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On Uniform Approximation by Cubic Splines

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Let

$$0 = x_0 < x_1 < \dots < x_n = 1, \qquad x_{n+k} = x_k + 1 \qquad (k = \pm 1, \pm 2, \dots) \quad (1.1)$$

be a subdivision of $(-\infty, \infty)$ with $\max_i (x_i - x_{i-1}) = \delta$. Let C_π denote the class of 1-periodic continuous functions on $(-\infty, \infty)$ and let S_π be the family of 1-periodic cubic splines with joints (1.1) having two continuous derivatives. In an earlier paper [6], we have shown that for every $f \in C_\pi$, the unique $\sigma(f) \in S_\pi$ which interpolates $f(x)$ at the points (1.1) satisfies the inequality

$$\|f - \sigma(f)\| \leqslant (1 + K^2)\,\omega(f; \delta)$$

where the norm is the usual uniform norm, $\omega(f;\delta)$ is the modulus of continuity of f and the mesh ratio K is given by $\max_i(x_i - x_{i-1})/\min_j(x_j - x_{j-1})$. Later Stig Nord showed [5] that there exists a sequence of joints with unbounded mesh ratios and a continuous function $f(x)$ such that the corresponding $\sigma(f)$ do not converge to f although the maximum mesh length tends to zero. The question then arises whether the boundedness of the K's is necessary for the convergence of the interpolatory splines. This problem is also related to a question raised by Cheney and Schurer [3] who seek to determine conditions on the sequence of points which would assure the boundedness of the norms of the sequence of interpolatory splines. In Theorem 1, we shall show that for the uniform convergence of the interpolatory splines it is sufficient that the ratios of consecutive mesh lengths remain between certain bounds.

Another result of Cheney and Schurer [3] states that for every $f \in C_\pi$, we have

$$\operatorname{dist}(f, S_\pi) \equiv \inf_{s \in S_\pi} \|f - s\| \leqslant 18\omega(f; \delta). \quad (1.2)$$

We shall show in Theorem 2 that the number 18 on the right side of (1.2) can be replaced by 5. This number can be further reduced if instead of S_π we consider the class S of twice continuously differentiable cubic splines with nodes (1.1) without periodic end conditions. In this case we have for every $f \in C[0, 1]$

$$\operatorname{dist}(f, S) \leqslant (2 + \tfrac{5}{48})\,\omega(f; \delta). \quad (1.3)$$

If moreover $f \in C_\pi$, then there exists $s \equiv s(f) \in S$ such that s and s' are 1-periodic and

$$\|f - s(f)\| \leqslant (2 + \tfrac{5}{48})\,\omega(f;\delta). \tag{1.4}$$

This result is the subject of Theorem 3 where we use a recent result of Hall [4].

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THEOREM 1. *Let $f \in C_\pi$ and let* (1.1) *be a subdivision of* [0, 1] *with $h_i = x_i - x_{i-1}$ $(i = 1,\ldots,n)$. Suppose*

$$\max_{|i-j|=1} h_i h_j^{-1} = P < \sqrt{2}. \tag{2.1}$$

Then for the interpolatory cubic spline $\sigma(x) \equiv \sigma(f;x)$ we have

$$\|f - \sigma\| \leqslant \left(1 + \tfrac{3}{4}\cdot\frac{P}{2-P^2}\right)\omega(f;\delta). \tag{2.2}$$

Proof. Let $x_{k-1} \leqslant x < x_k$ for a certain k and let

$$L_k(x) = \{f_k\cdot(x - x_{k-1}) + f_{k-1}\cdot(x_k - x)\}\,h_k^{-1} \qquad \text{where } f_i = f(x_i).$$

It was shown in [6] that

$$|\sigma(x) - L_k(x)| \leqslant \tfrac{1}{8}h_k^2 \max\{|\sigma''_{k-1}|, |\sigma_k''|\} \tag{2.3}$$

where $\sigma_i'' = \sigma''(x_i)$. If $\max_\nu h_\nu^2|\sigma_\nu''| = h_q^2|\sigma_q''|$, and $\max_\nu h_\nu^2|\sigma''_{\nu-1}| = h_p^2|\sigma''_{p-1}|$, then on using (3.0) of [6], we easily have

$$\begin{aligned}2(h_q^2 + h_q h_{q+1})\,|\sigma_q''| &\leqslant h_q^2|\sigma''_{q-1}| + h_q h_{q+1}|\sigma''_{q+1}| \\ &\quad + 6(h_q^2 + h_q h_{q+1})\,|[x_{q-1}, x_q, x_{q+1}; f]| \\ &\leqslant P^2 h_q^2|\sigma_q''| + P^2 h_q h_{q+1}|\sigma_q''| \\ &\quad + 6(h_q^2 + h_q h_{q+1})\,|[x_{q-1}, x_q, x_{q+1}; f]|. \end{aligned} \tag{2.4}$$

Observing that $|[x_{q-1}, x_q, x_{q+1}; f]| \leqslant \omega(f;\delta)/h_q h_{q+1}$, we have from (2.4) after simplification

$$h_q^2|\sigma_q''| \leqslant \frac{6P}{2-P^2}\,\omega(f;\delta)$$

and so, *a fortiori*, for all k,

$$h_k^2|\sigma_k''| \leqslant \frac{6P}{2-P^2}\,\omega(f;\delta).$$

Similarly we obtain for all k the inequality

$$h_k^2|\sigma''_{k-1}| \leqslant \frac{6P}{2-P^2}\,\omega(f;\delta).$$

Hence from (2.3) we have for $x_{k-1} \leqslant x < x_k$,

$$|\sigma(x) - f(x)| \leqslant |\sigma(x) - L_k(x)| + |L_k(x) - f(x)|$$
$$\leqslant \tfrac{3}{4} \cdot \frac{P}{2 - P^2} \omega(f; \delta) + \omega(f; \delta)$$

which proves (2.2).

Remark 1. This theorem throws light on the test case 3 treated by Cheney and Schurer in [3], p. 95.

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We now formulate the following

LEMMA. *For any subdivision* (1.1) *with* $\max_i(x_i - x_{i-1}) = \delta \leqslant \frac{1}{2}$ *there exist points* $0 \leqslant \xi_0 < \xi_1 < \ldots < \xi_{m-1} < \xi_m = 1 + \xi_0$, $\xi_{m+k} = 1 + \xi_k$ $(k = \pm 1, \pm 2, \ldots)$ *such that*

(i) *the* ξ_j's *form a subset of the* x_j's,
(ii) $\delta \leqslant \max_j(\xi_j - \xi_{j-1}) \leqslant 2\delta$.

Proof. Let ν be the smallest positive integer for which $x_\nu - x_{\nu-1} = \delta$ and define $\xi_0 = x_\nu$. Assume that $\xi_0 < \xi_1 < \ldots < \xi_{j-1}$ have been defined and that $\xi_{j-1} \leqslant 1 + \xi_0 - 2\delta$. Then we define

$$\xi_j = \min\{x_k | \xi_{j-1} + \delta \leqslant x_k < \xi_{j-1} + 2\delta\}. \tag{3.1}$$

If $m - 1$ is the first suffix for which $\xi_{m-1} > 1 + \xi_0 - 2\delta$ then we define $\xi_m = 1 + \xi_0$. Observe that because of (3.1), $\xi_{m-1} \leqslant x_{\nu+n-1} = 1 + \xi_0 - \delta$ and so $\delta \leqslant \xi_m - \xi_{m-1} \leqslant 2\delta$. Setting $\xi_{m+k} = 1 + \xi_k$ for $k = \pm 1, \pm 2, \ldots$ the lemma follows.

As a consequence we shall prove

THEOREM 2. *For any given subdivision* (1.1) *of* $(-\infty, +\infty)$ *there exists a linear operator* $L: C_\pi \to S_\pi$ *such that for* $f \in C_\pi$,

$$\|L(f) - f\| \leqslant 5\omega(f; \delta). \tag{3.2}$$

Proof. For $f \in C_\pi$, we define $L(f; x)$ to be the unique interpolatory 1-periodic cubic spline interpolating f at the joints $\{\xi_j\}_{-\infty}^{\infty}$. Then obviously $L(f; x)$ is a spline on the original given joints (1.1). By a theorem of Nord [5] (Theorem 6, p. 141),

$$|f(x) - L(f; x)| \leqslant (1 + \tfrac{3}{4}K')\,\omega(\delta') \tag{3.3}$$

where K' and δ' are the mesh ratio and maximum mesh length for the points $\{\xi_j\}$. But from (ii) of the above lemma $K' \leqslant 2$ and $\delta' \leqslant 2\delta$ so that (3.2) follows from (3.3).

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Let

$$0 = x_0 < x_1 < \ldots < x_n = 1 \tag{4.1}$$

be a subdivision of [0, 1] and denote by S_{2m-1} the set of splines of order $2m-1$ on [0, 1] with interior joints $x_1, x_2, \ldots, x_{n-1}$. Recently Birkhoff [*1*] and subsequently de Boor [*2*] have obtained estimates for the distance of a continuous function f from the set of splines of order $2m-1$ with joints (4.1). They show that if $f \in C^{2m}$, then $\operatorname{dist}(f, S_{2m-1}) = O(\delta^{2m})$ where δ is the maximum mesh length. The more precise estimate (1.3) when $f \in C[0,1]$ and $S = S_3$ is a consequence of

THEOREM 3. *For any given subdivision* (4.1) *of* [0, 1] *there exists a linear operator* $\Lambda: C \to S_3$ *such that for all* $f \in C[0,1]$,

$$\|\Lambda(f) - f\| \leqslant (2 + \tfrac{5}{48})\,\omega(f;\delta). \tag{4.2}$$

If f is 1-periodic, then $\Lambda(f;x)$ and its derivative are periodic.

For the proof we shall use the following result of Hall [*4*]:

Let $g \in C^4[0,1]$ and let $s(g;x) \in C^2[0,1]$ be the cubic spline interpolating g at the joints (4.1) and satisfying the end conditions $s'(0) = g'(0)$, $s'(1) = g'(1)$. Then $\|s - g\| \leqslant \frac{5}{348}\|g^{(\mathrm{iv})}\|\,\delta^4$.

We shall also need the following

LEMMA. *Let $g \in C(-\infty,\infty)$ and let $g_0(x) = g(x)$,*

$$g_k(x) = \frac{1}{\delta}\int_{x+(\delta/2)}^{x+(\delta/2)} g_{k-1}(t)\,dt, \qquad k = 1, 2, \ldots; \delta > 0. \tag{4.3}$$

Then $g_k \in C^k(-\infty,\infty)$ and

$$\|g_k^{(k)}\| \leqslant 2^{k-1}\,\delta^{-k}\,\omega(g;\delta) \tag{4.4}$$

$$\|g_k - g\| \leqslant \omega\left(g; \frac{k\delta}{2}\right). \tag{4.5}$$

The proof follows by easy induction.

Proof of Theorem 3. Given $f \in C[0,1]$, set

$$\begin{aligned} g(x) &= f(x), \quad && 0 \leqslant x \leqslant 1, \\ &= f(0), && x \leqslant 0, \\ &= f(1), && x \geqslant 1, \end{aligned}$$

and define for $0 \leqslant x \leqslant 1$, $\Lambda(f;x) = s(g_4;x)$ where $g_4(x)$ is given by (4.3) and $s(x) \equiv s(g_4;x)$ is the cubic spline interpolating $g_4(x)$ as in Hall's theorem. Obviously, Λ is a linear operator on $C[0,1]$ and we have

$$\begin{aligned} \|\Lambda(f) - f\| &= \|s(g_4) - f\| \\ &\leqslant \|s(g_4) - g_4\| + \|g_4 - g\|. \end{aligned}$$

Since $g \equiv f$ for $0 \leqslant x \leqslant 1$, it follows from (4.4), (4.5) and Hall's theorem that

$$\|\Lambda(f) - f\| \leqslant \tfrac{40}{384}\,\omega(f;\delta) + \omega(f;2\delta)$$

which yields (4.2).

References

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ADDED IN PROOF: From a personal communication of Cheney and Schurer we learn that they have been able to improve our Theorem 1 on using their earlier results in [3] and ous Theorem 2 above. In our notation their result reads: If $P < 2$, then for every $f \in C_\pi$ we have

$$\text{(i)} \quad \|\sigma(f)\| \leqslant \frac{7}{2-P}\|f\| \quad \text{and} \quad \text{(ii)} \quad \|f - \sigma\| \leqslant \frac{40}{2-P}\,\omega(f;\delta).$$