Structure of Riemann Solutions for 2-Dimensional Scalar Conservation Laws*

GUI-QIANG CHEN\textsuperscript{\dag}

Department of Mathematics, University of Chicago, Chicago, Illinois 60637

DENING LI\textsuperscript{\dag}

Mathematics Department, West Virginia University, Morgantown, West Virginia 26506-6310

AND

DECHUN TAN\textsuperscript{§}

Institute of Applied Mathematics, Beijing, China

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We study the structure of Riemann solutions for 2-dimensional scalar conservation laws. The Riemann data are three constants in three fan domains forming different angles. We study the dependence of the structure of the solution upon the value of the constants as well as the angles. © 1996 Academic Press, Inc.

1. Introduction

In this paper, we study the Riemann problem for a scalar conservation law

\begin{equation}
\frac{\partial u}{\partial t} + f(u)_x + g(u)_y = 0, \quad (1.1)
\end{equation}

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in two space variables with the initial data

\[ u(x, y, 0) = u_0(x, y). \tag{1.2} \]

The Riemann problem plays an important role in the study of one-dimensional conservation laws. It is much easier to study than the general Cauchy problem, but still reveals the basic properties of the Cauchy problem. Besides, the solution of Cauchy problem can be locally approached by the solution of the Riemann problem. Since the solution for the Riemann problem has explicit structure, it also serves as a touchstone for numerical schemes.

In the study of 2-dimensional hyperbolic conservation laws, the initial data with two constants on two sides of a straight line is a trivial generalization of the one-dimensional case. To get the nontrivial 2-D solution for Riemann problem, a natural choice is to assume that x-axis and y-axis are all initial discontinuities and the initial value \( u_0(x, y) \) in the four quadrants are different constants. The Riemann problem of scalar conservation law with such initial data was studied in [3, 6, 9]. In [4], the authors studied the Riemann problem for a 2-D \( 2 \times 2 \) system. Based on the previous analysis, [10] presented a series of conjectures for the structure of Riemann solutions for the gas dynamic system. The structure of the solution in [3, 6] is relatively simple since it was assumed that \( f(u) \equiv g(u) \) and thus it can be reduced to the discussion of combination of one-dimensional Riemann solutions. The structure of the solutions for the Riemann problem in [4, 9, 10] are quite complicated.

In order to study the general problem with piecewise smooth initial data, one has to deal with the intersection of initial discontinuity curves which does not form an angle of \( \pi/2 \). Therefore it is important to study the Riemann problem with straight initial discontinuity lines intersecting at an arbitrary angle. In this paper, we will consider the special case that the initial value \( u_0 \) takes three constants in three fan domains. We shown that the structure of solutions for the Riemann problem depends not only on the relative magnitude of the value \( u_0 \) in the three fan domains but also on the relative angles bounded by these domains. Since the flux function is no more convex at certain direction, we will have the contact discontinuity in the structure of solutions. For simplicity, we will discuss only the cases which are of general interest and show special features which are different from the previous results. The more general case of more than three initial constants can be discussed in a similar manner.

We make in (1.1) and (1.2) the following assumptions:

1. \( f, g \in C^3, f''(u) > 0, g''(u) > 0 \) and \( (f''/g') > 0, \forall u \in R^1 \);
2. \( u_0(x, y) = u_j \) for \( (x, y) \in \Omega_j, j = 1, 2, 3 \), with \( \Omega_1 \cup \Omega_2 \cup \Omega_3 = R^2 \).
$\Omega_i \cap \Omega_j = \emptyset, \ i \neq j$. Here $\Omega_i$ is the open domain bounded by $\Gamma_j, \Gamma_{j+1}$, $(\Gamma_2 = \Gamma_1)$, and $\Gamma_j (j = 1, 2, 3)$ are half straight lines starting from the origin.

We always assume $0 < \alpha < 2\pi$. (See Fig. 1.1.)

**Remark 1.1.** The restrictions on $f$ and $g$ in the above assumption 1 can be relaxed, e.g., we can assume only that $g''(u) > 0$ and $(f''/g') > 0$. In this case, it’s easy to see that $f''(u)$ is monotone increasing and consequently $f$ is convex in $u > u^*$, concave in $u < u^*$, for some $u^*$. Therefore, similar discussions will give the structure of the solutions.

In this paper, we are going to investigate the structure of solutions for the above Riemann problem and their dependence upon the initial values $u_j$ as well as the angles $\alpha$ and $\beta$.

As in [9], we seek self-similar solutions

$u(x, y, t) = u(x/t, y/t)$.

By the self-similar transformation $\zeta = x/t, \eta = y/t$, (1.1) is reduced to

$$-\zeta u_\zeta - \eta u_\eta + f(u)_\zeta + g(u)_\eta = 0. \tag{1.3}$$

In the following, we need only to discuss solutions for (1.3) which satisfy the entropy conditions.

The outline of the paper is as follows. In Section 2, we discuss some basic concepts about a slant discontinuity. In Section 3, we study the classification of the various combination of initial Riemann data. Section 4 and 5 are devoted to the construction of Riemann solutions and to the proof of entropy conditions for some typical situations.

### 2. SLANT DISCONTINUITY

We consider the solution structure near $\Gamma_2$ which forms angle $\alpha$ with positive direction of $\zeta$-axis.
Introduce the new coordinate \((\tau, n)\), such that \(\tau\) is parallel to \(I_2\) and \(n\) is normal to \(I_2\). We have
\[
\begin{align*}
(n &= \zeta \sin \alpha - \eta \cos \alpha, \\
\tau &= \zeta \cos \alpha + \eta \sin \alpha.
\end{align*}
\]
Therefore
\[
\partial_{\zeta} = \sin \alpha \partial_n + \cos \alpha \partial_{\tau}, \quad \partial_{\tau} = -\cos \alpha \partial_n + \sin \alpha \partial_{\zeta}.
\]
(1.3) can be rewritten as
\[
-mu_n - \tau u_r + \partial_{\tau} F(u) + \partial_n G(u) = 0,
\tag{2.1}
\]
where
\[
F(u) = f(u) \sin \alpha - g(u) \cos \alpha,
\]
\[
G(u) = f(u) \cos \alpha + g(u) \sin \alpha.
\]
In the neighborhood of \(I_2\) at infinity, the elementary waves are parallel to \(\tau\)-axis, hence the smooth solution \(u(n, \tau) = u(n)\) satisfies
\[
-mu_n + F(\tau, u) = 0. \tag{2.2}
\]
The Rankine-Hugoniot conditions for solutions of (2.1) with discontinuity \(n = \sigma \tau\) are
\[
(-\tau + \sigma \tau) [u] - \sigma [F(\tau, u)] + [G(\tau, u)] = 0.
\]
At infinity, the discontinuity should be parallel to \(\tau\)-axis, so
\[
n[u] - [F(\tau, u)] = 0. \tag{2.3}
\]
Combining (2.2) and (2.3), we see that finding the elementary wave of (1.1) at infinity is equivalent to solving the Riemann problem
\[
\begin{align*}
\partial_{t} u + \partial_{x} F(u) &= 0, \\
u(z, 0) &= u^\pm, \quad \pm z > 0
\end{align*}
\tag{2.4}
\]
with \(n = z/t\).

The elementary waves of (2.4) are the following.

1. Smooth solution: From (2.2), the smooth elementary wave satisfies \((F(\tau, u) - n) u_n = 0\). Since \(u_n \neq 0\), we have \(F'(\tau, u) - n = 0\) or
\[
\sin \alpha (f'(\tau) - \zeta) - \cos \alpha (g'(\tau) - \eta) = 0.
\]
This implies that on \((\xi, \eta)\) plane, the characteristic curves of elementary waves are straight lines which are parallel to \(F_2\). Along the characteristic line, \(u\) is constant which is the corresponding \(u\)-parameter value of the intersection point of the characteristic line and the base curve \(B\) (defined by \(\xi = f'(u), \eta = g'(u)\) as in [9]).

2. Discontinuity: From (2.3), we have
\[
\left( \frac{[f(u)]}{[u]} - \xi \right) \sin \alpha - \left( \frac{[g(u)]}{[u]} - \eta \right) \cos \alpha = 0.
\]
Therefore, the discontinuity is a straight line which starts from the point \(((f(u))/[u], (g(u))/[u])\) and parallel to \(F_2\).

For nonconvex \(F_\alpha(u)\), the Riemann solution for one-dimensional (2.4) can be obtained as follows (see [7, 8]):

- For \(u^- > u^+\), we draw concave closure curve \(C_s\) for the graph of \(F_\alpha(u)\) from \(u^-\) to \(u^+\). The straight line part in \(C_s\) corresponds a shock wave and the part of \(C_s\) coinciding with the graph of \(F_s\) corresponds a rarefaction wave.

- For \(u^- < u^+\), we draw convex closure curve \(C_s\) for the graph of \(F_\alpha(u)\) from \(u^-\) to \(u^+\). Again, the straight line part in \(C_s\) corresponds a shock wave and the part of \(C_s\) coinciding with the graph of \(F_s\) corresponds a rarefaction wave.

For convenience, we assume

\[\text{(H1)} \quad \lim_{u \rightarrow \pm \infty} f'(u) = \lim_{u \rightarrow \pm \infty} g'(u) = \pm \infty;\]
\[\text{(H2)} \quad \lim_{u \rightarrow + \infty} \frac{f''(u)}{G''(u)} = + \infty, \quad \lim_{u \rightarrow - \infty} \frac{f''(u)}{G''(u)} = 0.\]

It is easy to obtain the following properties.

**Proposition 2.1.** Under the assumptions (H1) and (H2),

1. \(F''_\alpha(u) > 0\) for \(\alpha \in (\pi/2, \pi)\);
2. \(F''_\alpha(u) < 0\) for \(\alpha \in (3\pi/2, 2\pi)\);
3. For \(\alpha \in (0, \pi/2)\), there exists a unique \(\bar{u}_\alpha\) such that
\[F''_\alpha(\bar{u}_\alpha) = 0, \quad \pm F''_\alpha(u) > 0, \quad \pm (u - \bar{u}_\alpha) > 0.
\]

Moreover, \(\bar{u}_\alpha\) is a monotone decreasing function in \(\alpha\) with \(\bar{u}_\alpha \rightarrow + \infty\) as \(\alpha \rightarrow +0\) and \(\bar{u}_\alpha \rightarrow - \infty\) as \(\alpha \rightarrow \pi/2 - 0\).
4. For \( \alpha \in (\pi, 3\pi/2) \), there also exists a unique \( \bar{\alpha} \) such that
\[
F^{\prime\prime}(\bar{\alpha}) = 0, \quad \pm F^{\prime}(\bar{\alpha}) > 0, \quad \pm (\bar{\alpha} - \bar{\alpha}) < 0.
\]
And \( \bar{\alpha} \) is a monotone decreasing function in \( \alpha \) with \( \bar{\alpha} \to +\infty \) as \( \alpha \to \pi \) and \( \bar{\alpha} \to -\infty \) as \( \alpha \to +3\pi/2 \).

The above properties are described in Fig. 2.1.

The inflection point \( \bar{\alpha} \) of \( F^{\prime}(\alpha) \) is important in the discussion of the structure of waves. For a given \( \alpha \), \( \bar{\alpha} \) decides the unique point \((\zeta, \eta)\) on the base curve. At this point, the slope of the tangent line for \( B \) is
\[
\frac{d\eta}{d\zeta} = \tan \alpha.
\]

Another point of importance is \( u = \bar{u} \) where the tangent line of \( F_s = F_s(u) \) passes the point \((u^{-}, F_s(u^{-}))\) \((u^{-} \neq \bar{\alpha})\). It is easy to see that \( \bar{\alpha} \) is uniquely determined by \( u^{-} \) and \( \alpha \).

For \( F^{\prime\prime}(u^{-}) \neq 0 \), the value \( \bar{\alpha} \) can be obtained from
\[
F^{\prime\prime}(\bar{\alpha}) = \frac{F_s(\bar{\alpha}) - F_s(u^{-})}{\bar{\alpha} - u^{-}}.
\]
Equation (2.5) can be rewritten as
\[ \sin x \left( f'(\hat{u}_s) - \frac{f(\hat{u}_s) - f(u^-)}{\hat{u}_s - u^-} \right) - \cos x \left( g'(\hat{u}_s) - \frac{g(\hat{u}_s) - g(u^-)}{\hat{u}_s - u^-} \right) = 0. \] (2.6)

Equation (2.6) means that the straight line passing \((f'(\hat{u}_s), g'(\hat{u}_s))\) and \((f'_{\hat{u}_s}, g'_{\hat{u}_s})\) is parallel to \(F_2\). From now on, we will systematically denote \(f_{\hat{u}_s} \equiv f(\hat{u}_s) - f(u^-) / \hat{u}_s - u^-\), \(g_{\hat{u}_s} \equiv g(\hat{u}_s) - g(u^-) / \hat{u}_s - u^-\).

\(f_{12}\) and \(f_{23}\) can be defined similarly.

Next consider various combinations of the initial data for different values of \(\alpha\). As in [9], for the elementary wave connecting two constant states, \(S\) denotes the shock (including, in particular, contact discontinuity when \(u\) on one side coincides with \(\hat{u}_s\)), and \(R\) denotes the rarefaction wave. From Fig. 2.2, we see that for \(\pi/2 < \alpha < \pi\) and \(3\pi/2 < \alpha < 2\pi\), the flux function \(F_\alpha(u)\) is convex (or concave) and the elementary wave is either a shock or a rarefaction wave. For \(\pi < \alpha < 3\pi/2\), the situation is similar to that of \(0 < \alpha < \pi/2\). Therefore, we will consider only the case \(0 < \alpha < \pi/2\). We have two cases for the relative locations of \(\hat{u}, u^-, \) and \(\hat{u}_s\).

1. \(u^- > \hat{u} > \hat{u}_s\); the wave structure for three different relative location of \(u^+\) are the following:
   - \(u^- < u^+ < \hat{u}_s\): \(R\);
   - \(\hat{u}_s < u^+ < u^-\): \(S\);
   - \(u^+ < \hat{u}_s < u^-\): \(S + R\).

2. \(u^- < \hat{u} < \hat{u}_s\); the wave structure for three different relative location of \(u^+\) are the following:
   - \(u^+ < u^- \): \(R\);
   - \(u^- < u^+ < \hat{u}_s\): \(S\);
   - \(u^- < \hat{u}_s < u^+ \): \(S + R\).

![Figure 2.2](image-url)
The above shock and rarefaction waves are the same as the one-dimensional waves. The general discontinuity of (1.3) is called a shock if it satisfies Rankine–Hugoniot condition

\[
\frac{d\eta}{d\xi} = \frac{[g]/[u] - \eta}{[f]/[u] - \xi}
\]

(2.7)
as well as the entropy condition

\[
\frac{F_s(u^-) - F_s(u^-)}{v^- - u^-} \geq \frac{F_s(u^+) - F_s(u^-)}{u^+ - u^-}, \quad \forall v \in (u^-, u^+).
\]

(2.8)
The equality sign in (2.8) implies that the contact discontinuity is included here as a special case of shock waves.

Since (2.8) is quite tedious to verify, it is useful to introduce the following proposition.

**Proposition 2.2.** Assume that the tangential direction of the discontinuity is \((\cos \alpha, \sin \alpha)\), \(F_s\) defined as above, and \(u^-, u^+\) denote the value of the solution on two sides of the discontinuity. Then the entropy condition (2.8) is equivalent to the following condition:

- when \(\alpha \in (0, \pi/2)\),

\[
F_s(u^+) \leq \frac{F_s(u^+) - F_s(u^-)}{u^+ - u^-};
\]

(2.9)

- when \(\alpha \in (\pi, 3\pi/2)\),

\[
F_s(u^-) \geq \frac{F_s(u^+) - F_s(u^-)}{u^+ - u^-}.
\]

(2.10)

The proof of the proposition proceeds by considering the equivalent tangential elementary waves and by discussing the corresponding one-dimensional Riemann problem. The conclusion follows readily from Fig. 2.2.

### 3. Classification of Initial Jump

Let's consider (1.1) and (1.2). Let \(\eta_j\) be the value of \(\eta_0\) in the domain \(\Omega_j\), \((j = 1, 2, 3)\). There are six combinations for the initial values \(u_1, u_2, u_3\). In this paper, we will focus our attention to the case \(u_3 > u_2 > u_1\) to show the relation between the structure of solutions and the angles \(\alpha, \beta\). All the other
combinations will produce similar structure of solutions and can be discussed in the same manner.

For the given $u_1 < u_2 < u_3$, as the angles $\alpha, \beta$ change, the structure of the elementary wave may be quite different. We will limit ourselves to considering the following cases of angles

- $0 < \alpha < \beta < \pi/2$;
- $0 < \alpha < \pi/2$, $\pi/2 < \beta < \pi$;
- $0 < \alpha < \pi/2, \pi < \beta < 3\pi/2$;
- $0 < \alpha < \pi/2, 3\pi/2 < \beta < 2\pi$.

All of the other cases can be obtained in the same way. In the following tables, $(\cdot, \cdot, \cdot)$ denotes the wave patterns near infinity, corresponding to the initial jumps $(\Gamma_1, \Gamma_2, \Gamma_3)$. And we will denote $\tilde{u}_1$ ($\tilde{u}_2$) the root of (2.5) with $u^- = u_3$ ($u^- = u_1$ and $\pi$ replaced by $\beta$). Also we denote $\tilde{u}_3$ ($\tilde{u}_4$) the root of (2.5) with $u^- = u_3$ ($u^- = u_1$), $\tilde{u}_{2\beta}$ the root of (2.5) with $u^- = u_2$ and $\pi$ replaced by $\beta$.

1. $0 < \alpha < \beta < \pi/2$: depending on the relative locations of $\tilde{u}_1$ and $\tilde{u}_3$, there are five subcases.

   $u_2 > \tilde{u}_1 > u_1 > \tilde{u}_3$ (so $u_2 > u_1 > \tilde{u}_3$) \hspace{1cm} (S, S, S)
   $u_2 > \tilde{u}_1 > u_1 > \tilde{u}_3$ (so $u_2 > u_1 > \tilde{u}_3$) \hspace{1cm} (S, S + R, S)
   $\tilde{u}_1 > u_2 > u_1$ and $u_3 > \tilde{u}_3 > u_2 > \tilde{u}_3$ \hspace{1cm} (S, R, S)
   $\tilde{u}_1 > u_2 > u_1$ and $u_3 > \tilde{u}_3 > u_2 > \tilde{u}_3$ \hspace{1cm} (S, R, S + R)
   $\tilde{u}_3 > u_2 > u_1$ \hspace{1cm} (S, R, R)

2. $0 < \alpha < \pi/2$ and $\pi/2 < \beta < \pi$: hence $F_\beta'(u) > 0$ and a shock always connect $u_2$ and $u_3$. There are three subcases.

   $u_2 > \tilde{u}_1 > u_1 > \tilde{u}_3$ \hspace{1cm} (S, S, S)
   $u_2 > \tilde{u}_1 > \tilde{u}_3 > u_1$ \hspace{1cm} (S, S + R, S)
   $u_2 < \tilde{u}_3$ \hspace{1cm} (S, R, S)

3. $0 < \alpha < \pi/2$ and $3\pi/2 < \beta < 2\pi$: hence $F_\beta'(u) < 0$ and a rarefaction wave always connect $u_2$ and $u_3$. There are three subcases.

   $u_2 > \tilde{u}_1 > u_1 > \tilde{u}_3$ \hspace{1cm} (S, S, R)
   $u_2 > \tilde{u}_1 > \tilde{u}_3 > u_1$ \hspace{1cm} (S, S + R, R)
   $u_2 < \tilde{u}_3$ \hspace{1cm} (S, R, R)
4. The Riemann Solutions for $0 < \alpha < \beta < \pi/2$

In this section, we construct the Riemann solutions of (1.1) and (1.2) for the initial value $u_3 > u_2 > u_1$ and the angles $0 < \alpha < \beta < \pi/2$.

4.1. Riemann Solution for $u_2 > u_1 > u_3$

From Section 3, the initial jumps away from the origin are $(S, S, S)$. We denote the shock (rarefaction) wave connecting $u_j$ and $u_{j+1}$ by $S_{jj+1}$ ($S_{jj+1}$), $j = 1, 2, 3$, with $S_{34} \equiv S_{13}$ and $u_4 \equiv u_1$.

From Section 2, we know $S_{12}$ (resp. $S_{13}$; $S_{12}$) is a half straight line which starts from $(f_{12}, g_{12})$ (resp. $(f_{13}, g_{13}); (f_{23}, g_{23})$) with direction $(\cos \alpha, \sin \alpha)$ (resp. $(1, 0)$; $(\cos \beta, \sin \beta)$). By the convexity of $f$ and $g$, we have $f_{12} < f_{13} < f_{23}, g_{12} < g_{13} < g_{23}$.

To construct the Riemann solution, we need the following facts:

F1. The intersection point $A(\xi_1, \eta_1)$ of $S_{12}$ and $S_{23}$ satisfies $\xi_1 > f_{23}, \eta_1 > g_{23}$.

F2. As a discontinuity line separating $u_1$ and $u_3$, the segment connecting $(\xi_1, \eta_1)$ and $(f_{13}, g_{13})$ satisfies the entropy condition.

From (F1) and (F2), we see the Riemann solution has the structure shown in Fig. 4.1.
Proof of (F1). From
\[
\begin{align*}
q_1 - g_{12} &= \tan \alpha (\xi_1 - f_{12}), \\
q_1 - g_{23} &= \tan \beta (\xi_1 - f_{23}),
\end{align*}
\]  
we have
\[
\xi_1 = f_{23} + \frac{\tan \alpha (f_{23} - f_{12}) - (g_{23} - g_{12})}{\tan \beta - \tan \alpha}.
\]  
Since \( \beta > \alpha \), we need only to show
\[
\tan \alpha > \frac{g_{23} - g_{12}}{f_{23} - f_{12}}.
\]  
Denote
\[
\begin{align*}
P(u) &= \frac{f(u) - f(u_2)}{u - u_2}, \quad Q(u) = \frac{g(u) - g(u_2)}{u - u_2}, \\
\phi(u, u_2) &= \frac{g'(u)(u - u_2) - (g(u) - g(u_2))}{f'(u)(u - u_2) - (f(u) - f(u_2))},
\end{align*}
\]  
so by the Cauchy mean value theorem, \( \exists v \in (u_1, u_3) \) such that
\[
\frac{g_{23} - g_{12}}{f_{23} - f_{12}} = \frac{Q(v)}{P(v)} = \phi(v, u_2). \tag{4.4}
\]  
We prove \( \partial_u \phi(u, u_2) < 0 \). In fact
\[
\phi_u(u, u_2) = \frac{J(u, u_2)(u - u_2) f(u)}{[f'(u)(u - u_2) - (f(u) - f(u_2))]}, \tag{4.5}
\]  
where
\[
J(u, u_2) = \frac{g(u)}{f(u)} \left[ f'(u)(u - u_2) - (f(u) - f(u_2)) \right] - \left[ g'(u)(u - u_2) - (g(u) - g(u_2)) \right].
\]  
Since \( J(u_2, u_2) = 0 \) and
\[
J_u(u, u_2) = \frac{g(u)}{f(u)} (u - u_2) \left( f'(u) - \frac{f(u) - f(u_2)}{u - u_2} \right) < 0, \quad \forall u \neq u_2,
\]
so \( J(u, u_2) > 0 \) for \( u < u_2 \) and \( J(u, u_2) < 0 \) for \( u > u_2 \). Hence \((u - u_2) J(u, u_2) < 0\) which gives \( \phi'(u, u_2) < 0 \).

From (2.6), we have \( \phi(\bar{u}, u_2) = \tan \alpha \). Since \( v > u_1 > \bar{u} \), hence \( \phi(v, u_1) < \phi(\bar{u}, u_2) = \tan \alpha \). Therefore, (4.2) gives \( \zeta_1 > f_{23} \). Then \( \eta_1 > g_{23} \) follows from (4.1).

**Proof of (F2).** From Proposition 2.2, we need only to verify

\[
F'(u_1) = \frac{F'(u_2) - F'(u_3)}{u_1 - u_3} \leq \tan \alpha.
\]

(4.6) can be rewritten as

\[
\frac{\eta_1 - g_{13}}{\zeta_1 - f_{13}} > \frac{g'(u_1)(u_1 - u_3) - (g(u_1) - g(u_3))}{f'(u_1)(u_1 - u_3) - (f(u_1) - f(u_3))} = \phi(u_1, u_3).
\]

Since \( \bar{u}_3 < \bar{u}_2 < u_1 \) and \( \phi(u, u_3) \) is a decreasing function of \( u \), from \( \phi(\bar{u}_3, u_3) = \tan \alpha \), we know \( \phi(u_1, u_3) < \tan \alpha \). Therefore, to prove (4.6), it suffices to prove

\[
\tan \gamma = \frac{\eta_1 - g_{13}}{\zeta_1 - f_{13}} > \tan \alpha.
\]

In fact, from (4.1), we know

\[
\tan \gamma = \frac{(g_{23} - g_{13})(\tan \beta - \tan \alpha) + \tan \alpha \tan \beta(f_{23} - f_{12}) - \tan \beta(g_{23} - g_{12})}{(f_{23} - f_{13})(\tan \beta - \tan \alpha) + \tan \alpha(f_{23} - f_{12}) - (g_{23} - g_{12})}.
\]

Therefore we need to show

\[
\tan \gamma - \tan \alpha = \frac{(\tan \beta - \tan \alpha)[\tan \alpha(f_{13} - f_{12}) - (g_{13} - g_{12})]}{(f_{23} - f_{13})(\tan \beta - \tan \alpha) + \tan \alpha(f_{23} - f_{12}) - (g_{23} - g_{12})} > 0.
\]

(4.7)

The convexity of \( f \) implies \( f_{23} - f_{13} > 0 \). From (4.3), \( \tan \alpha(f_{23} - f_{12}) - (g_{23} - g_{12}) > 0 \). Consequently, to prove (4.7), it suffices to show

\[
\tan \alpha > \frac{g_{13} - g_{12}}{f_{13} - f_{12}}.
\]

(4.8)

From the Cauchy mean value theorem, there exists \( v \in (u_2, u_3) \) such that

\[
\frac{g_{13} - g_{12}}{f_{13} - f_{12}} = \frac{g'(v)(v - u_3) - (g(v) - g(u_3))}{f'(v)(v - u_1) - (f(v) - f(u_1))}.
\]
By the definition of $\tilde{u}_{1\alpha}$ in (2.5),
\[ \tan \alpha = \phi(\tilde{u}_{1\alpha}, u_1), \]
it is easy to see $\tilde{u}_{1\alpha} < u_2 < v$. Since $\partial_v \phi(u, u_1) < 0$, hence $\phi(v, u_1) < \tan \alpha$. This completes the proof of (F2).

4.2. Riemann Solution for $u_2 > \tilde{u}_\alpha > \tilde{u}_a > u_1$

In this case, the initial jumps in the neighborhood of the infinity in the $(\xi, \eta)$-plane are $(S, S + R, S)$.

Let $(\xi_1, \eta_1)$ be the intersection point of $S_{12}$ and $S_{23}$. Then $F_1$ in Section 4.1 remains true. Starting from $(\xi_1, \eta_1)$ we solve (2.7) which describes the discontinuity curve $\eta = \eta(\xi)$ separating two values $u = u(\xi, \eta)$ and $u_3$. It can be written as follows
\[
\begin{align*}
\frac{d\eta}{d\xi} &= \frac{g(u) - g(u_3)}{u - u_3} - \frac{\eta}{\xi}, \\
\eta |_{\xi = \xi_1} &= \eta_1, \quad \text{or} \quad \eta |_{u = \tilde{u}_a} = \eta_1.
\end{align*}
\]
(4.9.a)

The value $u$ and $\xi, \eta$ are related by the rarefaction wave
\[
(f'(u) - \xi \sin \alpha) - (g'(u) - \eta) \cos \alpha = 0.
\]
(4.9.c)

From (4.9.c), we have
\[
F_\alpha(\eta) \frac{du}{d\eta} - \sin \alpha \frac{d\xi}{d\eta} + \cos \alpha = 0.
\]
Combining it with (4.9.a), we get
\[
\begin{align*}
\left( \frac{d\eta}{du} - a_s(u) \right) \left[ \frac{g(u) - g(u_3)}{u - u_3} - \eta \right], \quad &u \in (u_1, \tilde{u}_a), \\
\eta |_{u = \tilde{u}_a} = \eta_1.
\end{align*}
\]
(4.10)

Here
\[
a_s(u) = F_\alpha(\eta) \left( \frac{F_\alpha(u) - F_\alpha(u_3)}{u - u_3} - F_\alpha(u) \right)^{-1}.
\]

We consider two subcases: $u_1 > \tilde{u}_3$ and $u_1 < \tilde{u}_a$. 


1. \( u_i > \bar{u}_{3s} \): we know from the definition of \( \bar{u}_a, \bar{u}_{3s}, \) and \( \tilde{u}_a \) that \( a_s(u) < 0, \forall u \in (u_1, \tilde{u}_a) \). (4.10) can be solved as
\[
\eta(u) = \eta_1 \exp \left( \int_{\bar{u}_a}^u a_s(\tau) \, d\tau \right) + \int_{\bar{u}_a}^u a_s(s) \frac{g(s) - g(u_3)}{s - u_3} \exp \left( -\int_{\bar{u}_a}^u a_s(\tau) \, d\tau \right) \, ds. \tag{4.11}
\]
From integration by parts, (4.11) can be rewritten as
\[
\eta(u) = \frac{g(u) - g(u_3)}{u - u_3} + \left( \eta_1 - \frac{g(\bar{u}_a) - g(u_3)}{\bar{u}_a - u_3} \right) \exp \left( \int_{\bar{u}_a}^u a_s(\tau) \, d\tau \right) \frac{g(s) - g(u_3)}{s - u_3} \left( \int_{\bar{u}_a}^u a_s(s) \, ds \right) \tag{4.12}
\]
Since \( u \in (u_1, \bar{u}_a) \) and \( s < u_3 \), hence
\[
g'(s) < \frac{g(s) - g(u_3)}{s - u_3}.
\]
Therefore
\[
\eta(u) > \frac{g(u) - g(u_3)}{u - u_3} + \left( \eta_1 - \frac{g(\bar{u}_a) - g(u_3)}{\bar{u}_a - u_3} \right) \exp \left( \int_{\bar{u}_a}^u a_s(\tau) \, d\tau \right). \tag{4.13}
\]
We proceed to prove
\[
\eta_1 > \frac{g(\bar{u}_a) - g(u_3)}{\bar{u}_a - u_3} \equiv g_{3s}. \tag{4.14}
\]
In fact, from
\[
\eta_1 - g_{2s} = \tan \alpha (\xi_1 - f_{2s}), \quad \eta_1 - g_{2s} = \tan \beta (\xi_1 - f_{2s}),
\]
we have
\[
\eta_1 = g_{2s} + \frac{f_{2s} - f_{2s} - \cot \alpha (g_{2s} - g_{2s})}{\cot \alpha - \cot \beta}.
\]
Hence
\[
\eta_1 - g_{3s} = g_{2s} - g_{3s} + \frac{f_{2s} - f_{2s} - \cot \alpha (g_{2s} - g_{2s})}{\cot \alpha - \cot \beta}.
\]
Since $g_{23} > g_{38}$, we need only to show
\[ \tan \alpha > \frac{g_{23} - g_{28}}{f_{23} - f_{28}}. \] (4.15)

From the Cauchy mean value theorem, we have for some $v \in (\hat{u}_k, u_3)$
\[ \frac{g_{23} - g_{28}}{f_{23} - f_{28}} = \frac{g'(v)(v - u_2) - (g(v) - g(u_2))}{f'(v)(v - u_2) - (f(v) - f(u_2))} \]
\[ < \frac{g'(\hat{u}_k)(\hat{u}_k - u_2) - (g(\hat{u}_k) - g(u_2))}{f'(\hat{u}_k)(\hat{u}_k - u_2) - (f(\hat{u}_k) - f(u_2))} = \tan \alpha. \]

This completes the proof of (4.14). Therefore, (4.13) gives
\[ \eta(u) > \frac{g(u) - g(u_3)}{u - u_3}, \quad \forall u \in (u_1, \hat{u}_k). \] (4.16)

Similarly, we have
\[ \zeta(u) > \frac{f(u) - f(u_3)}{u - u_3}, \quad \forall u \in (u_1, \hat{u}_k). \] (4.17)

In particular, from (4.16) and (4.17), we have \( \zeta(u_1) > f_{13}, \eta(u_1) > g_{13} \). Connecting the points \((\zeta(u_1), \eta(u_1))\) and \((f_{13}, g_{13})\) with a shock wave, we find the solution has the structure shown in Fig. 4.2.

2. $u_1 < \hat{u}_3$: Similarly as above, we will solve the equation (4.9) for the discontinuity curve, first from intersection of $S_{12}$ and $S_{23}$ (corresponding to $\hat{u}_k$ downward) and then from intersection of $S_{13}$ and $S_{12}$ (corresponding to $u_1$ upward).

![Figure 4.2](image-url)
From the definition of \( \tilde{a}_S \) and \( \tilde{a}_{3S} \) we know that (4.16) and (4.17) hold for any \( u \in (\tilde{u}_{3S}, \tilde{u}_S) \). We are going to prove that the solution for (4.9) satisfies

\[
\lim_{u \to \tilde{u}_{3S}^+} \eta(u) = \frac{g(\tilde{u}_{3S}) - g(u)}{\tilde{u}_{3S} - u}, \quad \lim_{u \to \tilde{u}_{3S}^+} \zeta(u) = \frac{f(\tilde{u}_{3S}) - f(u)}{\tilde{u}_{3S} - u} \quad (4.18)
\]

In fact, from the definition of \( a_s(u) \) in (4.10) and the definition of \( \tilde{u}_{3S} \), we see that as \( u \to \tilde{u}_{3S}^+ \),

\[
\frac{F'_s(u) - F'_s(u_3)}{u - u_3} - F'_s(u) = \frac{F'_s(u) - F'_s(u_3)}{u - u_3} - \frac{F'_s(\tilde{u}_{3S}) - F'_s(u_3)}{\tilde{u}_{3S} - u_3}
\]

\[
+ F'_s(\tilde{u}_{3S}) - F'_s(u) = O(|u - \tilde{u}_{3S}|).
\]

Therefore

\[
\lim_{u \to \tilde{u}_{3S}^+} \exp \left( \int_u^{\tilde{u}_S} a_s(\tau) d\tau \right) = 0.
\]

Then the first part of (4.18) follows readily from (4.12). The second part of (4.18) can be proved similarly.

On the other hand, let the shock wave \( S_{13} \) intersect with the simple wave line in \( R_{12} \) on which \( u = u_1 \) at point \( B = (\xi_2, \eta_2) \). It is easy to show that \( \xi_2 > f_{13} \). Similar to (4.9), we solve the following ordinary differential equation:

\[
\frac{d\eta}{d\xi} = \frac{g(u) - g(u_1)}{u - u_1} - \eta \quad (4.19.a)
\]

\[
\eta \big|_{\xi = \xi_2} = \eta_2, \quad (4.19.b)
\]

\[
(f'(u) - \xi) \sin \alpha - (g'(u) - \eta) \cos \alpha = 0. \quad (4.19.c)
\]

Similar to (4.10), we have

\[
\eta'(u) = a_s(u) \left[ \frac{g(u) - g(u_1)}{u - u_1} - \eta \right], \quad \xi'(u) = a_s(u) \left[ \frac{f(u) - f(u_1)}{u - u_1} - \xi \right], \quad (4.20)
\]
for any \( u \in (u_1, \tilde{u}_{3\varepsilon}) \). And we can further show that

\[
\begin{align*}
\eta(u) &< \frac{g(u) - g(u_1)}{u - u_3}, \\
\zeta(u) &> \frac{f(u) - f(u_3)}{u - u_3}, \quad \text{for} \quad u \in (u_1, v_0), \\
\zeta(u) &< \frac{f(u) - f(u_3)}{u - u_3}, \quad \text{for} \quad u \in (v_0, \tilde{u}_{3\varepsilon}),
\end{align*}
\]

where \( v_0 \) is defined by \( \zeta(v_0)(v_0 - u_3) = f(v_0) - f(u_3) \), and

\[
\lim_{{u \to \tilde{u}_{3\varepsilon}^-}} \eta(u) = \frac{g(\tilde{u}_{3\varepsilon}) - g(u_3)}{\tilde{u}_{3\varepsilon} - u_3}, \quad \lim_{{u \to \tilde{u}_{3\varepsilon}^-}} \zeta(u) = \frac{f(\tilde{u}_{3\varepsilon}) - f(u_3)}{\tilde{u}_{3\varepsilon} - u_3}. \tag{4.21}
\]

These give the structure of solution as shown in Fig. 4.3.

It remains to prove the discontinuities in the above solutions satisfy entropy conditions. We will prove for the case \( u_1 < \tilde{u}_{3\varepsilon} \). The case \( u_1 > \tilde{u}_{3\varepsilon} \) can be proved similarly. First we notice that

\[
\zeta(u) > f'(u), \quad \forall u \in (u_1, \tilde{u}_{3\varepsilon}). \tag{4.22}
\]

In fact, (4.22) is true near \( u = \tilde{u}_{3\varepsilon} \) by (4.21). Let

\[
w = \inf\{u^* : \forall u \in (u^*, \tilde{u}_{3\varepsilon}), \zeta(u) > f'(u)\}.
\]

Since \( \zeta(w) = f'(w) \), we have \( \zeta'(w) > f(w) \), which implies, by (4.10),

\[
\left( \frac{f(w) - f(u_3)}{w - u_3} - f'(w) \right) g'(w) \leq \left( \frac{g(w) - g(u_3)}{w - u_3} - g'(w) \right) f'(w).
\]

Figure 4.3
Since \( w < u_3 \), we have

\[
\frac{g'(w) - g'(u_3)}{w - u_3} - \frac{g'(w)}{w - u_3} \leq \frac{f'(w) - f(u_3)}{w - u_3} - \frac{f'(w)}{w - u_3}. \tag{4.23}
\]

But the function

\[
G(u) \equiv \frac{g'(u)}{f'(u)} (f'(u)(u - u_3) - (f(u) - f(u_3))) - (g'(u)(u - u_3) - (g(u) - g(u_3)))
\]

satisfies \( G(u_3) = 0 \) and

\[
G'(u) = \left( \frac{g'}{f'} \right) ((u - u_3) f'(u) - f(u) + f(u_3)) < 0.
\]

Therefore \( G(u) > 0, \forall u < u_3 \), which contradicts to (4.23). This completes the proof of (4.22).

Now we prove the entropy condition is satisfied for \( u \in (u_1, \tilde{u}_{3b}) \). (2.8) is always true for \( \pi/2 < \alpha < \pi \) and \( u_- > u_+ \). By Proposition 2.2, we need to show that, \( \forall u \in (u_1, \tilde{u}_{3b}) \),

\[
F'_3(u) = \frac{F_3(u_3) - F_3(u)}{u_3 - u} < 0. \tag{4.24}
\]

At point \( B \) where \( u = u_1 \), (4.24) is true by (4.6). Let

\[
z = \sup \{ u^* \in (u_1, \tilde{u}_{3b}); (4.24) \text{ is true in } (u_1, u^*) \}.
\]

If \( z \neq \tilde{u}_{3b} \), we have

\[
F'_3(z) = \frac{F_3(u_3) - F_3(z)}{u_3 - z},
\]

from which we derive

\[
\left. \frac{d\eta}{dz} \right|_{u = z} = \frac{h(z) - g(u_3)}{z - u_3} - \frac{g'(z)}{z - u_3}.
\]
Combining with Rankine–Hugoniot conditions, we get
\[
\left( \frac{g(z) - g(u_3)}{z - u_3} - g'(z) \right) (\zeta - f'(z)) - \left( \frac{f(z) - f(u_3)}{z - u_3} - f'(z) \right) (\eta - g'(z)) = 0,
\]
\[
\sin \alpha (\zeta - f'(z)) - \cos \alpha (\eta - g'(z)) = 0.
\]
(4.25) implies that either
\[
\frac{g(z) - g(u_3)}{z - u_3} - g'(z) = \frac{f(z) - f(u_3)}{z - u_3} - f'(z),
\]
\[
\sin \alpha = \frac{g(z) - g(u_3)}{z - u_3} - g'(z), \quad \cos \alpha = \frac{f(z) - f(u_3)}{z - u_3} - f'(z), \quad (4.26)
\]
or
\[
\zeta = f'(z), \quad \eta = g'(z). \quad (4.27)
\]
(4.27) contradicts (4.22), so (4.26) is true, which by definition implies \( z = \tilde{u}_3 \). Therefore the solution of (4.19) satisfies the entropy condition for \( u \in (u_1, \tilde{u}_3) \).

The proof of the entropy condition for the solution of (4.19) in \( u \in (\tilde{u}_3, \tilde{u}_5) \) can be sketched as follows.

- As in Section 4.1, we see that the solution of (4.19) satisfies entropy condition (4.24) near \( u = \tilde{u}_3 \).

- Let
\[
w = \inf \{ u^* \in (\tilde{u}_3, \tilde{u}_5) \colon (4.24) \text{ is true in } (\tilde{u}_3, u^*) \}.
\]

Then from
\[
\zeta > \frac{f(u) - f(u_3)}{u - u_3} > f'(u), \quad \forall u \in (\tilde{u}_3, \tilde{u}_5),
\]
we derive \( w = \tilde{u}_3 \).

\[\text{Figure 4.4}\]
This completed the proof of the structure for solutions in the case $u_2 > \bar{u}_s > \bar{u}_t > u_1$.

4.3. Other Cases

For the other cases listed in Section 3, we give briefly the description of the structure of the solutions omitting the tedious but similar proof.

1. When $\bar{u}_s > u_2 > \bar{u}_p$ and $u_1 > \bar{u}_p > u_3$, the initial jumps away from the origin are $(S, R, S)$. The solution can be obtained by similar argument, with the structure shown in Fig. 4.4.

2. When $\bar{u}_s > u_2 > u_1$ and $u_1 > \bar{u}_p > \bar{u}_p > u_3$, the initial jumps away from the origin are $(S, R, S + R)$.

The structure of solution is shown in Fig. 4.5. Here $S_{13}$ penetrates $R_{12}$ and interacts with $R_{23}$ and then stops at the singular point $(f_3, g_3)$, where

$$f_3 = \frac{f(\bar{u}_p) - f(u_1)}{\bar{u}_p - u_1}, \quad g_3 = \frac{g(\bar{u}_p) - g(u_1)}{\bar{u}_p - u_1}.$$

3. When $\bar{u}_p > u_3$, The initial jumps away from the origin are $(S, R, R)$.

In this case, $S_{12}$ penetrates $R_{12}$ and then interacts with $R_{23}$ and vanishes at the point $(f'(u_3), g'(u_3))$, as shown in Fig. 4.6.
5. Riemann Solutions for $\beta \in (\pi/2, 2\pi)$

In this section we will give some examples for solutions of other combination of angles.

5.1. Riemann Solutions for $\pi/2 < \beta < \pi$

1. $u_1 > \tilde{u}_s > u_t > \tilde{u}_e$. The initial jumps away from the origin are (S, S, S). The structure of the solution can be determined as follows.

Denote the intersection point of $S_{12}$ and $S_{23}$ by $(\xi_1, \eta_1)$:

$$
\begin{align*}
\begin{cases}
g_1 - g_{12} = \tan \alpha (\xi_1 - f_{12}), \\
\eta_1 - g_{23} = \tan \beta (\xi_1 - f_{23}).
\end{cases}
\end{align*}
$$

(5.1)

From

$$(\tan \alpha - \tan \beta) (\xi_1 = (\tan \alpha - \tan \beta) f_{23} - \tan \alpha (f_{23} - f_{12}) + (g_{23} - g_{12})$$

$$(= (\tan \alpha - \tan \beta) f_{12} - \tan \beta (f_{23} - f_{12}) + (g_{23} - g_{12})$$

and noticing (4.3), we obtain

$$f_{12} < \xi_1 < f_{23}, \quad \eta_1 > g_{23}.$$  

There are two subcases of $\xi_1 < f_{13}$ and $\xi_1 > f_{13}$. Let $\gamma$ be the angle formed by the straight line connecting $A = (\xi_1, \eta_1)$ and $(f_{13}, g_{13})$.

- When $\xi_1 < f_{13}$, the corresponding $F = f(u) \sin \gamma - g(u) \cos \gamma$ with $\pi/2 < \gamma < \pi$. The discontinuity straight line connecting points $(\xi_1, \eta_1)$ and $(f_{13}, g_{13})$ with $u_t$ and $u_3$ on its two sides satisfies the entropy condition.

- When $\xi_1 > f_{13}$, similar to the case of Section 4.1, we can show $\gamma < \pi/2$. And the discontinuity line connecting points $(\xi_1, \eta_1)$ and $(f_{13}, g_{13})$ also satisfies the entropy condition. The structure of solution is shown in Fig. 5.1.

2. $u_2 > \tilde{u}_s > \tilde{u}_e > u_t$. The solution can be obtained in the same way as in Section 4.2. The structure of solution is illustrated in Fig. 5.2.

5.2. The Riemann Solutions for $3\pi/2 < \beta < 2\pi$

Omitting the tedious but similar exposition, we list the structure of solutions in the following figures.

1. $u_2 > \tilde{u}_s > u_1 > \tilde{u}_e$: Fig. 5.3.
2. $u_2 > \tilde{u}_s > \tilde{u}_e > u_t$: Fig. 5.4.
3. $u_2 < \tilde{u}_e$: Fig. 5.5.
RIEMANN PROBLEM FOR SCALAR CONSERVATION LAW

Figure 5.1

Figure 5.2

Figure 5.3

Figure 5.4
5.3. Riemann Solutions for $\beta \in (\pi, 3\pi/2)$

As we have seen in Section 3, there are nine cases in this situation. We list here three cases to be compared with the solutions obtained in Sections 4, 5.1, and 5.2.

1. $u_2 < \bar{u}_g$, $u_3 < \bar{u}_g$, and $u_4 > \bar{u}_a > u_1 > \bar{u}_a$. The Riemann solution has structure shown in Fig. 5.6. Here we notice that the solution has the basic features similar to the ones in Figs. 4.1 5.1, and being independent of the angles $\alpha$ and $\beta$.

2. $u_2 > \bar{u}_g$ and $u_3 > \bar{u}_c > u_1 > \bar{u}_c$. There are two subcases.
   - If $u_2 > \bar{u}_g$, then the solution has structure of Fig. 5.7, similar to the structure in Fig. 5.3.
   - If $u_2 < \bar{u}_g$, then the solution has the structure in Fig. 5.8.

References