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Bifurcation analysis for a regulated logistic growth model ☆

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Abstract

In this paper, we consider a regulated logistic growth model. We first consider the linear stability and the existence of a Hopf bifurcation. We show that Hopf bifurcations occur as the delay τ passes through critical values. Then, using the normal form theory and center manifold reduction, we derive the explicit algorithm determining the direction of Hopf bifurcations and the stability of the bifurcating periodic solutions. Finally, numerical simulation results are given to support the theoretical predictions.

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1. Introduction

It is well known that the delayed logistic differential equation

$$\dot{n}(t) = rn(t)\left(1 - \frac{n(t-\tau)}{K}\right) \tag{1.1}$$

is used to model the evolution of a single species $n(t)$. Often, the use of this model is not so much that it has any real microscopic justification. For instance, in some situations, one may need to adjust the size of the positive equilibrium (see, e.g., [1]). For this purpose, Gopalsamy et al. [2,3] first put forward a mechanism of "feedback regulation" to Eq. (1.1) by considering the following control system:

$$\begin{cases} \dot{n}(t) = rn(t)\left[1 - \dfrac{n(t-\tau)}{K} - cu(t)\right], \\ \dot{u}(t) = -au(t) + bn(t-\tau), \end{cases} \tag{1.2}$$

where function u is regarded as a "feedback control" variable, K, r, a, b, $\tau \in (0,\infty)$.

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Recently, there have been extensive literatures dealing with system (1.2) or systems similar to (1.2), regarding attractivity, global stabilities of positive equilibrium and other dynamics (see, e.g., [2–8] and references therein). For a long time, it has been recognized that delays can have very complicated impact on the dynamics of a system (see, e.g., [2,4,9]). For example, delays can cause the loss of stability and can induce various oscillations and periodic solutions. In the present paper, we are interested in the effect of delay on dynamics of Eq. (1.2). Taking the delay τ as a parameter, we show that a Hopf bifurcation will occurs as the delay τ passes through a critical value, i.e., a family of periodic solutions will be bifurcated from the positive equilibrium.

The rest of this paper is organized as follows. In the next section, we consider the stability and Hopf bifurcations. In Section 3, the stability of the bifurcating periodic solutions and the direction of the Hopf bifurcation at the critical values of τ are determined by using the normal form method and the center manifold reduction due to Hassard et al. [10]. Finally, a numerical example is given.

2. Stability of the positive equilibrium and existence of Hopf bifurcations

It is easy to verify that (1.2) has a unique positive equilibrium $E_* = (n_*, u_*)$, where

$$n^* = \frac{aK}{a + Kbc}, \qquad u^* = \frac{bK}{a + Kbc}.$$

We will begin by investigating its linearized stability. For the simplicity of notations, we define $x(t)$, $y(t)$, μ, α, β, γ as follows:

$$x(t) = \frac{n(t)}{n_*} - 1, \qquad y(t) = \frac{u(t)}{u_*} - 1, \tag{2.1}$$

$$\gamma = \frac{bcK}{a}, \qquad \alpha = \frac{r}{1+\gamma}, \qquad \beta = a. \tag{2.2}$$

Then (1.2) are transformed into

$$\begin{cases} \dot{x}(t) = -\alpha x(t-\tau) - \alpha\gamma y(t) - \alpha x(t)x(t-\tau) - \alpha\gamma x(t)y(t), \\ \dot{y}(t) = \beta x(t-\tau) - \beta y(t). \end{cases} \tag{2.3}$$

Then, the origin $(0,0)$ is a fixed point of Eq. (2.3), and the linearization of (2.3) about it is

$$\begin{cases} \dot{x}(t) = -\alpha x(t-\tau) - \alpha\gamma y(t), \\ \dot{y}(t) = \beta x(t-\tau) - \beta y(t). \end{cases} \tag{2.4}$$

The characteristic equation resulting from (2.4) is

$$\lambda^2 + \beta\lambda + \alpha[\lambda + \beta(1+\gamma)]\mathrm{e}^{-\lambda\tau} = 0. \tag{2.5}$$

It is well known that the zero steady state of system (2.4) is asymptotically stable if all roots of Eq. (2.5) have negative real parts, and unstable if Eq. (2.5) has a root with positive real part. In the sequel, we shall investigate the distribution of the roots of Eq. (2.5).

If $\mathrm{i}\omega$ ($\omega > 0$) is a root of Eq. (2.5), then

$$-\omega^2 + \beta\omega\mathrm{i} + \alpha[\lambda + \beta(1+\gamma)]\mathrm{e}^{-\mathrm{i}\omega\tau} = 0.$$

Separating the real and imaginary parts, we obtain

$$\begin{cases} \omega^2 = \alpha\beta(1+\gamma)\cos\omega\tau + \alpha\omega\sin\omega\tau, \\ \beta\omega = \alpha\beta(1+\gamma)\sin\omega\tau - \alpha\omega\cos\omega\tau, \end{cases} \tag{2.6}$$

from which we have

$$\omega^4 + (\beta^2 - \alpha^2)\omega^2 - \alpha^2\beta^2(1+\gamma)^2 = 0. \tag{2.7}$$

Clearly, Eq. (2.7) has only one positive root ω_0 defined by

$$\omega_0 = \left[\frac{1}{2}\left(\alpha^2 - \beta^2 + \sqrt{(\alpha^2+\beta^2)^2 + 4\gamma\alpha^2\beta^2(2+\gamma)}\right)\right]^{\frac{1}{2}}. \tag{2.8}$$

Define

$$\tau_j = \frac{1}{\omega_0}\left(\arcsin\frac{\omega_0(\omega_0^2+\beta^2(1+\gamma))}{\alpha(\omega_0^2+\beta^2(1+\gamma)^2)} + 2j\pi\right), \quad j = 0,1,2,\ldots. \tag{2.9}$$

Then (τ_j, ω_0) solves Eq. (2.6). This means that when $\tau = \tau_j$ Eq. (2.5) has a pair of purely imaginary roots $\pm i\omega_0$.

Now let us consider the behavior of the roots of Eq. (2.5) near τ_j. For the purpose of this, denote

$$\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$$

the root of Eq. (2.5) such that

$$\alpha(\tau_j) = 0, \qquad \omega(\tau_j) = \omega_0.$$

Substituting $\lambda(\tau)$ into Eq. (2.5) and differentiating both sides of it with respect to τ, we have

$$\left[\frac{d\lambda(\tau)}{d\tau}\right]^{-1} = \frac{(2\lambda+\beta)e^{\lambda\tau}}{\lambda(\alpha\lambda+\alpha\beta(1+\gamma))} + \frac{\alpha}{\lambda(\alpha\lambda+\alpha\beta(1+\gamma))} - \frac{\tau}{\lambda},$$

which leads to

$$\begin{aligned}
\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_j} &= \mathrm{Re}\left\{\frac{(2\lambda+\beta)e^{\lambda\tau}}{\lambda(\alpha\lambda+\alpha\beta(1+\gamma))}\right\}_{\tau=\tau_j} + \mathrm{Re}\left\{\frac{\alpha}{\lambda(\alpha\lambda+\alpha\beta(1+\gamma))} - \frac{\tau}{\lambda}\right\}_{\tau=\tau_j} \\
&= \mathrm{Re}\left\{\frac{\beta\cos\omega_0\tau_j - 2\omega_0\sin\omega_0\tau_j + i[2\omega_0\cos\omega_0\tau_j + \beta\sin\omega_0\tau_j]}{-\alpha\omega_0^2 + i\alpha\beta(1+\gamma)\omega_0}\right\} + \mathrm{Re}\left\{\frac{\alpha}{-\alpha\omega_0^2 + i\alpha\beta(1+\gamma)\omega_0}\right\} \\
&= \frac{1}{\Gamma}\{-\alpha\omega_0^2[\beta\cos\omega_0\tau_j - 2\omega_0\sin\omega_0\tau_j] + \alpha\beta(1+\gamma)\omega_0[2\omega_0\cos\omega_0\tau_j + \beta\sin\omega_0\tau_j] - \alpha^2\omega_0^2\} \\
&= \frac{1}{\Gamma}\{\beta\omega_0[\alpha\beta(1+\gamma)\sin\omega_0\tau_j - \alpha\omega_0\cos\omega_0\tau_j] + 2\omega_0^2[q\cos\omega_0\tau_j + \alpha\omega_0\sin\omega_0\tau_j] - \alpha^2\omega_0^2\} \\
&= \frac{\omega_0^2}{\Gamma}\{2\omega_0^2 + (\beta^2 - \alpha^2)\} = \frac{\omega_0^2}{\Gamma}\left\{\sqrt{(\alpha^2+\beta^2)^2 + 4\gamma\alpha^2\beta^2(1+\gamma)}\right\},
\end{aligned}$$

where we have used Eqs. (2.6) and (2.8), and $\Gamma = \alpha^2\omega_0^4 + \alpha^2\beta^2(1+\gamma)^2\omega_0^2 > 0$. Hence

$$\mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]_{\tau=\tau_j}\right\} = \mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_j}\right\} = \mathrm{sign}\left\{\frac{\omega_0^2}{\Gamma}\sqrt{(\alpha^2+\beta^2)^2 + 4\gamma\alpha^2\beta^2(1+\gamma)}\right\} > 0. \tag{2.10}$$

Therefore, when the delay τ near τ_j is increased, the root of Eq. (2.5) crosses the imaginary axis from left to right. In addition, note that when $\tau = 0$, Eq. (2.5) has only the roots with negative real parts. Thus, the well-known Rouché theorem means the following results about the distribution of roots of Eq. (2.5) hold.

Lemma 2.1. *Let τ_j ($j = 0,1,\ldots$) be defined as in (2.9). Then all roots of Eq. (2.5) have negative real parts for all $\tau \in [0,\tau_0)$. However, Eq. (2.5) has at least one root with positive real part when $\tau > \tau_0$, and Eq. (2.5) has a pair of purely imaginary root $\pm i\omega_0$ when $\tau = \tau_0$. More detail, for $\tau \in (\tau_j, \tau_{j+1}]$ ($j = 0,1,2,\ldots$), Eq. (2.5) has $2(j+1)$ roots with positive real parts. Moreover, all roots of Eq. (2.5) with $\tau = \tau_j$, $j = 0,1,2,\ldots$ have negative real parts except $\pm i\omega_0$.*

Applying Lemma 2.1, the transversality condition (2.10) and Theorem 11.1 [9], we easily obtain the following results on stability and bifurcation of system (1.2).

Theorem 2.1. *Let ω_0 and τ_j $(j = 0, 1, \ldots)$ be defined by (2.8) and (2.9), respectively.*

(i) *The positive equilibrium E_* of system (1.2) is asymptotically stable for all $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$.*
(ii) *System (1.2) undergoes a Hopf bifurcation at the positive equilibrium E_* when $\tau = \tau_j$, $j = 0, 1, 2, \ldots$*

3. Direction and stability of the Hopf bifurcation

In the previous section, we obtain that a family of periodic solutions bifurcate from the positive steady state E_* at the critical values of τ. We know that it is interesting to determine the direction, stability and period of these bifurcating periodic solutions. In this section, we shall derive the explicit formulae determining the properties of the Hopf bifurcation at the critical value τ_j using the normal form theory and center manifold reduction due to Hassard et al. [10].

Normalizing the delay τ by the time-scaling $t \to t/\tau$, system (2.3) is transformed into

$$\begin{cases} \dot{x}(t) = -\tau\alpha[x(t-1) + \gamma y(t) + x(t)x(t-1) + \gamma x(t)y(t)], \\ \dot{y}(t) = \tau\beta[x(t-1) - y(t)]. \end{cases} \tag{3.1}$$

Thus, we can work in the phase space $C = C([-1, 0], R^2)$. Without loss of generality, denote the critical value τ_j by $\tilde{\tau}$. Let $\tau = \tilde{\tau} + \mu$, then $\mu = 0$ is a Hopf bifurcation value of Eqs. (3.1). For the simplicity of notations, we rewrite (3.1) as

$$\dot{u}(t) = L_\mu(u_t) + f(\mu, u_t), \tag{3.2}$$

where $u(t) = (u_1(t), u_2(t))^{\mathrm{T}} = (x(t), y(t))^{\mathrm{T}} \in R^2$, $u_t(\theta) \in C$ is defined by $u_t(\theta) = u(t+\theta)$, and $L_\mu : C \to R$ and $f : R \times C \to R$ are given by

$$L_\mu u_t = (\tilde{\tau} + \mu)\begin{pmatrix} 0 & -\alpha\gamma \\ 0 & -\beta \end{pmatrix}\begin{pmatrix} x_{1t}(0) \\ x_{2t}(0) \end{pmatrix} + (\tilde{\tau} + \mu)\begin{pmatrix} -\alpha & 0 \\ \beta & 0 \end{pmatrix}\begin{pmatrix} x_{1t}(-1) \\ x_{2t}(-1) \end{pmatrix} \tag{3.3}$$

and

$$f(\mu, u_t) = (\tilde{\tau} + \mu)\begin{pmatrix} -\alpha u_{1t}(0)u_{1t}(-1) - \alpha\gamma u_{1t}(0)u_{2t}(0) \\ 0 \end{pmatrix}, \tag{3.4}$$

respectively. By the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$, such that

$$L_\mu \phi = \int_{-1}^{0} \mathrm{d}\eta(\theta, 0)\phi(\theta) \quad \text{for } \phi \in C. \tag{3.5}$$

In fact, we can choose

$$\eta(\theta, \mu) = (\tilde{\tau} + \mu)\begin{pmatrix} 0 & -\alpha\gamma \\ 0 & -\beta \end{pmatrix}\delta(\theta) - (\tilde{\tau} + \mu)\begin{pmatrix} -\alpha & 0 \\ \beta & 0 \end{pmatrix}\delta(\theta + 1), \tag{3.6}$$

where δ is the Dirac delta function. For $\phi \in C^1([-1, 0], R^2)$, define

$$A(\mu)\phi = \begin{cases} \dfrac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & \theta \in [-1, 0), \\ \int_{-1}^{0} \mathrm{d}\eta(\mu, s)\phi(s), & \theta = 0 \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ f(\mu, \phi), & \theta = 0. \end{cases}$$

Then system (3.2) is equivalent to

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \tag{3.7}$$

where $x_t(\theta) = x(t + \theta)$ for $\theta \in [-1, 0]$.

For $\psi \in C^1([0,1],(R^2)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s \in (0,1], \\ \int_{-1}^{0} \mathrm{d}\eta^{\mathrm{T}}(t,0)\psi(-t), & s = 0. \end{cases}$$

and a bilinear inner product

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)\,\mathrm{d}\eta(\theta)\phi(\xi)\,\mathrm{d}\xi, \tag{3.8}$$

where $\eta(\theta) = \eta(\theta, 0)$. Then $A(0)$ and A^* are adjoint operators. By the discussion in Section 2, we know that $\pm \mathrm{i}\omega_0\tilde{\tau}$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*. We first need to compute the eigenvector of $A(0)$ and A^* corresponding to $\mathrm{i}\omega_0\tilde{\tau}$ and $-\mathrm{i}\omega_0\tilde{\tau}$, respectively.

Suppose that $q(\theta) = (1,\rho)^{\mathrm{T}}\mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}\theta}$ is the eigenvector of $A(0)$ corresponding to $\mathrm{i}\omega_0\tilde{\tau}$. Then $A(0)q(\theta) = \mathrm{i}\omega_0\tilde{\tau}q(\theta)$. It follows from the definition of $A(0)$ and (3.5), (3.6) that

$$\tilde{\tau}\begin{pmatrix} \mathrm{i}\omega_0 + \alpha\mathrm{e}^{-\mathrm{i}\omega_0\tilde{\tau}} & \alpha\gamma \\ -\beta\mathrm{e}^{-\mathrm{i}\omega_0\tilde{\tau}} & \mathrm{i}\omega_0 + \beta \end{pmatrix} q(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we have

$$q(0) = (1,\rho)^{\mathrm{T}} = \left(1, \frac{\beta\mathrm{e}^{-\mathrm{i}\omega_0\tilde{\tau}}}{\beta + \mathrm{i}\omega_0}\right)^{\mathrm{T}}.$$

On the other hand, suppose that $q^*(s) = D(\sigma, 1)\mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}s}$ is the eigenvector of A^* corresponding to $-\mathrm{i}\omega_0\tilde{\tau}$. By direction computation we get

$$q^*(0) = D(1,\sigma) = D\left(1, \frac{\alpha\gamma}{\beta - \mathrm{i}\omega_0}\right).$$

In order to assure $\langle q^*(s), q(\theta)\rangle = 1$, we need to determine the value of D. From (3.7), we have

$$\begin{aligned} \langle q^*(s), q(\theta)\rangle &= \overline{D}\left\{(1,\bar{\sigma})(1,\rho)^{\mathrm{T}} - \int_{-1}^{0}\int_{\xi=0}^{\theta}(1,\bar{\sigma})\mathrm{e}^{-\mathrm{i}(\xi-\theta)\omega_0\tilde{\tau}}\,\mathrm{d}\eta(\theta)(1,\rho)^{\mathrm{T}}\mathrm{e}^{\mathrm{i}\xi\omega_0\tilde{\tau}}\,\mathrm{d}\xi\right\} \\ &= \overline{D}\left\{1 + \rho\bar{\sigma} - \int_{-1}^{0}(1,\bar{\sigma})\theta\mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}\theta}\,\mathrm{d}\eta(\theta)(1,\rho)^{\mathrm{T}}\right\} = \overline{D}\left\{1 + \rho\bar{\sigma} + \tilde{\tau}(\beta\bar{\sigma} - \alpha)\mathrm{e}^{-\mathrm{i}\omega_0\tilde{\tau}}\right\}. \end{aligned}$$

Thus, we can chose

$$D = \frac{1}{1 + \bar{\rho}\sigma + \tilde{\tau}(\beta\sigma - \alpha)\mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}}}$$

such that

$$\langle q^*(s), q(\theta)\rangle = 1, \qquad \langle q^*(s), \bar{q}(\theta)\rangle = 0.$$

Following the ideal of Hassard et al. [10], we first compute the coordinates to describe the center manifold $\mathbf{C}_0$ at $\mu = 0$. Let u_t be the solution of Eq. (3.2) with $\mu = 0$. Define

$$z(t) = \langle q^*, u_t\rangle, \qquad W(t,\theta) = u_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\}. \tag{3.9}$$

On the center manifold $\mathbf{C}_0$ we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta),$$

where

$$W(z,\bar{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots,$$

z and $\bar{z}$ are local coordinates for center manifold $\mathbf{C}_0$ in the direction of q^* and $\bar{q}^*$. Note that W is real if u_t is real. We consider only real solutions. For the solution $u_t \in \mathbf{C}_0$ of (3.2), since $\mu = 0$, we have

$$\begin{aligned}\dot{z} &= \mathrm{i}\omega_0\tilde{\tau}z + \langle \bar{q}^*(\theta), f(0, W(z,\bar{z},\theta) + 2\mathrm{Re}\{zq(\theta)\})\rangle \\ &= \mathrm{i}\omega_0\tilde{\tau}z + \bar{q}^*(0)f(0, W(z,\bar{z},0) + 2\mathrm{Re}\{zq(0)\}) \stackrel{\text{def}}{=} \mathrm{i}\omega_0\tilde{\tau}z + \bar{q}^*(0)f_0(z,\bar{z}).\end{aligned}$$

We rewrite this equation as

$$\dot{z} = \mathrm{i}\omega_0\tilde{\tau}z + g(z,\bar{z})$$

with

$$g(z,\bar{z}) = \bar{q}^*(0)f_0(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots \tag{3.10}$$

Noticing $u_t(\theta) = (u_{1t}(\theta), u_{2t}(\theta)) = W(t,\theta) + zq(\theta) + \bar{z}\bar{q}(\theta)$ and $q(\theta) = (1,\rho)^{\mathrm{T}}\mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}\theta}$, we have

$$\begin{aligned}u_{1t}(0) &= z + \bar{z} + W_{20}^{(1)}(0)\tfrac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\tfrac{\bar{z}^2}{2} + \cdots,\\ u_{1t}(-1) &= \mathrm{e}^{-\mathrm{i}\omega_0\tilde{\tau}}z + \mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}}\bar{z} + W_{20}^{(1)}(-1)\tfrac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\tfrac{\bar{z}^2}{2} + \cdots,\\ u_{2t}(0) &= \rho z + \overline{\rho z} + W_{20}^{(2)}(0)\tfrac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\tfrac{\bar{z}^2}{2} + \cdots\end{aligned}$$

Thus, from (3.4) and (3.10) we have

$$\begin{aligned}g(z,\bar{z}) &= \bar{q}^*(0)f_0(z,\bar{z}) = \tilde{\tau}\overline{D}(1,\bar{\sigma})\begin{pmatrix}-\alpha u_{1t}(0)u_{1t}(-1) - \alpha\gamma u_{1t}(0)u_{2t}(0)\\ 0\end{pmatrix}\\ &= -\tilde{\tau}\overline{D}\alpha\Bigg\{\left(z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots\right)\\ &\quad\times\left(\mathrm{e}^{-\mathrm{i}\omega_0\tilde{\tau}}z + \mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}}\bar{z} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + \cdots\right)\\ &\quad+\gamma\left(z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots\right)\\ &\quad\times\left(\rho z + \overline{\rho z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots\right)\Bigg\}\\ &= -\tilde{\tau}\overline{D}\alpha\Bigg\{2[\mathrm{e}^{-\mathrm{i}\omega_0\tilde{\tau}} + \gamma\rho]\frac{z^2}{2} + 2[\mathrm{Re}\{\mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}}\} + \gamma\mathrm{Re}\{\rho\}]z\bar{z} + 2[\mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}} + \gamma\bar{\rho}]\frac{\bar{z}^2}{2}\\ &\quad+\Big[\Big(2W_{11}^{(1)}(-1) + W_{20}^{(1)}(-1) + 2\mathrm{e}^{-\mathrm{i}\omega_0\tilde{\tau}}W_{11}^{(1)}(0) + \mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}}W_{20}^{(1)}(0)\Big)\\ &\quad+\gamma\Big(2W_{11}^{(2)}(0) + W_{20}^{(2)}(0) + 2\rho W_{11}^{(1)}(0) + \rho W_{20}^{(1)}(0)\Big)\Big]\frac{z^2\bar{z}}{2} + \cdots\Bigg\}.\end{aligned}$$

Comparing the coefficients with (3.10), we get

$$\begin{aligned}g_{20} &= -2\tilde{\tau}\overline{D}\alpha(\mathrm{e}^{-\mathrm{i}\omega_0\tilde{\tau}} + \gamma\rho);\\ g_{11} &= -2\tilde{\tau}\overline{D}\alpha(\mathrm{Re}\{\mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}}\} + \gamma\mathrm{Re}\{\rho\});\\ g_{02} &= -2\tilde{\tau}\overline{D}\alpha(\mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}} + \gamma\bar{\rho});\\ g_{21} &= -\tilde{\tau}\overline{D}\alpha\Big[\Big(2W_{11}^{(1)}(-1) + W_{20}^{(1)}(-1) + 2\mathrm{e}^{-\mathrm{i}\omega_0\tilde{\tau}}W_{11}^{(1)}(0) + \mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}}W_{20}^{(1)}(0)\Big)\\ &\quad+\gamma\Big(2W_{11}^{(2)}(0) + W_{20}^{(2)}(0) + 2\rho W_{11}^{(1)}(0) + \rho W_{20}^{(1)}(0)\Big)\Big].\end{aligned} \tag{3.11}$$

Since there are $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21}, we need to further determine them.

From (3.7) and (3.9), we have

$$\begin{aligned}\dot{W} &= \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} \\ &= \begin{cases} AW - 2\mathrm{Re}\{\bar{q}^*(0)f_0q(\theta)\}, & \theta \in [-1,0), \\ AW - 2\mathrm{Re}\{\bar{q}^*(0)f_0q(0)\} + f_0, & \theta = 0, \end{cases} \\ &\stackrel{\mathrm{def}}{=} AW + H(z,\bar{z},\theta), \end{aligned} \tag{3.12}$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{3.13}$$

On the other hand, on $\mathbf{C_0}$ near the origin

$$\dot{W} = W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}}. \tag{3.14}$$

It follows from (3.12)–(3.14) that

$$(A - 2\omega_0\tilde{\tau})W_{20}(\theta) = -H_{20}(\theta), \qquad AW_{11}(\theta) = -H_{11}(\theta), \cdots \tag{3.15}$$

From (3.10) and (3.12), we have, for $\theta \in [-1,0)$,

$$H(z,\bar{z},\theta) = -\bar{q}^*(0)f_0q(\theta) - q^*(0)\overline{f}_0\bar{q}(\theta) = -gq(\theta) - \overline{gq}(\theta). \tag{3.16}$$

Comparing the coefficients with (3.13) yields that for $\theta \in [-1,0)$,

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta) \tag{3.17}$$

and

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{3.18}$$

From (3.15), (3.17) and the definition of A we have

$$\dot{W}_{20}(\theta) = 2\mathrm{i}\omega_0\tilde{\tau}W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta).$$

Notice that $q(\theta) = q(0)\mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}\theta}$, hence

$$W_{20}(\theta) = \frac{\mathrm{i}g_{20}}{\omega_0\tilde{\tau}}q(0)\mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}\theta} + \frac{\mathrm{i}\bar{g}_{02}}{3\omega_0\tilde{\tau}}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\theta\omega_0\tilde{\tau}} + E_1\mathrm{e}^{2\mathrm{i}\omega_0\tilde{\tau}\theta}, \tag{3.19}$$

where $E_1 = \left(E_1^{(1)}, E_1^{(2)}\right) \in R^2$ is a constant vector.

In a similar way, combining (3.15) and (3.18) yields

$$W_{11}(\theta) = -\frac{\mathrm{i}g_{11}}{\omega_0\tilde{\tau}}q(0)\mathrm{e}^{\mathrm{i}\theta\omega_0\tilde{\tau}} + \frac{\mathrm{i}\bar{g}_{11}}{\omega_0\tilde{\tau}}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\theta\omega_0\tilde{\tau}} + E_2, \tag{3.20}$$

where $E_2 = \left(E_2^{(1)}, E_2^{(2)}\right) \in R^2$ is also a constant vector.

In what follows, we shall seek appropriate E_1 and E_2. From the definition of A and (3.15), we obtain

$$\int_{-1}^{0}\mathrm{d}\eta(\theta)W_{20}(\theta) = 2\mathrm{i}\omega_0\tilde{\tau}W_{20}(0) - H_{20}(0) \tag{3.21}$$

and

$$\int_{-1}^{0}\mathrm{d}\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{3.22}$$

where $\eta(\theta) = \eta(0,\theta)$. In addition, it follows from (3.10)–(3.12) that

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) - 2\tilde{\tau}\alpha\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\omega_0\tilde{\tau}} + \gamma\rho \\ 0\end{pmatrix} \tag{3.23}$$

and

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) - 2\tilde{\tau}\alpha\begin{pmatrix}\Re\{e^{i\omega_0\tilde{\tau}}\} + \gamma\Re\{\rho\} \\ 0\end{pmatrix}. \tag{3.24}$$

Substituting (3.19) and (3.23) into (3.21) and noticing that

$$\left(i\omega_0\tilde{\tau}I - \int_{-1}^{0} e^{i\theta\omega_0\tilde{\tau}}\,d\eta(\theta)\right)q(0) = 0$$

and

$$\left(-i\omega_0\tilde{\tau}I - \int_{-1}^{0} e^{-i\theta\omega_0\tilde{\tau}}\,d\eta(\theta)\right)\bar{q}(0) = 0,$$

we have

$$\left(2i\omega_0\tilde{\tau}I - \int_{-1}^{0} e^{2i\theta\omega_0\tilde{\tau}}\,d\eta(\theta)\right)E_1 = -2\alpha\begin{pmatrix}e^{-i\omega_0\tilde{\tau}} + \gamma\rho \\ 0\end{pmatrix},$$

which leads to

$$\begin{pmatrix}2i\omega_0 + \alpha e^{-2i\omega_0\tilde{\tau}} & \alpha\gamma \\ -\beta e^{-2i\omega_0\tilde{\tau}} & 2i\omega_0 + \beta\end{pmatrix}E_1 = -2\alpha\begin{pmatrix}e^{-i\omega_0\tilde{\tau}} + \gamma\rho \\ 0\end{pmatrix}.$$

It follows that

$$E_1^{(1)} = \frac{-2\alpha(e^{-i\omega_0\tilde{\tau}} + \gamma\rho)(2i\omega_0 + \beta)}{A}$$

and

$$E_1^{(2)} = \frac{-2\alpha\beta e^{-2i\omega_0\tilde{\tau}}(e^{-i\omega_0\tilde{\tau}} + \gamma\rho)}{A},$$

where

$$A = \begin{vmatrix}2i\omega_0 + \alpha e^{-2i\omega_0\tilde{\tau}} & \alpha\gamma \\ -\beta e^{-2i\omega_0\tilde{\tau}} & 2i\omega_0 + \beta\end{vmatrix}.$$

Similarly, it follows from (3.20), (3.22) and (3.24) that

$$\begin{pmatrix}\alpha & \alpha\gamma \\ -\beta & \beta\end{pmatrix}E_2 = -2\alpha\begin{pmatrix}\Re\{e^{i\omega_0\tilde{\tau}}\} + \gamma\Re\{\rho\} \\ 0\end{pmatrix}$$

and hence,

$$E_2^{(1)} = E_2^{(2)} = \frac{-2(\Re\{e^{i\omega_0\tilde{\tau}}\} + \gamma\Re\{\rho\})}{1+\gamma}.$$

Therefore, from (3.19) and (3.20) we can determine $W_{20}(0)$ and $W_{11}(0)$. Further, we can determine g_{21}. Therefore, each g_{ij} in (3.11) is determined by the parameters and delay in system (3.1). Thus, we can compute the following values:

$$\begin{aligned}
c_1(0) &= \frac{i}{2\tilde{\tau}\omega_0}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},\\
\mu_2 &= -\frac{\Re\{c_1(0)\}}{\Re\{\lambda'(\tilde{\tau})\}},\\
\beta_2 &= 2\Re\{c_1(0)\},\\
T_2 &= -\frac{\mathrm{Im}\{c_1(0)\} + \mu_2\mathrm{Im}\{\lambda'(\tilde{\tau})\}}{\tilde{\tau}\omega_0},
\end{aligned} \tag{3.25}$$

which determine the quantities of bifurcating periodic solutions in the center manifold at the critical value $\tilde{\tau}$, i.e., μ_2 determines the directions of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tilde{\tau}$ ($\tau < \tilde{\tau}$); β_2 determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); and T_2 determines the period of the bifurcating periodic solutions: the period increase (decrease) if $T_2 > 0$ ($T_2 < 0$).

It is worthy noting that when $c = 0$, the first equation of (1.2) reduces to Eq. (1.1), and we can easily obtain the following well-known results (see, e.g., [9]) for the delayed logistical equation (1.1).

Theorem 3.1. *For Eq.* (1.1), *we have*

(i) *the positive equilibrium $n \equiv K$ is asymptotically stable when $r\tau < \frac{\pi}{2}$, and Eq.* (1.1) *undergoes a Hopf bifurcation at the positive equilibrium $n \equiv K$ when $r\tau = \frac{\pi}{2} + 2j\pi$, $j = 0, 1, 2, \ldots$.*

(ii) *When $r\tau = \frac{\pi}{2}$, the direction of Hopf bifurcation is supercritical and the bifurcating periodic solution is orbitally asymptotically.*

Proof. Since $c = 0$, it follows from (2.2) that

$$\gamma = 0, \qquad \alpha = r, \qquad \beta = a,$$

which, together with (2.8) and (2.9), implies

$$\omega_0 = r, \qquad r\tau_0 = \frac{\pi}{2}.$$

Thus, we can obtain that

$$\rho = -\frac{a(r + a\mathrm{i})}{a^2 + r^2}, \qquad \sigma = 0, \qquad D = \frac{1 + \frac{\pi}{2}\mathrm{i}}{1 + \frac{\pi^2}{4}}.$$

From (2.9) we have

$$g_{11} = 0, \qquad g_{20} = \mathrm{i}\pi\overline{D}, \qquad g_{02} = -\mathrm{i}\pi\overline{D}. \tag{3.26}$$

Further, we can get

$$E_1^{(1)} = \tfrac{2}{5}(2 - \mathrm{i}), \qquad E_2^{(1)} = E_2^{(2)} = 0.$$

It follows from (3.20) that $W_{11}(\theta) \equiv 0$, and then

$$g_{21} = -\tilde{\tau}\overline{D}\alpha\left\{W_{20}^{(1)}(-1) + \mathrm{e}^{\mathrm{i}\omega_0\tilde{\tau}}\right\} - \frac{\pi}{5}\left(\frac{3\pi}{2} - 1\right)\Lambda + \mathrm{i}\frac{2\pi}{3}|D|^2\left(3 + \frac{\pi}{2}\right)\Lambda, \tag{3.27}$$

where $\Lambda = (1 + \frac{\pi^2}{4})$. Therefore, combining (3.25)–(3.27) leads to

$$\Re(c_1(0)) = \Re\left\{\frac{g_{21}}{2}\right\} = -\frac{\pi}{10}\Lambda\left(\frac{3\pi}{2} - 1\right) < 0.$$

It follows from (2.10) that

$$\mu_2 = -\frac{\Re(c_1(0))}{\Re\{\lambda'(\tilde{\tau}_0)\}} > 0$$

and

$$\beta_2 = 2\Re\{c_1(0)\} < 0.$$

This completes the proof. □

3.1. A numerical example

From the above algorithm, we know that if the values of r, K, a, b, c and τ are given, then we can determine the stability and direction of periodic solutions bifurcating from the positive equilibrium E_* at the critical point τ_j. In the rest of this section, we shall illustrate the validity of the results by considering the system:

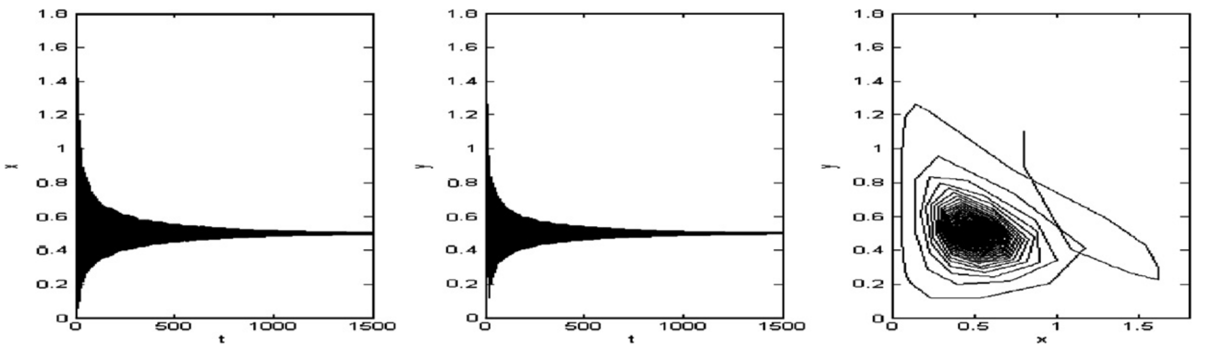

Fig. 1. The positive equilibrium E_* of system (3.28) is asymptotically stable when $\tau = 1.5 < \tau_0 \doteq 1.52678$. Here the initial value is (0.8, 1.1).

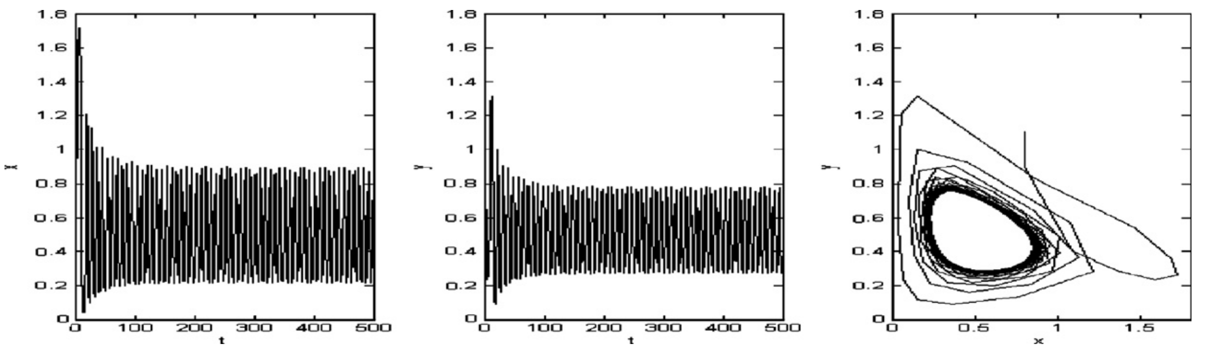

Fig. 2. When $\tau = 1.66 > \tau_0 \doteq 1.52678$, the positive equilibrium E_* of system (3.28) loses its stability and a Hopf bifurcation occurs. Further, the bifurcating periodic solution is orbitally, asymptotically stable. Here the initial value is (0.8, 1.1).

$$\begin{cases} \dot{n}(t) = n(t)[1 - n(t-\tau) - u(t)], \\ \dot{u}(t) = -u(t) + n(t-\tau), \end{cases} \tag{3.28}$$

which has a positive equilibrium $E_* = (\frac{1}{2}, \frac{1}{2})$. It follows from Section 2 and Theorem 2.1 that $\tau_j \doteq 1.52678 + \frac{2j\pi}{0.832466}$, the positive equilibrium E_* is stable when $\tau < \tau_0$ (see Fig. 1), and system (3.28) undergoes a Hopf bifurcation at τ_j. Further, from the above process, we can determine the stability and direction of periodic solutions bifurcating from the positive equilibrium at the critical point τ_j. For instance, when $\tau = \tau_0 \doteq 1.52678$, $c_1(0) \doteq -1.57608 + 1.30461I$. It follows from (3.25) that $\mu_2 > 0$ and $\beta_2 < 0$. Therefore, the bifurcation takes place when τ crosses τ_0 to the right ($\tau > \tau_0$), and the corresponding periodic orbits are orbitally asymptotically stable, as depicted in Fig. 2.

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