\sum |R_i| i^{-2}$ converges, and in this case we have*

$$S := \sum_{n\geqslant 1} r_n \left(\frac{1}{n} - \log\frac{n+1}{n}\right) = \sum_{i\geqslant 1} \frac{R_i}{2i(2i+1)}.$$

Proof. First note that if the series $\sum |R_i| i^{-2}$ converges, then so does the series $\sum |r_n| n^{-2}$: use the expression for r_n in terms of the R_i's in Lemma 5. Now suppose that the series $\sum |r_n| n^{-2}$ converges. As before, let $A_n := \frac{1}{n} - \log\frac{n+1}{n}$. Then $0 < A_n < \frac{1}{n} - \frac{1}{n+1}$. This implies that the series $S := \sum r_n (\frac{1}{n} - \log\frac{n+1}{n})$ is absolutely convergent. Now

$$A_n = \frac{1}{2n(2n+1)} + A_{2n} + A_{2n+1}$$

implies

$$A_n = \frac{1}{2n(2n+1)} + \frac{1}{4n(4n+1)} + \frac{1}{(4n+2)(4n+3)} + A_{4n} + A_{4n+1} + A_{4n+2} + A_{4n+3}.$$

Hence, repeating K times,

$$A_n = \sum_{1\leqslant k\leqslant K} \sum_{0\leqslant m<2^{k-1}} \frac{1}{(2^k n + 2m)(2^k n + 2m + 1)} + \sum_{0\leqslant q<2^K} A_{2^K n+q}.$$

Using the bounds $0 < A_n < \frac{1}{n} - \frac{1}{n+1}$ and telescoping, the last sum is less than 2^{-K}. Letting K tend to infinity, we obtain the (rapidly convergent) series

$$A_n = \sum_{k\geqslant 1} \sum_{0\leqslant m<2^{k-1}} \frac{1}{(2^k n + 2m)(2^k n + 2m + 1)}.$$

Substituting into the sum defining S yields the double series

$$S = \sum_{n\geqslant 1} \sum_{k\geqslant 1} \sum_{0\leqslant m<2^{k-1}} \frac{r_n}{(2^k n + 2m)(2^k n + 2m + 1)},$$

which converges absolutely. Thus we may collect terms with the same denominator, and we arrive at the series

$$S = \sum_{i\geqslant 1} \frac{R'_i}{2i(2i+1)},$$

where

$$R_i' := \sum_{n \in \mathcal{E}_i} r_n$$

with $\mathcal{E}_i := \{n \geqslant 1,\ \exists k \geqslant 1,\ \exists m \in [0, 2^{k-1}),\ 2^{k-1}n + m = i\}$. On the one hand, this proves that the series $\sum \frac{R_i'}{2i(2i+1)}$ is absolutely convergent (hence the series $\sum |R_i'| i^{-2}$ is convergent). On the other hand, R_i' can also be written as

$$R_i' := \sum_{1 \leqslant k \leqslant \frac{\log i}{\log 2}+1} r_{\lfloor \frac{i}{2^{k-1}} \rfloor} = \sum_{k \geqslant 0} r_{\lfloor \frac{i}{2^k} \rfloor}$$

(recall that we have set $r_0 := 0$). Thus $R_i' = R_i$ by the hypothesis, and the proof is complete. □

Example 3. Theorem 3 yields in particular the series for γ and $\log \frac{4}{\pi}$ in the introduction, in Example 1, and in [15, Corollary 1]. Namely,

If $r_1 = r_2 = \cdots = 1$, then the series defining S sums to γ from Lemma 4, and the formula defining $R_i' = R_i$ reduces to $R_i = \lfloor \frac{\log 2i}{\log 2} \rfloor = N_{1,2}(i) + N_{0,2}(i)$.

If $r_n = (-1)^{n-1}$, then $S = \log \frac{4}{\pi}$ (see [14] or decompose S into $\sum$(odd terms) $-$ $\sum$(even terms) and apply Lemma 4), and the formula defining $R_i' = R_i$ implies $R_i = N_{1,2}(i) - N_{0,2}(i)$. To see this equality, first note that if it holds for $i \geqslant 1$, then using Lemma 5 and looking at the cases n odd and n even,

$$r_n = R_n - R_{\lfloor \frac{n}{2} \rfloor} = (-1)^{n-1}$$

for $n \geqslant 1$ (compare [15, Lemma 2]). Now recall that properties (1) and (2) in Lemma 5 are equivalent.

Remark 5. Example 3 shows that it is possible to deduce the formula

$$\sum_{n \geqslant 1} \frac{N_{1,2}(n) - N_{0,2}(n)}{2n(2n+1)} = \log \frac{4}{\pi}$$

from Theorem 3 and Lemma 4 without using Lemma 1: this yields a proof of the formula that is different from those in [15] and Example 1. Similar reasoning applies for any ultimately periodic sequence $(r_n)_{n \geqslant 1}$. In particular, it is not hard to see that the relations giving r_{2n} and r_{2n+1} in terms of the R_i's imply that the sequence $(r_n)_{n \geqslant 1}$ is periodic whenever $R_i = N_{w,2}(i)$ for some fixed w and for every $i \geqslant 1$. Hence Theorem 1 can be deduced from Theorem 3 and Lemma 4 (along with the method for decomposing series employed in Example 3), without using Lemma 1. In the same vein, the generalization in Section 4.2 can be proved using a generalization of Theorem 3 to base B together with Lemma 4.

5. Future directions

Lemma 1 is the main tool for summing series in [2] and in the present paper. It might be possible to use the lemma to obtain the base B accelerated series for Euler's constant in [15, Theorem 2], and to sum more general series with $N_{w,B}(n)$. On the other hand, it might also be possible to extend the results of [2] and the present paper, and sum series where $(N_{w,B}(n))_{n\geqslant 1}$ is replaced by a more general integer sequence $(a_n)_{n\geqslant 1}$, using the decomposition in [12] of a sequence $(a_n)_{n\geqslant 1}$ into a (possibly infinite) linear combination of block-counting sequences $(N_{w,B}(n))_{n\geqslant 1}$ (see also [3]). Of course, since this may replace a series with an infinite sum, for the method to work the new series must be summable in closed form.

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