The Drinfel’d double of multiplier Hopf algebras

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Abstract

Let \langle A, B \rangle be a pairing of two regular multiplier Hopf algebras A and B. The Drinfel’d double associated to this pairing is constructed by using appropriate representations of A and B on the same vector space \( B \otimes A \). We realize the Drinfel’d double, denoted by \( D \), as an algebra of operators on the vector space \( B \otimes A \). In the case that \langle A, B \rangle is a multiplier Hopf \(^\ast\)-algebra pairing, we prove that \( D \) is again a multiplier Hopf \(^\ast\)-algebra. If A and B carry positive integrals, we prove that \( D \) also has a positive integral. This proof is not given before.

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Introduction

For an overview on the theory of multiplier Hopf algebras we refer to [13]. Throughout, we work over the field \( \mathbb{C} \). The Drinfel’d double construction is associated to a pairing of two multiplier Hopf algebras. We recall some essential ideas of the concept of a pairing. Start with two regular multiplier Hopf algebras \((A, \Delta_A)\) and \((B, \Delta_B)\) together with a non-degenerate bilinear map \( \langle \cdot, \cdot \rangle \) from \( A \times B \) to \( \mathbb{C} \) satisfying certain properties. The main property is that the coproduct in \( A \) is dual to the product in \( B \) and vice versa. There are however certain regularity conditions, needed to give a correct meaning to this statement. The investigation of these conditions is done in [3]. For \( a \in A \) and \( b \in B \), define \( a \triangleright b \), \( b \triangleleft a \), \( b \triangleright a \), and \( a \triangleleft b \) as multipliers in the following way. Take \( a' \in A \), \( b' \in B \) and define

\[ a \triangleright b = \langle a' \otimes b', b angle \]

\[ b \triangleleft a = \langle a' \otimes b', a angle \]

\[ b \triangleright a = \langle a' \otimes b', \Delta_B a angle \]

\[ a \triangleleft b = \langle b' \otimes \Delta_A b, a \rangle \]

\[ a \triangleright b = \langle a' \otimes b', \Delta_B a \rangle \]

\[ b \triangleleft a = \langle b' \otimes \Delta_A b, a \rangle \]

\[ b \triangleright a = \langle a' \otimes b', \Delta_B a \rangle \]

\[ a \triangleleft b = \langle b' \otimes \Delta_A b, a \rangle \]
Then it is possible to state that the product and the coproduct are dual to each other:

\[ (b \triangleright a)a' = \sum (a_{(2)}, b)a_{(1)}a', \quad (a \triangleright b)a' = \sum (a_{(1)}, b)a_{(2)}a', \]

\[ (a \triangleright b)b' = \sum (a, b_{(2)})b_{(1)}b', \quad (b \triangleright a)b' = \sum (a, b_{(1)})b_{(2)}b'. \]

The regularity conditions on the pairing say (among other things) that the multipliers \( b \triangleright a \) and \( a \triangleright b \) in \( M(A) \) (respectively \( a \triangleright b \) and \( b \triangleright a \) in \( M(B) \)) belong to \( A \) (respectively \( B \)). Then it is possible to state that the product and the coproduct are dual to each other:

\[ \langle a, bb' \rangle = \langle b' \triangleright a, b \rangle = \langle a \triangleright b, b' \rangle, \quad \langle a'a, b \rangle = \langle a, a' \triangleright b \rangle = \langle a', b \triangleright a \rangle. \]

In this way we get four modules. All these modules are unital. By definition, e.g., \( B \) is a unital left \( A \) module means that any element \( b \in B \) is a linear combination of elements of the form \( a \triangleright b' \) with \( a \in A \) and \( b' \in B \). A stronger result however is possible here, coming from the existence of local units. Indeed, for a finite set \( \{ R \} \subset A \), there is an \( e \in A \) such that \( ea_i = a_i \forall i \). Now, take \( b \in B \). Then there exist \( a_i \in A \) and \( b_j \in B \) such that \( b = \sum a_i \triangleright b_j = \sum ea_i \triangleright b_j = e \triangleright b = e \triangleright b. \) So,

\[ \forall b \in B \exists e \in A: \quad b = e \triangleright b. \]

As an important consequence of this last result, we can use the Sweedler notation in the framework of dual pairs in the following sense. Take \( a \in A \) and \( b \in B \), and, e.g., the element \( b \triangleright a = \sum (a_{(2)}, b)a_{(1)} \). In the right-hand side the element \( a_{(2)} \) is covered by \( b \) through the pairing because \( b = e \triangleright b \) for some \( e \in A \). Another consequence of the fact that the four modules are unital results in the feature that the maps \( R_1, R_2 : A \otimes B \rightarrow M(A \otimes B) \) defined by

\[ R_1(a \otimes b)(a' \otimes b') = \sum (a_{(2)}, b_{(1)})a_{(1)}a' \otimes b_{(2)}b', \]

\[ R_2(a \otimes b)(a' \otimes b') = \sum (a_{(1)}, b_{(2)})a_{(2)}a' \otimes b_{(1)}b', \]

for \( a, a' \in A \) and \( b, b' \in B \) are in fact maps from \( A \otimes B \) to \( A \otimes B \), see [3, Proposition 2.7]. Moreover, these two maps are one-to-one and onto and the inverses \( R_1^{-1}, R_2^{-1} \) are given by the formulas

\[ R_1^{-1}(a \otimes b)(a' \otimes b') = \langle S^{-1}(a_{(2)}), b_{(1)}a_{(1)}a' \otimes b_{(2)}b' \rangle, \]

\[ R_2^{-1}(a \otimes b)(a' \otimes b') = \langle S^{-1}(a_{(1)}), b_{(2)}a_{(2)}a' \otimes b_{(1)}b' \rangle. \]

Moreover, \( R_1 \circ R_2 = R_2 \circ R_1 \). The following formulas will be used in this paper:

\[ R_1((b \triangleright a) \otimes b') = \sum (b'_{(1)}b \triangleright a) \otimes b'_{(2)}, \]

\[ R_1^{-1}(((b \triangleright a) \otimes b') = \sum (S^{-1}(b'_{(1)})b \triangleright a) \otimes b'_{(2)}, \]

\[ R_2(a \otimes (a' \triangleright b)) = \sum (a_{(2)} \otimes (a_{(1)}a' \triangleright b)), \]

\[ R_2^{-1}(a \otimes (a' \triangleright b)) = \sum a_{(2)} \otimes (S^{-1}(a_{(1)}a' \triangleright b)). \]
We also mention that
\[ \langle S(a), b \rangle = \langle a, S(b) \rangle \]
as expected and
\[ \langle a, 1 \rangle = \varepsilon(a), \quad (1, b) = \varepsilon(b), \]
where \( a \in A \) and \( b \in B \). For these formulas, one has to extend the pairing to \( A \times M(B) \) and to \( M(A) \times B \). This can be done in a natural way using the fact that the four modules \( A \triangleright B, B \rhd A, A \rhd B, \) and \( B \rhd A \) are unital. These extensions where, e.g., introduced in [1].

We give some remarks concerning a multiplier Hopf \(^*\)-algebra pairing \( \langle A, B \rangle \). This means that
\[ \langle a^*, b \rangle = [a, S(b)^*]^{-}, \quad \langle a, b^* \rangle = [S(a)^*, b]^{-} \]
for all \( a \in A \) and \( b \in B \). The mappings \( R_1 \) and \( R_2 \) are related to the involutions “*” in the following way:
\[ R_1^{-1}(a^* \otimes b^*) = (R_1(a \otimes b))^*, \quad R_1(a^* \otimes b^*) = (R_1^{-1}(a \otimes b))^*, \]
\[ R_2^{-1}(a^* \otimes b^*) = (R_2(a \otimes b))^*, \quad R_2(a^* \otimes b^*) = (R_2^{-1}(a \otimes b))^* \]
for all \( a \in A \) and \( b \in B \).

In this paper we use that unital \( A \)-modules can be extended to modules over \( M(A) \) in a natural way, see [4, Proposition 3.3].

We often work with extensions of algebra morphisms which are non-degenerate. Recall that a morphism of algebras \( f : A \to M(B) \) is non-degenerate if \( f(A)B = Bf(A) = B \). From [11] we have that such a morphism can be extended in a unique way to a morphism from \( M(A) \) to \( M(B) \). The extension is still denoted by \( f \).

For \( A \) an algebra over \( \mathbb{C} \). Let \( L(A) \) denote the vector space of linear \( A \)-valued maps on \( A \). We denote the identity map on \( A \) by \( i_A \).

For a regular multiplier Hopf algebra \( (A, \Delta_A) \), we denote the opposite comultiplication by \( \Delta_A^{\text{op}} \).

A pairing of two (multiplier) Hopf algebras \( \langle A, B \rangle \) is a natural setting for the construction of the Drinfel’d double. This construction is done for Hopf algebras in [5,7,9,10]. For a multiplier Hopf algebra pairing, the Drinfel’d double construction can be done in a way similar as for Hopf algebras, see [2,3]. In [3] the Drinfel’d double is introduced as a twisted tensor product on \( A \otimes B \) with twist map \( R = R_1 \circ R_2^{-1} \circ \sigma \) (where \( \sigma \) is the usual flip map). In that framework one has to check that \( R \) defines an associative product. These calculations are quite involved and rely on the properties of the mappings \( R_1 \) and \( R_2 \). In the present paper we realize the Drinfel’d double as an algebra of operators on the vector space \( B \otimes A \). Also in this realization, the maps \( R_1 \) and \( R_2 \) play a crucial role. The advantage of this setting is that the associativity of the product of the Drinfel’d double is obvious.
The paper is organized as follows.

In Section 1, we start with algebra representations

\[ \pi : A \rightarrow L(B \otimes A) \quad \text{and} \quad \pi : B \rightarrow L(B \otimes A). \]

We investigate the commutation rules of the operators \( \pi(a) \) (with \( a \in A \)) and \( \pi(b) \) (with \( b \in B \)) in the algebra \( L(B \otimes A) \). Therefore, define linear maps \( \Gamma_1 \) and \( \Gamma_2 \) from \( A \otimes B \) to \( L(B \otimes A) \) by

\[ \Gamma_1(a \otimes b) = \pi(a)\pi(b), \quad \Gamma_2(a \otimes b) = \pi(b)\pi(a) \]

for \( a \in A \) and \( b \in B \).

Denote by \( \Gamma_1(A \otimes B) \) and \( \Gamma_2(A \otimes B) \) the span of the operators \( \pi(a)\pi(b) \), respectively \( \pi(b)\pi(a) \) with \( a \in A \) and \( b \in B \). We prove that \( \Gamma_1(A \otimes B) = \Gamma_2(A \otimes B) \); moreover, this span is a subalgebra of \( L(B \otimes A) \) with the commutation rules of a Drinfel’d double algebra, see Theorem 1.2. Furthermore, we prove that \( \Gamma_1 \) is an injective map, see Lemma 1.4. Therefore, we can define a product on \( A \otimes B \) in such a way that \( \Gamma_1 : A \otimes B \rightarrow L(B \otimes A) \) is an homomorphism. This algebra structure on \( A \otimes B \) is called the Drinfel’d double of the pair \( \langle A, B \rangle \) and is denoted as \( D = A \bowtie B \), see Definition 1.5. In the case that \( \langle A, B \rangle \) is a multiplier Hopf \(*\)-algebra pairing, we put a \(*\)-operation on \( D = A \bowtie B \) such that \( \Gamma_1 \) becomes a \(*\)-isomorphism, see Proposition 1.8.

In Section 2, we put a comultiplication on \( D = A \bowtie B \) as usual: it is given by the comultiplication on \( A \) and the opposite comultiplication on \( B \). Then the double becomes a multiplier Hopf (\(*\)-)algebra which is denoted as \( D = A \bowtie B^{\text{cop}} \), see Theorem 2.5 and Definition 2.6.

In Section 3, we suppose that \( \langle A, B \rangle \) is a multiplier Hopf \(*\)-algebra pairing and that \( A \) and \( B \) carry positive integrals. We prove that \( D = A \bowtie B^{\text{cop}} \) also carries a positive integral. More precisely, let \( \varphi_A \) be a positive left integral on \( A \) and \( \psi_B \) a positive right integral on \( B \). Then we search for an appropriate scalar \( \rho \in \mathbb{C} \) such that \( \rho(\varphi_A \otimes \psi_B) \) is a positive left integral on \( D = A \bowtie B^{\text{cop}} \). This result is not obvious and in fact relies on [6]. The scalar \( \rho \) is not given in [2,3].

1. The algebra structure of the Drinfel’d double

We start from a pairing \( \langle A, B \rangle \) between two regular multiplier Hopf algebras \( A \) and \( B \) as reviewed in the introduction.

Consider the following actions of \( A \) on \( A \), respectively \( B \). Let \( a \in A \). Define for \( x \in A \) and \( y \in B \)

\[ a \cdot x = a x, \quad a \triangleright y = \sum (a, y(2))y(1). \]

As both actions are unital, we obtain a unital action of \( A \otimes A \) on \( B \otimes A \), given by

\[ (a \otimes a') \cdot (y \otimes x) = (a \triangleright y) \otimes a' x \]
with \(a, a', x \in A\) and \(y \in B\). By extending this action to \(M(A \otimes A)\) and using the fact that the comultiplication \(\Delta\) is a homomorphism on \(A\), we end up with an action \(\pi\) of \(A\) on \(B \otimes A\) defined by

\[
\pi(a)(y \otimes x) = \sum (a(1) \triangleright y) \otimes a(2)x,
\]

where \(a, x \in A\) and \(y \in B\). Observe that \(B \otimes A\) is a faithful unital \(A\)-module under this action.

Analogously, we can construct an action of \(B\) on \(B \otimes A\), also denoted by \(\pi\), and defined by

\[
\pi(b)(y \otimes x) = \sum b(2)y \otimes (b(1) \triangleright x),
\]

where \(x \in A\) and \(b, y \in B\). Now, \(B \otimes A\) is also a faithful unital \(B\)-module. As the operators \(\pi(a)\) with \(a \in A\) and \(\pi(b)\) with \(b \in B\) operate on the same vector space \(B \otimes A\), we can look for their commutation rules. This is done in the theorem below. We first give some notations.

1.1. Notations. Recall that \(L(B \otimes A)\) denotes the vector space of linear \(B \otimes A\)-valued maps on \(B \otimes A\). Define linear maps \(\Gamma_1\) and \(\Gamma_2\) from \(A \otimes B\) to \(L(B \otimes A)\) by

\[
\Gamma_1(a \otimes b) = \pi(a)\pi(b), \quad \Gamma_2(a \otimes b) = \pi(b)\pi(a)
\]

for \(a \in A\) and \(b \in B\). We will denote by \(\Gamma_1(A \otimes B)\) and \(\Gamma_2(A \otimes B)\) the span of the operators \(\pi(a)\pi(b)\), respectively \(\pi(b)\pi(a)\) with \(a \in A\) and \(b \in B\).

1.2. Theorem. \(\Gamma_1(A \otimes B) = \Gamma_2(A \otimes B)\). Furthermore, this range is a subalgebra of \(L(B \otimes A)\) with non-degenerate product. The operators \(\pi(a)\) and \(\pi(b)\) with \(a \in A\) and \(b \in B\), respectively, belong to the multiplier algebra \(M(\Gamma_1(A \otimes B))\) of this algebra \(\Gamma_1(A \otimes B)\). Moreover, \(\pi : A \to M(\Gamma_1(A \otimes B))\) and \(\pi : B \to M(\Gamma_1(A \otimes B))\) are injective non-degenerate homomorphisms.

Proof. In this proof, the linear bijections \(R_1, R_2 : A \otimes B \to A \otimes B\) play a crucial role. Recall from the introduction that

\[
R_1(a \otimes b)(1 \otimes b') = \sum \langle a(2), b(1) \rangle a(1) \otimes b(2)b',
\]

\[
R_2(a \otimes b)(1 \otimes b') = \sum \langle a(1), b(2) \rangle a(2) \otimes b(1)b'
\]

for \(a, a' \in A\) and \(b, b' \in B\). We claim that \(\Gamma_1 \circ R_1 = \Gamma_2 \circ R_2\) on \(A \otimes B\). The calculation below uses the fact that \(B \triangleright A\) is a module algebra in the sense of [4]. Take \(x \in A\) and \(y \in B\). Then,
\[(\Gamma_1 \circ R_1)(a \otimes b))(y \otimes x) = \sum (a(2), b(1))\pi(a(1))(b(2) \triangleright (b(3)) \otimes (b(2) \triangleright x))
= \sum (a(3), b(1))a(1) \triangleright (b(1)) \otimes a(2)(b(2) \triangleright x)
= \sum (a(1) \triangleright (b(3))) \otimes (b(1) \triangleright a(2))(b(2) \triangleright x)
= \sum (a(1) \triangleright (b(2)) \otimes (b(1) \triangleright a(2)))
\]

Observe that all the decompositions are well covered because the modules involved are unital. Similarly, we also have
\[(\Gamma_2 \circ R_2)(a \otimes b))(y \otimes x) = \sum (a(1) \triangleright (b(2)) \otimes (b(1) \triangleright a(2))x).
\]

As \(R_1\) and \(R_2\) are bijections on \(A \otimes B\), the equality \(\Gamma_1 \circ R_1 = \Gamma_2 \circ R_2\) implies that \(\Gamma_1(A \otimes B) = \Gamma_2(A \otimes B)\). Moreover,
\[
\pi(b)\pi(a) = \Gamma_2(a \otimes b) = \Gamma_1((R_1 \circ R_2^{-1})(a \otimes b))
\]

with \(a \in A\) and \(b \in B\). This formula gives the commutation rule between the operators \(\pi(a)\) and \(\pi(b)\) in the algebra \(\Gamma_1(A \otimes B) = \Gamma_2(A \otimes B)\). From the fact that \(B \otimes A\) is a unital \(A\)-module as well as a unital \(B\)-module, it follows that the product in \(\Gamma_1(A \otimes B)\) is non-degenerate. Now the rest of the statement follows easily. \(\square\)

1.3. Remark. The commutation rule in \(\Gamma_1(A \otimes B)\) can be written explicitly as

(i) \[\sum (a(2), b(1))\pi(a(1))\pi(b(2)) = \sum (a(1), b(2))\pi(b(1))\pi(a(2))\]
with \(a \in A\) and \(b \in B\). The decompositions are well covered when we consider both sides of the equation as operators on the vector space \(B \otimes A\).

(ii) By rewriting the commutation rule to
\[
\Gamma_2 = \Gamma_1 \circ R_1 \circ R_2^{-1} \quad \text{or} \quad \Gamma_1 = \Gamma_2 \circ R_2 \circ R_1^{-1},
\]
we recover the formulas
\[
\pi(b)\pi(a) = \sum (a(1), S^{-1}(b(2)))|a(3), b(1))\pi(a(2))\pi(b(2)),
\]
\[
\pi(a)\pi(b) = \sum (a(3), S^{-1}(b(1)))|a(1), b(3))\pi(b(3))\pi(a(2))
\]

for \(a \in A\) and \(b \in B\).

The following result is needed for the definition of the Drinfel’d double that we will give below.

1.4. Lemma. The linear maps \(\Gamma_1, \Gamma_2 : A \otimes B \to L(B \otimes A)\) are injective.
Proof. We prove the injectivity of $\Gamma_1$; the injectivity of $\Gamma_2$ can be proven in a similar way. Suppose $\sum \pi(a_i) \pi(b_i) = 0$ in $L(B \otimes A)$. Thus, for all $x \in A$ and $y \in B$ we have

$$0 = \sum \pi(a_i) \pi(b_i)(y \otimes x) = \sum (a_{i(1)} \triangleright (b_{i(2)} y)) \otimes a_{i(2)} (b_{i(1)} \triangleright x)$$

$$= \sum (a_{i(1)} \triangleright (b_{i(2)} y)) \otimes (x_{(2)}, b_{i(1)}) a_{i(2)} x_{(1)}.$$

Now multiply the second factor to the right by any element $x' \in A$ and use the fact that elements of the form $\Delta(x)(x' \otimes 1)$ span all of $A \otimes A$. It follows that for all $x, x' \in A$ and $y \in B$, we will get

$$\sum (a_{i(1)} \triangleright (b_{i(2)} y)) \otimes x_{(2)} b_{i(1)} a_{i(2)} x' = 0.$$

As this is true for all $x \in A$ and because the pairing is non-degenerate, we get for all $x' \in A$ and $y \in B$ that

$$\sum (a_{i(1)} \triangleright (b_{i(2)} y)) \otimes b_{i(1)} \otimes a_{i(2)} x' = 0.$$

Now apply $i_B \otimes \varepsilon_B \otimes \varepsilon_A$ to obtain

$$\sum a_i \triangleright \langle b_i y \rangle = 0$$

for all $y \in B$

$$\sum (a_i, b_{i(2)} y_{(2)}) b_{i(1)} y_{(1)} = 0.$$

Now multiply to the right by any $y' \in B$ and use the fact that elements of the form $\Delta y(y' \otimes 1)$ span all of $B \otimes B$. It follows that for all $y, y' \in B$ we will get

$$\sum (a_i, b_{i(2)} y) b_{i(1)} y' = 0, \quad \sum (a_{i(1)}, b_{i(2)}) (a_{i(2)}, y) b_{i(1)} y' = 0.$$

As this is true for all $y \in A$ and because the pairing is non-degenerate, we get for all $y' \in B$ that

$$\sum \langle a_{i(1)}, b_{i(2)} \rangle a_{i(2)} \otimes b_{i(1)} y' = 0, \quad R_2(\sum a_i \otimes b_i) (1 \otimes y') = 0.$$

Thus $R_2(\sum a_i \otimes b_i) = 0$ and so $\sum a_i \otimes b_i = 0$. □

1.5. Definition. We make $A \otimes B$ into an algebra by defining the product so that the map $\Gamma_1 : A \otimes B \to L(B \otimes A)$ is a homomorphism. This is possible because $\Gamma_1$ is injective and its range $\Gamma_1(A \otimes B)$ is a subalgebra of $L(B \otimes A)$. This algebra structure on $A \otimes B$ is denoted by $D = A \bowtie B$ and is called the Drinfel’d double, associated to the pair $(A, B)$; elements $a \otimes b$, when considered in $A \bowtie B$, will be written as $a \triangleright b$. The algebra $\Gamma_1(A \otimes B) \subset L(B \otimes A)$ is a concrete realization of the Drinfel’d double as an algebra of operators on $B \otimes A$. 

1.6. Remarks on the product of the Drinfel’d double.

(1) In the algebra \( \Gamma_1(A \otimes B) = \Gamma_2(A \otimes B) \), the commutation rule is given by \( \Gamma_2 = \Gamma_1 \circ R_1 \circ R_2^{-1} \). As the Drinfel’d double is isomorphic as algebra to \( \Gamma_1(A \otimes B) \), the product in \( D = A \bowtie B \) can be expressed by the twist map \( R = R_1 \circ R_2^{-1} \circ \sigma \) (where \( \sigma \) is the usual tensor transposition) in the following way:

\[
(a \bowtie b)(a' \bowtie b') = (m_A \otimes m_B)(i_A \otimes R \otimes i_B)(a \otimes b \otimes a' \otimes b')
\]

with \( a, a' \in A \) and \( b, b' \in B \). An equivalent expression for the product in \( D = A \bowtie B \) is given by

\[
(a \bowtie b)(a' \bowtie b') = (a \otimes 1)R\left((b \otimes 1)R^{-1}(a' \otimes b')\right)
\]

for \( a, a' \in A \) and \( b, b' \in B \). This formula will be useful when we define the comultiplication on \( D = A \bowtie B \).

(2) We mention that in [3] the Drinfel’d double \( D = A \bowtie B \) is introduced as a twisted tensor product with twist map \( R = R_1 \circ R_2^{-1} \circ \sigma \). In that framework one has to check that \( R \) defines an associative product on \( A \otimes B \). This means that \( R \) must satisfy the following equations:

(i) \( R \circ (i_B \otimes m_A) = (m_A \otimes i_B)(i_A \otimes R)(R \otimes i_A) \);
(ii) \( R \circ (m_B \otimes i_A) = (i_A \otimes m_B)(R \otimes i_B)(i_B \otimes R) \).

The calculations of these equations are rather involved in this case. In the approach of the present paper, the associativity of the product on \( \Gamma_1(A \otimes B) \) (and so the associativity of the product on \( D = A \bowtie B \)) is obvious because \( \Gamma_1(A \otimes B) \) is a sub-algebra of \( L(B \otimes A) \).

1.7. Proposition. There exist non-degenerate injective homomorphisms \( \gamma : A \to M(A \bowtie B) \) and \( \gamma : B \to M(A \bowtie B) \) defined by

\[
\gamma(a')(a \bowtie b) = a'a \bowtie b, \quad (a \bowtie b)\gamma(b') = a \bowtie bb'
\]

with \( a, a' \in A \) and \( b, b' \in B \).

Proof. The statement follows easily from the fact that \( \pi : A \to M(\Gamma_1(A \otimes B)) \) and \( \pi : B \to M(\Gamma_1(A \otimes B)) \) are non-degenerate, injective homomorphisms. \( \square \)

By the nature of the above algebra embeddings, it is convenient to denote \( \gamma(a) = a \bowtie 1 \) and \( \gamma(b) = 1 \bowtie b \). Now the Drinfel’d double is spanned by the elements \( a \bowtie b = (a \bowtie 1)(1 \bowtie b) \) with \( a \in A \) and \( b \in B \). The product in \( D \) is determined by the commutation rule

\[
(1 \bowtie b)(a \bowtie 1) = R(b \otimes a) \quad \text{with} \quad R = R_1 \circ R_2^{-1} \circ \sigma.
\]
To finish this section, we investigate the \( * \)-case. Start from a multiplier Hopf \( * \)-algebra pairing \( \langle A, B \rangle \). This means that \( A \) and \( B \) are multiplier Hopf \( * \)-algebras such that the pairing \( \langle A, B \rangle \) satisfies furthermore the relations
\[
\langle a, b^* \rangle = \{ S(a)^*, b \}^- , \quad \langle a^*, b \rangle = \{ a, S(b)^* \}^- \n\]
for \( a \in A \) and \( b \in B \). We have the following proposition.

1.8. Proposition. Let \( \langle A, B \rangle \) be a multiplier Hopf \( * \)-algebra pairing. Then the Drinfel’d double \( D = A \bowtie B \) is a \( * \)-algebra with
\[
(a \bowtie b)^* = (1 \bowtie b^*)(a^* \bowtie 1) \n\]
for \( a \in A \) and \( b \in B \). The embeddings \( A \to M(D) \) and \( B \to M(D) \) are \( * \)-algebra embeddings.

Proof. We first put a \( * \)-operation on the algebra \( \Gamma_1(A \otimes B) \). From the introduction we know how the maps \( R_1 \) and \( R_2 \) relate to the \( * \)-operations on \( A \) and \( B \). If we combine these relations, we obtain that
\[
\sum \pi(b_i^*) \pi(a_i^*) = \pi(a^*) \pi(b^*) \quad \text{whenever} \quad \sum a_i \otimes b_i = (R_1 \circ R_2^{-1})(a \otimes b). \n\]
Now, define
\[
\left( \pi(a) \pi(b) \right)^* = \pi(b^*) \pi(a^*). \n\]
Then, the above relation says that
\[
\left( \pi(b) \pi(a) \right)^* = \pi(a^*) \pi(b^*) \n\]
for \( a \in A \) and \( b \in B \). We can easily check that this \( * \)-operation on \( D \) is anti-multiplicative and involutive. Clearly, the embeddings
\[
\pi : A \to M(\Gamma_1(A \otimes B)), \quad \pi : B \to M(\Gamma_1(A \otimes B)) \n\]
are \( * \)-embeddings. As expected, we put a \( * \)-operation on \( D \) such that \( \Gamma_1 : A \bowtie B \to \Gamma_1(A \otimes B) \) is an isomorphism of \( * \)-algebras. Now the statement of the proposition follows. \( \square \)

2. A multiplier Hopf algebra structure on \( D = A \bowtie B \)

Start from a multiplier Hopf algebra pairing \( \langle A, B \rangle \). In Section 1, we constructed the subalgebra \( \Gamma_1(A \otimes B) \) in \( L(B \otimes A) \) as a realization of the Drinfel’d double \( D = A \bowtie B \) associated to the pairing \( \langle A, B \rangle \). Recall that
\[ \Gamma_1(A \otimes B) = \text{Span}\{\pi(a)\pi(b) \mid a \in A \text{ and } b \in B\} \]
\[ = \text{Span}\{\pi(b)\pi(a) \mid a \in A \text{ and } b \in B\}, \]
where
\[ \pi(a)(y \otimes x) = \sum (a(1) \triangleright y) \otimes a(2)x, \quad \pi(b)(y \otimes x) = \sum b(2)y \otimes (b(1) \triangleright x) \]
for \( a, x \in A \) and \( b, y \in B \). As
\[ \pi(A)\Gamma_1(A \otimes B) = \Gamma_1(A \otimes B) = \Gamma_1(A \otimes B)\pi(A), \]
\[ \pi(B)\Gamma_1(A \otimes B) = \Gamma_1(A \otimes B) = \Gamma_1(A \otimes B)\pi(B), \]
the following homomorphisms
\[ \pi \otimes \pi : A \otimes A \to M\left(\Gamma_1(A \otimes B) \otimes \Gamma_1(A \otimes B)\right), \]
\[ \pi \otimes \pi : B \otimes B \to M\left(\Gamma_1(A \otimes B) \otimes \Gamma_1(A \otimes B)\right) \]
are non-degenerate and can be extended to \( M(A \otimes A) \), respectively \( M(B \otimes B) \). So, we can give a meaning to the multipliers \( (\pi \otimes \pi)(\Delta(a)) \) and \( (\pi \otimes \pi)(\Delta^{\text{cop}}(b)) \) in \( M(\Gamma_1(A \otimes B) \otimes \Gamma_1(A \otimes B)) \). We have, e.g., that
\[ (\pi \otimes \pi)(\Delta(a))(\pi(x)\pi(y) \otimes \pi(x')\pi(y')) = \sum \pi(a(1)x)\pi(y) \otimes \pi(a(2)x')\pi(y') \]
with \( a, x, x' \in A \) and \( y, y' \in B \). By a same reasoning, we can consider multipliers in \( M(\Gamma_1(A \otimes B) \otimes \Gamma_1(A \otimes B) \otimes \Gamma_1(A \otimes B)) \). We have, e.g.,
\[ (\pi \otimes \pi \otimes \pi)((i \otimes \Delta)\Delta(a)) = (\pi \otimes \pi \otimes \pi)((\Delta \otimes i)\Delta(a)). \]

To give a correct definition for a comultiplication on \( \Gamma_1(A \otimes B) \), we need the following lemma.

**2.1. Lemma.** For all \( a \in A \) and \( b \in B \)
\[ \sum (a(2), b(1))(\pi \otimes \pi)(\Delta(a(1)))(\pi \otimes \pi)(\Delta^{\text{cop}}(b(2))) \]
\[ = \sum (a(1), b(2))(\pi \otimes \pi)(\Delta^{\text{cop}}(b(1)))(\pi \otimes \pi)(\Delta(a(2))). \]

This is an equality between linear operators on \( (B \otimes A) \otimes (B \otimes A) \).

**Proof.** In the calculations below we make use of the fact that \( A \triangleright B \) and \( B \triangleright A \) are module algebras in the sense of [4]. Take \( x, x' \in A \) and \( y, y' \in B \).
\[ \sum (a_{(2)}, b_{(1)}) (\pi \otimes \pi) (\Delta(a_{(1)}))(\pi \otimes \pi) (\Delta^{\text{cop}}(b_{(2)})) \left( (y \otimes x) \otimes (y' \otimes x') \right) \\
= \sum \left( (a_{(1)} \triangleright b_{(4)} y) \otimes a_{(2)} (b_{(3)} \triangleright x) \right) \otimes \left( (a_{(3)} \triangleright b_{(2)} y') \otimes (b_{(1)} \triangleright a_{(4)} x') \right). \quad (1) \]

On the other hand,
\[ \sum (a_{(1)}, b_{(2)}) (\pi \otimes \pi) (\Delta^{\text{cop}}(b_{(1)}))(\pi \otimes \pi) (\Delta(a_{(2)})) \left( (y \otimes x) \otimes (y' \otimes x') \right) \\
= \sum \left( (a_{(1)} \triangleright b_{(4)} y) \otimes (b_{(3)} \triangleright a_{(2)} x) \right) \otimes \left( (b_{(2)}(a_{(3)} \triangleright y') \otimes (b_{(1)} \triangleright a_{(4)} x') \right) \\
= \sum \left( (a_{(1)} \triangleright b_{(4)} y) \otimes a_{(2)} (b_{(3)} \triangleright x) \right) \otimes \left( (a_{(3)} \triangleright b_{(2)} y') \otimes (b_{(1)} \triangleright a_{(4)} x') \right), \]

which is the expression (1). \(\Box\)

2.2. Definition. Start from a multiplier Hopf algebra pairing \((A, B)\). Let \(\Gamma_1(A \otimes B)\) be the realization of the Drinfel’d double as an algebra of operators on \(B \otimes A\). The comultiplication \(\overline{\Delta}\) on \(\Gamma_1(A \otimes B)\) is defined by
\[ \overline{\Delta}(\pi(a)\pi(b)) = (\pi \otimes \pi)(\Delta(a))(\pi \otimes \pi)(\Delta^{\text{cop}}(b)) \]
for \(a \in A\) and \(b \in B\). Remark that, because of Lemma 2.1 and because all maps involved are homomorphisms, \(\overline{\Delta}\) is a homomorphism on \(\Gamma_1(A \otimes B)\).

The homomorphism
\[ \overline{\Delta} : \Gamma_1(A \otimes B) \to M(\Gamma_1(A \otimes B) \otimes \Gamma_1(A \otimes B)) \]
is non-degenerate because
\[ \overline{\Delta}(\Gamma_1(A \otimes B))(1 \otimes \Gamma_1(A \otimes B)) = \Gamma_1(A \otimes B) \otimes \Gamma_1(A \otimes B). \]
To obtain this equation, we use that \(\Gamma_1(A \otimes B) = \text{Span}[\pi(a)\pi(b) \mid a \in A\text{ and } b \in B] = \text{Span}[\pi(b)\pi(a) \mid a \in A\text{ and } b \in B]\) and \(\Delta(B) (B \otimes 1) = B \otimes B\) and \(\Delta(A)(1 \otimes A) = A \otimes A\). Of course, \(\overline{\Delta}(\pi(a)) = (\pi \otimes \pi)(\Delta(a))\) and \(\overline{\Delta}(\pi(b)) = (\pi \otimes \pi)(\Delta^{\text{cop}}(b))\).

We now prove that the homomorphism
\[ \overline{\Delta} : \Gamma_1(A \otimes B) \to M(\Gamma_1(A \otimes B) \otimes \Gamma_1(A \otimes B)) \]
is coassociative in the sense of [11, Definition 2.2].

2.3. Proposition. For \(a \in A\) and \(b \in B\)
\[ (i \otimes \overline{\Delta})(\overline{\Delta}(\pi(a)\pi(b))) = (\overline{\Delta} \otimes i)(\overline{\Delta}(\pi(a)\pi(b))). \]
Proof. We first remark that both sides of the equation have a meaning as multipliers in \( M(\Gamma_1(A \otimes B) \otimes \Gamma_1(A \otimes B) \otimes \Gamma_1(A \otimes B)) \), e.g., to give a meaning to the left-hand side, we remark that the homomorphism

\[
i \otimes \overline{\Delta} : \Gamma_1(A \otimes B) \otimes \Gamma_1(A \otimes B) \to M(\Gamma_1(A \otimes B) \otimes \Gamma_1(A \otimes B) \otimes \Gamma_1(A \otimes B))
\]

is non-degenerate. By the definition of \( \overline{\Delta} \), we have the following equality of multipliers:

\[
(i \otimes \overline{\Delta})(\Delta(\pi(a))) = (\pi \otimes \pi \otimes \pi)((i \otimes \Delta)(\Delta(a))) = (\pi \otimes \pi \otimes \pi)((\Delta \otimes i)(\Delta(a)))
\]

for all \( a \in A \). Also

\[
(i \otimes \overline{\Delta})(\overline{\Delta}(\pi(b))) = (\overline{\Delta} \otimes i)(\overline{\Delta}(\pi(b)))
\]

for all \( b \in B \). Now the statement of the proposition follows, because all the maps involved are homomorphisms. \( \square \)

We now prove the main result of this section.

2.4. Theorem. Let \( \langle A, B \rangle \) be a multiplier Hopf algebra pairing. Let \( \Gamma_1(A \otimes B) \) be the realization of the Drinfel’d double as operators on \( B \otimes A \). Then \( (\Gamma_1(A \otimes B), \overline{\Delta}) \) is a regular multiplier Hopf algebra. The counit and the antipode on \( \Gamma_1(A \otimes B) \) are given by

\[
\varepsilon = (\varepsilon_A \otimes \varepsilon_B) \circ \Gamma_1^{-1}, \quad S = \Gamma_2 \circ (S_A \otimes S_B^{-1}) \circ \Gamma_1^{-1},
\]

where \( \sigma \) denotes the usual twist map on \( A \otimes B \). When \( \langle A, B \rangle \) is a multiplier Hopf* -algebra pairing, \( (\Gamma_1(A \otimes B), \overline{\Delta}) \) is a multiplier Hopf* -algebra.

Proof. We use [12, Proposition 2.9]. Recall from the foregoing that \( \Gamma_1(A \otimes B) \) is an associative algebra with non-degenerate product. Furthermore,

\[
\overline{\Delta} : \Gamma_1(A \otimes B) \to M(\Gamma_1(A \otimes B) \otimes \Gamma_1(A \otimes B))
\]

is a homomorphism. Moreover,

\[
\overline{\Delta}(\Gamma_1(A \otimes B))(1 \otimes \Gamma_1(A \otimes B)) = \Gamma_1(A \otimes B) \otimes \Gamma_1(A \otimes B)
\]

\[
= (\Gamma_1(A \otimes B) \otimes 1)\overline{\Delta}(\Gamma_1(A \otimes B))
\]

and \( \overline{\Delta} \) is coassociative in the sense of Proposition 2.4. Now define the counit \( \varepsilon \) on \( \Gamma_1(A \otimes B) \) by \( \varepsilon = (\varepsilon_A \otimes \varepsilon_B) \circ \Gamma_1^{-1} \). Then for \( a, a' \in A \) and \( b, b' \in B \),

\[
(i \otimes \varepsilon)(\pi(a)\pi(b)) = \pi(a)\pi(b)\pi(a')\pi(b'),
\]

\[
(i \otimes \varepsilon)(\pi(a)\pi(b) \otimes 1)\overline{\Delta}(\pi(a')\pi(b')) = \pi(a)\pi(b)\pi(a')\pi(b').
\]
By using the fact that $\Delta$ is a homomorphism and the equality
$$\Gamma_1(A \otimes B) \otimes \Gamma_1(A \otimes B) = (\Gamma_1(A \otimes B) \otimes 1)\Delta(\Gamma_1(A \otimes B)),$$
one can prove that $\pi$ is a homomorphism on $\Gamma_1(A \otimes B)$. Define the antipode $S$ on $\Gamma_1(A \otimes B)$ by $S = \Gamma_2 \circ (S_A \otimes S_B^{-1}) \circ \Gamma_1^{-1}$. Clearly, $S$ is an invertible map on $\Gamma_1(A \otimes B)$. For $a, a' \in A$ and $b, b' \in B$

$$m((S \otimes i)(\pi(a)\pi(b))(1 \otimes \pi(a')\pi(b'))) = \pi(\pi(a)\pi(b))\pi(a')\pi(b'),$$
$$m((i \otimes S)((\pi(a)\pi(b) \otimes 1)\Delta(\pi(a')\pi(b')))) = \pi(\pi(a')\pi(b'))\pi(a)\pi(b),$$

where $m$ denotes the product in $\Gamma_1(A \otimes B)$. By using the fact that $\Delta$ and $\pi$ are homomorphisms and the equality
$$\Gamma_1(A \otimes B) \otimes 1)\Delta(\Gamma_1(A \otimes B)) = \Gamma_1(A \otimes B) \otimes \Gamma_1(A \otimes B),$$
one can prove that $S$ in a anti-homomorphism. Here we obtain that $(\Gamma_1(A \otimes B), \Delta)$ is a multiplier Hopf algebra. To see that $(\Gamma_1(A \otimes B), \Delta)$ is regular, one could check that $(\Gamma_1(A \otimes B), \Delta^{\text{cop}})$ is again a multiplier Hopf algebra, where
$$\Delta^{\text{cop}}(\pi(a)\pi(b)) = (\pi \otimes \pi)(\Delta^{\text{cop}}(a))(\pi \otimes \pi)(\Delta(b)).$$

However, following [12, Proposition 2.9] the regularity follows from the foregoing and the equalities
$$\Gamma_1(A \otimes B) \otimes 1)\Delta(\Gamma_1(A \otimes B)) = \Gamma_1(A \otimes B) \otimes \Gamma_1(A \otimes B),$$
$$\Delta(\Gamma_1(A \otimes B))(\Gamma_1(A \otimes B) \otimes 1) = \Gamma_1(A \otimes B) \otimes \Gamma_1(A \otimes B).$$

To end, we consider the *-case. When $\langle A, B \rangle$ is a multiplier Hopf *-algebra pairing, we have from Proposition 1.8 that $\Gamma_1(A \otimes B)$ is a *-algebra with $(\pi(a)\pi(b))^* = \pi(b^*)\pi(a^*)$. It is not difficult to check that $\Delta$ is a *-homomorphism, making $\Gamma_1(A \otimes B)$ into a multiplier Hopf *-algebra. $\square$

Finally, we define the comultiplication on $D = A \prec B$, by using the comultiplication $\Delta$ on the operator algebra $\Gamma_1(A \otimes B)$. As $\Gamma_1 \otimes \Gamma_1 : D \otimes D \to \Gamma_1(A \otimes B) \otimes \Gamma_1(A \otimes B)$ is an isomorphism of (*-algebras, the following definition puts a comultiplication on $D = A \prec B$ in a natural way.

**2.5. Definition.** Let $\langle A, B \rangle$ be a multiplier Hopf (*-)algebra pairing with $D = A \prec B$ the associated Drinfel’d double algebra. The comultiplication on $D$ is given as follows. For $a \in A$ and $b \in B$, $\Delta(a \prec b)$ is the uniquely determined multiplier in $M(D \otimes D)$ such that
$$\Gamma_1 \otimes \Gamma_1)(\Delta(a \prec b)) = \Delta(\pi(a)\pi(b)).$$
The multiplier Hopf (\(^\ast\)-)algebra structure \((A \bowtie \triangleleft B, \Delta)\) follows from Theorem 2.4 and is denoted by \(D = A \bowtie \triangleleft B^{\text{cop}}\).

2.6. Remarks on the comultiplication of \(D = A \bowtie \triangleleft B^{\text{cop}}\)

(1) Recall from Remark 1.6(1) that the product in \(D = A \bowtie \triangleleft B^{\text{cop}}\) is determined by the twist map \(R = R_1 \circ R_2^{-1} \circ \sigma : B \otimes A \to A \otimes B\). For \(a \in A\) and \(b \in B\), the multiplier \(\overline{\Delta}(a \bowtie \triangleleft b) \in M(D \otimes D)\) is given by the following formula. Take \(a' \in A\) and \(b' \in B\), then
\[
\overline{\Delta}(a \bowtie \triangleleft b)((1 \bowtie \triangleleft 1) \otimes (a' \bowtie \triangleleft b')) = \sum (a(1) \bowtie \triangleleft b(2)) \otimes (a(2) \bowtimes 1)(R(b(1)b'_1 \otimes a'_1)),
\]
\[
((a' \bowtie \triangleleft b') \bowtimes (1 \bowtie \triangleleft 1))\overline{\Delta}(a \bowtie \triangleleft b) = \sum R(b'_1 \bowtimes a'_1a(1))(1 \bowtimes b(2)) \otimes a(2) \bowtie \triangleleft b(1),
\]
where \(R^{-1}(a' \bowtimes b') = \sum b'_j \bowtimes a'_j\). Observe that in the above formulas all decompositions are well-covered.

(2) Let \(A \to M(D) : a \mapsto a \bowtie \triangleleft 1\) and \(B \to M(D) : b \mapsto 1 \bowtie \triangleleft b\) be the algebra embeddings of Proposition 1.7. So, there are natural embeddings \(A \otimes A \to M(D \otimes D)\) and \(B \otimes B \to M(D \otimes D)\) which extend to the multiplier algebras. If we regard \(\Delta(a), \Delta^{\text{cop}}(b)\) as elements in \(M(D \otimes D)\), then \(\overline{\Delta}(a \bowtie \triangleleft b) = \Delta(a)\Delta^{\text{cop}}(b)\), just the product in \(M(D \otimes D)\).

3. Positive integrals on the Drinfel’d double

A special category of regular multiplier Hopf algebras is given by multiplier Hopf algebras that have integrals. Let \(A\) be a regular multiplier Hopf algebra. A linear functional \(\varphi\) on \(A\) is called a left integral on \(A\) if it is non-zero and \((i \otimes \varphi)\Delta(a) = \varphi(a)1\) for all \(a \in A\). Similarly, a linear functional \(\psi\) on \(A\) is called a right integral on \(A\) if it is non-zero and \((\psi \otimes i)\Delta(a) = \psi(a)1\) for all \(a \in A\). We will now formulate some results on integrals and define some data related to integrals. For more details we refer to [12]. If \(A\) admits a left integral, then it is unique (up to a scalar) and there is also a unique right integral. These functionals are faithful. There is a multiplier \(\delta\) in \(M(A)\) such that
\[
(\varphi \otimes i)\Delta(a) = \varphi(a)\delta
\]
for the left integral \(\varphi\) on \(A\). The multiplier \(\delta\) is called the modular element in \(M(A)\). Moreover, \(\delta\) is invertible and
\[
\Delta(\delta) = \delta \bowtimes \delta, \quad \varepsilon(\delta) = 1, \quad S(\delta) = \delta^{-1}.
\]
The multiplier \(\delta\) relates in the following way to a right integral \(\psi\) on \(A\):
\[
(i \otimes \psi)\Delta(a) = \psi(a)\delta^{-1}.
\]
In Hopf algebra theory, modular elements are called disguised group-like elements, see, e.g., [8]. Start from a multiplier Hopf algebra pairing \( \langle A, B \rangle \). In the foregoing sections we constructed the Drinfel’d double multiplier Hopf algebra \( D = A \bowtie B^{\text{op}} \) associated to this pairing. If we assume that \( \varphi_A \) is a left integral on \( A \) and \( \psi_B \) is a right integral on \( B \), it is quite obvious that \( \varphi_A \otimes \psi_B \) is a left integral on \( D = A \bowtie B^{\text{op}} \). In the case that \( \langle A, B \rangle \) is a multiplier Hopf \( ^* \)-algebra pairing, we proved in Section 1 that the associated Drinfel’d double is a multiplier Hopf \( ^* \)-algebra with \( ^* \)-operation

\[
(a \bowtie b)^* = (1 \bowtie b^*)(a^* \bowtie 1)
\]

for \( a \in A \) and \( b \in B \). If furthermore \( \varphi_A \) is positive on \( A \), i.e., \( \varphi(aa^*) \geq 0 \) for all \( a \in A \) and \( \psi_B \) is positive on \( B \), we prove that there is a scalar \( \rho \in \mathbb{C} \) such that \( \rho(\varphi_A \otimes \psi_B) \) is a positive left integral on \( D \). This result was announced in [2,3]. The proof of this result is not obvious and is given below. We first need some lemmas.

### 3.1. Lemma
Let \( \varphi_A \) be a left integral on a regular multiplier Hopf algebra \( A \). Let \( \delta_A \) denote the modular element in \( M(A) \). Then,

\[
\sum \varphi_A(ca(1))\delta_A S(a(2)) = \sum \varphi_A(c(1)a)c(2) \quad \text{for all } a, c \in A.
\]

**Proof.** For \( a, c \in A \) there exist \( a_j \) and \( c_j \) in \( A \) such that

\[
\Delta(c)(a \otimes 1) = \sum_j \Delta(c_j)(1 \otimes a_j).
\]

Following [11, Lemma 2.10] this implies that

\[
(c \otimes 1)\Delta(a) = \sum_j c_j \otimes S^{-1}(a_j).
\]

Now,

\[
(\varphi_A \otimes i)(\Delta(c)(a \otimes 1)) = \sum_j (\varphi_A \otimes i)\left(\Delta(c_j)(1 \otimes a_j)\right) = \sum_j \varphi_A(c_j)\delta_A a_j
\]

\[
= \delta_A S\left(\sum_j \varphi_A(c_j)S^{-1}(a_j)\right) = \delta_A S\left((\varphi_A \otimes i)((c \otimes 1)\Delta(a))\right)
\]

\[
= \sum \varphi_A(ca(1))\delta_A S(a(2)). \quad \Box
\]

### 3.2. Lemma
Let \( \langle A, B \rangle \) be a multiplier Hopf algebra pairing. Assume \( \psi_B \) is a right integral on \( B \). Then for \( a \in A \) and \( b \in B \)

\[
(i_A \otimes \psi_B)R(b \otimes a) = \psi_B(b)(\delta_B^{-1} \triangleright a),
\]
where $R$ is the twist map $R : B \otimes A \to A \otimes B$ defining the product on the Drinfel'd double $D = A \bowtie B^{\text{op}}$.

**Proof.** First observe that the right-hand side is well-defined because $A$ is a unital left $B$-module under the action $\triangleright$. As recalled in the introduction, this action can be extended to $M(B)$. From Remarks 1.6, we have that $R = R_1 \circ R_{-1}^2 \circ \sigma$. Recall from the introduction that

$$R_1((b \triangleright a) \otimes b') = \sum (b'_{(1)} b \triangleright a) \otimes b'_{(2)},$$

$$R_2^{-1}(a \otimes (a' \triangleright b)) = \sum a_{(2)} \otimes (S^{-1}(a_{(1)})a' \triangleright b)$$

with $a,a' \in A$ and $b,b' \in B$. A straightforward calculation shows that

$$(i_A \otimes \psi_B)R_1((b \triangleright a) \otimes b') = \psi_B(b')(\delta_B^{-1} \triangleright (b \triangleright a)).$$

As the module $B \triangleright A$ is unital, we obtain that

$$(i_A \otimes \psi_B)R_1(a \otimes b) = \psi_B(b)(\delta_B^{-1} \triangleright a).$$

Now,

$$(i_A \otimes \psi_B)R((a' \triangleright b) \otimes a) = \sum \psi_B(S^{-1}(a_{(1)})a' \triangleright b)(\delta_B^{-1} \triangleright a_{(2)})$$

$$= \sum \psi_B(S^{-1}(a_{(1)})a'_{(1)})(b_{(2)})\psi_B(b_{(1)})(\delta_B^{-1} \triangleright a_{(2)})$$

$$= \varepsilon(a')\psi_B(b)(\delta_B^{-1} \triangleright a) = \psi_B(a' \triangleright b)(\delta_B^{-1} \triangleright a).$$

As the module $A \triangleright B$ is unital, the formula stated in the lemma follows. $\square$

From [1] we recall that a multiplier Hopf algebra pairing $\langle A, B \rangle$ can be extended to $\langle A, M(B) \rangle$ or to $\langle M(A), B \rangle$ in the following way. For $a \in A$ and $m \in M(B)$,

$$\langle a, m \rangle = \langle a, bm \rangle = \langle a, mb' \rangle$$

with $b, b' \in B$ such that $a \triangleleft b = a$ and $b' \triangleright a = a$. One easily checks that this definition is independent of the choice of $b$ or $b'$. As expected, one also has that

$$\langle a \otimes a', \Delta(m) \rangle = \langle aa', m \rangle$$

with $a,a' \in A$ and $m \in M(B)$. If $n \in M(A)$ and $m \in M(B)$ such that $\Delta(m) = m \otimes m \in M(B \otimes B)$ we define $\langle n, m \rangle$ in the following way:

$$\langle n, m \rangle \langle a, m \rangle = \langle na, m \rangle$$

for all $a \in A$. Remark this is well-defined, because $\langle a, m \rangle = 0$ implies that $\langle na, m \rangle = 0$. 

In [6] is proven that for a multiplier Hopf $^*$-algebra $B$ with positive integral $\varphi_B$ the modular element $\delta_B \in M(B)$ is self-adjoint, i.e., $\delta_B^{\ast} = \delta_B$. Moreover, there exists a self-adjoint invertible element, denoted by $\delta_B^{1/2}$ such that

$$\delta_B^{1/2} \delta_B^{1/2} = \delta_B, \quad \Delta(\delta_B^{1/2}) = \delta_B^{1/2} \otimes \delta_B^{1/2}, \quad S(\delta_B^{1/2}) = \delta_B^{-1/2}$$

with $\delta_B^{-1/2} = (\delta_B^{1/2})^{-1}$. Let $\delta_A$ denote the modular element in $M(A)$, then $\langle \delta_A, \delta_B^{1/2} \rangle$ is defined by the formula

$$\langle \delta_A, \delta_B^{1/2} \rangle(a, \delta_B^{1/2}) = \langle \delta_A, \delta_B^{1/2} \rangle(a, \delta_B^{1/2})$$

for all $a \in A$.

The following lemma plays an important role in the search for the scalar $\rho$ such that $\rho(\varphi_A \otimes \psi_B)$ is positive left integral on $D$.

**3.3. Lemma.** Let $(A, B)$ be a multiplier Hopf $^*$-algebra pairing. Suppose there are positive integrals on $A$ and $B$. For a positive left integral $\varphi_A$ on $A$ we have that

$$\langle \delta_A, \delta_B^{1/2} \rangle \varphi_A(\delta_B^{1/2} \triangleright a^*) \geq 0 \quad \text{for all } a \in A.$$

**Proof.** Take $a, c \in A$. We claim that

$$\langle \delta_A, \delta_B^{1/2} \rangle \varphi_A(c(\delta_B^{1/2} \triangleright a)) = \varphi_A((\delta_B^{1/2} \triangleright c^*) a).$$

The left-hand side equals

$$\sum \langle \delta_A, \delta_B^{1/2} \rangle S(a(1)), \delta_B^{1/2} \rangle \varphi_A(c^* a(1)) = \sum \langle \delta_A S(a(1)), \delta_B^{1/2} \rangle \varphi_A(c^* a(1)).$$

By using Lemma 3.1, this expression becomes

$$\sum \varphi_A(c^* a(1) c^* (\delta_B^{1/2} \triangleright a)) = \varphi_A((\delta_B^{1/2} \triangleright c^*) a).$$

We now use that $(b \triangleright a)^* = S(b)^* \triangleright a^*$ for $a \in A$ and $b \in B$; the expression for the left-hand side in the claim becomes exactly the right-hand side. Now,

$$\langle \delta_A, \delta_B^{1/2} \rangle \varphi_A(a(\delta_B^{1/2} \triangleright a^*)) = \langle \delta_A, \delta_B^{1/2} \rangle \varphi_A(a(\delta_B^{1/2} \triangleright (\delta_B^{1/2} \triangleright a^*)))$$

By using the claim, this expression becomes

$$\varphi_A((\delta_B^{1/2} \triangleright a^*)^* (\delta_B^{1/2} \triangleright a^*)) \geq 0.$$

We now prove the main result of this section.
3.4. Theorem. Let \( \langle A, B \rangle \) be a multiplier Hopf*-algebra pairing. Suppose \( \varphi_A \) is a positive left integral on \( A \) and \( \psi_B \) is a positive right integral on \( B \). Then

\[
\langle \delta_A, \delta_B^{1/2} \rangle (\varphi_A \otimes \psi_B)
\]

is a positive left integral on the Drinfel’d double \( D = A \bowtie B^{\text{op}} \).

Proof. In this proof we stick to the notation \( \varphi = \langle \delta_A, \delta_B^{1/2} \rangle (\varphi_A \otimes \psi_B) \). An arbitrary element in \( D = A \bowtie B^{\text{op}} \) is given by \( \sum \ell a_\ell \bowtie b_\ell \) with \( a_\ell \in A \) and \( b_\ell \in B \). Then,

\[
\varphi \left( \left( \sum \ell a_\ell \bowtie b_\ell \right) \left( \sum \ell a_\ell \bowtie b_\ell \right)^* \right) = \varphi \left( \left( \sum \ell a_\ell \bowtie b_\ell \right) \left( \sum_j (1 \bowtie b_j^*) (a_j^* \bowtie 1) \right) \right) = \varphi \left( \sum_{\ell,j} (a_\ell \bowtie b_j^*) (a_j^* \bowtie 1) \right).
\]

Let \( R \) denote the twist map determining the product in \( D \), see Remarks 1.6. The expression becomes

\[
\langle \delta_A, \delta_B^{1/2} \rangle \sum_{\ell,j} \varphi_A (a_\ell \cdot) ((i_A \otimes \psi_B) R (b_\ell b_j^* \otimes a_j^*)).
\]

Now we use Lemma 3.2 to get

\[
\langle \delta_A, \delta_B^{1/2} \rangle \sum_{\ell,j} \varphi_A (a_\ell (\delta_B^{-1} \triangleright a_j^*)) \psi_B (b_\ell b_j^*).
\]

It is not difficult to show that the matrix \( M = (\psi_B (b_\ell b_j^*)) \) is a positive hermitian matrix; therefore \( M = U U^* \) for some complex matrix \( U \). So we get

\[
\langle \delta_A, \delta_B^{1/2} \rangle \varphi_A \left( \sum_{\ell,j} a_\ell (\delta_B^{-1} \triangleright a_j^*) \left( \sum_k u_{k\ell} \bar{u}_{jk} \right) \right) = \sum_k \langle \delta_A, \delta_B^{1/2} \rangle \varphi_A \left( \left( \sum_k u_{k\ell} a_\ell \right) (\delta_B^{-1} \triangleright \left( \sum_\ell u_{\ell k} a_\ell \right)^*) \right).
\]

By Lemma 3.3 this expression is positive. \( \square \)

3.5. The meaning of the number \( \langle \delta_A, \delta_B^{1/2} \rangle \) Using formulas from the paper by Johan Kustermans on the analytic structure of algebraic quantum groups [6], it is possible to calculate the scalar \( \langle \delta_A, \delta_B^{1/2} \rangle \). From [6, Proposition 7.10], we find that

\[
\langle \delta_A, \delta_B^{1/2} \rangle = \varepsilon \sigma_{1/2} (\delta_A),
\]
where $\sigma_z$ for $z \in \mathbb{C}$ is defined in [6, Proposition 5.8]. Observe that $\sigma_{-i}$ is the modular automorphism (usually denoted by $\sigma$), characterized by the formula $\varphi(ab) = \varphi(b)\sigma_{-i}(a)$ valid for all $a, b \in A$. It is found already in [12] that $\sigma_{-i}(\delta) = (1/\mu)\delta$ where the scalar $\mu$ is characterized by $\varphi \circ S^2 = \mu \varphi$. So one might expect that $\sigma_{i/2}(\delta) = \mu^{1/2}\delta$ for the right choice of the square root of the complex number $\mu$. It is also shown in [6] that a canonical choice for this square root is possible. In fact, the theory developed by Kustermans in [6] provides a positive number $\nu$ so that $\nu^2 = \mu$. The right choice of the square root of $\mu$ then appears to be $\nu^{1/2}$. See [6, Proposition 5.19].

References