ON KNOTS THAT ARE UNIVERSAL

HUGH M. HILDEN, MARIA TERESA LOZANO and JOSE MARIA MONTESINOS.

(Received 15 June 1984)

§1. INTRODUCTION

A link or knot \( L \) in \( S^3 \) is called universal if every closed, orientable 3-manifold can be represented as a covering of \( S^3 \) branched over \( L \). W. Thurston introduced this concept in his paper [6], where he also exhibited some universal links, and asked if the “figure-eight knot” was universal. The question for the trefoil knot, as well as for any torus knot or, more generally, for iterated torus knots or links, has negative answer, because these links, being fibers of a graph-manifold structure of \( S^3 \), can only be the branching set of graph-manifolds (compare [2]).

In this paper we answer the question of Thurston in the affirmative and we prove that every non toroidal 2-bridge knot or link is universal. The proof uses the fact that the Borromean rings are universal [4].

§2. DIHEDRAL COVERINGS BRANCHED OVER RATIONAL LINKS

To each rational number \( a/b \) there is associated the 2-bridge link \( L(a/b) \) shown in Figure 1. In (a), the central tangle consists of lines of slope \( \pm b/a \), which are drawn on a square “pillowcase” (compare [5], [3]). In (b) this link is isotoped to exhibit the two bridges. The link \( L(a/b) \) has one component if and only if \( a \) is odd. If \( L(a/b), L(a/b') \) are such that \( b \equiv b' \pmod{a} \) then they are equal and if \( b \equiv -b' \pmod{a} \) then \( L(a/b) \) is the mirror image of \( L(a/b') \). Thus we assume that \( a \) and \( b \) are relatively prime and that \( a > b \). The link \( L(a/b) \) is toroidal if and only if \( b \equiv \pm 1 \pmod{a} \).

According to [1, p. 1613] \( \pi_1(S^3 - L(a/b)) \) admits a representation \( f \) onto the dihedral group \( D_{2a} \) of \( 2a \) elements such that the image of a meridian is sent to a reflection. The group \( D_{2a} \) has the presentation \( \{x, y : x^2 = y^2 = (xy)^a = 1\} \). Let \( a \) be the transitive immersion of \( D_{2a} \) into the symmetric group \( S_a \) of \( a \) indices defined by

\[
\begin{align*}
\alpha(x) &= (12)(34)(56) \ldots \\
\alpha(y) &= (23)(45)(67) \ldots
\end{align*}
\]

We now describe the \( a \)-fold irregular dihedral covering of \( S^3 \) branched over \( L(a/b) \) associated to the representation \( sf: \pi_1(S^3 - L(a/b)) \to S_a \), where we can assume \( x, y \) are the meridians shown in Fig. 1(b).

![Figure 1. Figure-eight knot \( L(5/2) \).](image)
There is a sphere $S^2$ dividing $S^3$ in two balls $A$ and $B$, such that $(A, L(a/b) \cap A)$ and $(B, L(a/b) \cap B)$ are homeomorphic to a 3-ball with two properly embedded unknotted and unlinked arcs as the one shown in Fig. 2. We can imagine $S^2$ as a small “pillowcase” parallel to the one onto which lie the tangled part of $L(a/b)$.

Thus the covering of $S^3$ branched onto $L(a/b)$ with monodromy $\gamma_f$ is divided by the preimage of $S^2$ in two parts $\tilde{A}$ and $\tilde{B}$, covering $A$ and $B$ respectively. Both $\tilde{A}$ and $\tilde{B}$ are 3-balls and thus the covering we are describing is just $S^3$. The covering $\tilde{A} \to A$ for the knot $L(5/2)$ is shown in Fig. 3, where we have also depicted the intersection of the preimage of $D_1 \cup D_2 \subset B$ with $\partial \tilde{A}$. The preimage of the branching set in the covering $S^3 \to S^3$ branched over $L(a/b)$ is the union of the preimages of $L(a/b) \cap A$ and $L(a/b) \cap B$. The preimage of $I(a/b) \cap A$ are the $a + 1$ arcs properly imbedded in $\tilde{A}$ shown in Fig. 3. The preimage of $L(a/b) \cap B$ can be isotoped through the preimage of $D_1 \cup D_2 \subset B$ into $\partial \tilde{A}$, so that we can think of the preimage of $L(a/b) \cap B$ as a subset of the arcs lying on $\partial \tilde{A}$. Namely, if we delete just one of the two arcs that coalesce in the same point of the preimage of $L(a/b) \cap A$, for each such a point, we obtain the preimage of $L(a/b) \cap B$, up to isotopy in $\tilde{B}$. In Fig. 4 we have depicted the preimage of the branching set corresponding to the knot $L(5/2)$. Figure 4(b) shows the branching cover after puncturing the “pillowcase”, twisting one of its ends and flattening out onto the plane.

Similarly, Fig. 5 shows the branching cover corresponding to the link $L(12/5)$ under a dihedral covering of 6 sheets, obtained by an analogous procedure. This cover will be used later.

The preimage of the branching set of the figure eight knot $L(5/2)$ is the link of Fig. 6. This is an amphicheiral link to which we will refer as the “roman link”+.

† This link appears as part of the decoration of a roman mosaic found in the city of Zaragoza.
THEOREM 1. There is a covering $p:S^3 \to S^3$ branched over the roman link such that the preimage of the branching set contains the Borromean rings. Thus the roman link is universal.

Proof. The roman link of Fig. 6 is depicted in Fig. 7 with an assignment of permutations to the components $A$ and $B$. The corresponding dihedral covering of 4 sheets is described in Fig. 8(a). Figure 8(b) shows the preimage of the component $C$. This preimage is the rational link $L(12/5)$. In the dihedral cover of 6 sheets branched over $L(12/5)$, shown in Fig. 5, we extract the link $C_0 \cup C_2 \cup C_5 \cup C_6$ which we depict in Fig. 9. The preimage of $C_0 \cup C_6$ under the dihedral covering of 3 sheets branched along $C_2 \cup C_5$ is shown in Fig. 10. The preimage of $B$ under the 3-fold cyclic covering of $S^3$ branched over $A$ are the Borromean rings (Fig. 11).

Now in [4] it was shown that the Borromean rings are universal. This finishes the proof of the Theorem.

COROLLARY 2. The Figure-eight knot is universal.
§4. UNIVERSALITY OF RATIONAL KNOTS AND LINKS

THEOREM 3. The preimage of $L(a/b)$ under the $a$-fold covering $S^3 \to S^3$ branched over $L(a/b)$ and with monodromy group the dihedral group of $2a$ elements $D_{2a}$, contains the "roman link" as a sublink if and only if $b \neq \pm 1 \pmod{a}$, i.e. if $L(a/b)$ is not toroidal.

Proof. We first assume $b < \frac{a}{2}$. Figure 12 shows the preimage of $L(a/b)$ (= $L(8/3)$) under the $a$-fold covering $p:S^3 \to S^3$ that we are considering (compare with Fig. 4). The link $p^{-1}(L(a/b))$ has $1 + \left\lfloor \frac{a}{2} \right\rfloor$ components $C_0, C_1, \ldots, C_{\left\lfloor \frac{a}{2} \right\rfloor}$, and we claim that the link $C_0 \cup C_b \cup C_{a-(a/b)b}$, if $b \neq 1$, is the roman link $\tilde{C}_0 \cup \tilde{C}_2 \cup \tilde{C}_1$ of Fig. 4(b) and 6. It is evident that $C_0 \cup C_b$ is equal to $\tilde{C}_0 \cup \tilde{C}_2$. Now the partner of $C_1$ in Fig. 7 must be a $C_x$ such that $0 < x < b$ and with the additional property that traveling down along $C_x$, starting from the
left side of the pillowcase (see Fig. 12) we reach the first bridge of $C_x$ before touching the middle line of the pillowcase. This guarantees that $C_x$ lies on the plane (i.e. its projection has no double points), and behaves exactly like the curve $\tilde{C}_1$ of Fig. 4(b). Now $x_0 = a - \left[ \frac{a}{b} \right] b$ satisfies $0 < x_0 < b$ and the first bridge of $C_{x_0}$ is reached in the point of coordinates $(X, Y) = \left( \left[ \frac{a}{b} \right], b \right)$ with respect to the reference shown in Fig. 12. Since $\left( \left[ \frac{a}{b} \right], b \right)$ lies to the left of the middle line of the pillowcase, the component $C_{x_0}$ together with $C_0$ and $C_b$ is the roman link.

If $b > \frac{a}{2}$ the link $L\left( \frac{a}{b} \right)$ is the mirror image of $L\left( \frac{a}{a-b} \right)$. Thus $p^{-1}\left( L\left( \frac{a}{b} \right) \right)$ contains the mirror image of the roman link, i.e. the roman link.

Finally, since $L(a/1)$ is not universal, $p^{-1}\left( L(a/1) \right)$ cannot contain the roman link as a sublink. This finishes the proof of the Theorem.

**Corollary 4.** *Every rational knot or link which is not toroidal is universal.*

**Remark.** Using similar methods to that of [4] we can prove that many 3-bridge knots are universal.

**Question.** *Is every hyperbolic knot universal?*

**References**


Facultad de Ciencias,
Universidad de Zaragoza,
Spain.

Department of Mathematics,
University of Hawaii,
Honolulu, Hawaii 96822. USA.

Mathematical Sciences Research Institute,
1000 Centennial Drive,
Berkeley,
California 94720,
USA.