The possible cardinalities of global secure sets in cographs

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**ABSTRACT**

Let \( G = (V, E) \) be a graph. A global secure set \( SD \subseteq V \) is a dominating set which also satisfies a condition that \(|N[X] \cap SD| \geq |N[X] - SD|\) for every subset \( X \subseteq SD \). The minimum cardinality of the global secure set in the graph \( G \) is denoted by \( \gamma_s(G) \). In this paper, we introduce the notion of \( \gamma_s \)-monotone graphs. The graph \( G \) is \( \gamma_s \)-monotone if, for every \( k \in \{\gamma_s(G), \gamma_s(G) + 1, \ldots, n\} \), it has a global secure set of cardinality \( k \). We will also present the results concerning the minimum cardinality of the global secure sets in the class of cographs.

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1. Introduction

Let \( G = (V, E) \) be a graph: \( V \) denotes the set of vertices and \( E \) the set of edges. We will denote by \( n \) the order of \( G (n = |V|) \) and by \( m \) the number of edges. Throughout this paper, we consider only finite graphs without loops or multiple edges. An open neighbourhood of a vertex \( v \) is the set \( N_G(v) = \{x \in V : vx \in E\} \). The closed neighbourhood of the vertex \( v \) is the set \( N_C(v) = N_G(v) \cup \{v\} \). Similarly, we can define an open and closed neighbourhood of a set \( X \subseteq V \), i.e., \( N_G(X) = \bigcup_{v \in X} N_G(v) \) and \( N_C(X) = N_G(X) \cup X \). We write \( N(v) \), \( N[v] \), \( N(X) \), and \( N[X] \) if \( G \) is clear from the context. A set \( D \subseteq V \) is a dominating set if \( N[D] = V \). By \( K_n \) we denote a complete graph on \( n \) vertices and by \( K_n^c \) we denote its complement. For all undefined concepts we refer the reader to [4].

The secure sets were first presented by Brigham, Dutton, and Hedetniemi in [1].

**Definition 1 ([1]).** Let \( G = (V, E) \) be a graph. For any \( S = \{s_1, s_2, \ldots, s_k\} \subseteq V \), an attack on \( S \) is any \( k \) mutually disjoint sets \( A = \{A_1, A_2, \ldots, A_k\} \) for which \( A_i \subseteq N[s_i] - S, 1 \leq i \leq k \). A defense of \( S \) is any \( k \) mutually disjoint sets \( D = \{D_1, D_2, \ldots, D_k\} \) for which \( D_i \subseteq N[s_i] \cap S, 1 \leq i \leq k \). Attack \( A \) is defendable if there exists a defense \( D \) such that \( |D_i| \geq |A_i| \) for \( 1 \leq i \leq k \). Set \( S \) is secure if and only if every attack on \( S \) is defendable. The theorem presented below is fundamental for secure sets.

**Theorem 1 ([1]).** Set \( S \subseteq V \) is secure if and only if
\[
\forall X \subseteq S, \quad |N[X] \cap S| \geq |N[X] - S|.
\]
Following [1], the cardinality of a minimum secure set in a graph $G$ is the security number of $G$, and it is denoted by $s(G)$. The exact values and bounds on security number can be found in [1,9,10,16]. Clearly the defensive alliances, which were introduced by Kristiansen et al. in [17], are the basis of the notion of secure sets. A set $A \subseteq V$ is a defensive alliance if $|N[x] \cap A| \geq |N[x] - A|$ for every $x \in A$. The concept of an alliance has many variants and applications; see [17].

It was natural to ask about defensive alliances and secure sets that have an influence on the whole graph. By influence we mean that every vertex of $G$ is either a member of the alliance (secure set) or can be the attacker, i.e., is a neighbour of a member of the alliance (secure set). The global defensive alliance (global secure set) is a defensive alliance (secure set) which is also a dominating set. The minimum cardinality of a global defensive alliance (global secure set) will be denoted by $\gamma_s(G)$ ($\gamma(G)$). The complexity of finding $\gamma_s(G)$ was determined in [2]. The authors proved that this problem is NP-complete. The exact values of $\gamma_s(G)$ for cycles, paths, complete, and complete bipartite graphs were given in [13]. Also, upper and lower bounds on $\gamma_s(G)$ are known for general graphs, bipartite graphs, and trees. For more details and results, see [2,3,11,13,15]. Global secure sets, as a new research area, have not been so extensively studied. However, for some basic graph classes such as cycles, paths, and complete graphs, we can notice that $\gamma(G) = \gamma_s(G)$, and from the definition it follows immediately that $\gamma(G) \geq \lceil \frac{n}{2} \rceil$.

### 2. General graphs

Let $SD(G_1)$ and $S(G_2)$ denote a global secure set and secure set of cardinality $k$ in a graph $G$, respectively. We will simply write $SD_k$ and $S_k$ when no confusion can arise. By $\cup (G)$, we denote the set of the universal vertices of graph $G$, i.e., vertices of degree $n-1$. We say that the graph $G = G_1 + G_2$ is a join of a graph $G_1$ and $G_2$ if $E(G) = E(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{e = vu : u \in V(G_1) \text{ and } v \in V(G_2)\}$. The disjoint union of the graphs $G_1$ and $G_2$ is the graph $G = G_1 \cup G_2$ such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

The next three results play an important role in the proofs in Sections 3 and 4.

**Proposition 1 ([1]).** If $S_1$ and $S_2$ are vertex disjoint secure sets in the same graph, then $S_1 \cup S_2$ is a secure set.

**Lemma 1.** Let $k$ and $l$ be positive integers, and let $G_1$ and $G_2$ be disjoint graphs of order $n_1 \geq 0$ and $n_2 \geq 0$, respectively. If $k \geq \frac{n_1}{2}$ and $l \geq \frac{n_2}{2}$, then $S(G_1) \cup S(G_2)$ is a global secure set of cardinality $k + l$ in the graph $G = G_1 + G_2$.

**Proof.** Let $S_1$ and $S_2$ denote the sets $S(G_1)_k$ and $S(G_2)_l$, respectively. From the definition of the join of graphs and the fact that $k, l \geq 1$, it follows that the set $S = S_1 \cup S_2$ is a dominating set of the graph $G = G_1 + G_2$. We will show that $S$ is secure. From Theorem 1, it follows that it is sufficient to show that $\lvert N[X] \cap S \rvert \geq \lvert N[X] - S \rvert$, for every set $X \subseteq S$. If $X \subseteq S_1$, then $\lvert N[X] \cap S \rvert = \lvert N_{G_1}[X] \cap S_1 \rvert + l$ and $\lvert N[X] - S \rvert = \lvert N_{G_1}[X] - S_1 \rvert + n_2 - l$. Since $\lvert N_{G_1}[X] \cap S_1 \rvert \geq \lvert N_{G_1}[X] - S_1 \rvert$ and $l \geq \frac{n_2}{2}$, we have a desired inequality. Similarly, we can prove that $\lvert N[X] \cap S \rvert \geq \lvert N[X] - S \rvert$, for every $X \subseteq S_2$. Finally, we have to consider the case when $X \cap S_1 \neq \emptyset$ and $X \cap S_2 \neq \emptyset$. Now, $\lvert N[X] \cap S \rvert = k + l$ and $\lvert N[X] - S \rvert = n_1 + n_2 - k - l$, so clearly $\lvert N[X] \cap S \rvert \geq \lvert N[X] - S \rvert$. Thus the set $S$ is a global secure set in $G$. $\square$

**Lemma 2.** Let $G$ be a graph with a global secure set $SD$, and let $v$ be a vertex that belongs to $SD$. If there exists a vertex $v'$ such that $v' \notin SD$ and $v' \in U(G)$, then $SD' = (SD - \{v\}) \cup \{v'\}$ is a global secure set of $G$.

**Proof.** Because $v' \notin SD'$, it follows that $SD'$ is a dominating set, so it is sufficient to show that it is also a secure set. Once again we will use Theorem 1. Let $X \subseteq SD'$. If $v' \in X$, then obviously $\lvert N[X] \cap SD' \rvert \geq \lvert N[X] - SD' \rvert$. So let us suppose that $v' \notin X$. If $v \in N[X]$, then $\lvert N[X] \cap SD' \rvert = \lvert N[X] \cap SD \rvert + \lvert \{v'\} \rvert - \lvert \{v\} \rvert$ and $\lvert N[X] - SD' \rvert = \lvert N[X] - SD \rvert - \lvert \{v'\} \rvert + \lvert \{v\} \rvert$, so $\lvert N[X] \cap SD' \rvert \geq \lvert N[X] - SD' \rvert$. We have to consider the last case when $v \notin N[X]$. Then $\lvert N[X] \cap SD' \rvert = \lvert N[X] \cap SD \rvert + \lvert \{v'\} \rvert$ and $\lvert N[X] - SD' \rvert = \lvert N[X] - SD \rvert - \lvert \{v'\} \rvert$. Thus $\lvert N[X] \cap SD' \rvert \geq \lvert N[X] - SD' \rvert$, and this observation finishes the proof. $\square$

### 3. $\gamma_s$-monotone graphs

We will define a new notion of $\gamma_s$-monotone graphs. Namely, we say that a graph $G$ is $\gamma_s$-monotone if there exist sets $SD_{\gamma_s(G)}, SD_{\gamma_s(G)+1}, \ldots, SD_n$. Similarly, we can define $s$-monotone graphs, i.e., if a graph $G$ is $s$-monotone, then there exist sets $S_1, S_2, \ldots, S_n$.

**Lemma 3.** Complete graphs, paths, cycles, complete bipartite graphs, and trees are $s$-monotone and $\gamma_s$-monotone graphs.

The above lemma is easy to check, so we omit the proof. The following remarks will simplify some of our considerations.

**Remark 1.** If a graph $G$ has an order smaller or equal to 5, then it is $\gamma_s$-monotone.

**Remark 2.** Let $G$ be a connected graph of even order $n$. If $\gamma_s(G) \geq \frac{n}{2} + 1$, then $n \geq 6$.

In this paper, we will show that not all graphs are $\gamma_s$-monotone. For this reason, we will introduce two more graph classes. Namely, we will say that a graph $G$ belongs to $M_{\gamma_sD}$ if it is nearly $\gamma_s$-monotone, i.e., it has all the required global secure sets except the one of cardinality $k$, where $k > \gamma_s(G)$. Furthermore, let us by $s_m^k$ denote the class of all graphs which have a secure set of cardinality $k$, for any $k \in \{m, m+1, \ldots, n\}$. The forthcoming lemmata are rather technical, but they will be very useful in the proofs of the theorems presented in Section 4.
Lemma 4. Let both $G_1$ and $G_2$ be connected and $\gamma_*$-monotone graphs of order $n_1 > 0$ and $n_2 > 0$, respectively. If $\gamma_*(G_1) = \left\lceil \frac{n_1}{2} \right\rceil$ and $\gamma_*(G_2) = \left\lceil \frac{n_2}{2} \right\rceil$, then the graph $G = G_1 \cup G_2$ belongs to $s_k^{n_1+n_2}$, where $k = \left\lceil \frac{n_1+n_2}{2} \right\rceil$.

Proof. If graph $G_1$ or $G_2$ has even order, then clearly the lemma holds. If $G_1$ and $G_2$ have both odd order, then, without loss of generality, we can assume that $n_1 \geq n_2$. We obtain the two required sets of the smallest cardinality as follows: $S_{n_1+n_2} = SD(G_1) \cup S_{n_1+n_2+1} = SD(G_1) \cup SD(G_2)$. Now it is easy to see how to obtain the rest of the sets. □

Lemma 5. Let both $G_1$ and $G_2$ be connected and $\gamma_*$-monotone graphs of even order $n_1 > 0$ and $n_2 > 0$, respectively. If $\gamma_*(G_1) = \frac{n_1}{2} + 1$ and $\gamma_*(G_2) = \frac{n_2}{2} + 1$, then the graph $G = G_1 \cup G_2$ belongs to $s_k^{n_1+n_2}$, where $k = \frac{n_1+n_2}{2} + 2$ and there exists the set $S_{n_1+n_2}$.

Proof. Without loss of generality, we can assume that $n_1 \geq n_2$. From Remark 2, we know that $n_2 \geq 6$. Hence we can see that as the set $S_{n_1+n_2}$ we can choose the set $SD(G_1) \cup S_{n_1+n_2}$. The remaining sets are easy to obtain, so we omit this part of the proof. □

Lemma 6. Let both $G_1$ and $G_2$ be connected and $\gamma_*$-monotone graphs of order $n_1 > 0$ and $n_2 > 0$, respectively, where $n_1$ is even. If $\gamma_*(G_1) = \frac{n_1}{2} + 1$ and $\gamma_*(G_2) = \left\lceil \frac{n_2}{2} \right\rceil$, then the graph $G = G_1 \cup G_2$ belongs to $s_k^{n_1+n_2}$, where $k = \left\lceil \frac{n_1+n_2}{2} \right\rceil$.

Proof. One can see that, if $n_1 \geq n_2$, then the set $SD(G_1) \cup S_{n_1+n_2}$ gives us $S_{n_1+n_2}$; otherwise, we can choose $SD(G_2) \cup S_{n_1+n_2}$ as $S_{n_1+n_2}$. Now the other required sets can be easily computed. □

Lemma 7. Let both $G_1$ and $G_2$ be connected graphs of order $n_1 > 0$ and $n_2 > 0$, respectively, such that $G_1 \in M \Delta D_\gamma(G_1) + 1$, $n_1$ is even and $G_2$ is $\gamma_*$-monotone. If $\gamma_*(G_1) = \frac{n_1}{2}$ and $\gamma_*(G_2) = \left\lceil \frac{n_2}{2} \right\rceil$, then the graph $G = G_1 \cup G_2$ belongs to $s_{k}^{n_1+n_2}$, where $k = \left\lceil \frac{n_1+n_2}{2} \right\rceil$.

Proof. If $n_2 = 1$ or $n_2$ is even, then one can see that the lemma is true. So let us assume that $n_2 \geq 3$ and that $n_2$ is odd. From Remark 1, we know that $n_1 \geq 6$. We obtain the two required sets of the smallest cardinality as follows: if $n_1 \geq n_2$, then $S_{n_1+n_2} = SD(G_1) \cup S_{n_1+n_2+1}$. Since $n_2$ is odd, we have $\frac{n_1+n_2}{2} = \frac{n_1+n_2+1}{2} \geq \frac{n_1+4}{2} = \frac{n_1}{2} + 2$. So the indicated sets exist. On the other hand, if $n_1 < n_2$, then $S_{n_1+n_2} = SD(G_2) \cup S_{n_1+n_2+1}$. We can also see that the set $S_{n_1+n_2+1} \cup SD(G_1)$ can be chosen as the union of the sets $SD(G_1) \cup SD(G_2) \cup S_{n_1+n_2+1}$. Now it is easy to see how to obtain the rest of the required sets. □

Lemma 8. Let both $G_1$ and $G_2$ be connected graphs of order $n_1 > 0$ and $n_2 > 0$, respectively, such that $G_1 \in M \Delta D_\gamma(G_1) + 1$, $G_2$ is $\gamma_*$-monotone, and $n_2$ is even. If $\gamma_*(G_1) = \left\lceil \frac{n_1}{2} \right\rceil$ and $\gamma_*(G_2) = \frac{n_2}{2} + 1$, then the graph $G = G_1 \cup G_2$ belongs to $s_k^{n_1+n_2}$, where $k = \left\lceil \frac{n_1+n_2}{2} \right\rceil$.

Proof. From Remark 1, we know that $n_1 \geq 6$, and from Remark 2, it follows that also $n_2 \geq 6$. If $n_1 \geq n_2$, then $S_{\frac{n_1+n_2}{2}} = SD(G_1) \cup S_{n_1+n_2+1}$; otherwise, $S_{\frac{n_1+n_2}{2}} = SD(G_2) \cup S_{n_1+n_2+1}$. We can obtain the set $S_{\frac{n_1+n_2}{2}+1}$ as an union of the sets $SD(G_1) \cup SD(G_2) \cup S_{n_1+n_2+1}$. Furthermore, the set $S_{\frac{n_1+n_2}{2}+1}$ can be chosen as an union of the sets $SD(G_1) \cup SD(G_2) \cup S_{n_1+n_2+1}$. The remaining required sets are easy to obtain. □

Lemma 9. Let both $G_1$ and $G_2$ be connected graphs of even order $n_1 > 0$ and $n_2 > 0$, respectively, such that $G_1 \in M \Delta D_\gamma(G_1) + 1$ and $G_2 \in M \Delta D_\gamma(G_2) + 1$. If $\gamma_*(G_1) = \frac{n_1}{2}$ and $\gamma_*(G_2) = \frac{n_2}{2}$, then the graph $G = G_1 \cup G_2$ belongs to $s_k^{n_1+n_2}$, where $k = \frac{n_1+n_2}{2} + 2$, and there exists a set $S(G) \cup S_{n_1+n_2}$.

Proof. Without loss of generality, we can assume that $n_1 \geq n_2$. From Remark 1, it follows that $n_1, n_2 \geq 6$. Hence we have $S_2 = SD(G_1) \cup S_{n_1+n_2}, S_{n_1+n_2} = SD(G_1) \cup SD(G_2), S_{\frac{n_1+n_2}{2}+1} = SD(G_2) \cup SD(G_2) \cup S_{n_1+n_2+1}$. It is a simple matter to find the required secure sets of greater cardinalities. □

4. Cographs

Cographs are a well-known family of graphs with many applications. We define them recursively as follows:

1. the graph $K_1$ is a cograph,
2. the union of disjoint cographs $G_1$ and $G_2$ is a cograph,
3. the join of disjoint cographs $G_1$ and $G_2$ is a cograph.

Cographs are also graphs without induced path on four vertices [5]. Furthermore, they can be characterized by the cotree [6]. A cotree $T$ is a rooted tree of the modular decomposition of a cograph. A module in a graph is a set of vertices which have the same neighbourhood outside the set, and the modular decomposition is a decomposition of a graph into modules.
See [7,8,18,19] for more information. The cotree \( T \) can be described in the following way. The leaves of \( T \) are the vertices of \( G \), and every interior node is either a 0-node or a 1-node. A 0-node and a 1-node represent a union operation and a join operation, respectively. A cotree is uniquely determined, and no 0-node is a child of a 0-node and no 1-node is a child of a 1-node. What is more, every interior node has at least two children.

Now we will formulate our main theorem, which concerns the whole class of cographs.

**Theorem 2.** Let \( G \) be a cograph of order \( n \).

(1) If \( G \) is disconnected, then
   (a) if \( n \) is even, then \( G \in S^{n}_{2} \) and there exist set \( S_{2}^{n} \) or \( S_{2}^{n+1} \),
   (b) if \( n \) is odd, then \( G \in S^{n}_{2+1} \).

(2) If \( G \) is connected, then
   (a) if \( n \) is even, then \( G \) is \( \gamma_{s} \)-monotone and \( \gamma_{s} \in \left\{ \frac{n}{2}, \frac{n}{2} + 1 \right\} \) or \( G \in MSD_{2}^{n+1} \) and \( \gamma_{s} = \frac{n}{2} \),
   (b) if \( n \) is odd, then \( G \) is \( \gamma_{s} \)-monotone and \( \gamma_{s} = \frac{n+1}{2} \).

**Proof.** For the clarity of the proof, we will distinguish some remarks, so that we can easily refer to them. It is easy to verify that, for all cographs of order 1, 2, 3, and 4, our theorem and the next remark is true.

**Remark 3.** Every disconnected cograph of order 2 or 4 has \( S_{2}^{n} \) and \( S_{2}^{n+1} \), and every connected cograph of order 2 or 4 has \( SD_{2}^{n} \) and \( SD_{2}^{n+1} \).

We will use induction on the order of \( G \). Let us suppose that the theorem is true for all cographs of order \( k \), where \( 1 \leq k < n \). Let \( G \) be a cograph of order \( n \).

**Part 1.** Let us suppose that \( G \) is disconnected. We must consider two cases.

(a) Suppose that \( n \) is even. That implies that the number of components of odd order is even. So, if there exist components of odd order, we can join them in pairs. We do the same with the components of even order. There can be at most one component of even order without a pair. But to this component we can also apply our inductive hypothesis, and this fact together with **Lemma 4–9** and **Proposition 1** implies that \( G \) has a secure set of cardinality \( \frac{n}{2} \) or \( \frac{n}{2} + 1 \) and all the remaining sets \( S_{2}^{n+2}, \ldots, S_{n} \).

(b) Suppose that \( n \) is odd, which implies that we have an odd number of components of odd order. As previously, we join in pairs the components of the same parity of the order. If there is one component of even order without a pair, then we join it in a pair with one remaining component of odd order, and as in previous case we can prove that the theorem is true. The next remarks allow us to simplify some parts of the proof.

**Remark 4.** Every disconnected cograph \( G \) of even order \( n \) that has an even number of connected components has the set \( S_{2}^{n} \).

**Part 2.** Let us suppose that \( G \) is connected. If \( G \) is a complete graph, then our theorem is true, so we can assume that it is not the case. Let \( G = G_{1} + G_{2} \). The properties of the cotree allow us to assume that \( G_{1} \) or \( G_{2} \) is disconnected. Suppose the following.

(a) \( n_{1} \) and \( n_{2} \) are odd. Then from the inductive hypothesis and **Lemma 1** it follows that our theorem is true.

(b) \( n_{1} \) is even and \( n_{2} \) is odd. First, we suppose that there exists \( S(G_{1}) \in \frac{n_{1}}{2} \). If \( n_{2} > 1 \) or there exists \( S(G_{1}) \in \frac{n_{1}+1}{2} \), then clearly our theorem is true. If \( n_{2} = 1 \) and the set \( S(G_{1}) \in \frac{n_{1}}{2} \), then clearly our theorem is true. Then \( G_{2} \) is connected, then \( G_{1} \) is not, and from **Remark 1** we know that \( n_{1} \geq 6 \). Since \( S(G_{1}) \in \frac{n_{1}}{2} \) does not exist, in any partition into pairs of the components of \( G_{1} \) (as was described in Part 1) there does not exist a pair that satisfies the conditions of **Lemma 4** or **6–8**. Thus in \( G \) are only connected components of even order and to each pair in the partition we can apply either **Lemma 9** or **5**. Hence the existence of the missing set follows from following two claims, **Claims 1 and 2**.

**Claim 1.** If \( G \) is a disconnected cograph of order \( n \geq 12 \) such that it has two connected components \( G_{1} \) and \( G_{2} \) of even order, and \( G_{1} \in MSD_{\gamma_{s}(G_{1})+1}^{\gamma_{s}(G_{1})+1} \), \( G_{2} \in MSD_{\gamma_{s}(G_{2})+1}^{\gamma_{s}(G_{2})+1} \), \( \gamma_{s}(G_{1}) = \frac{n_{1}}{2} \), and \( \gamma_{s}(G_{2}) = \frac{n_{2}}{2} \), then \( G_{1} = G + K_{1} \) of order \( n_{1} + 1 \) is \( \gamma_{s} \)-monotone and \( \gamma_{s}(G) = \left\lceil \frac{n}{2} \right\rceil \).

**Proof.** Let \( V(K_{1}) = \{v\} \). From the inductive hypothesis of the theorem, we know that a cograph \( H \) induced by the vertex \( v \) and \( V(G_{1}) \) has \( SD(H) \in \frac{n_{1}+2}{2} \). Moreover, by **Lemma 2**, we can choose \( SD(H) \in \frac{n_{1}+2}{2} \) that contains \( v \), and so \( SD_{\frac{n_{1}+1}{2}} = SD(H) \cup SD_{\frac{n_{1}+2}{2}} \). The existence of the remaining required sets follows from **Lemmas 1 and 9**.  

**Claim 2.** If \( G \) is a disconnected cograph of order \( n \geq 4 \) such that it has two connected components \( G_{1} \) and \( G_{2} \) of even order, and both components are \( \gamma_{s} \)-monotone and \( \gamma_{s}(G_{1}) = \frac{n_{1}}{2} + 1 \) and \( \gamma_{s}(G_{2}) = \frac{n_{2}}{2} + 1 \), then \( G_{1} = G + K_{1} \) of order \( n_{1} + 1 \) is \( \gamma_{s} \)-monotone and \( \gamma_{s}(G) = \left\lceil \frac{n}{2} \right\rceil \).
Lemma 10. Let $\gamma$ belong to $G$ on a situation is not so clear in the case of cographs of even order. We know that there exist cographs with the procedure we described in Part 1. Otherwise, from the inductive hypothesis we have $S(G_1)_{\frac{n_1}{2}+1}$, Let $G_{1\|i}$ be the one component of even order which is not in a pair (if $G_1$ is connected then $G_{1\|i} = G_1$). Because $S(G_1)_{\frac{n_1}{2}}$ does not exist, then there does not exist $S(G_{1\|i})_{\frac{n_2}{2}+1}$. Let $H = G_1 + K_1$ and $V(K_1) = \{v\}$. From the inductive hypothesis and Lemma 2, we know that there exists $SD(H)_{\frac{n_1}{2}+1}$, that contains $v$ (please note that, if $G_{1\|i} = G_1$, then $G_2$ is disconnected and $n_2 > 1$; thus $|V(H)| < n$).

Remark 5. Vertex $v$ does not have to defend itself in any of the attacks of the vertices that belong to $V(H) - S(H)_{\frac{n_1}{2}+1}$.

Proof. Let $V(K_1) = \{v\}$. From the inductive hypothesis of the theorem, it follows that a cograph $H$ induced by the vertex $v$ and $V(G_1)$ has $SD(H)_{\frac{n_1}{2}+1}$ (n_1 \geq 6). Moreover, by Lemma 2, we can choose $SD(H)_{\frac{n_1}{2}+1}$ that contains $v$. Thus $SD_{\frac{n_1}{2}+1} = SD(H)_{\frac{n_1}{2}+1} \cup SD(G_2)_{\frac{n_2}{2}+1}$. The existence of the remaining required sets follows from Lemma 5.

So let us suppose that $S(G_1)_{\frac{n_1}{2}}$ does not exist. Thus, if $G_1$ is disconnected, then we can find the set $S(G_1)_{\frac{n_1}{2}+1}$ according to the procedure we described in Part 1. Otherwise, from the inductive hypothesis we have $S(G_1)_{\frac{n_1}{2}+1} = SD(G_1)_{\frac{n_1}{2}+1}$. Let $G_{1\|i}$ be the one component of even order which is not in a pair (if $G_1$ is connected then $G_{1\|i} = G_1$). Because $S(G_1)_{\frac{n_1}{2}}$ does not exist, then there does not exist $S(G_{1\|i})_{\frac{n_2}{2}+1}$. Let $H = G_1 + K_1$ and $V(K_1) = \{v\}$. From the inductive hypothesis and Lemma 2, we know that there exists $SD(H)_{\frac{n_1}{2}+1}$, that contains $v$ (please note that, if $G_{1\|i} = G_1$, then $G_2$ is disconnected and $n_2 > 1$; thus $|V(H)| < n$).

Of course there exists an attack in which $v$ must defend a vertex $x \in V(G_{1\|i}) \subset V(H)$; otherwise, there would exist set $S(G_{1\|i})_{\frac{n_2}{2}+1}$. Now we can see that, from Remarks 4 and 5 and Lemma 1, it follows that, if $W = G_1 - G_{1\|i}$, then $S(W)_{\frac{n_1}{2}} \cup (S(H)_{\frac{n_1}{2}+1} - \{v\}) \cup S(G_2)_{\frac{n_2}{2}+1}$ gives us $SD_{\frac{n_1}{2}+1}$ (if $G_{1\|i} = G_1$, then $S(W)_{\frac{n_1}{2}} = \emptyset$). It is a simple matter to see how to obtain the rest of the required global secure sets.

(c) $n_1$ and $n_2$ are even. First, let us suppose that there exists neither $S(G_1)_{\frac{n_1}{2}}$ nor $S(G_2)_{\frac{n_2}{2}}$. In this case, as in subcase (b), we can prove that there exist sets $SD_{\frac{n_1}{2}+1}$, . . . , $SD_n$. Finally, if there exists $S(G_1)_{\frac{n_1}{2}}$ or $S(G_2)_{\frac{n_2}{2}}$, then we can use the inductive hypothesis and Lemma 1 to complete the proof.

Cographs of odd order always have a global secure set of cardinality equal to the lower bound on $\gamma_1(G)$. Unfortunately the situation is not so clear in the case of cographs of even order. We know that there exist cographs with $\gamma_1(G) = \frac{n}{2}$ (for example complete graphs of even order), but the proof of Theorem 2 does not decide whether there exist cographs with $\gamma_1(G) = \frac{n}{2} + 1$. To prove this fact we need to present a special class of cographs. Let $Q_k$ denote an union of $k$ ($k \geq 1$) disjoint graphs $K_2$, and let $H_k = Q_k + Q_k$.

Let $G$ is a cograph : $G = (H_k \cup K_1) + (H_k \cup K_1)$ and $k$ is even} be a graph class denoted by $\mathcal{P}$. The example of the graph which belongs to $\mathcal{P}$ is presented in Fig. 1.

Lemma 10. If $G \in \mathcal{P}$, then $\gamma_1(G) = \frac{n}{2} + 1$.

Proof. Let $G = G_1 + G_2$, where $G_1 = (H_k \cup K_1)$, and $G_2 = (H_k \cup K_1)$, where $k$ is even and greater than or equal to 2. Furthermore, let $v$ denote the isolated vertex in $G_1$. It is easy to see that $\gamma_1(G) \leq \frac{n}{2} + 1$ (see Fig. 1 for an example). Thus it is sufficient to prove that there does not exist $SD_2$. Suppose for the sake of contradiction that there exists a set $D$ such that $D$ is secure, dominating, and $|D| = \frac{n}{2}$. From the construction of $G$ we know that $\frac{n}{2} = 4k + 1$ is an odd number. Let $D = D_1 \cup D_2$, where $D_1 = V(G_1) \cap D$ and $D_2 = V(G_2) \cap D$. Without loss of generality, we can assume that $|D_1| > |D_2|$. Let us denote this condition by $(s)$. We must also describe the graph $G_1$ more precisely. Let $H_k = Q^1_k + Q^2_k$, where the additional indices allow us to distinguish the vertices that belong to first and the second graphs which are joined in the construction of $H_k$. First, we assume that $v \in D_1$. Then $|D_2| = 2k$, or else $v$ would have more attackers than defenders. It follows that $|D_1| = 2k + 1$. Let us assume that $D_1 \cap (V(G_1) - \{v\}) \subseteq Q^1_k$. Then $|D_2| = 2k + 1$ (all the neighbours in $Q^1_k$ have two attackers). It follows that there exists a vertex $x \in D_1 \cap (V(G_1) - \{v\})$ such that $|N[x] \cap D_1| = 2k$. Thus $x$ has more attackers than defenders, which contradicts the security of $D$. So let us suppose that there exist $x, y \in D_1 \cap (V(G_1) - \{v\})$ such that $x \in Q^1_k$ and $y \in Q^2_k$. Then $|N[x, y] \cap D_1| = 2k + 2k$ (all the vertices that belong to $D_2$ and $D_1 - \{v\}$) and $|N[x, y] \cap D_2| = 2k + 2k + 1$ (all the vertices that belong to $V(G_1) - \{v\} - D_1$ are attackers).
and to $V(G_2) - D_2$. Once again we get a contradiction to the fact that $D$ is secure. It follows that, if there exists $SD_2^*$, then $v$ does not belong to it. Let $A = D \cap Q_1^+$ and $B = D \cap Q_2^+(A \cup B = D_1)$.

Without loss of generality, we can also assume that $|A| \leq |B|$. In our calculations we will use the following fact (**): $|A| + |B| + |D_2| = D = 4k + 1$.

Suppose that there exists a vertex $x \in B$ such that it has a neighbour $u \in Q_2^+$ and $u \notin B$. Thus $|N[x] \cap D| = |A| + |D_2| + |\{x\}|$ and $|N[x] \cap D| = 1 + 2k - |A| + 4k + 1 - |D_2|$ (u, the vertices in $V(Q_1^+) - A$ and $V(G_2) - D_2$). Since $D$ is secure, $|N[x] \cap D| \geq |N[x] - D|$, and from this inequality it follows that $|B| \leq k$ (see (1)).

If there does not exist a vertex $x$ that satisfies our requirements, then surely $|B| = k$. Let $x$ be any vertex in $B$. Thus $|N[x] \cap D| = |A| + |D_2| + |\{x, u\}|$ and $|N[x] - D| = 2k - |A| + 4k + 1 - |D_2|$. From the security of $D$ it follows that $|N[x] \cap D| \geq |N[x] - D|$, and from this inequality we get $|B| \leq k$ (see (2)).

So, in any case, $|B| \leq k$, which means that $|A| \leq |B| \leq k$. Therefore, $|D_1| \leq 2k$, which implies that $|D_2| \geq 2k + 1$. This contradicts our earlier assumption that $|D_1| > |D_2|$. \hfill \square

Theorem 2 does not give the necessary and sufficient conditions for a connected cograph to be $\gamma_\kappa$-monotone. This problem seems to be a hard task. Also, finding a class of cographs which are not $\gamma_\kappa$-monotone is not trivial. That is why we will give examples of such cographs. Let $\mathcal{P}$ denote a family of cographs such that for every graph $G \in \mathcal{P}$: $G = (G' \cup G'' + (G' \cup G'') \in \mathcal{P})$ and $G \in \mathcal{P}$.

**Theorem 3.** If $G \in \mathcal{P}$, then $\gamma_\kappa(G) = \frac{n}{2}$ and $G \in \mathcal{M}_\mathcal{S}D_{\gamma_\kappa(G)+1}$.

**Proof.** Let $G = (G_1 \cup G_2) + (G_3 \cup G_4)$, where $G_1 = G_2 = G_3 = G_4 = G'$ and $G \in \mathcal{P}$. Furthermore, let $G_1 = (H_k \cup K_1) + (H_k \cup K_1)$, for even $k > 0$. Clearly, $V(G_1) \cup V(G_2)$ gives us $SD(G_1^2)$. Suppose that the second part of the theorem is false. Then we can find a set $D$ such that $D$ is secure, dominating, and $|D| = \frac{n}{2} + 1 = 16k + 5$. Let $D_1 = V((G_1 \cup G_2)) \cap D$ and $D_2 = V((G_3 \cup G_4)) \cap D$. Because of the symmetry of the graph we can suppose, without loss of generality, that $|D_1| \geq |D_2|$. Additionally, let $A = V(G_1) \cap D$ and $B = V(G_2) \cap D$. We can assume that $|A| \geq |B|$. If $|B| = 0$, then $|D_1| \leq |V(G_1)| = 8k + 2$ and $|D_2| \geq 8k + 3$, which contradicts ($C^*_1$). So $A \neq \emptyset$ and $B \neq \emptyset$.

**Claim 3.** Let $G$ be a graph that belongs to $\mathcal{P}$ and let $G = G_1 + G_2$, where $G_1 = (H_k \cup K_1)$, $G_2 = (H_k \cup K_1)$, and $k$ is even and greater or equal to 2. If $S$ is a secure set in $G$, then $S$ is a dominating set that contains vertices of both $G_1$ and $G_2$.

**Proof.** Suppose that our claim is false. Then either $S \subseteq V(G_1)$ or $S \subseteq V(G_2)$. Without loss of generality, we can assume that $S \subseteq V(G_1)$. Let $v$ be an isolated vertex of $G_1$. If $v \notin S$, then $|N[v] \cap S| = 1$ and $|N[v] - S| = |V(G_2)| > 1$. Thus we have a contradiction to the fact that $S$ is secure. So $v \notin S$. Let $x$ be any vertex that belongs to $S$. It is easy to see that $|N[x] \cap S| \leq 2 + 2k$ and $|N[x] - S| \geq |V(G_1)| = 4k + 1$. Since $k \geq 2$, we have shown once again that $S$ cannot be a secure set. \hfill \square

Since, by Lemma 10, $\gamma_\kappa(G') = \frac{n}{2} + 1 = 4k + 2$, and by Claim 3 every secure set of $G'$ is a dominating set, it follows that, if $|A| \leq 4k + 1$ or $|B| \leq 4k + 1$, then $A$ or $B$ is not a secure set in $G_1$, $G_2$, respectively. Suppose, without loss of generality, that $|A| \leq 4k + 1$. Thus there exists a set $X \subseteq A$ such that $|N[X] \cap A| < |N[X] - A|$. Suppose that $|D_2| \leq 8k + 2$. Then $|N[X] \cap D| = |N[X] \cap A| + |D_2|$ and $|N[X] - D| = |N[X] - A| + |V((G_3 \cup G_4)) - D_2|$. Now from our assumptions it follows that $|N[X] \cap D| \leq |N[X] - D|$. Hence $|D_2| \geq 8k + 3$, $|D_1| \leq 8k + 2$, and we get a contradiction with ($C^*_1$).

Let us suppose that $|B| = z \geq 4k + 3$. Thus $|A| \geq 4k + 3$, $|D_1| \geq 8k + 6$, and $|D_2| \leq 8k - 1$. From the construction of $G_2$ we know that there exist vertices $x, y \in B$ such that $|N[x, y] \cap D| = z + |D_2|$ and $|N(x, y) - D| = |V(G_2^*) - z + V(G_3) \cup V(G_4^*) - |D_2|. Since $D$ is secure, $|N[x, y] \cap D| \geq |N(x, y) - D|$. (1) $|N[x, y] \cap D| \geq |N[x, y] - D| \Rightarrow z + |D_2| \geq 8k + 2 - z + 16k + 4 - |D_2| \Rightarrow z + |D_2| \geq 24k + 6 - z - |D_2| \Rightarrow z + |D_2| \geq 12k + 3 \Rightarrow |D| - |A| \geq 12k + 3 \Rightarrow 16k + 5 - |A| \geq 12k + 3 \Rightarrow |A| \leq 4k + 2$. From this inequality, it follows that $|A| \leq 4k + 2$ (see (1)), which contradicts our previous assumptions. Thus we have to assume that $|B| = 4k + 2$. Hence $|A| \geq 4k + 2$, $|D_1| \geq 8k + 4$, and $|D_2| \leq 8k + 1$. Again, from the construction of $G_2$ and the cardinality of $B$, it follows that there must exist vertices $x, y \in V(G_2)$ such that $N_{G_2}([x, y]) = V(G_2)$. Thus $|N(x, y) \cap D| = 4k + 2 + |D_2|$ and $|N(x, y) - D| = 4k + |V(G_3) \cup V(G_4)| - |D_2|$. One can see that the only possibility that $D$ could be secure is when $|D_2| = 8k + 1$ and $|A| = |B|$. Before we make the final conclusion, we must prove one more claim.

**Claim 4.** If $G \in \mathcal{P}$, then there does not exist $S^*_{\frac{n}{2} + 1}$ which is also secure in $G + K_2$ or $G + \overline{K_2}$.

**Proof.** Suppose that the assertion of the claim is false. Let $S'$ be a secure set in $G + K_2$ (the proof for $G + \overline{K_2}$ stays the same) and $|S'| = \frac{n}{2} + 1$. Furthermore, let $G = G_1 + G_2$, where $G_1 = H_k \cup K_1$, $V(K_1) = \{q\}$ and $G_2 = H_k \cup K_1$, for an even $k > 0$. Furthermore, let $S' = S'_1 \cup S'_2$, where $S'_1 = V(G_1) \cap S'$ and $S'_2 = V(G_2) \cap S'$. Without loss of generality, we can assume that $|S'_1| \geq |S'_2|$ (we will denote this condition by (*)). We must also describe the graph $G_1$.
more precisely. Let $H_k = Q_k^1 + Q_k^2$, where the additional indices allow us to distinguish the vertices that belong to the first and second graphs which are joined. Let $A_1 = Q_k^1 \cap S' \subset V(G_1)$ and $A_2 = Q_k^2 \cap S \subset V(G_2)$. Because of the symmetry of the graphs, we can assume that $|A_1| \geq |A_2|$. First, let us suppose that $q \in S_1$. From the degree of $q$ in $G + K_2$ and (*), it follows that $|S_2'| = |S_1'| = 2k + 1$ ($|S'| = 4k + 2$) and $|A_1| + |A_2| = 2k$. If $A_2 \neq \emptyset$, then, for any pair of vertices $x, y$ such that $x \in A_1$ and $y \in A_2$, we have $|N(x, y)| \cap S' = |A_1| + |A_2| + |S_2'| = |S'| - 1 = 4k + 1 + |N(x, y)' - S'| = |V(G_1) - \{q\} - |A_1| - |A_2| + |V(G_2)| - |S_2'| + 2 = 4k - 2k + 4k + 1 - (2k + 1) + 2 = 4k + 2$, which contradicts the security of $S'$. If $A_2 = \emptyset$, then, for any vertex $z \in A_1$, $|N(z) \cap S'| = 2 + 2k + 4k + 1 + |N(z)' - S'| = 2 + 2 + 2 + 2k$, and we have a contradiction to the fact that $S'$ is secure. Consequently, we can assume that $q \notin S'$. Suppose that there exists a vertex $x \in S_1$ such that it has a neighbour in $Q_k^1$ which does not belong to $A_1$. Then $|N(x)' \cap S'| = 1 + |A_2| + |S_2'|$ and $|N(x)' - S'| = 1 + 2k - |A_2| + 4k + 1 - |S_2'| + 2$. We will use this fact (**): $|A_1| + |A_2| + |S_2'| = |S'| = 4k + 2$.

(1) $|N(x)' \cap S'| \geq |N(x)' - S'| \Rightarrow 1 + |A_2| + |S_2'| \geq 1 + 2k - |A_2| + 4k + 1 - |S_2'| + 2 \Rightarrow 2(|A_2| + |S_2'|) \geq 6k + 3 \Rightarrow |A_2| + |S_2'| \geq 3k + \frac{3}{2} \Rightarrow |S' - |A_1| \geq 3k + \frac{3}{2} \Rightarrow |A_1| \leq k + \frac{1}{2} \Rightarrow |A_1| \leq k$.

(2) $|N(x)' \cap S'\cap S| \geq |N(x)' - S'| \Rightarrow 2 + |A_2| + |S_2'| \geq 2k - |A_2| + 4k + 1 - |S_2'| + 2 \Rightarrow 2(|A_2| + |S_2'|) \geq 6k + 3 \Rightarrow |A_2| + |S_2'| \geq 3k + \frac{3}{2} \Rightarrow |A_1| \leq k + \frac{3}{2} \Rightarrow |A_1| \leq k$. In both cases, we have shown that $|A_2| \leq |A_1| \leq k$ (see (1) and (2)). Thus $|S_1'| \leq 2k$ and $|S_2'| \geq 2k + 2$. This gives us a contradiction to (*). □

By the above claim, we know that neither $A$ nor $B$ is a secure set of cardinality $4k + 2$ that is also secure in $G + K_2$ or $G_2 + K_2$. Since $|V(G_3) \cup V(G_4)| - |D| = 8k + 3$, we conclude that $D$ is not secure.

The existence of the global secure sets of cardinality greater than $\gamma_2(G) + 1$ follows from Theorem 2. □

We will show that there exists a nontrivial subclass of cographs such that every graph from this subclass is $\gamma_2$-monotone. Let $C$ be a class of graphs defined as follows:

1. the graph $K_1$ belongs to $C$,
2. the union of disjoint graphs $G_1 \in C$ and $G_2 \in C$ belongs to $C$,
3. the join of disjoint graphs $G_1 \in C$ and $G_2 \in C$, where $G_1$ or $G_2$ is connected, belongs to $C$.

From the definition of the ctree and the class $C$, this remark follows.

**Remark 6.** If $G \in C$, then every 1-node of a ctree is adjacent to at least one leaf.

We can make a connection with another well-known graph class, i.e., trivially perfect graphs. A graph is trivially perfect if and only if it contains no vertex subset of cardinality 4 that induces a path or a cycle [12]. These graphs have also a ctree characterization.

**Lemma 11 ([14]).** A cograph $G$ is a trivially perfect graph if and only if, in the ctree $T$ of $G$, every 1-node has at most one child that is a 0-node.

From the above lemma and Remark 6 it follows that trivially perfect graphs form a subclass of $C$.

**Theorem 4.** Let $G \in C$ be a graph of order $n$.

(a) If $G$ is disconnected, then $G \in S_{\frac{n}{2}}$

(b) If $G$ is connected, then $\gamma_2(G) = \left\lceil \frac{n}{2}\right\rceil$, $G$ is $\gamma_2$-monotone, and for every $k > \frac{n}{2}$ there exists $SD_k \subseteq V(G)$ that is a global secure set in $G + K_1$.

**Proof.** Again we will use induction on the order of $G$. Let us suppose that the theorem is true for all graphs in $C$ of order $k$, where $1 \leq k < n$. Let $G \in C$ be a graph of order $n$. Suppose that $G$ is disconnected. Then, as in the proof of Theorem 2, we join its components in pairs with respect to their parity of orders. If we have an even number of components, then we create at most one mixed pair which contains one component of odd order and one of even order. To all pairs, even the mixed one (if it exists), we can apply Lemma 4. If the number of components is odd, then we have a component (at most one) without a pair. In this case, we can also use the inductive hypothesis, since every component is a graph from $C$. Please note that, if we have a component without a pair, then we do not have a mixed pair. From this considerations it follows that the first part of our theorem is true. We will prove one more property of the set $S_{\frac{n}{2}}$:

**Claim 5.** If a disconnected graph $G \in C$ has odd order, then it has a set $S_{\frac{n}{2}}$ that is secure in $G + K_1$.

**Proof.** We join in pairs the components of $G$ in accordance to the rules previously described. Since $G$ has odd order we have either one mixed pair or a component of odd order without a pair. Let $\{G', G''\}$ be a pair of components of orders $n'$ and $n''$, respectively, and let $n'' > n'$. By the inductive hypothesis we can choose the set $SD_{\left\lceil \frac{n''}{2}\right\rceil}(G')$ in such a way that it is secure in $G' + K_1$. If we repeat this procedure for every pair of components, and similarly we choose the global secure set for a component without a pair (if it exists), then the union of the chosen sets is $S_{\frac{n}{2}}$, which is secure in $G + K_1$. □
Now we suppose that $G$ is connected. From the definition of $C$ we know that join was the last operation in the creation of $G$. So let $G = G_1 + G_2$, where $G_1, G_2 \in C$ and $G_2$ is connected. If $G$ is a complete graph, then our theorem is true; otherwise, without loss of generality, we can assume that $G_1$ is disconnected. From the inductive hypothesis we know that $G_2$ is $\gamma_1$-monotone, $\gamma_1(G_2) = \left\lceil \frac{n_2}{2} \right\rceil$, and $G_1 \in \mathcal{C} \left\lceil \frac{n_1}{2} \right\rceil$, where $n_1$ and $n_2$ denote the order of $G_1$ and $G_2$, respectively.

Suppose that $G_1$ or $G_2$ has even order. By Lemma 1 and the inductive hypothesis we can obtain the set $SD_k$, for $k \geq \left\lceil \frac{n}{2} \right\rceil$, as an union of $S_p(G_1)$ and $S_d(G_2)$, where $p \geq \left\lceil \frac{n_1}{2} \right\rceil$, $t \geq \left\lceil \frac{n_2}{2} \right\rceil$, and $p + t = k$. Furthermore, if $k > \left\lceil \frac{n}{2} \right\rceil$ or $n_2$ is odd and $k = \left\lceil \frac{n}{2} \right\rceil$, then we formulate additional conditions, that $t$ must be greater than $\frac{n}{2}$ and $SD_1(G_2)$ chosen in such a way that it is a global secure set in $G_2 + K_1$. Then we can easily verify that the so-obtained set $SD_k$ is also a global secure set in $G + K_1$. We have one more case to consider, i.e., $n_1$ is odd ($n_2$ is even) and $k = \left\lceil \frac{n}{2} \right\rceil$. By Claim 5, $G_1$ has a secure set $S_{\left\lceil \frac{n_1}{2} \right\rceil}(G_1)$ that is secure in $G_1 + K_1$. Its union with $SD_{\left\lceil \frac{n}{2} \right\rceil}(G_2)$ gives us $SD_k$, which is a global secure set in $G + K_1$.

So let us assume that both $n_1$ and $n_2$ are odd. If we look for the set $SD_k$, where $k > \frac{n}{2}$, then we can obtain it by union of $S_p(G_1)$ and $S_d(G_2)$, which is secure in $G_2 + K_1$, where $p \geq \left\lceil \frac{n_1}{2} \right\rceil$, $t \geq \left\lceil \frac{n_2}{2} \right\rceil$, and $p + t = k$. We can see that the so-obtained set $SD_k$ is secure in $G + K_1$. To finish the proof, we need to show that there exists $SD_{\left\lceil \frac{n_1}{2} + \frac{n_2}{2} \right\rceil}$. Let $G'$ be any component of $G_1$ of odd order $n'$. Let $H$ be a graph that is isomorphic to $K_1$ and let $V(H) = \{v\}$. We will call $v$ an artificial vertex. From the inductive hypothesis and Lemma 2 we know that there exists a global secure set $SD_{\left\lceil \frac{n}{2} \right\rceil}(G' + H)$ that contains $v$. Let us denote it by $W'$. From the inductive hypothesis we also know that $G_1 - G'$ has a secure set $S_{\left\lceil \frac{n_1}{2} \right\rceil}(G_1 - G')$ and $G_2$ has a set $SD_{\left\lceil \frac{n_2}{2} \right\rceil}(G_2)$ that is a global secure set of $G + K_1$. Let us denote these sets by $W_1$ and $W_2$, respectively. Let $B = (W' - \{v\}) \cup W_1 \cup W_2$. Clearly, $B$ is a dominating set. Let us consider its security. Let $X$ be a subset of $B$. If $X \subseteq W_1$, then $|N(X) \cap B| = |N_{G_1 - G'}[X] \cap W_1| + |W_2|$, and $|N(X) \cap B| = |N_{G_1 - G'}[X] - W_1| + |W_2|$. Since $W_1$ is a secure set in $G_1 - G'$ and $\{W_2\} > |V(G_2)| - |W_2|$, we have that $|N(X) \cap B| > |N(X) \cap B|$. If $X \subseteq W_2$, then $|N(X) \cap B| = |N_{G_1 - G'}[X] \cap W_2| + |W_1| + |W_1| - 1 = |N_{G_1 - G'}[X] \cap W_2| + \left\lceil \frac{n_1}{2} \right\rceil - 1$ and $|N(X) - B| = |N_{G_1 - G'}[X] - W_2| + |V(G_2)| - |W_2| - (|W'\rangle - 1) = |N_{G_1 - G'}[X] - W_2| + \left\lceil \frac{n_1}{2} \right\rceil$. However, $W_2$ is a secure set in $G_2 + K_1$, so $|N_{G_2}[X] \cap W_2| \geq |N_{G_2}[X] - W_2| + 1$, which gives us $|N(X) \cap B| \geq |N_{G_2}[X] - W_2| + 1 + \left\lceil \frac{n_2}{2} \right\rceil - 1$, and thus $|N(X) \cap B| > |N(X) - B|$. So let us suppose that $X \subseteq (W' - \{v\})$. We have that $|N(X) \cap B| = |N_{G_1 + H}[X] \cap W'| - 1 + |W_2| = |N_{G_1 + H}[X] \cap W'| - 1 + \left\lceil \frac{n_1}{2} \right\rceil$ and $|N(X) - B| = |N_{G_1 + H}[X] - W'| + |V(G_2)| - |W_2| = |N_{G_1 + H}[X] - W'| + \left\lceil \frac{n_2}{2} \right\rceil - 1$. Since $W'$ is secure in $G' + H$, $|N(X) \cap B| \geq |N(X) - B|$. Let us consider the case when $X \subseteq W_1 \cup (W' - \{v\})$. Then there exist vertex disjoint sets $X_1 \subseteq W_1$ and $X_2 \subseteq W' - \{v\}$ such that $X = X_1 \cup X_2$. Thus $|N(X) \cap B| = |N_{G_1 - G'}[X_1] \cap W_1| + |N_{G_1 + H}[X_2] \cap W'| - 1 + |W_2|$. As, above, we have that $|N(X) \cap B| \geq |N(X) - B|$. To prove that $B$ is secure, we have to prove one more case when $X \cap V(G_2) \neq \emptyset$ and $X \cap V(G_1) \neq \emptyset$. It is easy to see that, since there exist vertices $x \in G_1$ and $y \in G_2$ that belong to $X$, $|N(X) \cap B| = |W'\rangle - 1 + |W_1| + |W_2| = \left\lceil \frac{n_1}{2} \right\rceil + \left\lceil \frac{n_2}{2} \right\rceil = \left\lceil \frac{n_1 + n_2}{2} \right\rceil$. Thus $|N(X) \cap B| = |N(X) - B|$. From the above considerations it follows that $B$ is a global secure set of cardinality $\frac{n_1 + n_2}{2}$, and the proof is complete. \hfill $\square$

**Theorem 5.** If $G \in C$, then we can find its global secure set of cardinality $k$, $\gamma_1(G) \leq k \leq n$, in $\mathcal{O}(n + m)$ time.

**Proof.** A cotree $T$ of $G$ can be computed in $\mathcal{O}(n + m)$, so we can assume that it is given. By using depth-first search algorithm we can compute for each node of $T$ the number of leaves among its descendants; we will call it a leaf number of a node. If a vertex is a leaf then we set its leaf number to 1. We prepare for each node the list of leaves that are adjacent to it, and additionally for each 1-node a list of 0-nodes and for each 0-node two lists of 1-nodes, one containing only nodes with odd leaf number and the second with nodes with even leaf number. These lists can be made during one execution of the breadth-first search algorithm. $T$ is now prepared to run an algorithm based on the proof of Theorem 4. In this recursive algorithm, if the root of the given tree is a 1-node, then, if all of its children are leaves, we can mark the required number of them as the vertices which belong to $SD_k$. If this is not the case, then in $\mathcal{O}(1)$ time we can split the tree and add an artificial vertex if it is required (if there is a situation that a subgraph to which we want to add the artificial vertex is $K_1$, then we do not modify $T$; we can simply omit the call of the algorithm for this subtree). We run the algorithm for the obtained subtrees. If the root of the given tree is a 0-node, then, after visiting every child only once (due to the previously prepared lists) we can join them in proper pairs and run the algorithm for chosen subtrees. If the root of the tree has no children (it is a leaf in $T$), then we simply mark it as the vertex that belongs to the global secure set that we search for. After the recursive algorithm stops, by using the depth-first search algorithm which we can delete all artificial vertices, and if the deleted vertex does not belong to the global secure set then we exclude from the global secure set a leaf from the proper subtree (the root node of this subtree is the parent of the removed artificial vertex). Since a 1-node can have at most one artificial vertex, we can store its address in the node. The above considerations show that the algorithm runs in $\mathcal{O}(n + m)$ time. \hfill $\square$

**5. Conclusion**

In this paper, we have defined $\gamma_1$-monotone graphs and have shown examples of such graphs. We have also proved that not every connected cograph is $\gamma_1$-monotone. However, we were able to give the best possible upper bound on the minimum
cardinality of global secure sets in this class of graphs. Furthermore, we have presented a linear-time algorithm which can find a global secure set of any proper cardinality in a special subclass of cographs. Namely, we have described a subclass of cographs such that every graph which belongs to it is $\gamma_s$-monotone. It is worth noticing that this subclass contains trivially perfect graphs.

References