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Incompressibility and least-area surfaces

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Abstract

We show that if F is a smooth, closed, orientable surface embedded in a closed, orientable 3-manifold M such that for each Riemannian metric g on M, F is isotopic to a least-area surface $F(g)$, then F is incompressible.

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1. Introduction

We assume throughout that all manifolds (and surfaces) we consider are orientable. Let M be a closed, smooth, 3-manifold and let F be a smoothly embedded surface in M.

For a Riemannian metric g on M, we can seek to minimise the area of embedded surfaces in the homotopy class of F. Here, we say two embedded surfaces F and F' in M are homotopic if there is a homeomorphism $\varphi : F \to F'$, so that if $i_F : F \to M$ and $i_{F'} : F' \to M$ denote the inclusion maps, then i_F is homotopic to $i'_F \circ \varphi : F \to M$. We consider the functional

$$A(g, F) = \inf\{\text{Area}_g(F') : F' \text{embedded surface homotopic to} F\}.$$

Definition 1.1. A surface F is said to be *least area* with respect to the metric g if $\text{Area}_g(F) = A(g, F)$.

We recall the concept of *incompressibility* of the surface F.

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Definition 1.2. A closed, smoothly embedded surface $F \subset M$ in a 3-manifold M is said to be incompressible if the following conditions hold:

- If $D \subset M$ is a smoothly embedded 2-disc with $\partial D \subset F$ and $\operatorname{int}(D) \cap F = \phi$, then there is a disc $E \subset F$ with $\partial E = \partial D$.
- If F is a 2-sphere, then F does not bound a 3-ball.

A disc $D \subset M$ such that $\partial D \subset F$ and $\operatorname{int}(D) \cap F = \phi$ so that D is transversal to F is called a *compressing disc* if there is no disc $E \subset F$ with $\partial E = \partial D$.

If F is *incompressible* and M is *irreducible*, then a fundamental result [11] (see also [4]) is that for each metric g, the above infimum is attained for some smooth, embedded surface $F(g)$, i.e., there is a least area surface. Recall that M is said to be irreducible if every embedded 2-sphere $S \subset M$ bounds a 3-ball in M. Least area surfaces enjoys several very useful properties [2]—for instance, leading to the equivariant Dehn's lemma of Meeks and Yau [12].

We show here that, conversely, the property of having least area representatives for each Riemannian metric characterises incompressibility. Fix henceforth a closed, smooth, orientable surface F embedded in a closed, orientable, 3-manifold M.

Theorem 1.3. *Suppose for each Riemannian metric g on M, there is a smooth, embedded surface $F(g)$ homotopic to F such that* $\mathrm{Area}_g(F(g)) = A(g, F)$, *then F is incompressible.*

Our result gives a geometric characterisation of incompressiblity, which may be useful in proving incompressibility for surfaces constructed as limits. In particular, this result was motivated by Tao Li's proof [10] of the Waldhausen conjecture in the non-Haken case, where an incompressible surface was constructed as a limit of strongly irreducible Heegaard splittings.

Henceforth assume, without loss of generality, that F is connected. If F is a 2-sphere, either F is incompressible or F bounds a 3-ball in M. In the first case, there is nothing to prove. In the second case, F is homotopically trivial and hence homotopic to the boundary of any 3-ball in M. By considering the boundaries of arbitrarily small balls, we see that $A(g, F) = 0$ for any metric g. Thus there is no embedded surface $F(g)$ with $\mathrm{Area}_g(F(g)) = A(g, F)$. Thus, we can assume henceforth that F is not a 2-sphere.

Suppose F is not incompressible (and F is not a 2-sphere), then there is a compressing disc D for F. A regular neighbourhood of $F \cup D$ has two boundary components, one of which is parallel to F. Denote the other by F'. We call F' the result of compressing F along D. Observe that F is obtained from F' by adding a 1-handle. If F is a 2-sphere that bounds a 3-ball, we declare the empty set to be the result of compressing F.

Given any surface F, we can inductively define a sequence of compressions. Namely, if F is not incompressible, then we compress F along some compressing disc D to get F'. We repeat this process for each component of F' which is not compressible. As the maximum of the genus of the components of the surface F' obtained by compressing F is less than the genus of F, this process terminates after finitely many steps. The result is a (possibly empty) surface $\hat{F}$, each component of which is incompressible.

Suppose the result of the compressions is empty, then F is homotopic to the boundary of a handlebody, i.e., the boundary of a regular neighbourhood of a graph $\Gamma \subset M$.

By considering arbitrarily small neighbourhoods of Γ, we see that for any metric g, $A(g, F) = 0$. Thus there is no embedded surface $F(g)$ with $\mathrm{Area}_g(F(g)) = A(g, F)$, i.e., the hypothesis of the theorem cannot be satisfied.

Thus, we can, and do, assume henceforth that F is not homotopic to the boundary of a handlebody and $\hat{F}$ is not empty. Then F is obtained from $\hat{F}$ by addition of 1-handles. Given $\varepsilon > 0$, the 1-handles can be attached to $\hat{F}$ so that the area of the resulting surface, which is homotopic to F, is at most $\mathrm{Area}_g(\hat{F}) + \varepsilon$. It follows that

$$A(g, F) \leqslant \mathrm{Area}_g(\hat{F}).$$

We construct a metric g which is a warped product in a neighbourhood $N(\hat{F})$ of $\hat{F}$ so that any least-area surface not contained in $N(\hat{F})$ has area greater than the area of $\hat{F}$. Thus, if a least-area surface homotopic to F exists, it must be contained in $N(\hat{F})$. The structure of the metric on $N(\hat{F})$ together with some topological arguments show that this cannot happen unless $F = \hat{F}$, i.e., F is incompressible.

2. Construction of the metric

In this section we construct the desired metric for which F has no least-area representative unless $F = \hat{F}$.

As $\hat{F}$ and M are orientable, a regular neighbourhood $N(\hat{F})$ of $\hat{F}$ is a product. We shall identify this with $\hat{F} \times [-T, T]$, with T to be specified later. We shall also consider the regular neighbourhood $n(\hat{F}) = \hat{F} \times [-1, 1] \subset N(\hat{F})$.

Choose and fix a metric of constant curvature 1, 0 or -1 on each component of $\hat{F}$ and denote this g_0. Let the area of $\hat{F}$ with respect to g_0 be A_0.

We shall use the *monotonicity lemma* of Geometric measure theory. We state this below in the form we need. For a stronger result in the Riemannian case, see [3].

Lemma 2.1 (*Monotonicity lemma*). *There exist constants* $\varepsilon > 0$ *and* $R > 0$ *such that if g is a Riemannian metric on M and x is a point so that the sectional curvature of g on the ball* $B_g(x, R)$ *of radius R around x (with respect to g) has sectional curvature satisfying* $|K| \leqslant \varepsilon$ *and F is a least-area surface with* $x \in F$, *then* $\mathrm{Area}_g(F \cap B(x, R)) > A_0$.

We shall construct the desired metric in the following lemma.

Lemma 2.2. *There is a Riemannian metric g on M satisfying the following properties.*

- *On* $N(\hat{F})$, *g is of the form* $g = f(t)g_0 \oplus dt^2$, *with f a smooth function with* $f(0) = 1$ *and* $f(t) > 1$ *for* $t \neq 0$.
- *For* $x \in M - \mathrm{int}(N(\hat{F}))$, *the sectional curvature of g on M satisfies* $|K| \leqslant \varepsilon$.
- *For* $x \in M - \mathrm{int}(N\hat{F}))$, *the injectivity radius at x is greater than R.*

Proof. Observe that $N(\hat{F}) - \mathrm{int}(n(\hat{F}))$ has two components for each component $\hat{F}_0$ of $\hat{F}$, each of which can be identified with $\hat{F}_0 \times [0, 1]$ with $\hat{F} \times \{1\}$ a component of $\partial N(\hat{F})$ and $\hat{F} \times \{0\}$ a component of $\partial n(\hat{F})$. On each such component consider the product Riemannian metric $g_0 \oplus dt^2$. Extend this smoothly to a metric on the complement $M - \mathrm{int}(n(\hat{F}))$ of the interior of $n(\hat{F})$. Rescale the metric by a constant $s > 1$ to ensure that it has sectional curvature satisfying $|K| \leqslant \varepsilon$ and the injectivity radius at each point outside $\mathrm{int}(N(\hat{F}))$ is at least R. We choose the constant s to be greater than 1 even if this is not necessary to ensure the bounds on curvature and the injectivity radius. We denote the rescaled metric, defined on $M - \mathrm{int}(n(\hat{F}))$, by g. The restriction of g to each component of $N(T)$ can be identified with the product metric $sg_0 + dt^2$.

Let $T = 1 + s$. Then there is a natural identification of $N(\hat{F})$ with $\hat{F} \times [-T, T]$, with $n(\hat{F})$ identified with $\hat{F} \times [-1, 1]$ and with the restriction of the metric g to $N(\hat{F}) - \mathrm{int}(n(\hat{F}))$ given by $sg_0 \oplus dt^2$.

We extend the constant function $f(t) = s$ on $[-T, -1] \cup [1, T]$ to a smooth function on $[-T, T]$ with $f(0) = 1$ and $f(t) > 1$ if $t \neq 0$. The Riemannian metric on the complement of $n(\hat{F})$ extends smoothly to one given by $g = f(t)g_0 \oplus dt^2$ on $N(\hat{F})$. This satisfies all the conditions of the lemma. □

Note that by construction $\hat{F} \times \{0\}$ is isometric to $\hat{F}$ with the metric g_0. Further the projection map $p : F \times [-T, T] \to F$ is (weakly) distance decreasing, and strictly distance decreasing outside $\hat{F} \times \{0\}$.

3. Proof of incompressibility

Suppose now that there is a surface $F(g)$ homotopic to F with area $A(g, F)$.

Lemma 3.1. *We have $F(g) \subset \hat{F}_0 \times (-T, T)$ for some component $\hat{F}_0$ of $\hat{F}$.*

Proof. If there is a point $x \in F(g) - \hat{F} \times (-T, T)$, the monotonicity lemma applied to $F(g) \cap B(x, R)$ shows that the area of $F(g)$ is greater than A_0, a contradiction. Further, as $F(g)$ is connected, for some component $\hat{F}_0$ of $\hat{F}$, $F(g) \subset \hat{F}_0 \times [-T, T]$. □

To simplify notation, we henceforth denote the surface $F(g)$ by F. We consider the restriction of the projection map from $F_0 \times [-T, T]$ to F, which we denote, by abuse of notation, by $p : F \to \hat{F}_0$. We have seen that this is strictly distance decreasing unless $F = \hat{F}_0$.

We have two cases, depending on whether the embedded surface $F \subset \hat{F}_0 \times (-T, T)$ separates the boundary components of $\hat{F}_0 \times [-T, T]$.

In case F does not separate the boundary components of $\hat{F}_0 \times [-T, T]$, there is a curve γ joining the boundary components disjoint from F. By considering cup products, it follows that $p : F \to \hat{F}_0$ has degree zero. On the other hand, if F does separate the boundary components of $\hat{F}_0 \times [-T, T]$, as F is connected there is a curve γ joining the boundary components intersecting F transversely in one point. It follows that we can choose an orientation on F so that $p : F \to \hat{F}_0$ has degree one.

Recall that F is a connected, orientable surface that is not a 2-sphere. As $\hat{F}_0 \times (-T, T)$ deformation retracts to $\hat{F}_0$, the homotopy class of the inclusion map is determined by the homotopy class of p. If $\hat{F}_0$ is not a 2-sphere, then the homotopy class of p is determined by the induced map on fundamental groups. If $\hat{F}_0$ is a 2-sphere, then the homotopy class of p is determined by the degree of p.

We show first that the case where F does not separate the boundary components of $\hat{F}_0 \times [-T, T]$ cannot occur.

Lemma 3.2. *The surface F must separate the boundary components of* $\hat{F}_0 \times [-T, T]$.

Proof. We have seen that $p : F \to \hat{F}_0$ has degree zero. Suppose first that $\hat{F}_0$ is not a 2-sphere. By a theorem of Hopf and Knesser [5,6,8,9], it follows that p is homotopic to a map whose image does not contain some point $p \in F_0$. It follows that $G = p_*(\pi_1(F))$ (which is finitely generated) is conjugate to the subgroup of a free group and hence is a finitely generated free group. The projection $p : \hat{F}_0 \times [-T, T] \to \hat{F}_0$ induces an isomorphism of fundamental groups. This gives an identification of G with the image of $\pi_1(F)$ under the homomorphism induced by the inclusion $i : F \to \hat{F}_0 \times [-T, T]$.

By a theorem of Jaco [7], there is a finite graph Γ and $\pi_1(\Gamma)$ isomorphic to G and a map $f : F \to \Gamma$ so that the mapping cylinder $M(f)$ of f is a handlebody. Moreover, we have an identification of $\pi_1(\Gamma)$ with G with respect to which the homomorphism from $\pi_1(F)$ to $G \subset \pi_1(\hat{F}_0 \times [-T, T])$ induced by the inclusion corresponds to the map induced by inclusion from $\pi_1(F)$ to $\pi_1(M(f)) = \pi_1(\Gamma)$.

Choose an embedding of Γ in $\hat{F}_0 \times [-T, T]$ with induced map on fundamental groups $\pi_1(\Gamma) \to G \subset \pi_1(\hat{F}_0 \times [-T, T])$ corresponding to the above identification of $\pi_1(\Gamma)$ with G. Then the surface F is homotopic to the boundary of a regular neighbourhood of the image of Γ, which is a handlebody in $\hat{F}_0 \times [-T, T]$. We have seen that in this case the hypothesis of the theorem cannot be satisfied.

Finally consider the case when $\hat{F}_0$ is a 2-sphere. As p has degree zero, p is homotopic to a constant map. Hence the inclusion map $i : F \to \hat{F}_0 \times (-T, T)$ is also homotopic to a constant map. It follows that F is homotopic to the boundary of a handlebody. As above, the hypothesis of the theorem cannot be satisfied in this case. □

It thus suffices to consider the case when F does separate the boundary components of $\hat{F}_0 \times [-T, T]$.

Lemma 3.3. *Suppose F does separate the boundary components of* $\hat{F}_0 \times [-T, T]$, *then* $F = \hat{F}_0$.

Proof. We use a theorem of Edmonds [1] regarding degree-one maps $\varphi : F \to F'$ between closed surfaces. Namely, there is a map ψ homotopic to φ, a compact, connected subsurface Σ in F and a disc $D \subset F'$ such that $\psi(\Sigma) \subset D$ and ψ maps $F - \mathrm{int}(\Sigma)$ homeomorphically onto $F' - \mathrm{int}(D)$. We can regard F as obtained from F' by attaching 1-handles to D. Further, all the cores and co-cores of these 1-handles are mapped to homotopically trivial curves by ψ.

It follows that an embedded surface homotopic to F is obtained from $\hat{F}_0$ by adding 1-handles (corresponding to those required to obtain F from $\hat{F}_0$). Thus, if A_1 is the area of $\hat{F}_0 \times \{0\}$, then $A(g, F) \leqslant A_1$, hence $\mathrm{Area}_g(F) \leqslant A_1$.

Let $\mathrm{d}\omega$ and $\mathrm{d}A$ denote the area forms on $\hat{F}_0 = \hat{F}_0 \times \{0\}$ and F, respectively. Then for a smooth function h on F, $p^*(\mathrm{d}\omega) = h\,\mathrm{d}A$. As the projection map is distance-decreasing, $h(p) \leqslant 1$ for all p, with equality at all points only in the case where $F = \hat{F}_0$.

Observe that $A_1 = \int_{\hat{F}_0} \mathrm{d}\omega = \int_F p^*(\mathrm{d}\omega) = \int_F h\, \mathrm{d}A$, where the second equality holds as p has degree one. Further, if $F \neq \hat{F}_0$, then $\int_F h\, \mathrm{d}A < \int_F \mathrm{d}A = \mathrm{Area}(F)$. Thus, $A_1 < \mathrm{Area}_g(F)$, a contradiction.

It follows that $F = \hat{F}_0$. □

Thus, the surface F must be homotopic to $\hat{F}_0$, which is incompressible. Recall that we can assume that F is not a 2-sphere. It follows that F is incompressible by the characterisation of incompressible surfaces as those for which the induced map on fundamental groups is injective (if F is not the 2-sphere). This contradicts our assumption that F is compressible, completing the proof of the theorem. □

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