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A NON-LUKACSIAN REGRESSIONAL CHARACTERIZATION OF THE GAMMA DISTRIBUTION

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Abstract—This paper adds to the numerous investigations of second order conditional structure of linear forms in independent random variables. A new characterization of the gamma law is obtained.

1. INTRODUCTION

We begin with the celebrated Lukacs [1] Theorem: *If random variables X, Y are positive non-degenerate and independent, then X/Y and $X+Y$ are independent iff X and Y have gamma distributions with the same scale parameter.* Recall that the gamma law is defined by the density

$$f(x) = \begin{cases} \frac{a^p}{\Gamma(p)} x^{p-1} e^{-ax} & \text{for } x > 0, \\ 0 & \text{for } x \le 0, \end{cases} \tag{1}$$

where $a > 0$, $p > 0$ (a is called a scale parameter).

Numerous regressional versions of this characterization are known. One of the most recent results and references are given in [2]. In all such results, the essential assumptions are imposed on forms of conditional moments of scale invariant statistics in X and Y given $X+Y$—see also [3]. In this note we try to free ourselves of this Lukacsian scheme of independence by considering the second order conditional structure of an arbitrary linear transformation of (X, Y). More precisely, we are interested in the form of $E(U \mid V)$ and $E(U^2 \mid V)$, where $U = aX + bY$, $V = cX + dY$ and a, b, c, d are real numbers.

It is well-known that these conditional moments characterize the normal distribution—see, for example, [4, Theorem 5.7.1]. However, in this case $E(U \mid V)$ and $E(U^2 \mid V)$ have very simple forms: they are linear and quadratic functions of the condition, respectively.

In the gamma case, the conditional moments have a much more complicated form. They are given in Section 2. Section 3 contains the main result which is a characterization of the gamma law in terms of the properties of conditional moments of linear forms in i.i.d. random variables. Let us emphasize that we assume slightly weaker assumptions than the exact form of conditional moments obtained in Section 2.

2. CONDITIONAL MOMENTS OF LINEAR FORMS IN GAMMA RANDOM VARIABLES

Let X, Y be independent r.v.'s with the common density (1). We compute the conditional moments $E(U \mid V)$ and $E(U^2 \mid V)$. Observe that, without any loss of generality, it suffices to consider $U = Y - \alpha X$, $V = Y - \beta X$, $\alpha \neq \beta \neq 0$.

After some elementary but laborious calculations we get: For $\beta < 0$,

$$E(U \mid V) = V(1 + F_0(V)),$$
$$E(U^2 \mid V) = V^2(1 + 2AF_0(V) + A^2F_1(V)),$$

where $A = \frac{\alpha-\beta}{\beta}$ and

$$F_k(v) \int_0^1 x^{p-1}(1-x)^{p-1}\exp(\delta v x)\,dx = \int_0^1 x^{p+k}(1-x)^{p-1}\exp(\delta v x)\,dx$$

for $v \in R$, $k = 0, 1$, $\delta = \frac{a(1+\beta)}{\beta}$. (Observe that for $\beta = -1$, the r.v.'s $F_k(V)$, $k = 0, 1$, are degenerate.)

For $\beta > 0$, the formulas are even more complicated:

$$E(U \mid V) = V\{1 + A(G_0(V)I(V<0) - H_0(V)I(V>0))\},$$
$$E(U^2 \mid V) = V^2\{1 + 2A(G_0(V)I(V<0) - H_0(V)I(V>0)) + A^2(G_1(V)I(V<0) + H_1 I(V>0))\}$$

where

$$G_k(v) \int_1^\infty x^{p-1}(x-1)^{p-1}\exp(\delta v x)\,dx = \int_1^\infty x^{p+k}(x-1)^{p-1}\exp(\delta v x)\,dx$$

for $v < 0$, and

$$H_k(v) \int_0^\infty x^{p-1}(x+1)^{p-1}\exp(-\delta v x)\,dx = \int_0^\infty x^{p+k}(1+x)^{p-1}\exp(-\delta v x)\,dx$$

for $v > 0$, $k = 0, 1$.

3. CHARACTERIZATION

Let X, Y be positive non-degenerate i.i.d. r.v.'s and $EX^2 < \infty$, $U = Y - \alpha X$, $V = Y - \beta X$, $\alpha \neq \beta \neq 0$. We are interested in characterizing the gamma distribution by the form of $E(U \mid V)$ and $E(U^2 \mid V)$.

THEOREM 1. *Assume that $\beta \neq \pm 1$ and*

$$E(U \mid V) = V(1 + AZ), \tag{2}$$
$$E(U^2 \mid V) = V^2(1 + 2AZ + A^2 Z) + bA^2 V(1 - 2Z), \tag{3}$$

where Z is a $\sigma(V)$-measurable r.v. and b is a non-zero real number. Then, X is a gamma random variable.

The result for $\beta = -1$ is known—see for example [3] or [5, Lemma 2.3]. The method of the proof is based on a translation of conditions (2) and (3) into a differential equation of the first order for the characteristic functions. We make it by a modification of an idea originally applied in [6]. The problem in the case $\beta = 1$ remains open.

Observe that if X and Y are gamma with the density (1), then (2) and (3) hold with

$$Z = \begin{cases} F_0(V) & \text{for } \beta < 0, \\ G_0(V)I(V<0) - H_0(V)I(V>0) & \text{for } \beta > 0, \end{cases}$$

and

$$b = \frac{\beta p}{(1+\beta)a}.$$

This observation follows from the equation

$$VT_1(V) = (V - 2b)T_0(V) + b, \tag{4}$$

where, for $k = 0, 1$, T_k is equal F_k or G_k or H_k. Each version of the formula (4) follows from an elementary calculation involving integration by parts.

PROOF. By (2) and (3) for any real t we have

$$EUe^{itV} = EVe^{itV} + AEVZe^{itV}, \tag{5}$$
$$EU^2e^{itV} = EV^2e^{itV} + A^2bEVe^{itV} + A(A+2)\,EV^2Ze^{itV} - 2A^2bEVZe^{itV}. \tag{6}$$

We differentiate (5) and obtain

$$EUVe^{itV} = EV^2e^{itV} + AEV^2Ze^{itV}. \tag{7}$$

Now, application of (5) and (7) to (6) leads to the formula

$$EU^2e^{itV} + (A+1)EV^2e^{itV} = (A+2)EUVe^{itV} + A(A+2)bEVe^{itV} - 2AbEUe^{itV}.$$

From the definitions of U, V and A, we find

$$EYe^{itY}EXe^{i(-\beta t)X} = b(\beta^{-1}EYe^{itY}Ee^{i(-\beta t)X} + Ee^{itY}EXe^{i(-\beta t)X}) \tag{8}$$

Observe that $b > 0$—it suffices to put $t = 0$ in (8). The above equation yields in a neighbourhood N of the origin

$$i\frac{\phi(t)}{\phi'(t)} + i\beta^{-1}\frac{\phi(-\beta t)}{\phi'(-\beta t)} = b^{-1} = 2c,$$

where ϕ is a characteristic function of the r.v. X. Now, by Rao's theorem (see [4, Theorem 1.5.7]), we conclude that

$$i\frac{\phi(t)}{\phi'(t)} = c_1t + c, \qquad t \in N. \tag{9}$$

Taking derivatives of the both sides of (9) at 0, we get $c_1 = -i/\rho$ for some $\rho > 0$. On the other hand, from (9), it follows that

$$\ln\phi = \frac{i}{c_1}\ln(c_1t + c) + c_2.$$

Consequently,

$$\phi(t) = \left(1 - \frac{it}{c\rho}\right)^{-\rho}$$

for $t \in N$.

Now the result follows from the fact that the gamma distribution is uniquely determined by all its moments. ∎

Laha and Lukacs [7] obtained a characterization of the gamma law by constancy of regression of a polynomial of the second order on a sum of i.i.d. r.v.'s—see also [4, Theorem 6.2.8]. However, the method applied there does not fit the situation, when in conditioning we have an arbitrary linear form. Therefore, it is of some interest to observe that from the proof of Theorem 1 the following result may be extracted easily (consider equation (8)):

THEOREM 2. *Assume that X, Y are i.i.d. positive, non-degenerate r.v.'s, $EX^2 < \infty$. If for some $\beta, 0 \neq \beta \neq \pm 1$, and b, $b \neq 0$,*

$$E(XY - 2bX \mid Y - \beta X) = \frac{b}{\beta}(Y - \beta X),$$

then X is a gamma r.v. ∎

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