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On M-theory fourfold vacua with higher curvature terms

Thomas W. Grimm, Tom G. Pugh *, Matthias Weißenbacher

Max Planck Institute for Physics, Föhringer Ring 6, 80805 Munich, Germany

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ABSTRACT

We study solutions to the eleven-dimensional supergravity action, including terms quartic and cubic in the Riemann curvature, that admit an eight-dimensional compact space. The internal background is found to be a conformally Kähler manifold with vanishing first Chern class. The metric solution, however, is non-Ricci-flat even when allowing for a conformal rescaling including the warp factor. This deviation is due to the possible non-harmonicity of the third Chern-form in the leading order Ricci-flat metric. We present a systematic derivation of the background solution by solving the Killing spinor conditions including higher curvature terms. These are translated into first-order differential equations for a globally defined real two-form and complex four-form on the fourfold. We comment on the supersymmetry properties of the described solutions.

1. Introduction and summary

The study of M-theory on eight-dimensional compact manifolds is of both conceptional as well as phenomenological interest. On the one hand, this compactifications allow the dynamics of three-dimensional effective theories with various amounts of supersymmetry to be investigated. On the other hand, the M-theory to F-theory limit can be used to lift the three-dimensional theories to four space–time dimensions for a certain class of eight-dimensional manifolds [1]. From a phenomenological point of view, compactifications in which the effective theory preserves only small amounts of supersymmetry are of particular interest. For example, compactifications of M-theory and F-theory preserving four supercharges allow for background fluxes that can induce a four-dimensional chiral spectrum.

The aim of this note is to study vacua of eleven-dimensional supergravity on compact eight-dimensional manifolds $\mathcal{M}_8$ including the known higher derivative terms to the action. More precisely, our starting point will include terms admitting eight derivatives and are fourth and third order in the eleven-dimensional Riemann curvature $\hat{\mathcal{R}}$, i.e. schematically of the form $\hat{\mathcal{R}}^4$ and $\hat{\mathcal{R}}^3\hat{G}^2$, where $\hat{G}$ is the field strength of the M-theory three-form. The terms fourth order in $\hat{\mathcal{R}}$ are known since the works [2–8], while recently the third order terms involving $\hat{G}$ have been analyzed in [9]. Given this action we introduce an Ansatz for the background metric and fluxes capturing corrections expanded in powers of $\alpha \propto \ell_M^3$, where ℓ_M is the eleven-dimensional Planck length. This Ansatz includes a warp-factor as well as a shift of the internal metric at order α^2 [10]. The field equations pose second order differential constraints on the shifted internal metric which we are able to solve explicitly. The internal manifold turns out to have still vanishing first Chern class, but the metric background has to be chosen to no longer be Ricci flat. At order α^2 the deviation from Ricci-flatness is measured by the warp-factor and the non-harmonic part of the third Chern form $c_3^{(0)}$ on $\mathcal{M}_8$ evaluated in the zeroth order, Ricci-flat metric.

In order to systematically find an explicit solution and analyze its supersymmetry properties we also study the eleven-dimensional supersymmetry variations. Unfortunately, these are not known to the required order to give a complete check of the preservation of three-dimensional $\mathcal{N}=2$ supersymmetry corresponding to four supercharges. It was, however, argued in [11,12] that the eleven-dimensional gravitino variations have to include certain seven-derivative couplings involving three Riemann curvature tensors. Evaluated for the background Ansatz this induces modified Killing spinor equations for a globally defined spinor on $\mathcal{M}_8$ that has to exist in order to have a supersymmetric solution. We show that the integrability condition on these Killing spinor equations yields the modified Einstein equations at order α^2. Furthermore, we use the globally defined spinor to introduce a globally defined real two-form J and complex four-form Ω. The Killing spinor equations translate into first order differential constraints

* Corresponding author.
E-mail addresses: grimm@mpp.mpg.de (T.W. Grimm), pught@mpp.mpg.de (T.G. Pugh), mweisse@mpp.mpg.de (M. Weißenbacher).

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on these forms, which imply that the metric is (conformally) Kähler. In fact, this formulation allows us to give a simple derivation of the α^2 correction to the internal metric found by solving the Einstein equations. Our results can also be reformulated in terms of torsion classes on an $SU(4)$ structure manifold. We find that, upon separating the conformal rescaling of the internal metric, only the torsion form $\mathcal{W}_5$ in $d\Omega = \overline{\mathcal{W}}_5 \wedge \Omega$ is non-vanishing but exact. At the two-derivative level eleven-dimensional supergravity on $SU(4)$ structure manifolds has recently been studied in [13].

It should be stressed that the first part of our analysis closely parallels the seminal papers [14,10]. In particular, the derivation of the equations of motion satisfied by the background is in accordance with [10]. We are, in addition, able to explicitly solve these conditions and give a geometric interpretation of the result. The fact that the metric is no longer Ricci flat when higher derivative couplings and α'-corrections are taken into account is a classical result for Calabi–Yau manifolds without background fluxes in string theory [15] and has been recently investigated for *Spin*(7) and G_2 compactifications [16]. It is gratifying to observe that this result indeed carries over to warped Calabi–Yau fourfold compactifications with fluxes of eleven-dimensional supergravity. To fully check supersymmetry, however, it would be interesting to show that the proposed gravitino variation is complete. Furthermore, it is still an open problem to derive the three-dimensional effective action including fluctuations around the presented background. If the resulting three-dimensional action carries the properties of an $\mathcal{N}=2$ supergravity theory, this would give a further test for the supersymmetry of this background. We hope to present the derivation of the effective action in a forthcoming publication [17] extending the results of [18–20].

The paper is organized as follows. In Section 2 we present the Ansatz for the metric and the background fluxes and give the equations satisfied by the appearing functions. We then solve the internal Einstein equations finding corrections to the metric. The gravitino variations are analyzed in Section 3. We derive the modified Killing spinor equations and translate the conditions into first order differential equations for J, Ω. We comment on the compatibility with the Einstein equations and the implications for supersymmetry. Useful identities and a summary of our conventions are supplemented in Appendix A.

2. Warped background solutions to eleven-dimensional supergravity

In the following we will determine a bosonic solution to eleven-dimensional Einstein equations in the presence of higher curvature corrections and background fluxes. We will explicitly solve the Einstein equations finding a correction to the internal Calabi–Yau metric. Supersymmetry properties of this solution will be discussed in Section 3.

2.1. The eleven-dimensional action

Recall that the bosonic spectrum of eleven-dimensional $\mathcal{N}=1$ supergravity consists only of the metric $\hat{g}_{MN}$ and a three-form $\hat{C}$. We denote the field strength of $\hat{C}$ by $\hat{G}=d\hat{C}$ and note that the hats on the symbols indicate that we are dealing with eleven-dimensional fields, with indices raised and lowered with $\hat{g}_{MN}$.

The dynamics of the fields is determined by the bosonic part of the $\mathcal{N}=1$ supergravity action given by

$$S^{(11)} = S_{\text{class}} + \alpha^2 S_{\hat{R}^4} + \alpha^2 S_{\hat{R}^3\hat{G}^2} + \alpha^2 S_{\hat{R}^2(\nabla\hat{G})^2} + \dots \,. \tag{2.1}$$

Here we have introduced the expansion parameter α given by

$$\alpha^2 = \frac{(4\pi\kappa_{11}^2)^{\frac{2}{3}}}{(2\pi)^4 3^2 2^{13}} \,, \tag{2.2}$$

which is proportional to sixth power of the eleven-dimensional Planck length. For the following analysis the relevant terms in (2.1) are, firstly, the classical two-derivative action [21]

$$S_{\text{class}} = \frac{1}{2\kappa_{11}^2}\int \hat{R}\hat{*}1 - \frac{1}{2}\hat{G}\wedge\hat{*}\hat{G} - \frac{1}{6}\hat{C}\wedge\hat{G}\wedge\hat{G} \,, \tag{2.3}$$

where $\hat{R}$ is the Ricci scalar. Secondly, $S_{\hat{R}^4}$ denotes the terms quartic in the Riemann curvature and given by [2–8]

$$S_{\hat{R}^4} = \frac{1}{2\kappa_{11}^2}\int (\hat{t}_8\hat{t}_8 - \frac{1}{24}\hat{\epsilon}_{11}\hat{\epsilon}_{11})\hat{R}^4\hat{*}1 + 3^2 2^{13}\hat{C}\wedge\hat{X}_8 \,. \tag{2.4}$$

The explicit form of the various terms in (2.4) is given in Appendix A. It is believed that these are all terms quartic in the Riemann tensor at this order in α. The terms at higher order in $\hat{G}$ and α, such as $S_{\hat{R}^3\hat{G}^2}$ and $S_{\hat{R}^2(\nabla\hat{G})^2}$, will not be needed in what follows as their contribution is higher order in α when evaluated on the ansatz we will make. In particular we will see that the ansatz for $\hat{G}$ contains only $\mathcal{O}(\alpha)$ and higher pieces, which put these terms beyond the order we will study.

2.2. Ansatz for the vacuum solution

We now consider solutions for which the internal space is a compact eight-dimensional manifold $\mathcal{M}_8$ and the external space is $\mathbb{R}^{2,1}$. At lowest order in α the solution takes the form

$$\begin{aligned} d\hat{s}^2 &= \hat{g}_{MN}dx^M dx^N = \eta_{\mu\nu}dx^\mu dx^\nu + g^{(0)}_{mn}dy^m dy^n + \mathcal{O}(\alpha) \,, \\ \hat{G} &= 0 + \mathcal{O}(\alpha) \,, \end{aligned} \tag{2.5}$$

where $\mu = 0,\dots,2$ and $m=1,\dots,8$. The Einstein equations imply Ricci-flatness of the internal space $R^{(0)}_{mn}=0$. In fact, together with the supersymmetry conditions requiring the preservation of four supercharges, one infers that the internal manifold is Calabi–Yau and thus admits a nowhere vanishing Kähler form $J^{(0)}_{mn}$ and a holomorphic $(4,0)$-form $\Omega^{(0)}_{mnrs}$ that are harmonic.

Having deduced this lowest order solution we can then work to second order in α by considering the field equations of the α corrected action. To solve the corrected Einstein equations we make an Ansatz for the metric[1]

$$\begin{aligned} d\hat{s}^2 = {} & e^{\alpha^2\Phi^{(2)}}(e^{-2\alpha^2 W^{(2)}}\eta_{\mu\nu}dx^\mu dx^\nu + e^{\alpha^2 W^{(2)}}g_{mn}dy^m dy^n) \\ & + \mathcal{O}(\alpha^3), \end{aligned} \tag{2.6}$$

where

$$g_{mn} = g^{(0)}_{mn} + \alpha^2 g^{(2)}_{mn} + \mathcal{O}(\alpha^3) \,. \tag{2.7}$$

Here $\Phi^{(2)}$, $W^{(2)}$, $g^{(0)}_{mn}$ and $g^{(2)}_{mn}$ depend only on the internal coordinates y^m in the background. The function $\Phi^{(2)}$ is an overall Weyl rescaling that we will discuss in more detail below, while $W^{(2)}$ is known as the warp-factor. At this order in α a background four-form field strength must also be included. Following [10] we make the Ansatz

$$\begin{aligned} \hat{G}_{mnrs} &= \alpha G^{(1)}_{mnrs} + \mathcal{O}(\alpha^3) \,, \\ \hat{G}_{\mu\nu\rho m} &= \epsilon_{\mu\nu\rho}\partial_m e^{-3\alpha^2 W^{(2)}} + \mathcal{O}(\alpha^3) \,, \end{aligned} \tag{2.8}$$

[1] Note that an alternative ansatz with AdS external space can also be analyzed. However, this is not compatible with the lowest order supersymmetry conditions on the flux combined with the second order equations of motion.

where $G^{(1)}$ is a background four-form flux on the internal manifold $\mathcal{M}_8$ that is harmonic with respect to $g^{(0)}_{mn}$. Let us note that the term linear in α appearing in $\hat{G}_{mnrs}$ has the correct mass dimensions such that the background flux $G^{(1)}_{mnrs}$ integrates to a dimensionless number. In fact $T_{\rm M2}\int_{C_4}\hat{G}$ has to be dimensionless and the inverse M2-brane tension $T^{-1}_{\rm M2}$ is proportional to α. We do not include an α^2 term in the Ansatz for $\hat{G}_{mnrs}$, since it can be shown to either decouple or to give contributions at only $\mathcal{O}(\alpha^3)$ in the following evaluations.

2.3. Equations determining the solution

The functions appearing in our ansatz may then be constrained by substituting into the eleven-dimensional equations of motion. The solution is found by expanding each of the equations of motion in powers of α and inferring the respective constraints [10].

To begin with, we note that the equations of motion of $\hat{C}$ and the eleven-dimensional Einstein equations derived from (2.1) do not decouple at first. However, combining the $\hat{C}$ equation with the external Einstein equations one infers that $G^{(1)}$ in the Ansatz (2.8) is self-dual in the Calabi–Yau background, i.e.

$$\alpha G^{(1)} = \alpha *^{(0)} G^{(1)} + \mathcal{O}(\alpha^3)\,, \tag{2.9}$$

where one uses that $\mathcal{M}_8$ is compact. By using (2.9) the second order equation of motion of $\hat{C}$ implies the warp-factor equation

$$\Delta e^{3\alpha^2 W^{(2)}} + \tfrac{1}{4!2}\alpha^2 G^{(1)}_{mnrs}G^{(1)mnrs} - \tfrac{3^2 2^{13}}{8!}\alpha^2 \epsilon^{m_1\ldots m_8}X_{8\,m_1\ldots m_8} + \mathcal{O}(\alpha^3) = 0\,, \tag{2.10}$$

where the Laplacian $\Delta = \nabla_m\nabla^m$, the X_8, and the contractions of $G^{(1)}_{mnrs}$ are evaluated using g_{mn} given in (2.7). We stress that with the above Ansatz (2.8) the corrections to the $\hat{C}$ equation of motion (2.9) and (2.10) from $S_{\hat{R}^3\hat{G}^2}$ in (2.1) give contributions at least of order α^3. At this order not all higher curvature contributions are known. Therefore, these conditions give constraints only to order α^2. This indicates consistency of our Ansatz for the warp-factor and implies that lower α powers in the solution to (2.10) would be constants. Moreover, at this order in α the metric used in (2.10) is only $g^{(0)}_{mn}$. Integrating (2.10) over the internal manifold $\mathcal{M}_8$ one infers that, in the absence of localized sources, a non-trivial background flux $\hat{G}^{(1)}_{mnrs}$ is required by consistency for a manifold with $\int_{\mathcal{M}_8} X^{(0)}_8 \neq 0$.

Next we use the Ansatz (2.6) and (2.8), along with the constraints (2.9) and (2.10), to rewrite the Einstein equations into a simple form. Firstly, we expand

$$R_{mn} \equiv R(g^{(0)}_{rs} + \alpha^2 g^{(2)}_{rs})_{mn} = R^{(0)}_{mn} + \alpha^2 R^{(2)}_{mn} \tag{2.11}$$

which defines $R^{(2)}_{mn}$. Using this abbreviation the internal part of the eleven-dimensional Einstein equations can be rewritten as

$$R^{(2)}_{mn} - \tfrac{1}{2} g^{(0)}_{mn} g^{(0)\,rs} R^{(2)}_{rs} + 768\, J^{(0)r}_m J^{(0)s}_n \nabla_r\nabla_s Z - \tfrac{9}{2}\nabla_m\nabla_n\Phi^{(2)} + \tfrac{9}{2} g^{(0)}_{mn} g^{(0)\,rs}\nabla_r\nabla_s\Phi^{(2)} = 0\,, \tag{2.12}$$

where $J^{(0)n}_m = J^{(0)}_{mp} g^{(0)pn}$ is the complex structure on the underlying Calabi–Yau manifold. The conditions (2.9) and (2.10) are used to cancel all flux dependence in (2.12) and ensure that the Einstein equations involving $\hat{R}_{m\mu}$ are automatically satisfied at the order considered. The external part of the Einstein equations takes the form

$$R^{(2)}_{mn} g^{(0)\,mn} - 9 g^{(0)\,mn}\nabla_m\nabla_n\Phi^{(2)} = 0\,. \tag{2.13}$$

The derivation of (2.12) and (2.13) is rather lengthy and requires the use of the identities summarized in Appendix A. Furthermore, we have used Ricci-flatness $R^{(0)}_{mn} = 0$ for the lowest order part of the Riemann tensor to simplify the result. In these expressions the scalar Z is proportional to the six-dimensional Euler density and is given by

$$\begin{aligned} Z &= *^{(0)}(J^{(0)} \wedge c^{(0)}_3) \\ &= \tfrac{1}{12}(R^{(0)rs}_{mn} R^{(0)tu}_{rs} R^{(0)mn}_{tu} - 2 R^{(0)r}_{m}{}_{n}{}^{s} R^{(0)t}_{r}{}_{s}{}^{u} R^{(0)m}_{t}{}_{u}{}^{n})\,, \end{aligned} \tag{2.14}$$

where $c^{(0)}_3$ the third Chern form evaluated in the metric $g^{(0)}_{mn}$ given explicitly in (A.7). Tracing the internal part of the Einstein equation and demanding compatibility with the external part then fixes

$$\begin{aligned} \Phi^{(2)} &= -\tfrac{512}{3} Z\,, \\ R^{(2)}_{mn} &= -768(J^{(0)r}_m J^{(0)s}_n \nabla_r\nabla_s Z + \nabla_m\nabla_n Z)\,. \end{aligned} \tag{2.15}$$

In other words, the solution indeed requires the presence of a non-trivial eleven-dimensional Weyl rescaling involving the higher curvature terms.

2.4. Solving the modified Einstein equation

In order to solve (2.15) we follow a technique equivalent to that shown in [15]. We begin by noting that as $c^{(0)}_3$ is real and closed but not co-closed with respect to the Kähler metric $g^{(0)}_{mn}$. This means that it may be expanded as

$$c^{(0)}_3 = H c^{(0)}_3 + i\partial^{(0)}\bar{\partial}^{(0)} F \tag{2.16}$$

where H indicates the projection to the harmonic part with respect to the metric $g^{(0)}_{mn}$. This equation defines a co-closed $(2,2)$-form F that will be key to the following discussions.[2] Then by using (2.14) we see that

$$Z = *^{(0)}(J^{(0)} \wedge H c^{(0)}_3) + \tfrac{1}{4}\Delta^{(0)} *^{(0)} (J^{(0)} \wedge J^{(0)} \wedge F) \tag{2.17}$$

where $*^{(0)}(J^{(0)} \wedge H c^{(0)}_3)$ is constant over the internal space as a result of the harmonic projection. We are now in the position to use these quantities to solve (2.15) for a metric correction at order α^2. The explicit solution is given by

$$g^{(2)}_{mn} = 384(J^{(0)r}_m J^{(0)s}_n \nabla^{(0)}_r\nabla^{(0)}_s + \nabla^{(0)}_m\nabla^{(0)}_n) *^{(0)} (J^{(0)} \wedge J^{(0)} \wedge F)\,, \tag{2.18}$$

where F is the four-form introduced in (2.16). Clearly, one can now explicitly check that (2.18) solves (2.15).[3] In the next section we will show by introducing globally defined forms on $\mathcal{M}_8$ how one is naturally lead to the solution (2.18).

3. Killing spinor equations and globally defined forms

In this section we comment on the supersymmetry properties of the solution introduced in Section 2. This is a challenging task, since the supersymmetry variations are not fully known at the desired order α^2. Following a strategy used in [11,12] we will be

[2] The harmonicity of Chern forms has been also discussed in the mathematical literature and lead to the introduction of the Bando–Futaki character [22], which is however trivially vanishing in the Calabi–Yau case.

[3] Recently, it was pointed out in [20] that a redefinition of the metric background $g_{mn} = g^{(0)}_{mn} - 768\alpha^2 J^{(0)r}_m (*^{(0)} c^{(0)}_3)_{rn}$ trivializes the kinetic terms for the vectors obtained from $\hat{G}$ in the three-dimensional effective action. This interesting observation, however, has to be contrasted with the fact that this shift is not a solution to the Einstein equations at order α^2.

able to extract at least partial information about the supersymmetry properties by studying the Killing spinor equations at order α^2. Furthermore, we will then translate these equations into differential conditions on the globally defined forms J and Ω on $\mathcal{M}_8$. This will lead to a stepwise derivation of the correction (2.18).

To set the stage of our study, let us note that we assert that at quadratic order in α the eleven-dimensional gravitino variation is given by

$$\begin{aligned}\delta\hat{\psi}_M = {} & \hat{\nabla}_M\hat{\epsilon} - \tfrac{1}{288}\hat{G}_{NRST}\hat{\Gamma}_M{}^{NRST}\hat{\epsilon} + \tfrac{1}{36}\hat{G}_{MNRS}\hat{\Gamma}^{NRS}\hat{\epsilon} \\ & + \tfrac{128}{3}\alpha^2\hat{\nabla}_N\hat{Z}\hat{\Gamma}_M{}^N\hat{\epsilon} \\ & - 48\alpha^2\hat{\nabla}^N\hat{R}_{MRN_1N_2}\hat{R}_{NSN_3N_4}\hat{R}^{RS}{}_{N_5N_6}\hat{\Gamma}^{N_1\ldots N_6}\hat{\epsilon} \\ & + \mathcal{O}(\alpha^2)\,, \end{aligned} \tag{3.1}$$

where the remaining order α^2 terms vanish on the backgrounds we consider. Here $\hat{Z}$ is proportional to the six-dimensional Euler density in eleven dimensions and is given by

$$\hat{Z} = \tfrac{1}{12}(\hat{R}_{MN}{}^{RS}\hat{R}_{RS}{}^{TU}\hat{R}_{TU}{}^{MN} - 2\hat{R}_M{}^R{}_N{}^S\hat{R}_R{}^T{}_S{}^U\hat{R}_T{}^M{}_U{}^N)\,. \tag{3.2}$$

This form of the gravitino variation is compatible with the terms that are necessary in [11,12]. In other words, we will see below that the Killing spinor equations derived from (3.1) are compatible with the Einstein equations up to order α^2. Remarkably, the terms in (3.1) also appear in the gravitino variations deduced by eleven-dimensional Noether coupling in [23].

3.1. Dimensional reduction of the supergravity variations

We next dimensionally reduce the supersymmetry variations (3.1) on the background introduced in Section 2. To begin with, we decompose the eleven-dimensional supersymmetry parameter and gamma matrices in a way that is compatible with our Ansatz as

$$\begin{aligned}&\hat{\epsilon} = e^{-\frac{1}{2}\alpha^2 W^{(2)}}\epsilon\otimes\eta\,, \qquad \hat{\Gamma}_\mu = e^{\frac{1}{2}\alpha^2\Phi^{(2)}-\alpha^2 W^{(2)}}\gamma_\mu\otimes\gamma^9\,, \\ &\hat{\Gamma}_m = e^{\frac{1}{2}\alpha^2\Phi^{(2)}+\frac{1}{2}\alpha^2 W^{(2)}}\mathbb{1}\otimes\gamma_m\,,\end{aligned} \tag{3.3}$$

where ϵ is a spinor in the three-dimensional external space and η is a no-where vanishing spinor on $\mathcal{M}_8$. The spinor η is chosen to satisfy $\gamma^9\eta = \eta$, $\eta^\dagger\eta = 1$ and $\eta^T\eta = 0$.

Substituting this decomposition along with the reduction ansatz (2.6) and (2.8) into (3.1) we find for the internal gravitino variation

$$\begin{aligned}\delta\hat{\psi}_m = {} & e^{-\frac{1}{2}\alpha^2 W^{(2)}}\epsilon\otimes\nabla_m\eta - \tfrac{1}{288}\alpha G^{(1)}_{nrst}\epsilon\otimes\gamma_m{}^{nrst}\eta \\ & + \tfrac{1}{36}\alpha G^{(1)}_{mnpq}\epsilon\otimes\gamma^{npq}\eta \\ & - 48\alpha^2\nabla^n R_{mrm_1m_2}R_{nsm_3m_4}R^{rs}{}_{m_5m_6}\epsilon\otimes\gamma^{m_1\ldots m_6}\eta \\ & + \tfrac{128}{3}\alpha^2\nabla_n Z\epsilon\otimes\gamma_m{}^n\eta + \tfrac{1}{4}\alpha^2\nabla_n\Phi^{(2)}\epsilon\otimes\gamma_m{}^n\eta \\ & + \mathcal{O}(\alpha^3) = 0\,,\end{aligned} \tag{3.4}$$

and for the external gravitino variation

$$\begin{aligned}\delta\hat{\psi}_\mu = {} & e^{-\frac{1}{2}\alpha^2 W^{(2)}}\nabla_\mu\epsilon\otimes\eta - \alpha\tfrac{1}{288}G^{(1)}_{mnpq}\gamma_\mu\epsilon\otimes\gamma^{mnpq}\eta \\ & - \tfrac{128}{3}\alpha^2\nabla_n Z\gamma_\mu\epsilon\otimes\gamma^n\eta - \tfrac{1}{4}\alpha^2\nabla_n\Phi^{(2)}\gamma_\mu\epsilon\otimes\gamma^n\eta \\ & + \mathcal{O}(\alpha^3) = 0\,.\end{aligned} \tag{3.5}$$

These equations can then be satisfied if at lowest order in α the background is Calabi–Yau, as already noted at the beginning of Section 2.2, and one has $\nabla_\mu\epsilon = 0$. At linear order in α one finds the condition

$$G^{(1)}_{mnrs}\gamma^{nrs}\eta = 0 \tag{3.6}$$

Finally, at second order in α one finds that (2.15) has to be satisfied and η obeys the Killing spinor equation

$$\nabla_m\eta = -384\alpha^2 J^{(0)}{}_m{}^n\nabla_n Z_{rs}\gamma^{rs}\eta + \mathcal{O}(\alpha^3)\,, \qquad Z_{rs} = \tfrac{1}{2}(*c_3^{(0)})_{rs} \tag{3.7}$$

where $J^{(0)rs}Z_{rs} = Z$.

3.2. Differential conditions on the globally defined forms

Using the spinor η one can introduce a globally defined nowhere vanishing real two-form J and a complex four-form Ω. This is a familiar strategy for manifolds with reduced structure group. The case of having $SU(4)$ structure was discussed in [13,24]. Concretely, we use η to construct the forms

$$J_{mn} = i\eta^\dagger\gamma_{mn}\eta\,, \qquad \Omega_{mnrs} = \eta^T\gamma_{mnrs}\eta\,. \tag{3.8}$$

By using Fierz identities we see that these forms satisfy

$$J\wedge\Omega = 0\,, \qquad J\wedge J\wedge J\wedge J = \tfrac{3}{2}\Omega\wedge\bar{\Omega}\,. \tag{3.9}$$

The Kähler form $J^{(0)}_{mn}$ corresponding to the Ricci flat metric $g^{(0)}_{mn}$ is then the lowest order part of J_{mn}.

We can now rewrite the supersymmetry conditions (3.6) and (3.7) using J and Ω. The constraint on the flux (3.6) implies that

$$G^{(1)}\wedge J^{(0)} = 0\,, \qquad G^{(1)} \text{ is of type } (2,2) \text{ in } J^{(0)\,n}_m \tag{3.10}$$

where $J^{(0)\,n}_m$ is the complex structure of the underlying Calabi–Yau fourfold. Furthermore, the Killing spinor equation (3.7) satisfied by η translates to the differential conditions

$$\begin{aligned}&\nabla_m J_{nr} = 0 + \mathcal{O}(\alpha^3)\,, \\ &\nabla_m\Omega_{nrst} = 6144\alpha^2 J^{(0)\,p}_m\nabla^{(0)}_p Z_{[n}{}^q\Omega^{(0)}_{rst]q} + \mathcal{O}(\alpha^3)\end{aligned} \tag{3.11}$$

Antisymmetrizing in the indices then gives

$$dJ = 0 + \mathcal{O}(\alpha^3)\,, \qquad d\Omega = -768\alpha^2 dZ\wedge\Omega^{(0)} + \mathcal{O}(\alpha^3)\,. \tag{3.12}$$

We can thus infer that the metric g_{mn} including α^2 corrections is still Kähler. In fact, the higher curvature terms only amount to introducing the non-closedness of Ω with a result proportional to Ω itself. In fact, translated into torsion forms for an $SU(4)$-structure manifold (see, for example, [13,24]), the only non-trivial torsion form is $\overline{\mathcal{W}}_5 = -768\alpha^2\bar{\partial}^{(0)}Z$, which is exact.

Let us stress that the derivation of the Killing spinor equation makes use of the full internal space metric $\hat{g}_{MN}$. However, the overall Weyl rescaling and warp-factor terms precisely cancel and the resulting equation (3.7) depends only on the metric g_{mn} appearing in (2.6). The J and Ω appearing (3.12) are thus related to the metric g_{mn}. Clearly one could introduce an alternative $\tilde{J}$ and $\tilde{\Omega}$ related to rescaled metric $\hat{g}_{mn}$. This would induce new terms proportional to $\tilde{J}$ in $d\tilde{J}$ and $\tilde{\Omega}$ in $d\tilde{\Omega}$ will then be induced, since the gamma-matrices in (3.8) are rescaled.

We can now use the condition that g_{mn} is a Kähler metric and study the integrability condition of (3.7). Here the commutator $[\nabla_m, \nabla_n]\eta = \frac{1}{4}R_{mnrs}\gamma^{rs}\eta$ can be compared with the result obtained form (3.7). This simply results in the condition

$$\frac{1}{4}R_{mnrs}\gamma^{rs}\eta - 768\alpha^2 J^{(0)}{}_{[m}{}^r\nabla^{(0)}_{n]}\nabla^{(0)}_r Z_{pq}\gamma^{pq}\eta + \mathcal{O}(\alpha^3) = 0\,. \tag{3.13}$$

Contracting with $\eta^\dagger$ we see that this implies

$$\tfrac{1}{4}R_{mnrs}J^{rs} - 768\alpha^2 J^{(0)}{}_{[m}{}^r\nabla^{(0)}_{n]}\nabla^{(0)}_r Z + \mathcal{O}(\alpha^3) = 0\,. \tag{3.14}$$

As we know that $R_{mnrs}J^{rs} = 2R_{mrns}J^{rs}$ by the first Bianchi identity and that for a Kähler manifold $J_m{}^p R_{pnrs} = J_n{}^p R_{mprs}$ we then see that (3.14) implies $R^{(0)}_{mn} = 0$ at zeroth α order and the Einstein equations (2.15) at order α^2.

3.3. Solving the equations for J and Ω

We now wish to solve Eqs. (3.12) subject to the algebraic constraints (3.9). To do this we begin by expanding these equations in α to find

$$dJ^{(2)} = 0\,, \qquad d\Omega^{(2)} = -768 dZ \wedge \Omega^{(0)}\,. \tag{3.15}$$

We may solve the constraint on $\Omega^{(2)}$ by letting

$$\Omega^{(2)} = \phi\Omega^{(0)} + \rho\,, \quad \text{where} \quad d\phi = -768 dZ\,, \quad d\rho = 0\,. \tag{3.16}$$

The (4, 0) part of ρ can be absorbed into $\phi\Omega^{(0)}$ so we may assume that $\rho \wedge \bar{\Omega}^{(0)} = 0$. Similarly as $J^{(2)}$ is a real d-closed 2-form on a Kahler manifold

$$J^{(2)} = \sigma + i\partial^{(0)}\bar{\partial}^{(0)}\psi\,, \quad \text{where} \quad d\sigma = d^{(0)\dagger}\sigma = 0\,. \tag{3.17}$$

Then considering the expansion of (3.9) we see that

$$4J^{(2)} \wedge J^{(0)} \wedge J^{(0)} \wedge J^{(0)} = \frac{3}{2}(\Omega^{(2)} \wedge \bar{\Omega}^{(0)} + \Omega^{(0)} \wedge \bar{\Omega}^{(2)})\,, \tag{3.18}$$

and substituting (3.16) and (3.17) into (3.18) we find

$$\frac{1}{3} * (\sigma \wedge J^{(0)} \wedge J^{(0)} \wedge J^{(0)}) - \Delta^{(0)}\psi = 2(\phi + \bar{\phi})\,, \tag{3.19}$$

which implies that $d\Delta^{(0)}\psi = 3072 dZ$. Considering this along with (3.16) and using the expansion of Z given by (2.17) we see that we are lead to a solution for $J^{(2)}$ and $\Omega^{(2)}$ where

$$\begin{aligned} J^{(2)} &= i786\, \partial^{(0)}\bar{\partial}^{(0)} *^{(0)} (F \wedge J^{(0)} \wedge J^{(0)})\,, \\ \Omega^{(2)} &= -192\, \Delta^{(0)} *^{(0)} (F \wedge J^{(0)} \wedge J^{(0)})\Omega^{(0)}\,. \end{aligned} \tag{3.20}$$

This shows that the internal space Kähler potential is shifted by a term proportional to F. The remaining forms ρ and σ correspond to moduli which will be studied in [17]. Expanding the relationship

$$g_{mn} = \tfrac{i}{48}\Omega_{(m|rpt}\bar{\Omega}_{|n)squ} J^{rs} J^{pq} J^{tu}\,, \tag{3.21}$$

which may be demonstrated by using the results of Appendix A, we find

$$\begin{aligned} g^{(2)}_{mn} = &-J^{(0)r}_{(m} J^{(2)}_{n)r} + \frac{1}{2} J^{(0)rs} J^{(2)}_{rs} g^{(0)}_{mn} - \frac{1}{48}\Omega^{(2)}_{(m|rst}\bar{\Omega}^{(0)rst}_{|n)} \\ &- \frac{1}{48}\bar{\Omega}^{(2)}_{(m|rst}\Omega^{(0)rst}_{|n)}\,, \end{aligned} \tag{3.22}$$

and using this we see that the correction to J and Ω implies the metric correction (2.18) that solves (2.15).

The analysis presented here shows that the first order equations (3.12) on J and Ω, which are derived from the Killing spinor equations (3.7) are economically solved by (3.20). This then provides a solution to the second order equations (2.15) arising from the internal space Einstein equations. While we have no complete proof of the supersymmetry of this solution this result provides a necessary condition. Furthermore, as we expect that the lowest order supersymmetry carries over to the higher order analysis and we have made a general analysis of the corrections to the eleven-dimensional field equations, it seems natural to expect that further corrections to the gravitino variation (3.1) vanish in the background presented. It would be interesting to continue to develop the Noether coupling analysis of [23] to find the complete expression for the gravitino variation at order α^2.

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Appendix A. Conventions, definitions and identities

We denote the total eleven-dimensional space indices by capital Latin letters $M, N, R, S, \dots$, the external ones by $\mu, \nu = 0, 1, 2$ and real indices of the internal space by $m, n, r, s = 1, \dots, 8$. Quantities for which the indices are raised and lower with the total space metric carry a hat e.g. $\hat{G}$, $\hat{R}$. Furthermore, the convention for the totally antisymmetric tensor in Lorentzian space in an orthonormal frame is $\epsilon_{012\dots10} = \epsilon_{012} = +1$. The epsilon tensor in d dimensions then satisfies

$$\begin{aligned} &\epsilon^{R_1\cdots R_p N_1\dots N_{d-p}}\epsilon_{R_1\dots R_p M_1\dots M_{d-p}} \\ &\quad = (-1)^s (d-p)! p! \delta^{N_1}{}_{[M_1}\dots\delta^{N_{d-p}}{}_{M_{d-p}]}\,, \end{aligned} \tag{A.1}$$

where $s = 0$ if the metric has Riemannian signature and $s = 1$ for a Lorentzian metric.

We adopt the following conventions for the Riemann tensor of the internal space

$$\begin{aligned} &\Gamma^r{}_{mn} = \frac{1}{2} g^{rs}(\partial_m g_{ns} + \partial_n g_{ms} - \partial_s g_{mn})\,, \qquad R_{mn} = R^r{}_{mrn}\,, \\ &R^m{}_{nrs} = \partial_r \Gamma^m{}_{sn} - \partial_s \Gamma^m{}_{rn} + \Gamma^m{}_{rt}\Gamma^t{}_{sn} - \Gamma^m{}_{st}\Gamma^t{}_{rn}\,, \\ &R = R_{mn} g^{mn}\,, \end{aligned} \tag{A.2}$$

with equivalent definitions for the Riemann tensor on the total and external spaces. Perturbing the internal metric by $g_{mn} = g^{(0)}_{mn} + \alpha^2 g^{(2)}_{mn}$ the correction to the internal Ricci tensor at $\mathcal{O}(\alpha^2)$ is then given by

$$\begin{aligned} R^{(2)}_{mn} = &\alpha^2 \nabla^{(0)}_r \nabla^{(0)}_{(m} g^{(2)r}_{n)} - \alpha^2 \frac{1}{2}\nabla^{(0)r}\nabla^{(0)}_r g^{(2)}_{mn} \\ &- \alpha^2 \frac{1}{2}\nabla^{(0)}_m \nabla^{(0)}_n g^{(2)r}_r\,. \end{aligned} \tag{A.3}$$

The scalar functions $\hat{t}_8\hat{t}_8\hat{R}^4$ and $\hat{\epsilon}_{11}\hat{\epsilon}_{11}\hat{R}^4$ are given by

$$\begin{aligned} \hat{\epsilon}_{11}\hat{\epsilon}_{11}\hat{R}^4 = &\epsilon_{R_1R_2R_3N_1\dots N_8}\epsilon^{R_1R_2R_3M_1\dots M_8}\hat{R}^{N_1N_2}{}_{M_1M_2}\hat{R}^{N_3N_4}{}_{M_3M_4} \\ &\times \hat{R}^{N_5N_6}{}_{M_5M_6}\hat{R}^{N_7N_8}{}_{M_7M_8}\,, \\ \hat{t}_8\hat{t}_8\hat{R}^4 = &\hat{t}_{8N_1\dots N_8}\hat{t}_8{}^{R_3M_1\dots M_8}\hat{R}^{N_1N_2}{}_{M_1M_2}\hat{R}^{N_3N_4}{}_{M_3M_4}\hat{R}^{N_5N_6}{}_{M_5M_6} \\ &\times \hat{R}^{N_7N_8}{}_{M_7M_8}\,, \end{aligned} \tag{A.4}$$

with

$$\begin{aligned} \hat{t}_8^{N_1\dots N_8} = \frac{1}{16}\Big(&-2\Big(\hat{g}^{N_1N_3}\hat{g}^{N_2N_4}\hat{g}^{N_5N_7}\hat{g}^{N_6N_8} \\ &+ \hat{g}^{N_1N_5}\hat{g}^{N_2N_6}\hat{g}^{N_3N_7}\hat{g}^{N_4N_8} + \hat{g}^{N_1N_7}\hat{g}^{N_2N_8}\hat{g}^{N_3N_5}\hat{g}^{N_4N_6}\Big) \\ &+ 8\Big(\hat{g}^{N_2N_3}\hat{g}^{N_4N_5}\hat{g}^{N_6N_7}\hat{g}^{N_8N_1} \\ &+ \hat{g}^{N_2N_5}\hat{g}^{N_6N_3}\hat{g}^{N_4N_7}\hat{g}^{N_8N_1} + \hat{g}^{N_2N_5}\hat{g}^{N_6N_7}\hat{g}^{N_8N_3}\hat{g}^{N_4N_1}\Big) \\ &- (N_1 \leftrightarrow N_2) - (N_3 \leftrightarrow N_4) - (N_5 \leftrightarrow N_6) \\ &- (N_7 \leftrightarrow N_8)\Big)\,. \end{aligned} \tag{A.5}$$

While the 8-form X_8 is given by

$$X_8 = \frac{1}{192}\left[\mathrm{Tr}(\hat{\mathcal{R}}^4) - \frac{1}{4}\left(\mathrm{Tr}(\hat{\mathcal{R}}^2)\right)^2\right]\,, \tag{A.6}$$

where $\hat{\mathcal{R}}^M{}_N = \frac{1}{2}\hat{R}^M{}_{NRS}dx^R \wedge dx^S$ and the 3rd Chern form on the internal space may be expressed as

$$c_3 = -\frac{1}{48} R_{m_1 m_2 n_1 n_2} R_{m_3 m_4 n_3 n_4} R_{m_5 m_6 n_5 n_7} J^{n_2 n_3} J^{n_4 n_5} \times J^{n_6 n_1} dx^{m_1} \wedge \ldots \wedge dx^{m_6} .$$

From the spinor bilinear J we may form the projectors

$$\Pi^{\pm}{}_m{}^n = \frac{1}{2}\left(\delta_m{}^n \mp i J_m{}^n\right) , \quad \text{where} \quad \Pi^-{}_m{}^i \Omega_{inrs} = \Omega_{mnrs} , \qquad \Pi^+{}_m{}^i \Omega_{inrs} = 0 , \tag{A.7}$$

which satisfy

$$\Omega_{mnrs}\bar{\Omega}^{tuvw} = 4!\,2^4 \Pi^-{}_{[m}{}^t \Pi^-{}_n{}^u \Pi^-{}_r{}^v \Pi^-{}_{s]}{}^w \tag{A.8}$$

as may be shown by using Fierz identities [25]. Using these techniques we can also show that the remaining spinor bilinears on the internal space can be written as

$$\eta^\dagger \gamma_{mnrs}\eta = -3 J_{[mn}J_{rs]} , \qquad \eta^\dagger \gamma_{mnrstu}\eta = 15 i J_{[mn}J_{rs}J_{tu]} ,$$
$$\eta^\dagger \gamma_{mnrstuvw}\eta = 105 J_{[mn}J_{rs}J_{tu}J_{vw]} ,$$
$$\eta^T \gamma_{p_1 \ldots p_d}\eta = 0 \quad \text{where } d \neq 4 ,$$
$$\eta^\dagger \gamma_{p_1 \ldots p_d}\eta = 0 \quad \text{where } d = \text{odd} . \tag{A.9}$$

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