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0-Bisimple Inverse Semigroups

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In this paper a structure theory for 0-bisimple inverse semigroups is given in terms of groups and semilattices. The semilattices concerned are 0-uniform; that is, they are of the form $E = E^0$, where any two nonzero principal ideals of E are isomorphic.

Let S^* be a 0-bisimple inverse semigroup with semilattice E and let μ denote the greatest congruence on S^* contained in $\mathscr{H}$. Then $S = S^*/\mu$ is a 0-bisimple inverse semigroup with semilattice isomorphic to E and with no nontrivial congruences contained in $\mathscr{H}$ (Theorem 1.1). Moreover, E is 0-uniform and S can be expressed as a certain semigroup of partial isomorphisms of E (Theorems 1.2, 1.3). These results are taken from an earlier paper [10]. Now let G be any μ-class of S^* containing a nonzero idempotent. Then G is a subgroup of S^*. The main problem is to express the structure of S^* in terms of G and S. A solution is provided by Theorem 2.2, which generalises the Schreier extension theorem for groups. To conclude, the theory is applied to a class of 0-bisimple inverse semigroups characterised by a particular type of semilattice (Theorem 4.2).

From the results for 0-bisimple inverse semigroups we can easily deduce corresponding results for bisimple inverse semigroups. An alternative structure theory for the latter has already been developed by Reilly and Clifford [14]; this rests on the notion of an RP-system introduced by Reilly [12], in a paper generalising the work of Clifford [2] on bisimple inverse semigroups with an identity. The principal result of [14] gives a representation of the elements of a bisimple inverse semigroup S^* by equivalence classes of $\bar{R} \times G \times \bar{R}$, where G is a group consisting of any idempotent μ-class of S^* and $\bar{R}$ is a certain RP-system. Moreover, these equivalence classes are singletons if and only if $\mu = \mathscr{H}$ on S^*.

Finally, Reilly [13] makes explicit the connection between the approach of [14] and that of the present paper.

1. Fundamental 0-Bisimple Inverse Semigroups

We shall follow the notation and terminology of [3].

By an *inverse semigroup* we mean a semigroup S in which to each element a there corresponds a unique element a^{-1} (the inverse of a) such that

$$a\,a^{-1}a = a, \qquad a^{-1}a\,a^{-1} = a^{-1}$$

[3, Section 1.9]. Any two idempotents of S commute and so the set of all idempotents is a subsemigroup of S; we call this *the semilattice of* S and denote it by E_S. For $a, b \in S$ and $e \in E_S$ we have that

$$(a^{-1})^{-1} = a, \qquad (ab)^{-1} = b^{-1}a^{-1}, \qquad e^{-1} = e.$$

It is easy to verify that an inverse semigroup $S = S^0$ is 0-bisimple if and only if, to each pair $(e, f) \in E_S\backslash 0 \times E_S\backslash 0$, there corresponds $a \in S$ such that

$$aa^{-1} = e, \qquad a^{-1}a = f.$$

Let ρ be a congruence on an inverse semigroup S. Then every idempotent ρ-class contains an idempotent of S and so S/ρ is an inverse semigroup; moreover, $(x\rho)^{-1} = x^{-1}\rho$ for all $x \in S$ [3, Theorem 7.36]. We shall be concerned only with those congruences ρ such that $\rho \subseteq \mathscr{H}$; these are precisely the *idempotent-separating congruences* on S. There is a greatest such congruence μ, which can be described as follows [4]:

$$(a, b) \in \mu \Leftrightarrow a^{-1}\,ea = b^{-1}\,eb \qquad \text{for all} \quad e \in E_S\,.$$

As in [10], we say that S is *fundamental* if and only if the only congruence on S contained in $\mathscr{H}$ is the identity congruence ι. Thus S is fundamental if and only if $\mu = \iota$.

In this section we restate several results from [10] concerning fundamental 0-bisimple inverse semigroups. Green's relations $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$ on a semigroup S will be denoted by $\mathscr{R}_S$, $\mathscr{L}_S$, $\mathscr{H}_S$ when we want to emphasise the particular semigroup involved. Likewise, the identity relation ι on S will sometimes be denoted by ι_S.

The first theorem lists some of the basic properties of idempotent-separating congruences.

THEOREM 1.1. *Let S^* be an inverse semigroup, let ρ be an idempotent-separating congruence on S^* and let μ be the greatest such congruence. Let S denote the inverse semigroup S^*/ρ. Then*

(i) $E_S \cong E_{S^*}$;

(ii) $(a\rho, b\rho) \in \mathscr{R}_S[\mathscr{H}_S] \Leftrightarrow (a, b) \in \mathscr{R}_{S^*}[\mathscr{H}_{S^*}](a, b \in S^*)$;

(iii) *S is fundamental* $\Leftrightarrow \rho = \mu$;

(iv) $\mathscr{H}_S = \iota_S \Leftrightarrow \rho = \mathscr{H}_{S^*}(=\mu)$;

(v) *S is 0-bisimple* $\Leftrightarrow$ *S^* is 0-bisimple.*

Proof. (i) This is immediate from [3, Lemma 7.34].

(ii) It is clear that if $(a, b) \in \mathscr{R}_{S^*}$, then $(a\rho, b\rho) \in \mathscr{R}_S$. Assume, conversely, that $(a\rho, b\rho) \in \mathscr{R}_S$. Then there exist $x, y \in S^*$ such that $(ax)\rho = b\rho$, $a\rho = (by)\rho$. Hence, since $\rho \subseteq \mathscr{H}_{S^*} \subseteq \mathscr{R}_{S^*}$, there exist $u, v \in S^*$ such that $(ax)u = b$, $a = (by)v$ and this shows that $(a, b) \in \mathscr{R}_{S^*}$. A similar argument applies to $\mathscr{L}$ and the result for $\mathscr{H}$ then follows.

(iii) If $\rho = \mu$, then S is fundamental [4, p. 75; 10, Theorem 2.4]. Now suppose, conversely, that S is fundamental. Since $\rho \subseteq \mu$ we can define a congruence μ/ρ on S by the rule that

$$(a\rho, b\rho) \in \mu/\rho \Leftrightarrow (a, b) \in \mu \qquad (a, b \in S^*).$$

Let $(a\rho, b\rho) \in \mu/\rho$. Then $(a, b) \in \mu \subseteq \mathscr{H}_{S^*}$ and so, from (ii), $(a\rho, b\rho) \in \mathscr{H}_S$. Thus $\mu/\rho \subseteq \mathscr{H}_S$ and so $\mu/\rho = \iota_S$, since S is fundamental. Hence $\rho = \mu$.

(iv) This is readily deduced from (ii).

(v) Let S^* be 0-bisimple. Since the homomorphism $\rho^\natural$ is 0-restricted, it follows that S is also 0-bisimple. Conversely, let S be 0-bisimple and let $(e, f) \in E_{S^*}\backslash 0 \times E_{S^*}\backslash 0$. Then $e\rho$, $f\rho$ are nonzero idempotents of S and so there exists $a \in S^*$ such that

$$(a\rho)(a\rho)^{-1} = e\rho, (a\rho)^{-1}(a\rho) = f\rho.$$

Hence

$$(aa^{-1}, e) \in \rho, (a^{-1}a, f) \in \rho.$$

Thus, since ρ is idempotent-separating, we have that $aa^{-1} = e$, $a^{-1}a = f$ and this shows that S^* is 0-bisimple.

Note, in particular, that if S^* is a 0-bisimple inverse semigroup, then S^*/μ is a fundamental 0-bisimple inverse semigroup with semilattice isomorphic to E_{S^*}.

A semilattice $E = E^0$ is said to be *0-uniform* if and only if $Ee \simeq Ef$ for any two nonzero idempotents e, f of E. The connection between such semilattices and 0-bisimple inverse semigroups is given by the following result [10, Theorem 3.1].

THEOREM 1.2. *A semilattice is 0-uniform if and only if it is isomorphic to the semilattice of a 0-bisimple inverse semigroup.*

The proof of Theorem 1.2 depends on the following concept. Let E be a semilattice and let T_E denote the subset of $\mathscr{I}_E$, the symmetric inverse semigroup on E [3, p. 29], consisting of all isomorphisms between principal ideals of E. Then T_E is an inverse subsemigroup of $\mathscr{I}_E$ and $E_{T_E} \cong E$; moreover, if E is 0-uniform, then T_E is 0-bisimple [7, Lemma 2.1; 10, Theorem 3.1].

Let $E = E^0$. An inverse subsemigroup S of T_E is said to be *0-transitive* [10] if and only if the following two conditions hold:

(i) S contains the zero of T_E (the identity mapping of the zero ideal of E),

(ii) for all $(e, f) \in E\backslash 0 \times E\backslash 0$ there exists an element α in S whose domain is Ee and whose codomain is Ef.

It is clear that if T_E contains a 0-transitive inverse subsemigroup, then T_E itself is 0-transitive and E is 0-uniform.

The following theorem provides a characterisation of fundamental 0-bisimple inverse semigroups in terms of mappings [10, Theorem 3.1].

THEOREM 1.3. *Let $S = S^0$ be an inverse semigroup. Then S is fundamental and 0-bisimple if and only if it is isomorphic to a 0-transitive inverse subsemigroup of T_{E_S}.*

Let E be any 0-uniform semilattice. Then, in particular, T_E itself is a fundamental 0-bisimple inverse semigroup. The problem of finding all fundamental 0-bisimple inverse semigroups with semilattices isomorphic to E is reduced, by Theorem 1.3, to that of finding all 0-transitive inverse subsemigroups of T_E. A sufficient condition for T_E to contain no proper inverse subsemigroups of this type is that $\mathscr{H} = \iota$ on T_E: for two elements of T_E are $\mathscr{H}$-equivalent if and only if they have the same domain and the same codomain [10, Lemma 1.2]; hence if $\mathscr{H} = \iota$ and $(e, f) \in E\backslash 0 \times E\backslash 0$ then there exists exactly one element of T_E with domain Ee and codomain Ef.

A semilattice F is said to be *inversely well-ordered* if and only if, with respect to the natural ordering on F ($e \geqslant f \Leftrightarrow ef = f$), every subset of F has a greatest element. Such a semilattice is necessarily a chain.

A slightly modified version of the proof of [7, Theorem 3.2] then establishes the following result.

THEOREM 1.4. *Let S^* be an inverse semigroup and let $E = E_{S^*}$. Suppose that every principal ideal of E is inversely well-ordered. Then $\mathscr{H} = \iota$ on T_E and $\mu = \mathscr{H}$ on S^*. Moreover, if S^* is 0-bisimple, then $S^*/\mathscr{H} \cong T_E$.*

It follows, in particular, that, if S is a fundamental 0-bisimple inverse semigroup such that E_S contains a nonzero inversely well-ordered principal ideal, then $S \cong T_{E_S}$; for, since E_S is 0-uniform (Theorem 1.2), every principal ideal of E_S is inversely well-ordered and so $\mathscr{H} = \mu = \iota$ on S. An example is given in Section 4.

2. The Extension Theorem

We turn now to the problem of recovering a 0-bisimple inverse semigroup S^* from the fundamental 0-bisimple inverse semigroup S^*/μ and the group $e^*\mu$, where e^* is any nonzero idempotent of S^*. An answer is provided by Theorem 2.2 below. This differs from the extension theorem of D'Alarcao [1] in that we do not require to use the whole kernel of μ.

Let S be a 0-bisimple inverse semigroup, let e be a nonzero idempotent of S and let X be a transversal (cross-section) of the $\mathscr{H}$-classes of S that are contained in R_e. For each $a \in S\backslash 0$ let a_1 denote the unique element of X lying in $L_{aa^{-1}}$. Since $a_1 a_1^{-1} \in R_e$ and $a_1^{-1} a_1 \in L_{aa^{-1}}$ it follows that

$$a_1 a_1^{-1} = e, \qquad a_1^{-1} a_1 = aa^{-1}.$$

Hence

$$a_1 aa^{-1} = a_1, \qquad a a^{-1} a_1^{-1} = a_1^{-1}.$$

These properties of a_1 will be used without further comment.

Let P_e be defined by

$$P_e = R_e \cap eSe = \{x \in R_e : xe = x\}.$$

This is the right unit subsemigroup of eSe [12, Lemma 1.2]. It is therefore a right cancellative subsemigroup of S.

The following technical lemma (first given in [9], Appendix) is required for the proof of Theorem 2.2.

Lemma 2.1. *Let a, b be elements of S such that $ab \neq 0$. Define p, q by*

$$p = (ab)_1 a_1^{-1}, \qquad q = (ab)_1 ab_1^{-1}.$$

Then

$$\text{(i)} \quad pa_1 = (ab)_1, \qquad \text{(ii)} \quad qb_1 = (ab)_1 a, \qquad \text{(iii)} \quad p, q \in P_e.$$

Proof. We first note that $(ab)_1 = (ab)_1(ab)(ab)^{-1} = (ab)_1 abb^{-1}a^{-1}$; this is used in all three parts below.

(i) $pa_1 = (ab)_1 a_1^{-1} a_1 = (ab)_1 ab b^{-1}a^{-1} \cdot aa^{-1} = (ab)_1 ab b^{-1}a^{-1} = (ab)_1$.

(ii) $qb_1 = (ab)_1 ab_1^{-1} b_1 = (ab)_1 ab b^{-1}a^{-1} \cdot abb^{-1} = (ab)_1 ab b^{-1}a^{-1}a = (ab)_1 a$.

(iii) From (i), $pp^{-1} = pa_1(ab)_1^{-1} = (ab)_1(ab)_1^{-1} = e$.

Also $qq^{-1} = (ab)_1\, ab_1^{-1}\, b_1a^{-1}(ab)_1^{-1} = (ab)_1\, abb^{-1}a^{-1}(ab)_1^{-1} = (ab)_1(ab)_1^{-1} = e$. Hence $p, q \in R_e$. Further, $e = a_1a_1^{-1} = b_1b_1^{-1}$ and so $pe = p$, $qe = q$. Thus $p, q \in P_e$.

We now state the main result of this section.

THEOREM 2.2. (A) *Let S be a 0-bisimple inverse semigroup and let G be a group. Let e be a nonzero idempotent of S and let $P_e = R_e \cap eSe$, the right unit subsemigroup of eSe. Let X be a transversal of the $\mathscr{H}$-classes of S that are contained in R_e and, for all $a \in S\backslash 0$, let a_1 denote the unique element of X in $L_{aa^{-1}}$. For all $p \in P_e$ let θ_p be an endomorphism of G and for all $(p, r) \in P_e \times R_e$ let $f_{p,r}$ be an element of G. Suppose, further, that the θ_p and the $f_{p,r}$ satisfy the following four conditions*:

(C1) $f_{p,r}\theta_q = f_{q,p}f_{qp,r}f_{q,pr}^{-1} \quad (p, q \in P_e;\, r \in R_e)$,

(C2) $g\theta_p\theta_q = f_{q,p}(g\theta_{qp})f_{q,p}^{-1} \quad (p, q \in P_e;\, g \in G)$,

(C3) $g\theta_e = g \quad (g \in G)$,

(C4) $f_{p,e} = f_{e,r} = 1$, *the identity of G* $(p \in P_e;\, r \in R_e)$.

Let $S^* = (G \times S\backslash 0) \cup 0$ and define a multiplication in S^* by the rule that, for all $(g, a), (h, b) \in G \times S\backslash 0$,

$$0(g, a) = (g, a)0 = 0^2 = 0,$$

and

$$(g, a)(h, b) = \begin{cases} (f_{p,a_1}^{-1}(g\theta_p)f_{p,a_1a}f_{q,b_1}^{-1}(h\theta_q)f_{q,b_1b}\,,\ ab) & \text{if } ab \neq 0, \\ 0, & \text{if } ab = 0, \end{cases}$$

where $p = (ab)_1\, a_1^{-1}$ and $q = (ab)_1\, a\, b_1^{-1}$. (By Lemma 2.1 (iii), $p, q \in P_e$.) *Then S^* is a 0-bisimple inverse semigroup. Moreover, there exists an idempotent-separating congruence ρ on S^* such that $S^*/\rho \cong S$ and $e^*\rho \cong G$ for any nonzero idempotent e^* in S^*.*

(B) *Conversely, if S^* is a 0-bisimple inverse semigroup and ρ an idempotent-separating congruence on S^* then, to within isomorphism, S^* has the structure described above, where $S = S^*/\rho$ and $G = e^*\rho$ for any nonzero idempotent e^* of S^*.*

Proof. (A) We first show that S^* is a semigroup under the prescribed multiplication. Consider the elements (g, a), (h, b), (k, c) of $G \times S\backslash 0$. To establish associativity it is enough to show that

$$[(g, a)(h, b)](k, c) = (g, a)[(h, b)(k, c)].$$

Clearly this holds if $abc = 0$; hence we assume that $abc \neq 0$.

The G-component of $[(g, a)(h, b)](k, c)$ is

$$f^{-1}_{r,(ab)_1}[f^{-1}_{p,a_1}(g\theta_p)f_{p,a_1a}f^{-1}_{q,b_1}(h\theta_q)f_{q,b_1b}]\,\theta_r f_{r,(ab)_1ab}f^{-1}_{s,c_1}(k\theta_s)f_{s,c_1c}\,, \qquad (1)$$

where

$$p = (ab)_1\,a_1^{-1}, \qquad q = (ab)_1\,ab_1^{-1}, \qquad r = (abc)_1\,(ab)_1^{-1}, \qquad s = (abc)_1\,ab\,c_1^{-1}.$$

From (C1), (C2) and Lemma 2.1 (i), (ii) we have that

$$(f_{p,a_1}\theta_r)^{-1} = (f_{r,p}f_{rp,a_1}f^{-1}_{r,pa_1})^{-1} = f_{r,(ab)_1}f^{-1}_{rp,a_1}f^{-1}_{r,p}\,,$$

$$g\theta_p\theta_r = f_{r,p}(g\theta_{rp})f^{-1}_{r,p}\,,$$

$$f_{p,a_1a}\theta_r = f_{r,p}f_{rp,a_1a}f^{-1}_{r,pa_1a} = f_{r,p}f_{rp,a_1a}f^{-1}_{r,(ab)_1a}\,,$$

$$(f_{q,b_1}\theta_r)^{-1} = (f_{r,q}f_{rq,b_1}f^{-1}_{r,qb_1})^{-1} = f_{r,(ab)_1a}f^{-1}_{rq,b_1}f^{-1}_{r,q}\,,$$

$$h\theta_q\theta_r = f_{r,q}(h\theta_{rq})f^{-1}_{r,q}\,,$$

$$f_{q,b_1b}\theta_r = f_{r,q}f_{rq,b_1b}f^{-1}_{r,qb_1b} = f_{r,q}f_{rq,b_1b}f^{-1}_{r,(ab)_1ab}\,.$$

Substituting the expressions on the right in (1) we find that the G-component of $[(g, a)(h, b)](k, c)$ reduces to

$$f^{-1}_{rp,a_1}(g\theta_{rp})f_{rp,a_1a}f^{-1}_{rq,b_1}(h\theta_{rq})f_{rq,b_1b}f^{-1}_{s,c_1}(k\theta_s)f_{s,c_1c}\,. \qquad (2)$$

Similarly it can be shown that the G-component of $(g, a)[(h, b)(k, c)]$ is

$$f^{-1}_{v,a_1}(g\theta_v)f_{v,a_1a}f^{-1}_{wt,b_1}(h\theta_{wt})f_{wt,b_1b}f^{-1}_{wu,c_1}(k\theta_{wu})f_{wu,c_1c}\,,$$

where

$$t = (bc)_1\,b_1^{-1}, \qquad u = (bc)_1\,bc_1^{-1}, \qquad v = (abc)_1\,a_1^{-1}, \qquad w = (abc)_1\,a(bc)_1^{-1}.$$

The proof of associativity will therefore be complete if we can show that

$$rp = v, \qquad rq = wt, \qquad s = wu.$$

Now, by Lemma 2.1 (i), $r(ab)_1 = (abc)_1$. Hence

$$rp = r(ab)_1\,a_1^{-1} = (abc)_1\,a_1^{-1} = v.$$

Similarly,

$$rq = r(ab)_1\,ab_1^{-1} = (abc)_1\,ab_1^{-1}.$$

But, by Lemma 2.2 (ii), $w(bc)_1 = (abc)_1\, a$. Hence

$$wt = w(bc)_1\, b_1^{-1} = (abc)_1\, ab_1^{-1} = rq.$$

Furthermore,

$$wu = w(bc)_1\, bc_1^{-1} = (abc)_1\, abc_1^{-1} = s.$$

Thus S^* is a semigroup.

Next we show that S^* is regular. Consider the element $(g, a)(h, a^{-1})(g, a)$, where $g, h \in G$ and $a \in S\backslash 0$. Taking $b = a^{-1}$, $c = a$ above, we have that

$$rp = (aa^{-1}a)_1 a_1^{-1} = a_1\, a_1^{-1} = e,$$
$$rq = (aa^{-1}a)_1\, a(a^{-1})_1^{-1} = a_1\, a(a^{-1})_1^{-1},$$
$$s = (aa^{-1}a)_1\, a\, a^{-1}a_1^{-1} = a_1\, a_1^{-1}\, a_1 a_1^{-1} = e.$$

Hence, using (C3) and (C4), we see from (2) that the G-component of $(g, a)(h, a^{-1})(g, a)$ is

$$g f_{d,b_1}^{-1}(h\theta_d)\, f_{d,b_1 b}\, g,$$

where $b = a^{-1}$, $d = a_1 a b_1^{-1}$. Thus to show that (g, a) is a regular element of S^* it will be enough to prove that we can choose h such that

$$h\theta_d = f_{d,b_1} g^{-1} f_{d,b_1 b}^{-1}\,. \tag{3}$$

Now $d \in P_e$ and

$$d^{-1}d = b_1\, a^{-1}\, a_1^{-1}\, a_1\, a\, b_1^{-1} = b_1\, a^{-1}\, aa^{-1}a\, b_1^{-1} = b_1\, b_1^{-1} = e;$$

hence $d \in H_e$ and so $d^{-1} \in H_e$. Define h by

$$h = (f_{d,d^{-1}}^{-1} f_{d,b_1} g^{-1} f_{d,b_1 b}^{-1} f_{d,d^{-1}})\, \theta_{d^{-1}}\,.$$

Then from (C2) and (C3) we see that (3) holds. Hence S^* is regular.

We now examine the idempotents of S^*. Let $(g, a) \in S^*\backslash 0$ and suppose that $(g, a)^2 = (g, a)$. Then $a^2 = a\ (=a^{-1})$ and

$$f_{p,a_1}^{-1}(g\theta_p)\, f_{p,a_1 a}\, f_{q,a_1}^{-1}(g\theta_q)\, f_{q,a_1 a} = g,$$

where

$$p = (a^2)_1\, a_1^{-1} = a_1 a_1^{-1} = e, \qquad q = (a^2)_1\, aa_1^{-1} = a_1(aa^{-1})a_1^{-1} = a_1 a_1^{-1} = e.$$

Hence, from (C3) and (C4), we have that $g^2 = g$ and so $g = 1$. Conversely, if $a^2 = a$, then $(1, a)^2 = (1, a)$. Thus

$$E_{S^*}\backslash 0 = \{(1, a) : a^2 = a\}.$$

We prove next that the idempotents of S^* commute. Let $a, b \in E_S\backslash 0$. Suppose first that $ab \neq 0$. Then

$$(1, a)(1, b) = (f^{-1}_{p,a_1} f_{p,a_1 a} f^{-1}_{q,b_1} f_{q,b_1 b}, ab),$$

where

$$p = (ab)_1\, a_1^{-1}, \qquad q = (ab)_1\, a\, b_1^{-1}.$$

Since $a^2 = a$ it follows that $a_1 a = a_1\, aa^{-1} = a_1$. Similarly $b_1 b = b_1$. Hence $(1, a)(1, b) = (1, ab)$. But the idempotents of S commute and so $ba = ab \neq 0$. A similar argument then shows that $(1, b)(1, a) = (1, ba)$. Thus $(1, a)(1, b) = (1, b)(1, a)$. Further, this result still holds if $ab = 0$. Hence S^* is an inverse semigroup. Note if $(g, a) \in S^*\backslash 0$, then the $S\backslash 0$-component of $(g, a)^{-1}$ must be a^{-1}.

To see that S^* is 0-bisimple, consider any two nonzero idempotents $(1, a)$, $(1, b)$ of S^*. It will suffice to show that there is an element $(1, c) \in S^*$ for which

$$(1, c)(1, c)^{-1} = (1, a), \qquad (1, c)^{-1}(1, c) = (1, b).$$

Since $a, b \in E_S\backslash 0$ and S is 0-bisimple we can find $c \in S\backslash 0$ such that

$$cc^{-1} = a, \qquad c^{-1}c = b.$$

Now $(1, c)(1, c)^{-1}$ is a nonzero idempotent of S^* and its $S\backslash 0$-component is cc^{-1}. Hence

$$(1, c)(1, c)^{-1} = (1, cc^{-1}) = (1, a).$$

Similarly,

$$(1, c)^{-1}(1, c) = (1, c^{-1}c) = (1, b).$$

Thus S^* is 0-bisimple.

Define an equivalence ρ on $S^*\backslash 0$ by the rule that

$$((g, a), (h, b)) \in \rho \Leftrightarrow a = b$$

and extend this to an equivalence on S^* by taking $\{0\}$ to be a ρ-class. Clearly ρ is a congruence on S^*. Also, if $((1, a), (1, b)) \in \rho$, then $(1, a) = (1, b)$; that is, ρ is idempotent-separating. Further, $S^*/\rho \simeq S$.

Let $e^* = (1, a) \in E_{S^*}\backslash 0$. Then

$$e^*\rho = \{(g, a) : g \in G\}.$$

Then since $a^2 = a$, it follows, as in the discussion of the idempotents of S^*, that

$$\begin{aligned}(g, a)(h, a) &= (f^{-1}_{e,a_1}(g\theta_e) f_{e,a_1 a} f^{-1}_{e,a_1}(h\theta_e) f_{e,a_1 a}, a)\\ &= (gh, a).\end{aligned}$$

Hence the mapping $g \mapsto (g, a)$ is an isomorphism from G to $e^*\rho$. This completes the proof of (A).

(B) Let S^* be a 0-bisimple inverse semigroup and let ρ be an idempotent-separating congruence on S^*. Denote S^*/ρ by S. By Theorem 1.1 (v) S is a 0-bisimple inverse semigroup. Let e^* be any nonzero idempotent of S^*. For brevity, write $R^* = R_{e^*}$, $H^* = H_{e^*}$ (the $\mathscr{R}$- and $\mathscr{H}$-classes of S^* containing e^*). Since $\rho \subseteq \mathscr{H}_{S^*}$ it follows that $e^*\rho$ is a normal subgroup of the group H^*. Write

$$e = e^*\rho = G.$$

We shall denote $e^*\rho$ by e when we regard it as an element of S and by G when we regard it as a subgroup of S^*.

Now select a transversal of those ρ-classes of S^* that are contained in R^*. By Theorem 1.1 (ii) we can label the elements of this transversal by the elements of R_e. Let u_r denote the representative of the ρ-class r for all $r \in R_e$. We also adopt the convention that $u_e = e^*$. Note that $u_r u_r^{-1} = e^*$ for all $r \in R_e$.

Let $P_e = R_e \cap eSe = \{x \in R_e : xe = x\}$ and let $p \in P_e$, $r \in R_e$. Then

$$pr\,(pr)^{-1} = pr\,r^{-1}\,p^{-1} = p^{-1}\,ep = pp^{-1} = e$$

and so $pr \in R_e$; thus u_{pr} is defined. Hence

$$(u_p\,u_r)\,\rho^\natural = pr = u_{pr}\,\rho^\natural.$$

But $\rho \subseteq \mathscr{H}_{S^*}$ and so there is an element x in S^* such that $u_p\,u_r = xu_{pr}$. Write $f_{p,r} = xe^*$. Then, since $xu_{pr} = x(e^*u_{pr}) = f_{p,r}\,u_{pr}$, we have that

$$u_p\,u_r = f_{p,r}\,u_{pr}\,. \tag{4}$$

Moreover, $f_{p,r} = f_{p,r}\,e^* = f_{p,r}\,u_{pr}\,u_{pr}^{-1} = u_p\,u_r\,u_{pr}^{-1}$ and so

$$f_{p,r}\,\rho^\natural = pr(pr)^{-1} = e.$$

Thus, $f_{p,r} \in G$.

We now show that (C4) holds. First let $r \in R_e$. Then $u_r = u_e\,u_r = f_{e,r}\,u_{er} = f_{e,r}\,u_r$ and so

$$f_{e,r} = f_{e,r}u_r\,u_r^{-1} = u_r\,u_r^{-1} = e^*.$$

Next let $p \in P_e$. Then $u_p\,e^* = u_p\,u_e = f_{p,e}\,u_{pe} = f_{p,e}\,u_p$. Hence $f_{p,e} = u_p\,e^*\,u_p^{-1}$, which is an idempotent. Thus $f_{p,e} = e^*$, the identity of G. Therefore (C4) holds (with $1 = e^*$).

Let $P^* = R^* \cap e^*S^*e^* = \{x \in R^* : xe^* = x\}$ and let $p \in P_e$. Then, from above,

$$u_p\, e^* = f_{p,e}\, u_{pe} = u_p$$

and so $u_p \in P^*$. It follows that $u_p\, G \subseteq G\, u_p$ [14, Section 2; also 8, Lemma 3]; moreover, we can define an endomorphism θ_p of G by the rule that, for all $g \in G$,

$$u_p\, g = (g\theta_p)\, u_p\,.$$

We now verify that (C1), (C2) and (C3) hold. Let $p, q \in P_e$ and let $r \in R_e$. Then

$$u_q\,(u_p\, u_r) = u_q\,(f_{p,r}\, u_{pr}) = (f_{p,r}\theta_q)\, u_q\, u_{pr} = (f_{p,r}\theta_q) f_{q,pr}\, u_{qpr};$$

also

$$(u_q\, u_p)\, u_r = (f_{q,p}\, u_{qp})\, u_r = f_{q,p} f_{qp,r}\, u_{qpr}\,.$$

Hence, equating these expressions and postmultiplying by u_{qpr}^{-1}, we find that $(f_{p,r}\,\theta_q) f_{q,pr} = f_{q,p} f_{qp,r}$. Thus (C1) holds. In the same way, from the equation $u_q\,(u_p\, g) = (u_q\, u_p)\, g$ ($p, q \in P_e$; $g \in G$) we deduce that (C2) holds. Further, for all $g \in G$,

$$g\theta_e = (g\theta_e)\, u_e = u_e\, g = g$$

and this establishes (C3).

Let X be a transversal of those $\mathscr{H}$-classes of S that lie in R_e. For each a in $S\backslash 0$ define a_1 to be the (unique) element of X in $L_{aa^{-1}}$. Since

$$a_1 a(a_1 a)^{-1} = a_1\, aa^{-1}\, a_1^{-1} = a_1\, a_1^{-1}\, a_1\, a_1^{-1} = e$$

it follows that $a_1 a \in R_e$ and so $u_{a_1 a}$ is defined. The next stage in the proof consists of showing that each element of $S^*\backslash 0$ is uniquely expressible in the form

$$u_{a_1}^{-1}\, g\, u_{a_1 a} \qquad (a \in S\backslash 0;\ g \in G).$$

Let $a \in S\backslash 0$, $g \in G$. Then

$$(u_{a_1}^{-1}\, g\, u_{a_1 a})\rho^{\natural} = a_1^{-1}\, e\, a_1 a = a_1^{-1}\, a_1\, a = a\, a^{-1} a = a. \tag{5}$$

Hence, in particular, $u_{a_1}^{-1}\, g\, u_{a_1 a} \neq 0$. This enables us to define a mapping $\phi : G \times S\backslash 0 \to S^*\backslash 0$ by the rule that

$$(g, a)\phi = u_{a_1}^{-1}\, g\, u_{a_1 a}\,.$$

We now show that ϕ is bijective.

First let $(g, a)\phi = (h, b)\phi$. Then, taking images under $\rho^\natural$, we see from (5) that $a = b$. Hence

$$g = u_{a_1} (u_{a_1}^{-1} g\, u_{a_1 a})\, u_{a_1 a}^{-1} = u_{b_1} (u_{b_1}^{-1} h\, u_{b_1 b})\, u_{b_1 b}^{-1} = h$$

and so ϕ is injective.

Let $a^* \in S^*\backslash 0$. Since X is a transversal of the $\mathscr{H}$-classes of S in R_e it follows from Theorem 1.1 (ii) that $\{u_x : x \in X\}$ is a transversal of the $\mathscr{H}$-classes of S^* in R^*. Hence, by [3, Theorem 3.4],

$$a^* = u_x^{-1}\, k\, u_y$$

for some $x, y \in X$ and some $k \in H^*$. Now, by Theorem 1.1 (ii), $\{u_z : z \in H_e\}$ is a transversal of those ρ-classes of S^* that are contained in H^*, that is, a transversal of the cosets of G in H^*. Thus $k = h\, u_z$ for some $h \in G$, $z \in H_e$. Hence

$$a^* = u_x^{-1}\, h\, u_z\, u_y = u_x^{-1}\, h f_{z,y}\, u_{zy} = u_x^{-1}\, g\, u_{zy}\,,$$

where $g = hf_{z,y} \in G$. Write $a = x^{-1}zy$. Then $aa^{-1} = x^{-1}\, zy\, y^{-1}\, z^{-1}x = x^{-1}z\, e\, z^{-1}x = x^{-1}z\, z^{-1}x = x^{-1}ex = x^{-1}x$. Since $x^{-1}x \neq 0$ is follows that $a \neq 0$ and so $aa^{-1} = a_1^{-1}\, a_1$. Thus $x^{-1}x = a_1^{-1}\, a_1$ and so $x = a_1$, since both x and a_1 lie in X. Furthermore,

$$zy = (ez)y = (xx^{-1})zy = x(x^{-1}zy) = a_1 a\,.$$

Hence $a^* = u_{a_1}^{-1}\, g\, u_{a_1 a}$. This shows that ϕ is surjective.

Now extend ϕ to a bijection from $(G \times S\backslash 0) \cup 0$ to S^* by taking $0\phi = 0$. Define a multiplication on $(G \times S\backslash 0) \cup 0$ as in (A). We complete the proof of (B) by showing that ϕ is an isomorphism. Let $(g, a), (h, b) \in G \times S\backslash 0$. It will be enough to show that

$$(g, a)\phi\, (h, b)\phi = [(g, a)(h, b)]\phi.$$

We separate two cases.

Suppose first that $ab = 0$. Then

$$(u_{a_1 a}\, u_{b_1}^{-1})\rho^\natural = a_1\, ab_1^{-1} = a_1\, a\, bb^{-1}\, b_1^{-1} = 0$$

and so $u_{a_1 a}\, u_{b_1}^{-1} = 0$ since $\rho^\natural$ is 0-restricted. Thus

$$(g, a)\phi\, (h, b)\phi = 0 = 0\phi = [(g, a)(h, b)]\phi.$$

Now suppose that $ab \neq 0$. Then

$$\begin{aligned} a_1\, a\, b_1^{-1} &= a_1\, aa^{-1}a\, bb^{-1}\, b_1^{-1} \\ &= a_1\, abb^{-1}\, a^{-1}a\, b_1^{-1} \\ &= a_1\, ab\, (ab)^{-1}\, ab_1^{-1} \\ &= a_1\, (ab)_1^{-1}\, (ab)_1\, ab_1^{-1} \\ &= p^{-1}q, \end{aligned}$$

where $p = (ab)_1\, a_1^{-1}$, $q = (ab)_1\, a\, b_1^{-1}$. By Lemma 2.1 (iii), $p,\ q \in P_e$ and so u_p, u_q are defined. Hence

$$(u_{a_1 a}\, u_{b_1}^{-1})\rho^{\natural} = a_1\, a\, b_1^{-1} = p^{-1}q = (u_p^{-1}\, u_q)\rho^{\natural}.$$

Then since $\rho \subseteq \mathscr{H}_{S^*}$ there exist elements x, y in S^* such that $u_{a_1 a}\, u_{b_1}^{-1} = u_p^{-1}\, u_q\, x = y\, u_p^{-1}\, u_q$. It follows that

$$\begin{aligned} u_{a_1 a}\, u_{b_1}^{-1} &= (u_p^{-1}\, u_p)\, u_{a_1 a}\, u_{b_1}^{-1}\, (u_q^{-1}\, u_q) \\ &= u_p^{-1}\, (f_{p,a_1 a}\, u_{pa_1 a})(u_{qb_1}^{-1}\, f_{q,b_1}^{-1})\, u_q \\ &= u_p^{-1}\, f_{p,a_1 a}\, f_{q,b_1}^{-1}\, u_q\,, \end{aligned} \tag{6}$$

since $pa_1a = qb_1$ by Lemma 2.1 (i), (ii). Further,

$$g\, u_p^{-1} = (u_p\, g^{-1})^{-1} = [(g^{-1}\theta_p)\, u_p]^{-1} = u_p^{-1}\, (g^{-1}\theta_p)^{-1} = u_p^{-1}\, (g\theta_p). \tag{7}$$

Finally,

$$\begin{aligned} u_{a_1}^{-1}\, g\, u_{a_1 a} \cdot u_{b_1}^{-1}\, h\, u_{b_1 b} &= u_{a_1}^{-1}\, g\, u_p^{-1}\, f_{p,a_1 a}\, f_{q,b_1}^{-1}\, u_q\, h\, u_{b_1 b} \qquad \text{by (6)}, \\ &= u_{a_1}^{-1}\, u_p^{-1}\, (g\theta_p)\, f_{p,a_1 a}\, f_{q,b_1}^{-1}\, (h\theta_q)\, u_q\, u_{b_1 b} \qquad \text{by (7)}, \\ &= u_{pa_1}^{-1}\, f_{p,a_1}^{-1}\, (g\theta_p)\, f_{p,a_1 a}\, f_{q,b_1}^{-1}\, (h\theta_q)\, f_{q,b_1 b}\, u_{qb_1 b} \\ &= u_{(ab)_1}^{-1}[f_{p,a_1}^{-1}\, (g\theta_p)\, f_{p,a_1 a}\, f_{q,b_1}^{-1}\, (h\theta_q)\, f_{q,b_1 b}]u_{(ab)_1 ab}\,, \end{aligned}$$

since $pa_1 = (ab)_1$, $qb_1 = (ab)_1 a$ by Lemma 2.1 (i), (ii). Thus

$$(g,\, a)\phi\, (h,\, b)\phi = [(g,\, a)(h,\, b)]\phi,$$

as required. This completes the proof of (B).

We call S^* an *extension of G by S*.

Remarks on Theorem 2.2

(2.3). In part (A) let $S = K^0$, where K is a group, and let e be the identity of K. Then $P_e = R_e = H_e = K$. Choose $X = \{e\}$; thus $a_1 = e$ for all $a \in K$. Since S is a group with zero so also is S^*, by Theorem 1.1 (i). Write $K^* = S^* \backslash 0$. For all $a, b \in K$ we have that

$$p = (ab)_1 \, a_1^{-1} = e, \qquad q = (ab)_1 a \, b_1^{-1} = a$$

and so the multiplication in K^* is according to the rule that

$$(g, a)(h, b) = (g\,(h\theta_a) f_{a,b}\,,\, ab).$$

In this case K^* is an extension of G by K in the usual group-theoretic sense [5, Section 48]. Conversely, every extension of G by K is, to within isomorphism, of this type.

(2.4). Let S be a fundamental 0-bisimple inverse semigroup and let E be a semilattice isomorphic to E_S. By Theorem 1.2, E is 0-uniform and by Theorem 1.3 we can regard S as a 0-transitive inverse subsemigroup of T_E. For all $e \in E$ let ϵ_e denote the identity mapping of Ee. Then $E_S = E_{T_E} = \{\epsilon_e : e \in E\}$ [7, 10]. Denote the domain and codomain of an element α of S by $\Delta(\alpha)$ and $\nabla(\alpha)$, respectively. Choose $e \in E \backslash 0$. Then it is easily verified [10, Lemma 1.2] that

$$R_{\epsilon_e} = \{\alpha \in S : \Delta(\alpha) = Ee\},$$

$$H_{\epsilon_e} = \{\alpha \in S : \Delta(\alpha) = Ee, \nabla(\alpha) = Ee\},$$

$$P_{\epsilon_e} = \{\alpha \in S : \Delta(\alpha) = Ee, \nabla(\alpha) \subseteq Ee\}.$$

Take S as above in part (A) and let ϵ_e be the chosen nonzero idempotent of S. For each x in $E \backslash 0$ select one element ξ_x of S such that $\Delta(\xi_x) = Ee$, $\nabla(\xi_x) = Ex$. (This is possible since S is 0-transitive.) Then the set

$$X = \{\xi_x : x \in E \backslash 0\}$$

is a transversal of the $\mathscr{H}$-classes of S that are contained in R_{ϵ_e} and every such transversal can be obtained in this way. We now interpret the mapping $\alpha \mapsto \alpha_1$, where α is a nonzero element of S. Let $\Delta(\alpha) = Ey$ ($y \in E \backslash 0$); then $\alpha\alpha^{-1} = \epsilon_y$ and so $\alpha_1 = \xi_y$. Finally, the congruence ρ defined in (A) must coincide with μ, since S is fundamental [Theorem 1.1 (iii)]. Note that if we make the further assumption that $\mathscr{H}_S = \iota_S$, then $\mu = \mathscr{H}_{S^*}$ [Theorem 1.1 (iv)]: in this case the set X coincides with R_{ϵ_e}.

3. The Splitting Case

A considerable simplification occurs in Theorem 2.2(A) if we choose each $f_{p,r}$ to be the identity of G. The conditions (C1) and (C4) become redundant, while (C2) asserts that the mapping $\theta : p \mapsto \theta_p$ of P_e into *end* G, the semigroup of endomorphisms of G, is an antihomomorphism. The multiplication in S^* then reduces to the following: for $(g, a), (h, b) \in S^*\backslash 0$,

$$\left.\begin{aligned} 0(g, a) &= (g, a)0 = 0^2 = 0, \\ \text{and} \qquad (g, a)(h, b) &= \begin{cases} (g\theta_p \, h\theta_q \, , ab) & \text{if} \quad ab \neq 0, \\ 0 & \text{if} \quad ab = 0, \end{cases} \end{aligned}\right\} \tag{3.1}$$

where p, q are defined as before. In this case the set

$$S' = \{(1, a) : a \in S\backslash 0\} \cup 0$$

is a transversal of the ρ-classes of S^* and is clearly a subsemigroup of S^* isomorphic to S.

Conversely, if we add to the hypotheses of Theorem 2.2(B) the additional requirement that there exists a transversal S' of the ρ-classes of S^* which is also a subsemigroup of S^* then, by selecting the set $\{u_r : r \in R_e\}$ to be a subset of S', we see that Eq. (4) in the proof can be replaced by

$$u_p \, u_r = u_{pr} \, .$$

It follows that $f_{p,r} = 1$ for all $p \in P_e$, $r \in R_e$ and so S^* is isomorphic to $(G \times S\backslash 0) \cup 0$, with multiplication defined as in (3.1).

In this case we say that S^* *splits over* ρ and that S^* is a *split extension of* G *by* S. The result corresponding to the above for bisimple inverse semigroups is given in [9, Theorem 7].

(3.2). It is clear that a necessary and sufficient condition for S^* to split over ρ is that there exists a transversal $\{u_r : r \in R_e\}$ of the ρ-classes of S^* contained in R^* with the property that $u_r\rho^\natural = r$ and $u_p \, u_r = u_{pr}$ for all $p \in P_e$ and $r \in R_e$.

4. An Application

In this final section we apply the foregoing results to a particular class of 0-bisimple inverse semigroups first discussed in [6]. The set of all non-negative integers will be denoted throughout by N.

DEFINITION. Let I be a nonempty set. A semilattice $E = E^0$ will be said to be of *type* (ω, I) if and only if $E\backslash 0 = \{e_{m,i} : m \in N, i \in I\}$ and

$$e_{m,i}\, e_{n,j} = \begin{cases} e_{t,i} & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where $t = \max\{m, n\}$.

Such a semilattice is 0-uniform and each of its principal ideals is inversely well-ordered.

DEFINITION. An inverse semigroup $S = S^0$ is an (ω, I) *inverse semigroup* if and only if its semilattice E_S is of type (ω, I).

In particular, if $S = S^0$ is an (ω, I) inverse semigroup with $|I| = 1$, then E_S is a chain and so S has no proper divisors of zero; in this case S is an inverse ω-semigroup (in the sense of Reilly [11]) with a zero element adjoined.

The structure of 0-bisimple (ω, I) inverse semigroups has been determined by Lallement and Petrich [6, Corollary 5.7, also Section 6]. Their method makes use of Reilly's structure theorem for bisimple inverse ω-semigroups [11]. In Theorem 4.2 below we give a proof based on Theorem 2.2. Reilly's theorem can then be deduced by taking $|I| = 1$.

We require a lemma.

LEMMA 4.1. *Let I be a nonempty set and let $E = E^0$ be a semilattice of type (ω, I). Then T_E is a fundamental 0-bisimple (ω, I) inverse semigroup on which $\mathscr{H}$ is trivial. Further, to within isomorphism, $T_E = [(N \times N) \times (I \times I)] \cup 0$, with multiplication according to the following rule: 0 is the zero element and the product of nonzero elements is given by*

$$[(m, n), (i, j)][(r, s), (k, l)] = \begin{cases} [(m - n + t, s - r + t), (i, l)] & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

where $t = \max\{n, r\}$.

Proof. Let $E\backslash 0 = \{e_{m,i} : m \in N, i \in I\}$, as in the definition. Since E is 0-uniform it follows from Theorem 1.3 that T_E is a fundamental 0-bisimple (ω, I) inverse semigroup. Moreover, every principal ideal of E is inversely well-ordered and so, by Theorem 1.4, $\mathscr{H} = \iota$ on T_E. There is therefore one and only one element of T_E with domain $Ee_{m,i}$ and codomain $Ee_{n,j}$; denote this by $[(m, n), (i, j)]$. Let 0 denote the identity mapping of the zero ideal of E. It is then easy to verify that the multiplication in T_E is as stated.

Note that $E_{T_E}\backslash 0 = \{[(m, m), (i, i)] : m \in N, i \in I\}$ and that $[(m, n), (i, j)]^{-1} = [(n, m), (j, i)]$.

We then have

THEOREM 4.2. (A) *Let G be a group, let α be an endomorphism of G and let I be a nonempty set. Let $S^* = [G \times (N \times N) \times (I \times I)] \cup 0$ and define a multiplication in S^* by the following rule*: 0 *is the zero element and the product of nonzero elements is given by*

$$[g, (m, n), (i, j)][h, (r, s), (k, l)]$$

$$= \begin{cases} [g\alpha^{t-n}\, h\alpha^{t-r}, (m - n + t, s - r + t), (i, l)] & \text{if } j = k \\ 0 \quad \text{if } j \neq k, \end{cases}$$

where $t = \max\{n, r\}$ *and* α^0 *denotes the identity automorphism of* G. *Then* S^* *is a* 0-*bisimple* (ω, I) *inverse semigroup.*

(B) *Conversely, every* 0-*bisimple* (ω, I) *inverse semigroup is, to within isomorphism, of the type described in* (A).

Proof. (A) Let $E = E^0$ be a semilattice of type (ω, I). By Lemma 4.1, T_E is a fundamental 0-bisimple (ω, I) inverse semigroup on which $\mathscr{H}$ is trivial. Take $S = T_E$ and suppose that it is represented in terms of N and I as in the lemma. We shall establish the result by showing that S^* is a split extension of G by S (Section 3).

Choose an element 0 in I and keep it fixed. Let $e = [(0, 0), (0, 0)] \in S$. Then $e^2 = e$ and it is easily seen that $R_e = \{[(0, n), (0, i)] : n \in N,\ i \in I\}$, $P_e = \{[(0, n), (0, 0)] : n \in N\}$. Since $\mathscr{H}_S = \iota_S$ it follows that the only transversal X of the $\mathscr{H}$-classes of S in R_e is R_e itself; hence if $a \in S\backslash 0$, then a_1 is the unique element of R_e such that $a_1^{-1} a_1 = aa^{-1}$.

Write $y = [(0, 1), (0, 0)]$ and take $y^0 = e$. Then $y^n = [(0, n), (0, 0)]$ for all $n \in N$ and so $P_e = \{y^n : y \in N\}$. Define $\theta : p \mapsto \theta_p$ of P_e into *end* G by the rule that

$$\theta_{y^n} = \alpha^n \qquad (n \in N).$$

Then θ is an antihomomorphism and $\theta_e = \alpha^0$, the identity automorphism of G. To complete the proof of (A) we show that the multiplication in S^* is as in (3.1). Let

$$a = [(m, n), (i, j)], \qquad b = [(r, s), (j, k)] \in S\backslash 0$$

and let $p = (ab)_1\, a_1^{-1}$, $q = (ab)_1\, a\, b_1^{-1}$. We need only prove that

$$\theta_p = \alpha^{t-n}, \qquad \theta_q = \alpha^{t-r},$$

where $t = \max\{n, r\}$. First, $aa^{-1} = [(m, m), (i, i)]$ and so $a_1 = [(0, m), (0, i)]$. In the same way we obtain b_1, $(ab)_1$ and a simple calculation then shows that

$$p = [(0, t - n), (0, 0)] = y^{t-n},$$
$$q = [(0, t - r), (0, 0)] = y^{t-r}.$$

This gives the required result.

(B) Let S^* be a 0-bisimple (ω, I) inverse semigroup. Write $E = E_{S^*}$. By Theorem 1.4, $\mathscr{H}$ is a congruence on S^* and $S^*/\mathscr{H} \cong T_E$. Take $S = S^*/\mathscr{H}$ and assume that S is represented in terms of N and I as in Lemma 4.1. We show that S^* splits over $\mathscr{H}$.

Let e, y be defined as in part (A) above. Write $z_i = [(0, 0), (0, i)]$ for all $i \in I$; in particular, $z_0 = e$. Then

$$y^n z_i = [(0, n), (0, 0)][(0, 0), (0, i)] = [(0, n), (0, i)]$$

and so

$$R_e = \{y^n z_i : n \in N, i \in I\}, \qquad P_e = \{y^n z_0 : n \in N\}.$$

Now let e^* be that idempotent of S^* whose image under $\mathscr{H}^\natural$ is e. Let $R^* = R_{e^*}$. Choose elements v, w_i in R^* such that

$$v\mathscr{H}^\natural = y, \qquad w_i\mathscr{H}^\natural = z_i \qquad (i \in I)$$

and take $w_0 = e^*$. Then

$$(v^n w_i)\mathscr{H}^\natural = y^n z_i \in R_e .$$

Hence $\{v^n w_i : n \in N, i \in I\}$ is a transversal of the $\mathscr{H}$-classes of S^* contained in R^* and $\{v^n w_0 : n \in N\}$ is the subset of this transversal that maps onto P_e under $\mathscr{H}^\natural$. But

$$(v^m w_0)(v^n w_i) = v^{m+n} w_i$$

for all $m, n \in N$ and all $i \in I$. Hence, by (3.2), S^* splits over $\mathscr{H}$. Let $G = H_{e^*}$. It follows that S^* is isomorphic to $(G \times S\backslash 0) \cup 0$ with multiplication defined as in (3.1), where $\theta : p \mapsto \theta_p$ is an antihomomorphism of P_e into *end* G with the property that θ_e is the identity automorphism of G. Write $\alpha = \theta_y$, $\alpha^0 = \theta_e$. Then $\theta_{y^n} = \alpha^n$ for all $n \in N$. Further, for $a \in S\backslash 0$, a_1 is the unique element of R_e such that $a_1^{-1} a_1 = aa^{-1}$. Finally, let

$$a = [(m, n), (i, j)], \qquad b = [(r, s), (j, k)] \in S\backslash 0$$

and let $p = (ab)_1 a_1^{-1}$, $q = (ab)_1 a b_1^{-1}$. Then, as in (A), it follows that

$$\theta_p = \alpha^{t-n}, \qquad \theta_q = \alpha^{t-r},$$

where $t = \max\{n, r\}$. This completes the proof.

References

1. H. D'Alarcao, Idempotent-separating extensions of inverse semigroups, *J. Austral. Math. Soc.* **9** (1969), 211–217.
2. A. H. Clifford, A class of *d*-simple semigroups, *Amer. J. Math.* **75** (1953), 547–556.
3. A. H. Clifford and G. B. Preston, "The Algebraic Theory of Semigroups," Math. Surveys of the Amer. Math. Soc., 7, Vols. I and II, Amer. Math. Soc., Providence, R.I., 1961 and 1967.
4. J. M. Howie, The maximum idempotent-separating congruence on an inverse semigroup, *Proc. Edinburgh Math. Soc.* **14** (1964), 71–79.
5. A. G. Kurosh, "The Theory of Groups," Vol. II, 2nd English edn., Chelsea, New York, 1960.
6. G. Lallement and M. Petrich, A generalization of the Rees theorem in semigroups, *Acta Sci. Math. (Szeged)* **30** (1969), 113–132.
7. W. D. Munn, Uniform semilattices and bisimple inverse semigroups, *Quart. J. Math. Oxford Ser.* **17** (1966), 151–159.
8. W. D. Munn, The idempotent-separating congruences on a regular 0-bisimple semigroup, *Proc. Edinburgh Math. Soc.* **15** (1967), 233–240.
9. W. D. Munn, Some recent results on the structure of inverse semigroups, "Semigroups: Proceedings of a symposium," Academic Press, New York, 1969.
10. W. D. Munn, Fundamental inverse semigroups, *Quart. J. Math. Oxford Ser.* **21** (1970), 157–170.
11. N. R. Reilly, Bisimple ω-semigroups, *Proc. Glasgow Math. Assoc.* **7** (1966), 160–167.
12. N. R. Reilly, Bisimple inverse semigroups, *Trans. Amer. Math. Soc.* **132** (1968), 101–114.
13. N. R. Reilly, An isomorphism for two extension theorems for bisimple inverse semigroups (unpublished).
14. N. R. Reilly and A. H. Clifford, Bisimple inverse semigroups as semigroups of ordered triples, *Canad. J. Math.* **20** (1968), 25–39.