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Discrete Mathematics 308 (2008) 3491–3507

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Bounds on total domination in claw-free cubic graphs ☆

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Received 25 January 2006; received in revised form 25 June 2007; accepted 4 July 2007

Available online 8 August 2007

Abstract

A set S of vertices in a graph G is a total dominating set, denoted by TDS, of G if every vertex of G is adjacent to some vertex in S (other than itself). The minimum cardinality of a TDS of G is the total domination number of G, denoted by $\gamma_t(G)$. If G does not contain $K_{1,3}$ as an induced subgraph, then G is said to be claw-free. It is shown in [D. Archdeacon, J. Ellis-Monaghan, D. Fischer, D. Froncek, P.C.B. Lam, S. Seager, B. Wei, R. Yuster, Some remarks on domination, J. Graph Theory 46 (2004) 207–210.] that if G is a graph of order n with minimum degree at least three, then $\gamma_t(G) \leqslant n/2$. Two infinite families of connected cubic graphs with total domination number one-half their orders are constructed in [O. Favaron, M.A. Henning, C.M. Mynhardt, J. Puech, Total domination in graphs with minimum degree three, J. Graph Theory 34(1) (2000) 9–19.] which shows that this bound of $n/2$ is sharp. However, every graph in these two families, except for K_4 and a cubic graph of order eight, contains a claw. It is therefore a natural question to ask whether this upper bound of $n/2$ can be improved if we restrict G to be a connected cubic claw-free graph of order at least 10. In this paper, we answer this question in the affirmative. We prove that if G is a connected claw-free cubic graph of order $n \geqslant 10$, then $\gamma_t(G) \leqslant 5n/11$.

MSC: 05C69

Keywords: Bounds; Claw-free cubic graphs; Total domination

1. Introduction

Total domination in graphs was introduced by Cockayne et al. [4] and is now well studied in graph theory (see, for example, 3,7,11]). The literature on this subject has been surveyed and detailed in the two books by Haynes et al. [9,10].

Let $G = (V, E)$ be a graph with vertex set V and edge set E. A *total dominating set*, denoted by TDS, of G with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S (other than itself). Every graph without isolated vertices has a TDS, since $S = V$ is such a set. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS. We call a TDS of G of cardinality $\gamma_t(G)$ a $\gamma_t(G)$-set.

☆ Research supported in part by the South African National Research Foundation and the University of KwaZulu-Natal.

[1] This paper was started while the author was visiting the Laboratoire de Recherche en Informatique (LRI) at the Université de Paris-Sud in July 2002.

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doi:10.1016/j.disc.2007.07.007

For notation and graph theory terminology we in general follow [9]. Specifically, let $G=(V,E)$ be a graph with vertex set V of order n and edge set E, and let v be a vertex in V. The *open neighborhood* of v is $N(v)=\{u \in V \mid uv \in E\}$ and the *closed neighborhood of* v is $N[v]=\{v\} \cup N(v)$. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. A vertex $w \in V \setminus S$ is an *external private neighbor of* v (with respect to S) if $N(w) \cap S = \{v\}$; and the *external private neighbor set of* v *with respect to* S, denoted $\mathrm{epn}(v,S)$, is the set of all external private neighbors of v. For subsets $S, T \subseteq V$, S *totally dominates* T if $T \subseteq N(S)$. A *cycle* on n vertices is denoted by C_n and a *path* on n vertices by P_n. The minimum degree (resp., maximum degree) among the vertices of G is denoted by $\delta(G)$ (resp., $\Delta(G)$).

We say that a graph is F*-free* if it does not contain F as an induced subgraph. In particular, if $F=K_{1,3}$, then we say that the graph is *claw-free*. An excellent survey of claw-free graphs has been written by Flandrin et al. [8].

2. Known results on total domination

The following result establishes a property of minimum TDSs in graphs.

Theorem 1 (*Henning [11]*). *If G is a connected graph of order* $n \geqslant 3$, *and* $G \not\cong K_n$, *then G has a* $\gamma_t(G)$*-set S in which every vertex v has one of the following two properties*:

P_1 : $|\mathrm{epn}(v,S)| \geqslant 1$;
P_2 : *v is adjacent to a vertex of degree one in* $G[S]$ *that has property* P_1.

The decision problem to determine the total domination number of a graph is known to be NP-complete. Hence, it is of interest to determine upper bounds on the total domination number of a graph. Cockayne et al. [4] obtained the following upper bound on the total domination number of a connected graph in terms of the order of the graph.

Theorem 2 (*Cockayne et al. [4]*). *If G is a connected graph of order* $n \geqslant 3$, *then* $\gamma_t(G) \leqslant 2n/3$.

Brigham et al. [3] characterized the connected graphs of order at least three with total domination number exactly two-thirds their order. If we restrict G to be a connected claw-free graph, then the upper bound of Theorem 2 cannot be improved since the graph G obtained from a complete graph H by attaching a path of length 2 to each vertex of H so that the resulting paths are vertex disjoint (the graph G is called the 2-*corona* of H) is a connected claw-free graph with total domination number two-thirds its order.

If we restrict the minimum degree to be at least two, then the upper bound in Theorem 2 can be improved.

Theorem 3 (*Henning [11]*). *If G is a connected graph of order n with* $\delta(G) \geqslant 2$ *and* $G \notin \{C_3, C_5, C_6, C_{10}\}$, *then* $\gamma_t(G) \leqslant 4n/7$.

It is shown in [6] that the upper bound of Theorem 3 can be improved if we restrict G to be a claw-free graph.

Theorem 4 (*Favaron and Henning [6]*). *If G is a connected claw-free graph of order n with* $\delta(G) \geqslant 2$, *then* $\gamma_t(G) \leqslant (n+2)/2$ *with equality if and only if G is a cycle of length congruent to* 2 *modulo* 4.

It was shown in [7] that if G is a connected graph of order n with $\delta(G) \geqslant 3$, then $\gamma_t(G) \leqslant 7n/13$ and conjectured that this upper bound could be improved to $n/2$. Archdeacon et al. [1] recently found an elegant one page proof of this conjecture.

Theorem 5 (*Archdeacon et al. [1]*). *If G is a graph of order n with* $\delta(G) \geqslant 3$, *then* $\gamma_t(G) \leqslant n/2$.

The generalized Petersen graph of order 16 shown in Fig. 1 achieves equality in Theorem 5.

Two infinite families $\mathscr{G}$ and $\mathscr{H}$ of connected cubic graphs (described below) with total domination number one-half their orders are constructed in [7] which shows that the bound of Theorem 5 is sharp. For $k \geqslant 2$ consider two copies of the path P_{2k} with respective vertex sequences $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ and $c_1, d_1, c_2, d_2, \ldots, c_k, d_k$. For each $i \in \{1, 2, \ldots, k\}$, join a_i to d_i and b_i to c_i. To complete the construction of graphs in $\mathscr{G}$ ($\mathscr{H}$, respectively), join a_1 to c_1 and b_k to d_k (a_1 to b_k and c_1 to d_k). Two graphs G and H in the families $\mathscr{G}$ and $\mathscr{H}$ are illustrated in Fig. 2.

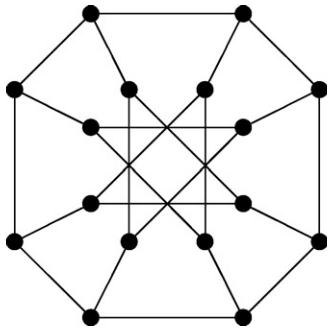

Fig. 1. A generalized Petersen graph of order 16.

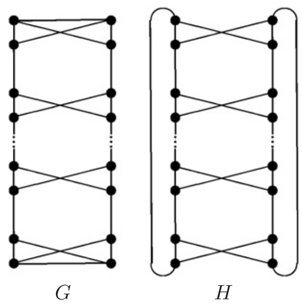

Fig. 2. Cubic graphs $G \in \mathscr{G}$ and $H \in \mathscr{H}$ of order n with $\gamma_t(G) = n/2$.

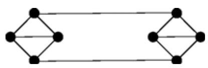

Fig. 3. A claw-free cubic graph G_1 with $\gamma_t(G_1) = n/2$.

The connected graphs with minimum degree at least three that achieve equality in the bound of Theorem 5 are characterized in [12].

Theorem 6 (*Henning and Yeo [12]*). *If G is a connected graph with minimum degree at least three and total domination number one-half its order, then $G \in \mathscr{G} \cup \mathscr{H}$ or G is the generalized Petersen graph of order* 16 *shown in Fig.* 1.

Every graph in the two families $\mathscr{G}$ and $\mathscr{H}$, except for K_4 and the cubic graph G_1 shown in Fig. 3, contains a claw, as does the generalized Petersen graph shown above. Hence, as a consequence of Theorem 6, the connected claw-free cubic graphs achieving equality in Theorem 5 contain at most eight vertices. (This result is also established in [5].)

Theorem 7 (*Favaron and Henning [5], Henning and Yeo [12]*). *If G is a connected claw-free cubic graph of order n, then $\gamma_t(G) \leqslant n/2$ with equality if and only if $G = K_4$ or $G = G_1$ where G_1 is the graph shown in Fig.* 3.

It is therefore a natural question to ask whether the upper bound of Theorem 5 can be improved if we restrict G to be a connected claw-free cubic graph of order at least 10. In this paper, we show that under these conditions the upper bound on the total domination number of G in Theorem 5 decreases from one-half its order to five-elevenths its order.

3. Main result

We shall prove:

Theorem 8. *If G is a connected claw-free cubic graph of order $n \geqslant 6$, then either $G = G_1$ where G_1 is the graph shown in Fig.* 3 *or $\gamma_t(G) \leqslant 5n/11$.*

As an immediate consequence of Theorem 8, we have the following result.

Corollary 9. *If G is a connected claw-free cubic graph of order* $n \geqslant 10$, *then* $\gamma_t(G) \leqslant 5n/11$.

4. Cost function

Before presenting a proof of Theorem 8 we introduce the concept of a cost function of a TDS in a claw-free graph. Let S be a TDS of a claw-free graph $G=(V,E)$. Let $I(S)$ denote the number of isolated vertices in $G[V\setminus S]$. Let $P_2(S)$ and $P_4(S)$ denote the number of components in $G[S]$ isomorphic to a path P_2 and P_4, respectively. Let $P(S)$ denote the number of external private neighbors of vertices of S. Let $T(S)$ denote the number of triangles in $G[V\setminus S]$.

We define a *bad vertex* of $V\setminus S$ as a vertex of $V\setminus S$ that is adjacent to exactly one vertex in a P_2-component of $G[S]$ and exactly one vertex (necessarily, an end-vertex since G is claw-free) in a P_3-component of $G[S]$. We observe that if $\delta(G)\geqslant 3$, then by the claw-freeness of G a bad vertex of $V\setminus S$ is not an isolated vertex of $G[V\setminus S]$. We let $B(S)$ denote the number of bad vertices in $V\setminus S$.

We define the *cost function* of S, denoted by $c(S)$, in the graph G by

$$c(S)=7I(S)+4P_4(S)+2B(S)-2P_2(S)-2P(S)-2T(S).$$

Intuitively, an isolated vertex in $G[V\setminus S]$ costs us \$7, a P_4-component in $G[S]$ costs us \$4 and a bad vertex of $V\setminus S$ costs us \$2. On the other hand, for each P_2-component in $G[S]$ or external private neighbor of a vertex of S or triangle in $G[V\setminus S]$ we receive a \$2 rebate.

5. Proof of Theorem 8

Let $G=(V,E)$ be a connected claw-free cubic graph of order $n\geqslant 6$. Among all $\gamma_t(G)$-sets, let S be chosen so that:

(1) Every vertex in S has property P_1 or P_2 given in Theorem 1.
(2) Subject to (1), the number of K_3's in $G[S]$ is minimized.
(3) Subject to (2), the cost function $c(S)$ is minimized.

The existence of the set S is guaranteed by Theorem 1. Throughout our proof, whenever we give a diagram of a subgraph of G we indicate vertices of S by darkened vertices and vertices of $V\setminus S$ by circled vertices.

We proceed further with series of lemmas. The proofs of these lemmas follow from the way in which the set S is chosen. Since these proofs are technical in nature, we present them in later subsections. We begin with the following lemma, a proof of which is presented in Section 5.1.

Lemma 10. *Every component of* $G[S]$ *is a path* P_2, P_3 *or* P_4.

To simplify the notation in what follows, we shall use the following notation. Let $u\in V$ and let G_u be a subgraph of G containing u. We define $S_u=S\cap V(G_u)$. A proof of the following lemma is presented in Section 5.2.

Lemma 11. *If u is an isolated vertex of* $G[V\setminus S]$, *then either* $G=G_1$ *or we can uniquely associate with u the connected subgraph* G_u *of G shown in Fig. 4(a) or (b) where the vertices in* $V(G_u)$ *are not adjacent in G to any vertex of* $S\setminus S_u$ *and where in Fig. 4(b) either* G_u *or* G_u+ab *is an induced subgraph of G.*

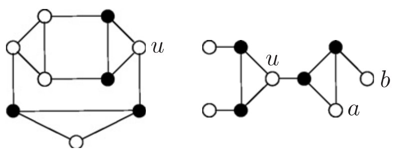

Fig. 4. The two subgraphs uniquely associated with an isolated vertex u of $G[V\setminus S]$. (a) G_u and (b) G_u.

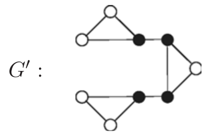

Fig. 5. The subgraph uniquely associated with a P_4-component in $G[S_2]$.

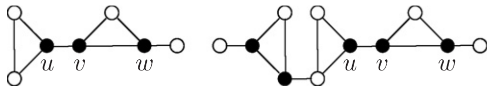

Fig. 6. The two subgraphs uniquely associated with a P_3-component in $G[S_2]$. (a) G' and (b) G'.

Let $V_1 = \cup V(G_u)$ where the union is taken over all isolated vertices u in $G[V\backslash S]$ and where G_u is the subgraph of G defined in the statement of Lemma 11. Let $|V_1| = n_1$. Let $S_1 = S \cap V_1$ and let $S_2 = S\backslash S_1$. Then, $N[S_1] = V_1$. Notice that the set S_u defined in Lemma 11 is a TDS of G_u of cardinality four-ninths the order of G_u. Thus we have the following immediate consequence of Lemma 11.

Lemma 12. $|S_1| = 4n_1/9$, *and the vertices in* $N[S_1]$ *are not adjacent in G to any vertex of* $S\backslash N[S_1]$.

If $S_2 = \emptyset$, then $S = S_1$ and $n = n_1$, and so $\gamma_t(G) \leqslant 4n/9 < 5n/11$. Hence, we may assume $S_2 \neq \emptyset$, for otherwise the desired result follows. Since $N_G(v) \cap S \subset S_1$ for every vertex $v \in V_1$, every edge joining a vertex in $N[S_1]$ with a vertex in $N[S_2]$ belongs to $G[V\backslash S]$. Hence, letting $V_2 = N[S_2]$, V can be written as disjoint union of V_1 and V_2. In particular, if both S_1 and S_2 are nonempty, then V_1 and V_2 is a partition of V. Let $|V_2| = n_2$, and so $n = n_1 + n_2$.

Since V_1 contains all the isolated vertices of $G[V\backslash S]$, every vertex of $V\backslash S$ not in V_1 (and therefore not dominated by S_1) is adjacent to at most two vertices of S_2 and at least one vertex of $V\backslash S$. A proof of the following lemma is presented in Section 5.3.

Lemma 13. *If* $S' \subseteq S_2$ *induces a* P_4*-component in* $G[S]$, *then we can uniquely associate with* S' *the subgraph* G' *of G shown in Fig.* 5 *where the vertices in* $V(G')$ *are not adjacent in G to any vertex of* $S\backslash S'$.

By Lemma 13, if P_4 is a component in $G[S_2]$, then there are five vertices of $V\backslash S$ that are dominated by at least one of the four vertices of this P_4 but by no other vertex of S.

A proof of the following lemma is presented in Section 5.4.

Lemma 14. *If* u, v, w *is a* P_3*-component in* $G[S_2]$, *then we can uniquely associate with this* P_3*-component the subgraph* G' *of G shown in either Fig.* 6(a) *or* (b) *where the* (*circled*) *vertices in* $V(G')$ *are not adjacent in G to any vertex of* $S\backslash V(G')$.

We say that two components of $G[S]$ are at distance k apart if the length of a shortest path in G joining a vertex from one component to a vertex of the other has length k. In particular, two components of $G[S]$ are at distance two apart if there exists a vertex of $V\backslash S$ that is adjacent with a vertex from each component. By Lemma 14, if P_3 is a component in $G[S_2]$, then either (i) there are four vertices of $V\backslash S$ that are dominated by at least one of the three vertices of this P_3 but by no other vertex of S, or (ii) there is a (*unique*) P_2-component at distance two from this P_3-component and there are six vertices of $V\backslash S$ that are dominated by at least one of the five vertices from these two components but by no other vertex of S.

Let S^* be the set of all vertices of S_2 that belong to a P_2-component of $G[S_2]$ that is at distance at least three from every P_3-component of $G[S_2]$. If $S^* \neq \emptyset$, then $G[S^*]$ is the disjoint union of copies of P_2. A proof of the following lemma is presented in Section 5.5.

Lemma 15. $|S^*| \leqslant 4|N[S^*]|/9$ *and the vertices in* $N[S^*]$ *are not adjacent in G to any vertex of* $S\backslash N[S^*]$.

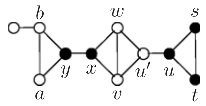

Fig. 7. A subgraph of G.

The following result is an immediate consequence of Lemmas 13–15.

Lemma 16. $|S_2| \leqslant 5n_2/11$, *and the vertices in* $N[S_2]$ *are not adjacent in* G *to any vertex of* $S \backslash N[S_2]$.

By Lemmas 12 and 16, $\gamma_t(G) = |S_1| + |S_2| \leqslant 4n_1/9 + 5n_2/11 = 5n/11$. This completes the proof of the theorem.

5.1. *Proof of Lemma 10*

To prove Lemma 10, we first prove two claims.

Claim 1. *If* $u \in S$ *belongs to a* K_3 *in* $G[S]$, *then* $N[u] \subset S$.

Proof. Let $X = \{s, t, u\}$ be a subset of S such that $G[X] = K_3$. Suppose that $N[u] \not\subset S$. Since both neighbors of u in X have degree at least two in $G[S]$, the vertex u has property P_1 by condition (1). Let $\mathrm{epn}(u, S) = \{u'\}$. Let $N(u') = \{u, v, w\}$. Since $\mathrm{epn}(u, S) = \{u'\}$, $\{u', v, w\} \cap S = \emptyset$. Since G is claw-free, $G[\{u', v, w\}] = K_3$.

Claim 1.1. *The vertex* v *does not belong to a* $K_4 - e$.

Proof. Suppose that v belongs to a $K_4 - e$. Let x be the common neighbor of v and w, different from u', and let y be the remaining neighbor of x. To totally dominate v and w, $\{x, y\} \subset S$.

Suppose $y \in X$, say $y = t$. If $N[s] \subset S$, then $(S \backslash \{u, t\}) \cup \{v\}$ is a TDS of G of cardinality less than $\gamma_t(G)$, which is impossible. Hence, $N(s) \cap S = \{t, u\}$. Since S satisfies condition (1), $|\mathrm{epn}(s, S)| = 1$. But then $(S \backslash \{x, t\}) \cup \{u', v\}$ is a TDS of G that satisfies condition (1) but induces fewer K_3's than does $G[S]$, contradicting our choice of S. Hence, $y \notin X$.

If y is adjacent to a vertex of X, then, since G is claw-free, $N(y) = \{s, t, x\}$. Thus, G is the graph G_1 shown in Fig. 3, a contradiction since then $\gamma_t(G) = 4$ but $|S| = 5$. Hence, $N(y) \cap X = \emptyset$. Let $N(y) = \{a, b\}$. Since G is claw-free, $G[\{a, b, y\}] = K_3$. If $\{a, b\} \subset S$, then $(S \backslash \{u, y\}) \cup \{v\}$ is a TDS of G, a contradiction. Hence, $|\{a, b\} \cap S| \leqslant 1$.

Suppose $a \in S$. Then, $b \notin S$. If a has degree two in $G[S]$, then $(S \backslash \{u, y\}) \cup \{v\}$ is a TDS of G, a contradiction. Hence, a has degree one in $G[S]$. Since S satisfies condition (1), $|\mathrm{epn}(a, S)| = 1$. But then $(S \backslash \{u\}) \cup \{v\}$ is a TDS of G that satisfies condition (1) but induces fewer K_3's than does $G[S]$, contradicting our choice of S. Hence, $a \notin S$. Similarly, $b \notin S$. Hence, $G[\{x, y\}]$ is a component in $G[S]$.

If $\mathrm{epn}(y, S) = \emptyset$, then $(S \backslash \{u, y\}) \cup \{w\}$ is a TDS of G, which is impossible. Hence, $|\mathrm{epn}(y, S)| \geqslant 1$. We may assume $b \in \mathrm{epn}(y, S)$. Thus the graph shown in Fig. 7 is a subgraph of G. But then $(S \backslash \{u\}) \cup \{v\}$ is a TDS of G that satisfies condition (1) but induces fewer K_3's than does $G[S]$, contradicting our choice of S. □

Let v' and w' be the neighbors of v and w, respectively, not in the triangle $G[\{u', v, w\}]$. By Claim 1.1, $v' \neq w'$. Then, $\{v', w'\} \subset S$ to dominate v and w.

Claim 1.2. $\{v', w'\} \cap \{s, t\} = \emptyset$.

Proof. If $\{v', w'\} = \{s, t\}$, say if $v' = s$ and $w' = t$, then $G = K_2 \times K_3$ and $n = 6$, a contradiction since then $\gamma_t(G) = 2$ but $|S| = 3$. Suppose $|\{v', w'\} \cap \{s, t\}| = 1$. We may assume $w' = t$. If $N[s] \subset S$, then $(S \backslash \{u, t\}) \cup \{v\}$ is a TDS of G, a contradiction. Hence, $N(s) \cap S = \{t, u\}$. Since S satisfies condition (1), $|\mathrm{epn}(s, S)| = 1$. Let $N(v') = \{a, b, v\}$. Then, $G[\{a, b, v'\}] = K_3$. To totally dominate v', we may assume $a \in S$. If a has degree two or three in $G[S]$, then $(S \backslash \{u, v'\}) \cup \{w\}$ is a TDS of G, a contradiction. Hence, a has degree one in $G[S]$, and so $G[\{a, v'\}]$ is a component of $G[S]$. If $\mathrm{epn}(a, S) = \emptyset$, then $(S \backslash \{a, t\}) \cup \{v\}$ is a TDS of G, a contradiction. Hence, $|\mathrm{epn}(a, S)| = 1$. But then

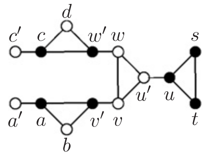

Fig. 8. A subgraph of G where $\text{epn}(a, S) = \{a'\}$ and $\text{epn}(c, S) = \{c'\}$.

$(S\backslash\{u\}) \cup \{w\}$ is a TDS of G that satisfies condition (1) but induces fewer K_3's than does $G[S]$, contradicting our choice of S. □

If $v'w' \in E(G)$, then since G is claw-free, v' and w' have a common neighbor. But then $(S\backslash\{u, w'\}) \cup \{v\}$ is a TDS of G, a contradiction. Hence, $v'w' \notin E(G)$. Let $N(v') = \{a, b, v\}$ and let $N(w') = \{c, d, w\}$. Since G is claw-free, $G[\{a, b, v'\}] = K_3$ and $G[\{c, d, w'\}] = K_3$. If $\{a, b\} = \{s, t\}$, then $(S\backslash\{u, v'\}) \cup \{w\}$ is a TDS of G, a contradiction. Hence, $\{a, b\} \cap X = \emptyset$. Similarly, $\{c, d\} \cap X = \emptyset$.

In order to dominate v', we may assume that $a \in S$. If a has degree two or three in $G[S]$, then $(S\backslash\{u, v'\}) \cup \{w\}$ is a TDS of G, a contradiction. Hence, a has degree one in $G[S]$, thus implying $\{a, b\} \neq \{c, d\}$, and so $G[\{a, v'\}]$ is a component of $G[S]$. If $\text{epn}(a, S) = \emptyset$, then $(S\backslash\{a, u\}) \cup \{v\}$ is a TDS of G, a contradiction. Hence, $|\text{epn}(a, S)| = 1$ and so a is a vertex of degree one in $G[S]$ that has property P_1. Similarly, to dominate w' we may assume that c is a vertex of degree one in $G[S]$ that has property P_1. Thus the graph shown in Fig. 8 is a subgraph of G. But then $(S\backslash\{u\}) \cup \{v\}$ is a TDS of G that satisfies condition (1) but induces fewer K_3's than does $G[S]$, contradicting our choice of S. □

Claim 2. *The maximum degree in $G[S]$ is at most two.*

Proof. Suppose that $N[u] \subset S$ for some vertex $u \in S$. Then $\text{epn}(u, S) = \emptyset$, and so, by condition (1), u has property P_2 and therefore has a neighbor v of degree one in $G[S]$ that has property P_1. Let $N(u) = \{s, t, v\}$. Then, $G[\{s, t, u\}] = K_3$. Let $X = \{s, t, u\}$. Let s' and t' be the neighbors of s and t, respectively, not in X. By Claim 1, $s' \in S$ and $t' \in S$. Since S satisfies condition (1), $s' \neq t'$ and s', t' are vertices of degree one in $G[S]$ that have property P_1. Let $N(v) = \{u, w, x\}$. Since G is claw-free, $G[\{v, w, x\}] = K_3$. Since $|\text{epn}(v, S)| \geqslant 1$, we may assume that $w \in \text{epn}(v, S)$. If $x \notin \text{epn}(v, S)$, then $(S\backslash\{u, v\}) \cup \{x\}$ is a TDS, a contradiction. Hence, $\text{epn}(v, S) = \{w, x\}$.

Suppose that w belongs to a $K_4 - e$. Let y be the common neighbor of w and x different from v, and let z be the remaining neighbor of y. Since $\text{epn}(v, S) = \{w, x\}$, $y \notin S$, and so $z \in S$. Since G is claw-free, $z \notin \{s', t'\}$. Let z' be a neighbor of z in S. If z' has degree two or three in $G[S]$, then $(S\backslash\{u, v, z\}) \cup \{y, w\}$ is a TDS in G. If z' has degree one in $G[S]$ and $\text{epn}(z', S) = \emptyset$, then $(S\backslash\{u, v, z'\}) \cup \{y, w\}$ is a TDS in G. If z' has degree one in $G[S]$ and $\text{epn}(z', S) \neq \emptyset$, then $(S\backslash\{u, v\}) \cup \{y, w\}$ is a TDS in G that satisfies condition (1) but induces fewer K_3's than does $G[S]$. All these cases lead to a contradiction. Hence, w does not belong to a $K_4 - e$. Let $w' = N(w)\backslash\{v, x\}$ and let $x' = N(x)\backslash\{v, w\}$. Then, $x' \neq w'$. Since $\text{epn}(v, S) = \{w, x\}$, $\{w', x'\} \cap S = \emptyset$.

Claim 2.1. $w'x' \notin E(G)$.

Proof. Suppose $w'x' \in E(G)$. Let c be the common neighbor of w' and x', and let d be the remaining neighbor of c. Since G is claw-free, $G[N[d]\backslash\{c\}] = K_3$. In order to totally dominate w' and x', $\{c, d\} \subset S$. If d has degree two or three in $G[S]$, then $(S\backslash\{u, v, c\}) \cup \{w, w'\}$ is a TDS of G, a contradiction. Hence, d has degree one in $G[S]$, and so $G[\{c, d\}]$ is a component of $G[S]$. If $\text{epn}(d, S) = \emptyset$, then $(S\backslash\{d, u, v\}) \cup \{w, w'\}$ is a TDS of G, a contradiction. Hence, $|\text{epn}(d, S)| \geqslant 1$, and so d is a vertex of degree one in $G[S]$ that has property P_1. Thus the graph shown in Fig. 9 is a subgraph of G. But then $(S\backslash\{u, v\}) \cup \{w, w'\}$ is a TDS of G that satisfies condition (1) but induces fewer K_3's than does $G[S]$, contradicting our choice of S. □

Let $N(w') = \{e, f, w\}$ and let $N(x') = \{g, h, x\}$. Since G is claw-free, $G[\{e, f, w'\}] = K_3$ and $G[\{g, h, x'\}] = K_3$. By condition (1), $\{e, f\} \neq \{g, h\}$ and thus $\{e, f\} \cap \{g, h\} = \emptyset$. In order to dominate w', we may assume that $e \in S$. Let e' be a neighbor of e in $G[S]$ different from f if such a neighbor exists (possibly, $e = s'$ and $e' = s$, but e' is necessarily different from s'). If e' has degree two or three in $G[S]$, in particular if $e = s'$, then $(S\backslash\{e, u, v\}) \cup \{w, w'\}$ is a TDS of

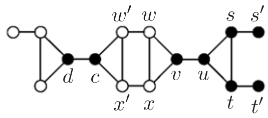

Fig. 9. A subgraph of G.

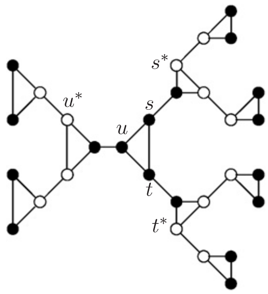

Fig. 10. A subgraph of G.

G, a contradiction. Hence, e' has degree one in $G[S]$. If $\mathrm{epn}(e', S) = \emptyset$, then $(S \backslash \{e', u, v\}) \cup \{w, w'\}$ is a TDS of G, a contradiction. Hence, $|\mathrm{epn}(e', S)| \geqslant 1$, and so e' is a vertex of degree one in $G[S]$ that has property P_1.

If $f \notin S$, then $(S \backslash \{u, v\}) \cup \{w, w'\}$ is a TDS of G that satisfies condition (1) but induces fewer K_3's than does $G[S]$, contradicting our choice of S. Hence $f \in S$. Similarly, $\{g, h\} \in S$.

Repeating the argument with the vertex u replaced by s or t shows that the graph shown in Fig. 10 is a subgraph of G. But then with the vertices s^*, t^* and u^* as indicated in Fig. 10, $(S \backslash \{u, s, t\}) \cup \{u^*, s^*, t^*\}$ is a TDS of G that satisfies condition (1) but induces fewer K_3's than does $G[S]$, contradicting our choice of S. □

We can now return to the proof of Lemma 10. As an immediate consequence of Claims 1 and 2, every component of $G[S]$ is an induced path or cycle different from K_3. Suppose that $G[S]$ contains a path P_5 on five vertices or a cycle C_p with $p \geqslant 4$. Let v denote the central vertex of the P_5 or any vertex of C_p, and let v_1 and v_2 be the neighbors of v in S. Let v' (v_1', v_2', respectively) be the neighbor of v (v_1, v_2) in $V \backslash S$. Since G is claw-free, v', v_1', v_2' are not S-external private neighbors of v, v_1, v_2, and so v does not have property P_1 nor P_2. This contradicts the fact that the set S satisfies condition (1). Hence each component of $G[S]$ is a path of length at most 3.

5.2. *Proof of Lemma 11*

Since u is an isolated vertex in $G[V \backslash S]$, $N(u) \subset S$. Let $N(u) = \{v, w, x\}$ where $vw \in E(G)$. To prove Lemma 11, we first prove six claims.

Claim 3.1. *The vertex u does not belong to a $K_4 - e$, except if $G = G_1$.*

Proof. Suppose that u belongs to a $K_4 - e$. Then, u is a vertex of degree three in this $K_4 - e$ since S satisfies condition (1). We may assume that $wx \in E(G)$. Let v' and x' be the neighbors of v and x, respectively, not in this $K_4 - e$. Since G is claw-free, $v' \neq x'$. Since S satisfies condition (1), w must have property P_2, and so we may assume that v has property P_1, i.e., $\mathrm{epn}(v, S) = \{v'\}$. Moreover, if $x' \notin S$ then $\mathrm{epn}(x, S) = \{x'\}$.

Claim 3.1.1. $v'x' \notin E(G)$.

Proof. Suppose $v'x' \in E(G)$. Let y be the common neighbor of v' and x' and let z denote the remaining neighbor of y. Let $N(z) = \{a, b, y\}$. Then, $G[\{a, b, z\}] = K_3$. Since $\mathrm{epn}(v, S) = \{v'\}$, $\{x', y\} \cap S = \emptyset$, and so x has property P_1 and $\mathrm{epn}(x, S) = \{x'\}$. In order to totally dominate y, we may assume that $\{a, z\} \subset S$. If a has degree two or three in $G[S]$, then $(S \backslash \{x, z\}) \cup \{v'\}$ is a TDS of G, a contradiction. Hence, a has degree one in $G[S]$, and so $G[\{a, z\}]$ is a component

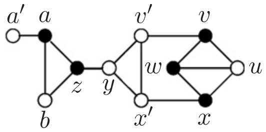

Fig. 11. A subgraph of G where $\mathrm{epn}(a, S) = \{a'\}$.

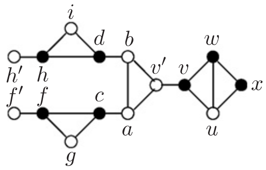

Fig. 12. A subgraph of G where $\mathrm{epn}(f, S) = \{f'\}$ and $\mathrm{epn}(h, S) = \{h'\}$.

of $G[S]$. If $\mathrm{epn}(a, S) = \emptyset$, then $(S\backslash\{a, v, x\}) \cup \{u, y\}$ is a TDS of G, a contradiction. Hence, $|\mathrm{epn}(a, S)| = 1$, and so a is a vertex of degree one in $G[S]$ that has property P_1. Thus the graph shown in Fig. 11 is a subgraph of G. But then $S' = (S\backslash\{x\}) \cup \{v'\}$ is a TDS of G that satisfies conditions (1) and (2) but with $c(S') < c(S)$, contradicting our choice of S. □

Let $N(v') = \{a, b, v\}$. Since $\mathrm{epn}(v, S) = \{v'\}$, $\{a, b\} \cap S = \emptyset$.

Claim 3.1.2. *If a belongs to a $K_4 - e$, then $G = G_1$.*

Proof. Suppose a belongs to a $K_4 - e$. Let f be the second common neighbor of a and b. Let g be the remaining neighbor of f. Then, $\{f, g\} \subset S$. Note that $f \neq x$ and $g \neq x'$. Suppose $f \neq x'$. We then consider the component $\mathscr{C}$ of $G[S]$ containing $\{f, g\}$. If $\mathscr{C}$ is a P_4 or a P_2 such that $\mathrm{epn}(g, S) = \emptyset$, then $(S\backslash\{g, v\}) \cup \{a\}$ is a TDS of G, a contradiction. If $\mathscr{C}$ is a P_3 or a P_2 such that $|\mathrm{epn}(g, S)| \geqslant 1$, then $S' = (S\backslash\{v\}) \cup \{a\}$ is a TDS of G that satisfies conditions (1) and (2) but with $c(S') < c(S)$ (irrespective of whether or not $x' \in S$), contradicting our choice of S. Hence, $f = x'$, and so $g = x$. Thus, $G = G_1$. □

Let c and d be the neighbors of a and b, respectively, not in the triangle $G[\{a, b, v'\}]$. Since $v'x' \notin E(G)$ by Claim 3.1.1, $x' \notin \{a, b\}$ and thus $x \notin \{c, d\}$. Since G is claw-free, $x' \notin \{c, d\}$. To dominate a and b, $\{c, d\} \subset S$. If $cd \in E(G)$, then c and d have a common neighbor and $(S\backslash\{d, v\}) \cup \{a\}$ is a TDS of G, a contradiction. Hence, $cd \notin E(G)$. Let $N(c) = \{a, f, g\}$ and let $N(d) = \{b, h, i\}$. Note that $\{f, g\} \neq \{h, i\}$ by symmetry with Claim 3.1.1, and thus $\{f, g\} \cap \{h, i\} = \emptyset$. To totally dominate c and d, we may assume $\{f, h\} \subset S$. Hence since $G[S]$ is K_3-free, $g \notin S$ and $i \notin S$. If f has degree two in $G[S]$, then $(S\backslash\{c, v\}) \cup \{b\}$ is a TDS of G, a contradiction. Hence, $G[\{c, f\}]$ is a component if $G[S]$. If $\mathrm{epn}(f, S) = \emptyset$, then $(S\backslash\{f, v\}) \cup \{a\}$ is a TDS of G, a contradiction. Hence, $|\mathrm{epn}(f, S)| = 1$ and f is a vertex of degree one in $G[S]$ that has property P_1. Similarly, $|\mathrm{epn}(h, S)| = 1$ and h is a vertex of degree one in $G[S]$ that has property P_1. Thus the graph shown in Fig. 12 is a subgraph of G. But then $S' = (S\backslash\{v\}) \cup \{a\}$ is a TDS of G that satisfies conditions (1) and (2) but with $c(S') < c(S)$ (irrespective of whether or not $x' \in S$), contradicting our choice of S. This completes the proof of Claim 3.1. □

By Claim 3.1, we may assume that $G[\{v, w, x\}] = K_2 \cup K_1$.

Claim 3.2. *The vertex u does not belong to a 4-cycle.*

Proof. Suppose that u belongs to a 4-cycle u, x, y, w, u. Let z be the common neighbor of x and y. Since S satisfies condition (1), $y \notin S$ and so $z \in S$. Let $N(v) = \{u, w, v'\}$ and let $N(z) = \{x, y, z'\}$. Since S satisfies condition (1), each of v and z has property P_1, and so $\mathrm{epn}(v, S) = \{v'\}$ and $\mathrm{epn}(z, S) = \{z'\}$. Thus the graph shown in Fig. 13 is a

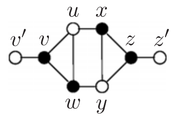

Fig. 13. A subgraph of G.

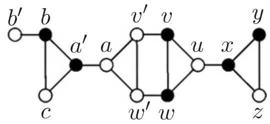

Fig. 14. A subgraph of G where $\mathrm{epn}(b, S) = \{b'\}$.

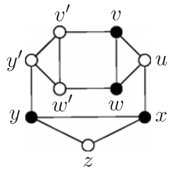

Fig. 15. The subgraph G_u.

subgraph of G. But then $S' = (S \backslash \{w\}) \cup \{u\}$ is a TDS of G that satisfies conditions (1) and (2) but with $c(S') < c(S)$, contradicting our choice of S. $\square$

Let $N(x) = \{u, y, z\}$. Since G is claw-free, $G[\{x, y, z\}] = K_3$. To dominate x, we may assume $y \in S$. Since $G[S]$ is K_3-free, $z \notin S$. Since S satisfies condition (1), $|\mathrm{epn}(y, S)| = 1$. Let $\mathrm{epn}(y, S) = \{y'\}$ and let $N(z) = \{x, y, z'\}$ (possibly, $y' = z'$). By Claim 3.2, $z' \notin \{v, w\}$.

Claim 3.3. $N(z) \cap S = \{x, y\}$.

Proof. Suppose $z' \in S$. Then, $z' \neq y'$ and $y'z' \notin E(G)$. Let $N(z') = \{g, f, z\}$. Then, $G[\{g, f, z'\}] = K_3$. To totally dominate z', we may assume $g \in S$. Since $G[S]$ is K_3-free, $f \notin S$. Since $\mathrm{epn}(z', S) = \emptyset$, g is a vertex of degree one in $G[S]$ that has property P_1. But then $S' = (S \backslash \{x\}) \cup \{z\}$ is a TDS of G that satisfies conditions (1) and (2) but with $c(S') < c(S)$, contradicting our choice of S. Hence, $z' \notin S$ (possibly, $y' = z'$). $\square$

Let $N(v) = \{u, w, v'\}$ and let $N(w) = \{u, v, w'\}$. Since S satisfies condition (1), $v' \neq w'$.

Claim 3.4. *If $v'w' \in E(G)$, then the desired result follows.*

Proof. Let a be the common neighbor of v' and w', and let a' be the remaining neighbor of a. By Lemma 10 and since S satisfies condition (1), we may assume that $\mathrm{epn}(v, S) = \{v'\}$. Thus, $\{a, w'\} \cap S = \emptyset$, and so $\mathrm{epn}(w, S) = \{w'\}$. To dominate $a, a' \in S$. Hence, $a \neq z'$. Suppose $a \neq y'$. Let $N(a') = \{a, b, c\}$. Since G is claw-free, $G[\{a', b, c\}] = K_3$. To totally dominate a', we may assume $b \in S$ and so $c \notin S$. If b has degree two in $G[S]$, then $(S \backslash \{a', w\}) \cup \{v'\}$ is a TDS of G, a contradiction. Hence, b has degree one in $G[S]$, and so $G[\{a', b\}]$ is a component of $G[S]$. If $\mathrm{epn}(b, S) = \emptyset$, then $(S \backslash \{b, v, w\}) \cup \{a, u\}$ is a TDS, a contradiction. Hence, $|\mathrm{epn}(b, S)| = 1$, and so b is a vertex of degree one in $G[S]$ that has property P_1.The graph shown in Fig. 14 is therefore a subgraph of G. But then $S' = (S \backslash \{w\}) \cup \{v'\}$ is a TDS that satisfies conditions (1) and (2), but with $c(S') < c(S)$, contradicting our choice of S. Hence, $a = y'$ and $a' = y$.

Let $V_u = \{u, v, v', w, w', x, y, y', z\}$ and let $G_u = G[V_u]$ (see Fig. 15). Further, let $S_u = \{v, w, x, y\}$. Then S_u is a TDS of G_u of cardinality four-ninths the order of G_u. Since in G, $N(t) \cap S \subset S_u$ for every vertex $t \in V(G_u)$ (including the vertex z by Claim 3.3), we uniquely associate u with the connected subgraph G_u, as desired. $\square$

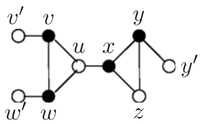

Fig. 16. The subgraph G_u where zy' may or may not be an edge.

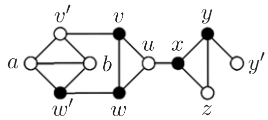

Fig. 17. A subgraph of G.

By Claim 3.4, we may assume that $v'w' \notin E(G)$, for otherwise the desired result follows. Let $N(v') = \{a, b, v\}$ and let $N(w') = \{c, d, w\}$. Since G is claw-free, $G[\{a, b, v'\}] = K_3$ and $G[\{c, d, w'\}] = K_3$.

Claim 3.5. *If $G[\{v, w\}]$ is a component in $G[S]$, then the desired result follows.*

Proof. Since S satisfies condition (1), at least one of v or w has property P_1. We may assume $\text{epn}(v, S) = \{v'\}$. Then $\{a, b\} \cap S = \emptyset$ and $\{a, b\} \neq \{c, d\}$ for otherwise a and b are not dominated. Suppose w does not have property P_1. Then, w' is also dominated by a vertex of $S \backslash \{w\}$. We may assume $c \in S$. Then, irrespective of whether or not $d \in S$, $S' = (S \backslash \{w\}) \cup \{u\}$ is a TDS that satisfies conditions (1) and (2), but with $c(S') < c(S)$, contradicting our choice of S. Hence, $\text{epn}(w, S) = \{w'\}$. Thus, $\{a, b, c, d\} \cap S = \emptyset$.

Let $V_u = \{u, v, v', w, w', x, y, y', z\}$ and let $G_u = G[V_u]$ (see Fig. 16). Further, let $S_u = \{v, w, x, y\}$. Then S_u is a TDS of G_u of cardinality four-ninths the order of G_u. Since $N(t) \cap S \subset S_u$ for every vertex $t \in V(G_u)$ (including the vertex z by Claim 3.3), we uniquely associate u with the subgraph G_u, as desired. □

By Claim 3.5, we may assume that the component of $G[S]$ containing v and w is either P_3 or P_4. The next result shows that in fact this component must be a P_4.

Claim 3.6. *The vertices v and w are internal vertices of a P_4 in $G[S]$.*

Proof. Suppose that v has degree one in $G[S]$. Then, by assumption, w has degree two in $G[S]$, and so $w' \in S$. Since S satisfies condition (1), $\text{epn}(v, S) = \{v'\}$ and so $\{a, b\} \cap S = \emptyset$. We consider two possibilities.

Case 1: w' has degree one in $G[S]$. Since S satisfies condition (1), w' has property P_1 and so $|\text{epn}(w', S)| \geqslant 1$. If $\{a, b\} = \{c, d\}$, then the graph shown in Fig. 17 is a subgraph of G. But then $(S \backslash \{v\}) \cup \{a\}$ is a TDS of G that satisfies conditions (1) and (2) but with $c(S') < c(S)$, contradicting our choice of S. Hence, $\{a, b\} \cap \{c, d\} = \emptyset$.

Suppose that a and b have a common neighbor h. Let i be the remaining neighbor of h and let $N(i) = \{h, j, k\}$. Then, $G[\{i, j, k\}] = K_3$. To totally dominate a and b, $\{h, i\} \subset S$. Since at least one of c and d belongs to the set $\text{epn}(w', S)$, $\{c, d\} \cap \{j, k\} = \emptyset$. Suppose $\{j, k\} \cap S = \emptyset$. If $\text{epn}(i, S) = \emptyset$, then $(S \backslash \{i, v\}) \cup \{a\}$ is a TDS of G, a contradiction. Hence, $|\text{epn}(i, S)| \geqslant 1$ and i is a vertex of degree one in $G[S]$ that has property P_1. But then $S' = (S \backslash \{v\}) \cup \{a\}$ is a TDS of G that satisfies conditions (1) and (2) but with $c(S') < c(S)$, contradicting our choice of S. Hence, since $G[S]$ is K_3-free, $|\{j, k\} \cap S| = 1$. We may assume that $j \in S$ and $k \notin S$. If j has degree two in $G[S]$, then $(S \backslash \{i, v\}) \cup \{a\}$ is a TDS of G, a contradiction. Hence, j is a vertex of degree one in $G[S]$. Since S satisfies condition (1), j has property P_1 and so $|\text{epn}(j, S)| = 1$. The graph shown in Fig. 18 is therefore a subgraph of G. But then $S' = (S \backslash \{v\}) \cup \{a\}$ is a TDS of G that satisfies conditions (1) and (2) but with $c(S') < c(S)$, contradicting our choice of S. Hence, a and b have no common neighbor.

Let a' and b' be the neighbors of a and b, respectively, not in the triangle $G[\{a, b, v'\}]$. In order to dominate a and b, $a' \in S$ and $b' \in S$, respectively. If $a'b' \in E(G)$, then a' and b' have a common neighbor, and $\text{epn}(a', S) = \{a\}$ and $\text{epn}(b', S) = \{b\}$. The graph shown in Fig. 19 is therefore a subgraph of G. But then $S' = (S \backslash \{b', v\}) \cup \{a\}$ is a TDS of G, a contradiction. Hence, $a'b' \notin E(G)$.

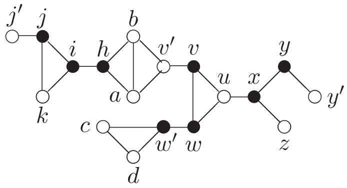

Fig. 18. A subgraph of G where $\mathrm{epn}(j, S) = \{j'\}$ and $\mathrm{epn}(w', S) = \{c, d\}$.

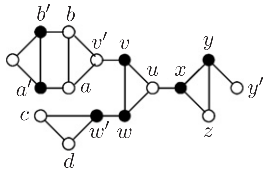

Fig. 19. A subgraph of G.

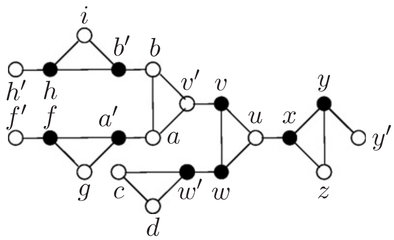

Fig. 20. A subgraph of G where $\mathrm{epn}(f, S) = \{f'\}$ and $\mathrm{epn}(h, S) = \{h'\}$.

Let $N(a') = \{a, f, g\}$ and let $N(b') = \{b, h, i\}$. Then, $G[\{a', f, g\}] = K_3$ and $G[\{b', h, i\}] = K_3$. To totally dominate a' (resp., b'), we may assume that $f \in S$ (resp., $h \in S$). Since $G[S]$ is K_3-free, $g \notin S$ and $i \notin S$. If $\{f, g\} = \{h, i\}$, then g would be an isolated vertex in $G[V \setminus S]$ contained in a $K_4 - e$, contradicting Claim 3.1. Hence, $\{f, g\} \cap \{h, i\} = \emptyset$. If f has degree two in $G[S]$, then $(S \setminus \{a', v\}) \cup \{b\}$ is a TDS of G, a contradiction. Hence, f is a vertex of degree one in $G[S]$. If $\mathrm{epn}(f, S) = \emptyset$, then $(S \setminus \{f, v\}) \cup \{a\}$ is a TDS of G, a contradiction. Hence, $|\mathrm{epn}(f, S)| = 1$ and f is a vertex of degree one in $G[S]$ that has property P_1. Similarly, $|\mathrm{epn}(h, S)| = 1$ and h is a vertex of degree one in $G[S]$ that has property P_1. The graph shown in Fig. 20 is therefore a subgraph of G. But then $S' = (S \setminus \{v\}) \cup \{a\}$ is a TDS of G that satisfies conditions (1) and (2) but with $c(S') < c(S)$, contradicting our choice of S. Hence Case 1 cannot occur.

Case 2: w' has degree two in $G[S]$. Since $G[S]$ is K_3-free, we may assume that $c \in S$ and $d \notin S$. Since S satisfies condition (1), c has property P_1 and so $|\mathrm{epn}(c, S)| = 1$. Since $\{a, b\} \cap S = \emptyset$, $\{a, b\} \cap \{c, d\} = \emptyset$ and therefore the triangles $G[\{a, b, v'\}]$ and $G[\{c, d, w'\}]$ are disjoint. Proceeding now exactly as in Case 1 (except that the first situation, $\{a, b\} = \{c, d\}$, cannot occur), we can contradict our choice of S. This completes the proof of Claim 3.6. □

We now return to our proof of Lemma 11. By Claim 3.6, we have $\{v', w'\} \subset S$. Thus, $G[\{v, v', w, w'\}] = P_4$ is a component of $G[S]$. Since S satisfies condition (1), $|\mathrm{epn}(v', S)| \geqslant 1$ and $|\mathrm{epn}(w', S)| \geqslant 1$. We may assume $b \in \mathrm{epn}(v', S)$. If $a \notin \mathrm{epn}(v', S)$, then a is dominated by two vertices of S. But then $(S \setminus \{v, v'\}) \cup \{a\}$ is a TDS of G, a contradiction. Hence, $\mathrm{epn}(v', S) = \{a, b\}$. Similarly, $\mathrm{epn}(w', S) = \{c, d\}$.

Suppose a and b have a common neighbor f. Let g be the remaining neighbor of f and let $N(g) = \{f, h, i\}$. Since $a \in \mathrm{epn}(v', S)$, $f \notin S$. To totally dominate f, we may assume that $\{g, h\} \subset S$, and so $i \notin S$. If h has degree two in $G[S]$, then $(S \setminus \{g, v\}) \cup \{a\}$ is a TDS of G, a contradiction. Hence, h is a vertex of degree one in $G[S]$. If $\mathrm{epn}(h, S) = \emptyset$, then $(S \setminus \{h, v'\}) \cup \{f\}$ is a TDS of G, a contradiction. Hence, $|\mathrm{epn}(h, S)| = 1$ and h is a vertex of degree one in $G[S]$ that has property P_1. The graph shown in Fig. 21 is therefore a subgraph of G. But then $S' = (S \setminus \{v, v'\}) \cup \{a, f\}$ is a

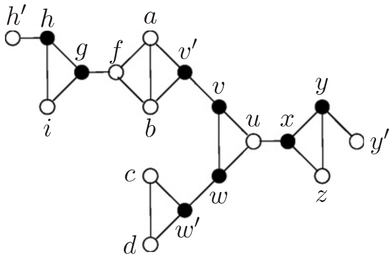

Fig. 21. A subgraph of G where $\mathrm{epn}(h, S) = \{h'\}$.

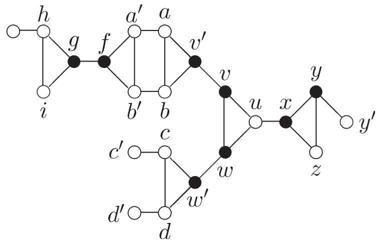

Fig. 22. A subgraph of G.

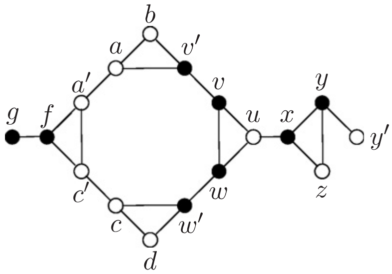

Fig. 23. A subgraph of G.

TDS of G that satisfies conditions (1) and (2) but with $c(S') < c(S)$, contradicting our choice of S. Hence, a and b do not have a common neighbor. Similarly, c and d do not have a common neighbor.

Let a' and b' be the neighbors of a and b, respectively, that do not belong to the triangle $G[\{a, b, v'\}]$. Further, let c' and d' be the neighbors of c and d, respectively, that do not belong to the triangle $G[\{c, d, w'\}]$. Since $\mathrm{epn}(v', S) = \{a, b\}$ and $\mathrm{epn}(w', S) = \{c, d\}$, $\{a', b', c', d'\} \cap S = \emptyset$.

Suppose that $a'b' \in E(G)$. Let f be the common neighbor of a' and b', and let g be the remaining neighbor of f. Let $N(g) = \{f, h, i\}$. To totally dominate a' and b', $\{f, g\} \subset S$. If g has degree two in $G[S]$, then $(S \backslash \{f, v, v'\}) \cup \{a, a'\}$ is a TDS of G, a contradiction. Hence, $h \notin S$ and $i \notin S$. If $\mathrm{epn}(g, S) = \emptyset$, then $(S \backslash \{g, v, v'\}) \cup \{a, a'\}$ is a TDS of G, a contradiction. Hence, $|\mathrm{epn}(g, S)| \geqslant 1$ and g is a vertex of degree one in $G[S]$ that has property P_1. The graph shown in Fig. 22 is therefore a subgraph of G. But then $S' = (S \backslash \{v, v'\}) \cup \{a, a'\}$ is a TDS of G that satisfies conditions (1) and (2) but with $c(S') < c(S)$, contradicting our choice of S. Hence, $a'b' \notin E(G)$. Similarly, $c'd' \notin E(G)$.

Suppose that $a'c' \in E(G)$. Let f be the common neighbor of a' and c', and let g be the remaining neighbor of f. Then, $\{f, g\} \subset S$ and $\mathrm{epn}(f, S) = \{a', c'\}$. Since $|\mathrm{epn}(y, S)| = 1$, $f \neq y$ (and clearly, $f \neq x$). The graph shown in Fig. 23 is therefore a subgraph of G. If g has degree two in $G[S]$, then $(S \backslash \{f, v, w\}) \cup \{a, c\}$ is a TDS of G, a contradiction. Hence g has degree one in $G[S]$. If $\mathrm{epn}(g, S) = \emptyset$, then $(S \backslash \{g, v, v'\}) \cup \{a, a'\}$ is a TDS of G, a contradiction. Hence, $|\mathrm{epn}(g, S)| \geqslant 1$ and g is a vertex of degree one in $G[S]$ that has property P_1. But then $S' = (S \backslash \{v, v'\}) \cup \{a, a'\}$ is a

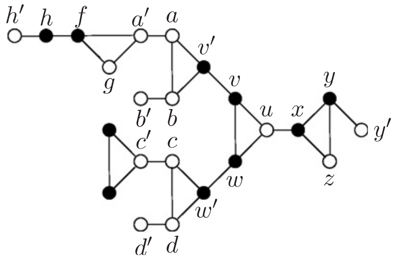

Fig. 24. A subgraph of G where h has degree one in $G[S]$ and $h' \in \mathrm{epn}(h, S)$.

TDS of G that satisfies conditions (1) and (2) but with $c(S') < c(S)$, contradicting our choice of S. Hence, $a'c' \notin E(G)$. Similarly, there is no edge joining a vertex in $\{a', b'\}$ and a vertex in $\{c', d'\}$.

Let $N(a') = \{a, f, g\}$. Then $G[\{a', f, g\}] = K_3$. If a' belongs to a common $K_4 - e$ with b', c' or d', then $\{f, g\} \subseteq S$ and $(S \backslash \{f, v, v'\}) \cup \{a, a'\}$ is a TDS of G, a contradiction. Hence no $K_4 - e$ in G contains a' and a vertex in $\{b', c', d'\}$. Similarly, no $K_4 - e$ in G contains two vertices from $\{a', b', c', d'\}$. To dominate a', we may assume that $f \in S$. Let h be the neighbor of f not in the triangle $G[\{a', f, g\}]$. Let $N(c') = \{c, i, j\}$. Then $G[\{c', i, j\}] = K_3$. To dominate c', we may assume that $i \in S$. Let k be the neighbor of i not in the triangle $G[\{c', i, j\}]$.

Suppose $g \notin S$. Then, $h \in S$ to totally dominate f. If h has degree two in $G[S]$, then $(S \backslash \{f, v, v'\}) \cup \{a, a'\}$ is a TDS of G, a contradiction. Hence, $G[\{f, h\}]$ is a component in $G[S]$. If $\mathrm{epn}(h, S) = \emptyset$, then $(S \backslash \{h, v, v'\}) \cup \{a, a'\}$ is a TDS of G, a contradiction. Hence, $|\mathrm{epn}(h, S)| \geqslant 1$. Therefore, h is a vertex of degree one in $G[S]$ that has property P_1. Similarly, if $j \notin S$, then k is a vertex of degree one in $G[S]$ that has property P_1. But then $S' = (S \backslash \{v, w\}) \cup \{a, c\}$ is a TDS of G that satisfies conditions (1) and (2) but with $c(S') < c(S)$, contradicting our choice of S. Hence, $j \in S$. Thus the graph shown in Fig. 24 is a subgraph of G. But once again $S' = (S \backslash \{v, w\}) \cup \{a, c\}$ is a TDS of G that satisfies conditions (1) and (2) but with $c(S') < c(S)$, contradicting our choice of S. Hence, $g \in S$.

We have shown that $N(a') \backslash \{a\} \subset S$. Similarly, $N(b') \backslash \{b\} \subset S$, $N(c') \backslash \{c\} \subset S$ and $N(d') \backslash \{d\} \subset S$. But once again $S' = (S \backslash \{v, w\}) \cup \{a, c\}$ is a TDS of G that satisfies conditions (1) and (2) but with $c(S') < c(S)$, contradicting our choice of S. This completes the proof of Lemma 11.

5.3. Proof of Lemma 13

Let u, v, w, x be a P_4 in $G[S_2]$. To prove Lemma 13, we first prove the following claim.

Claim 4. *v and w have a common neighbor.*

Proof. Suppose that v and w do *not* have a common neighbor. Let y and z be the neighbors of v and w, respectively, in $V \backslash S$. Then, $y \neq z$. Since G is claw-free and since every vertex of $V(G_2) \backslash S_2$ is adjacent to at most two vertices of S_2, $N(y) \cap S = \{u, v\}$ and the neighbor y' of y is not in S (y' is possibly equal to z). Similarly, $N(z) \cap S = \{w, x\}$. Let u' and x' be the neighbors of u and x in $V \backslash S$ different from y and z, respectively. Since S satisfies condition (1), $\mathrm{epn}(u, S) = \{u'\}$ and $\mathrm{epn}(x, S) = \{x'\}$.

Suppose $u'x' \in E(G)$. Let a be the common neighbor of u' and x', and let b be the remaining neighbor of a. Let $N(b) = \{a, c, d\}$. Then, $G[\{b, c, d\}] = K_3$. Since $\mathrm{epn}(u, S) = \{u'\}$, $a \notin S$, and so $b \in S$ to dominate a. But then $(S \backslash \{u, x\}) \cup \{a\}$ is a TDS of G, a contradiction. Hence, $u'x' \notin E(G)$.

Let $N(u') = \{a, b, u\}$ and $N(x') = \{c, d, x\}$. Then, $G[\{a, b, u'\}] = K_3$ and $G[\{c, d, x'\}] = K_3$. Since $\mathrm{epn}(u, S) = \{u'\}$ and $\mathrm{epn}(x, S) = \{x'\}$, $\{a, b\} \cap S = \emptyset$ and $\{c, d\} \cap S = \emptyset$. Hence, $\{a, b\} \cap \{c, d\} = \emptyset$.

Suppose a and b have a common neighbor f, different from u'. Let g be the remaining neighbor of f. To totally dominate a and b, $\{f, g\} \subset S$. If g has degree two in $G[S]$, then $(S \backslash \{f, v\}) \cup \{u'\}$ is a TDS of G, a contradiction. Hence, g has degree one in $G[S]$. If $\mathrm{epn}(g, S) = \emptyset$, then $(S \backslash \{u, g\}) \cup \{a\}$ is a TDS of G, a contradiction. Hence, $|\mathrm{epn}(g, S)| \geqslant 1$, and so g is a vertex of degree one in $G[S]$ that has property P_1. But then $S' = (S \backslash \{v\}) \cup \{u'\}$ is a TDS of G that satisfies conditions (1) and (2) but with $c(S') < c(S)$, contradicting our choice of S. Hence, a and b do not have a common neighbor. Similarly, c and d do not have a common neighbor.

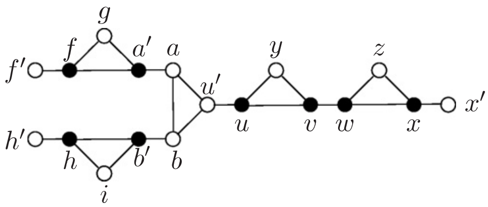

Fig. 25. A subgraph of G where $f' \in \text{epn}(f, S)$ and $h' \in \text{epn}(h, S)$.

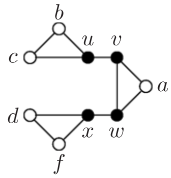

Fig. 26. A P_4-component in $G[S]$ where $\text{epn}(u, S) = \{b, c\}$ and $\text{epn}(x, S) = \{d, f\}$.

Let a' and b' be the neighbors of a and b, respectively, that do not belong to the triangle $G[\{a, b, u'\}]$. Further, let c' and d' be the neighbors of c and d, respectively, that do not belong to the triangle $G[\{c, d, x'\}]$. Since G is claw-free, $\{a', b'\} \cap \{c', d'\} = \emptyset$. To dominate $\{a, b, c, d\}$, $\{a', b', c', d'\} \subset S$. If $a'b' \in E(G)$, then a' and b' have a common neighbor and $(S \backslash \{b', u\}) \cup \{a\}$ is a TDS of G, a contradiction. Hence, $a'b' \notin E(G)$.

Let $N(a') = \{a, f, g\}$ and let $N(b') = \{b, h, i\}$. Then, $G[\{a', f, g\}] = K_3$ and $G[\{b', h, i\}] = K_3$. To totally dominate a' and b', we may assume that $f \in S$ and $h \in S$, respectively. Thus, since $G[S]$ is K_3-free, $g \notin S$ and $i \notin S$. If $\{f, g\} = \{h, i\}$, then g would be an isolated vertex in $G[V \backslash S]$ contained in a $K_4 - e$, contradicting Claim 3.1. Hence, $\{f, g\} \cap \{h, i\} = \emptyset$. If f has degree two in $G[S]$, then $(S \backslash \{a', v\}) \cup \{u'\}$ is a TDS of G, a contradiction. Hence, f has degree one in $G[S]$. If $\text{epn}(f, S) = \emptyset$, then $(S \backslash \{f, u\}) \cup \{a\}$ is a TDS of G, a contradiction. Hence, $|\text{epn}(f, S)| = 1$, and so f is a vertex of degree one in $G[S]$ that has property P_1. Similarly, $|\text{epn}(h, S)| = 1$ and h is a vertex of degree one in $G[S]$ that has property P_1. Hence the graph shown in Fig. 25 is a subgraph of G. But then $S' = (S \backslash \{v\}) \cup \{u'\}$ is a TDS of G that satisfies conditions (1) and (2) but with $c(S') < c(S)$, contradicting our choice of S. $\square$

We now return to the proof of Lemma 13. By Claim 4, v and w have a common neighbor, a say. We show now that each of u and x has two external private neighbors. Let $N(u) = \{b, c, v\}$ and let $N(x) = \{d, f, w\}$. Since S satisfies condition (1), $|\text{epn}(u, S)| \geqslant 1$ and $|\text{epn}(x, S)| \geqslant 1$. We may assume $b \in \text{epn}(u, S)$. If $c \notin \text{epn}(u, S)$, then c is dominated by two vertices of S. But then $(S \backslash \{u, v\}) \cup \{c\}$ is a TDS of G, a contradiction. Hence, $\text{epn}(u, S) = \{b, c\}$. Similarly, $\text{epn}(x, S) = \{d, f\}$. Thus the graph shown in Fig. 26 is a subgraph of G and the third neighbor a' of a is in $V \backslash S$ by the definition of S_2. This completes the proof of Lemma 13.

5.4. *Proof of Lemma 14*

Let u, v, w be a P_3-component in $G[S_2]$, and let $S' = \{u, v, w\}$. Since G is claw-free, we may assume that v and w have a common neighbor, say a. Since S satisfies condition (1), $|\text{epn}(u)| \geqslant 1$ and $|\text{epn}(w)| = 1$. Let $\text{epn}(w, S) = \{b\}$. Let $N(u) = \{c, d, v\}$. Then, $G[\{c, d, u\}] = K_3$. We may assume that $c \in \text{epn}(u, S)$. If $d \in \text{epn}(u, S)$, then the graph G' shown in Fig. 6(a) is a subgraph of G with $V(G') = N[S']$ and where the vertices in $V(G') \backslash S'$ are not adjacent in G to any vertex of $S \backslash S'$.

Suppose then that $d \notin \text{epn}(u, S)$. Then, d is dominated by a vertex of $S \backslash \{u\}$, say x. Let y be a vertex of S adjacent to x. Since G is claw-free, x and y have a common neighbor, say f. Further, since $G[S]$ is K_3-free, $f \in V \backslash S$, and so x has no external private neighbor. Thus, x must have property P_2. Consequently, $|\text{epn}(y, S)| = 1$. Let $\text{epn}(y, S) = \{g\}$. Hence the graph G' shown in Fig. 27 is a subgraph of G where the vertices in $V(G')$ are not adjacent in G to any vertex of $S \backslash V(G')$. This completes the proof of Lemma 14.

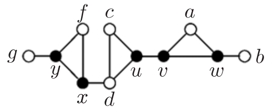

Fig. 27. A subgraph of G where $\text{epn}(u, S) = \{c\}$, $\text{epn}(w, S) = \{b\}$ and $\text{epn}(y, S) = \{g\}$.

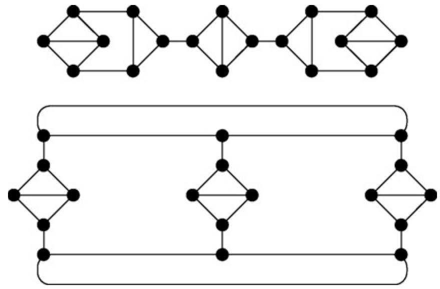

Fig. 28. Claw-free cubic graphs with total domination numbers four-ninths their orders.

5.5. *Proof of Lemma 15*

Let $|S^*| = 2k$. Let T be the set of all vertices of $V \setminus S$ that are dominated by S^* and let $|T| = t$. Let $n^* = |S^*| + |T|$. Let $[S^*, T]$ denote the set of all edges with one end in S^* and the other in T. Since each vertex of S^* is adjacent to exactly two vertices of T, $|[S^*, T]| = 2|S^*| = 4k$. On the other hand, let ℓ denote the number of vertices in T that are dominated by a unique vertex of S^*. Since S satisfies condition (1), at least one vertex in every P_2-component of $G[S^*]$ has property P_1. Hence at least k vertices in S^* have an external private neighbor, and so $\ell \geqslant k$. Thus, since every vertex of T is adjacent to at most two vertices of S by the definition of S_2, $|[S^*, T]| = \ell + 2(t - \ell) = 2(n^* - 2k) - \ell \leqslant 2n^* - 5k$. Consequently, $k \leqslant 2n^*/9$, and so $|S^*| \leqslant 4n^*/9$, as desired.

6. Conclusion

We remark that our proof of Theorem 8 shows that if G has no subgraph G' shown in Fig. 6(b) where the vertices in $V(G')$ are not adjacent in G to any vertex of $S \setminus V(G')$, then $\gamma_t(G) \leqslant 4n/9$. We believe that the bound of five-elevenths the order is not sharp, and we close with the following conjecture.

Conjecture 1. Every connected claw-free cubic graph of order at least 10 has total domination number at most four-ninths its order.

If Conjecture 1 is true, then the bound is tight as may be seen by considering the connected claw-free cubic graphs F and H shown in Fig. 28 with total domination number four-ninths their orders.

Final remark (concerning paired domination): In a previous paper [5] we proved that if a connected claw-free cubic graph of order $n \geqslant 6$ does not contain $K_4 - e$ nor C_4 as an induced subgraph, then its paired domination number satisfies $\gamma_{\text{pr}}(G) \leqslant 3n/8$ and the unique extremal graph has 48 vertices. The proof used the property established by Hobbs and Schmeichel that the matching number $\nu(H)$ of a cubic graph H of order N is at least $7N/16$. This property was recently improved (see [2]) for $N > 16$ to $\nu(H) \geqslant (4N - 1)/9$. Using this new result, our bound on $\gamma_{\text{pr}}(G)$ in connected cubic $(K_{1,3}, K_4 - 3, C_4)$-free graphs improves for $n \geqslant 48$ to $(10n + 6)/27$ with infinitely many extremal graphs.

Acknowledgment

The second author thanks the LRI for their warm hospitality.

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