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Investigation of clustered actuation in tensegrity structures

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ABSTRACT

As tensegrity research is moving away from static structures toward active structures it is becoming critical that new actuation strategies and comprehensive active structures theories are developed to fully exploit the properties of tensegrity structures. In this paper a new general tensegrity paradigm is presented that incorporates a concept referred to as clustered actuation. Clustered actuation exploits the existence of cable elements in a tensegrity structure by allowing cables to be run over frictionless pulleys or through frictionless loops at the nodes. This actuation strategy is a scalable solution that can be utilized for active structures that incorporate many active elements and can reduce the number of actuators necessary for complex shape changes. Clustered actuation also has secondary benefits, specifically reducing the force requirements of actuators in dynamic structures, reducing the number of pre-stress modes to potentially one global mode and relieving element size limitations that occur with embedded actuation. Newly formulated clustered equilibrium equations are developed using energy methods and are shown to be a generalization of the classic tensegrity governing equations. Pre-stress analysis, mechanism analysis and stability of clustered structures are discussed. Lastly, examples compare the mechanics of a clustered structure to an equivalent classic structure and the utility of clustering is highlighted by allowing for actuation throughout a class 1 (no bar-to-bar connections) tensegrity while not embedding the actuators into the structure.

1. Introduction and motivation

Research into active structures has been at the forefront of aerospace, mechanical and civil engineering in recent years. Tensegrity structures (Fig. 1a) are well suited as active structures and, more specifically, they are advantageous for large amplitude, low inertia applications. Tensegrity structures are a subset of pin-jointed truss structures of the statically indeterminate type (Pellegrino and Calladine, 1986). The integrity of a tensegrity structure is due to the balance of tensional forces in the cables with compressive forces in the struts. The cable elements are unilaterally rigid meaning that they cannot carry a compressive load due to buckling instability, but they give tensegrity systems high strength-to-weight ratios by reducing the structural mass. This can reduce the energetic cost to accelerate the system compared to conventional pin-jointed frameworks.

The goal of this paper is to present an actuation strategy for tensegrity structures that is *scalable*, from active structures with a few active elements (<10) to active structures with many active elements (>10). This strategy is known as clustered actuation which is the actuation of a cable element *cluster*. A cluster is a group of individual cable elements in a structure that are combined into one continuous cable element that runs over frictionless pulleys or through frictionless loops at the nodes (Fig. 1b). The concept of clustered actuation has been investigated in the context of deployable pantographic structures (Kwan et al., 1993; Kwan and Pellegrino, 1994; You and Pellegrino, 1996, 1997) and in some deployable tensegrity structures (Smaili and Motro, 2005) as a method for the application and removal of self-stress to deploy and collapse a structure. This current study formulates the potential energy, equilibrium equations and stiffness matrix for a general tensegrity structure with clustered actuation and it is shown that the classic tensegrity equations are a subset of the clustered tensegrity equilibrium equations.

Currently there are two actuation strategies for introducing actuators into a tensegrity system. The first strategy, which will be referred to as *embedded actuation*, directly places actuators into the structure by replacing individual elements of the structure and has been used by many researchers (Moored and Bart-Smith, 2007; Djouadi et al., 1998; Fest et al., 2004). Embedded actuation is the simplest solution to envision and has the benefit of working in any type of tensegrity structure. However, embedded actuation has many limitations. As an active prototype with a few active elements is scaled up to an active structure with many active elements the number of necessary embedded actuators scales up as well. Thus this strategy becomes impractical due to increased controls complexity, energy consumption, added mass, and cost.

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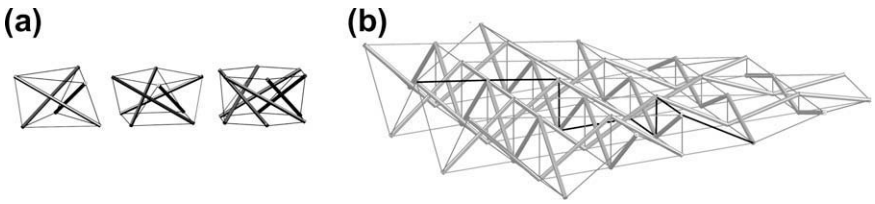

Fig. 1. (a) A three strut, a four strut and a six strut tensegrity prism, and (b) a tensegrity wing structure with a cluster (black) routed through the structure.

Moreover, the element sizes are constrained by the size of the actuators and in dynamic actuation the added inertia increases the power consumption. The second actuation strategy which will be referred to as *strut-routed actuation* allows an actuator to attach to an active element by running a cable through a series of connected struts that make a path from the active element to the actuator location (Motro, 2003). This strategy has the advantage of migrating the actuators outside of the structure to reverse the added mass and element size constraints of embedded actuation. This strategy, though, has two main limitations. First, when this strategy is applied to active structures with many active elements it quickly becomes infeasible to fabricate since numerous active cables are routed through the same strut paths and spacing becomes critical. Secondly, strut-routed actuation only works for tensegrity structures with bar-to-bar connections. In small-scale structures (<30 elements) this limitation tends not to matter, but for large-scale (>30 elements) class 1 structures[1] there is a localized region where actuation can be applied leaving regions of the structure that cannot be reached using a strut-routed actuation strategy.

Embedded actuation and strut-routed actuation do not significantly change the mechanics of the classical tensegrity governing equations. They only add mass to the members in the case of embedded actuation or compressive loads to the struts in the case of strut-routed actuation, but the response can be predicted with the classic tensegrity equations. However, as soon as two cables are clustered together there can be a dramatic shift in the mechanics of a tensegrity structure. Specifically the number of self-stress states can decrease and the mechanisms can increase. Thus new governing equations must be developed that include the mechanics of clustered elements and the implications must be explored.

This paper is organized in the following manner. Section 2 builds up the definitions that are used in clustered equilibrium and stability. Section 3 follows the energy approach to derive the potential energy of a clustered active tensegrity structure. Section 4 derives the equilibrium equations from the potential energy and discusses the physical interpretations of clustered actuation. Section 5 discusses the pre-stress and mechanism analysis of clustered tensegrity structures. Section 6 presents the clustered stiffness matrix and discusses the stability of a clustered structure. Section 7 gives examples highlighting the change in the structural response and the utility of clustered actuation. Since clustered equilibrium is a generalization of classical equilibrium then all results and equations are valid for any classical tensegrity structure in the classic limit, which will be shown.

2. Definitions

Much of the tensegrity notation and many of the definitions come from Masic et al. (2005). However, to mitigate any confusion, the complete set of relevant definitions, notation and identities are discussed in this section.

[1] Class k structures have at most k struts attached to any node of the structure (Williamson et al., 2003).

2.1. Geometry

The mathematical tools to describe the configuration of a tensegrity structure must first be defined.

Definition 2.1. $\mathbf{p} \in \mathbb{R}^{3n\times 1}$ is a vector of the nodal point positions, where n is the number of nodes in the structure. For a Cartesian coordinate system $\mathbf{p} = [\mathbf{p_1}\,\mathbf{p_2}\cdots\mathbf{p_n}]^{\mathbf{T}} = [x_1\,y_1\,z_1\,x_2\,y_2\,z_2\cdots x_n\,y_n\,z_n]^{\mathbf{T}}$.

Given the nodal points of the structure, the set of elements for a particular structural configuration can be defined. First, though, the element types must be defined.

Definition 2.2. $\mathbf{z} \in \mathbb{R}^{e\times 1}$ is an element identifier vector where, e, is the number of elements and

$$z_i = \begin{cases} 1 & \text{if element } i \text{ is a cable} \\ -1 & \text{if element } i \text{ is a strut} \end{cases}$$

Definition 2.3. $e_i = \{[j,k], z_i\}$ is an element in a specific structure, where j and k are number identifiers of the nodes. The element set $\mathbb{E}(\mathbf{e})$ can be broken into two sets, $\mathbb{E}_c$ which is the set of cables for a structure and $\mathbb{E}_s$ which is the set of struts for a structure.

Definition 2.4. The connectivity matrix, $\mathbf{C}^1 \in \mathbb{R}^{n\times e}$ is a matrix parameterization of the element set of a tensegrity structure. The matrix defines the element-node connectivity. For every element in the structure a direction vector, $\boldsymbol{\nu}_\mathbf{i}$ should be drawn along its axis, where the direction is arbitrary but must remain consistent throughout analysis.

$$C^1_{ji} = \begin{cases} z_i & \text{if the vector } \boldsymbol{\nu}_\mathbf{i} \text{ of element } i \text{ radiates from node } j \\ -z_i & \text{if the vector } \boldsymbol{\nu}_\mathbf{i} \text{ of element } i \text{ terminates at node } j \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, $\mathbf{C} \in \mathbb{R}^{3n\times 3e}$ and $\mathbf{C} = \mathbf{C^1} \otimes \mathbf{I_d}$, where $\otimes$ is the Kronecker tensor product and $\mathbf{I_d}$ is an identity matrix of size $\mathbf{d}$ which is the dimension of the space, typically 2 or 3.

Definition 2.5. The mapping matrix, $\mathbf{M} \in \mathbb{R}^{3e\times 3n}$, maps the nodal point positions to the projection of the element lengths in each Cartesian direction, $\mathbf{g}$.

$$\mathbf{g} = \mathbf{M}\mathbf{p} \tag{1}$$

The mapping matrix is found from the connectivity matrix and the member identifier vector.

$$\mathbf{M} = \hat{\mathbf{z}}\mathbf{C}^{\mathbf{T}}$$

The *hat* operator, $\hat{(\cdot)}$, is defined as $\hat{(\cdot)} = \tilde{(\cdot)} \otimes \mathbf{I_d}$. The *tilde* operator, $\tilde{(\cdot)}$, is defined as the block diagonal of a vector, *blockdiag* $(\cdot)$ when each entry has multiple components associated with it. Otherwise, the *tilde* operator reduces to a diagonal operator.

The element lengths, $\mathbf{l}$, can now be written in either an Euclidean form or a quadratic form.

$$\mathbf{l} = \sqrt{\tilde{\mathbf{g}}^{\mathbf{T}}\mathbf{g}}, \quad \tilde{\mathbf{l}}\mathbf{l} = \tilde{\mathbf{g}}^{\mathbf{T}}\mathbf{g} \tag{2}$$

Since a tensegrity structure is required to always be in a state of self-stress to keep its integrity, there is a vector of element rest lengths or manufacturing lengths, $\mathbf{l_0}$, that must be shorter than the element lengths for cables and are typically longer for struts. The rest lengths are chosen based on the pre-stress analysis of a structure discussed in Section 5.

2.2. Material properties

The material properties for a tensegrity structure can be fully defined with the element stiffness vector.

Definition 2.6. The element stiffness vector, $\mathbf{k}$, is composed of the element's Young's modulus, $\mathbf{y}$, cross-sectional area, $\mathbf{a}$ and rest length, $\mathbf{l_0}$.

$$\mathbf{k} = \tilde{\mathbf{y}}\tilde{\mathbf{l}}_{\mathbf{0}}^{-1}\mathbf{a} \tag{3}$$

In this paper a linear elastic constitutive law is assumed.

2.3. Force densities

Schek (1974) first proposed describing the equilibrium of a cable-net structure in terms of the force densities in the elements instead of the forces in the elements. Later, Vassart and Motro (1999) introduced the force density method as a form finding method for tensegrity structures. The force density substitution converts the appearance of the geometrically nonlinear equilibrium equations into a bilinear form. The bilinear form can be truly utilized in form finding, but for the purposes of this paper, the substitution allows for the equilibrium equations to be cast into a compact form.

Definition 2.7. The force density in an element is the force in that element divided by its length.

$$\boldsymbol{\lambda} = \tilde{\mathbf{z}}\tilde{\mathbf{k}}\tilde{\mathbf{l}}^{-1}(\mathbf{l}-\mathbf{l_0}) = \tilde{\mathbf{z}}\tilde{\mathbf{y}}\tilde{\mathbf{a}}\tilde{\mathbf{l}}_{\mathbf{0}}^{-1}\tilde{\mathbf{l}}^{-1}(\mathbf{l}-\mathbf{l_0}) \tag{4}$$

Definition 2.8. The stress in an element can be defined from the force density.

$$\boldsymbol{\sigma} = \tilde{\mathbf{l}}\tilde{\mathbf{a}}^{-1}\boldsymbol{\lambda} \tag{5}$$

Definition 2.9. The contribution of force density that an actuator supplies to a structure is defined as the force applied by the actuator divided by the current length of the element the actuator is acting on.

$$\boldsymbol{\lambda}_{\mathbf{a}} = \tilde{\mathbf{z}}\tilde{\mathbf{k}}\tilde{\mathbf{l}}^{-1}\tilde{\mathbf{l}}_{\mathbf{0}}\boldsymbol{\epsilon}_{\mathbf{a}} = \tilde{\mathbf{z}}\tilde{\mathbf{y}}\tilde{\mathbf{a}}\tilde{\mathbf{l}}^{-1}\boldsymbol{\epsilon}_{\mathbf{a}} \tag{6}$$

Current actuator technology falls into two broad categories: force actuators and displacement actuators. With force actuators the force, $\mathbf{f_a}$, is directly controlled. The actuator force can be normalized by the current length of the element that the actuator is working on to define the actuator force density, $\boldsymbol{\lambda}_{\mathbf{a}}$. For displacement actuators the displacement, $\mathbf{u_a}$, is the controlled variable. The actuator displacement can be normalized by the rest length of the element that it is acting on to define an actuator strain, $\boldsymbol{\epsilon}_{\mathbf{a}} = \tilde{\mathbf{l}}_{\mathbf{0}}^{-1}\mathbf{u_a}$. An actuator force density can also be defined using the actuator strain.

2.4. Clustered actuation

Clustered actuation is a strategy that can only be applied to the cables in a structural system. When two elements in a structure are theoretically clustered together they become a single element that is assumed to run frictionlessly through a nodal point (Fig. 2). This can be achieved by having a cable element wrap once around a pulley pinned to a node while allowing the cable to exit in two directions thereby replacing two cables in the structure with one. To fully define the state of a clustered tensegrity structure new information is supplied in the form of the clustering matrix.

Definition 2.10. The clustering matrix must follow two rules: (1) only cable elements are allowed to be clustered and (2) only adjacent elements are allowed to be clustered.[2] The clustering matrix, $\mathbf{S} \in \mathbb{R}^{\bar{e}\times e}$, relates the clustered set of elements, $\bar{e}$, to the classic set of elements, e,

$$S_{ij} = \begin{cases} 1 & \text{if the clustered element } \bar{e}_i \text{ is composed} \\ & \quad \text{of the classic element } e_j \\ 0 & \text{otherwise} \end{cases}$$

Fig. 2 shows an example of applying Definition 2.10 to parameterize a clustering matrix for a planar tensegrity structure with a cluster being composed of elements 1 and 3. As can be seen from the example, if no cable elements were clustered in the planar structure then the clustering matrix would identically be equal to the identity matrix, $\mathbf{I} \in \mathbb{R}^{e\times e}$. Therefore the classical limit is when $\mathbf{S} \equiv \mathbf{I}$.

Clustering changes the mechanics of a tensegrity structure. There are two physical interpretations for understanding how the mechanics are affected by clustering: (1) the kinematic perspective and (2) the static perspective. Each perspective alludes to a different mathematical condition that can be used to derive the clustered equilibrium equations.

The kinematic perspective views every member in a tensegrity structure as a kinematic constraint applied to a node. For every cable element that is added to a cluster, a kinematic constraint on the system is lost. A reduction in the number of constraints can change or create mechanisms that may or may not be stabilized by a state of pre-stress. This concept facilitates the need to have accurate equilibrium equations for pre-stress analysis. The kinematic condition converts the vector of classic lengths, $\mathbf{l}$, to a reduced set of clustered lengths, $\bar{\mathbf{l}}$, through the clustering matrix $\mathbf{S}$.

$$\bar{\mathbf{l}} = \mathbf{S}\mathbf{l} \tag{7}$$

Equivalently the rest lengths of the members may be defined in the same way.

$$\bar{\mathbf{l}}_{\mathbf{0}} = \mathbf{S}\mathbf{l}_{\mathbf{0}} \tag{8}$$

The second physical interpretation of clustering is that the forces in the elements that are clustered must equilibrate to the same tension, assuming no friction at the joints. This static interpretation can be written into the following mathematical condition where the forces in the elements of the clustered structure, $\bar{\mathbf{f}}$, are related to the equivalent forces in the elements, $\mathbf{f}$, of the equivalent classic structure through the transpose of the clustering matrix, $\mathbf{S}^{\mathbf{T}}$.

$$\mathbf{f} = \mathbf{S}^{\mathbf{T}}\bar{\mathbf{f}} \tag{9}$$

The kinematic condition (7) and the static condition (9) convert classic variables into clustered variables and give a physical interpretation of clustering, but the other element vectors (e.g. Young's modulus vector) also need to be reduced in size. The mappings from classic to clustered element vectors are the following.

[2] If part of a cluster is routed through a bar then the connectivity matrix should reflect the existence of a cable element occupying the same space as the bar.

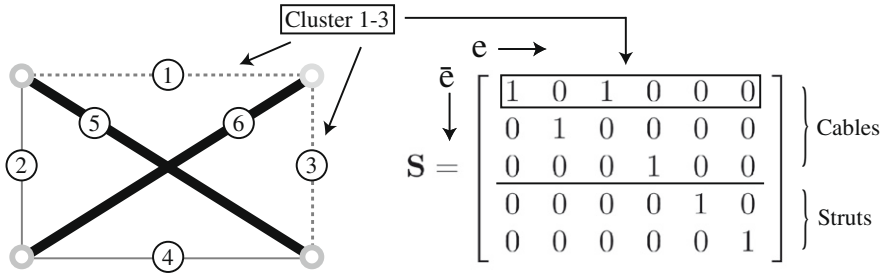

Fig. 2. Example shows the clustering matrix for a planar tensegrity with one cluster. The thick solid lines are the struts, the thin solid lines are discrete cables and the thin dashed line is the cluster.

$$\begin{aligned}\bar{\mathbf{z}} &= (\mathbf{S}\mathbf{S}^{\mathbf{T}})^{-1}\mathbf{S}\mathbf{z}\\ \bar{\mathbf{y}} &= (\mathbf{S}\mathbf{S}^{\mathbf{T}})^{-1}\mathbf{S}\mathbf{y}\\ \bar{\mathbf{a}} &= (\mathbf{S}\mathbf{S}^{\mathbf{T}})^{-1}\mathbf{S}\mathbf{a}\\ \bar{\epsilon}_{\mathbf{a}} &= (\mathbf{S}\mathbf{S}^{\mathbf{T}})^{-1}\mathbf{S}\epsilon_{\mathbf{a}}\end{aligned} \tag{10}$$

These mappings average the values of the properties which in practice is usually not of consequence since the cables of a tensegrity structure tend to be the same diameter and modulus. Also, the vector of clustered element stiffnesses can be written as, $\bar{\mathbf{k}} = \tilde{\bar{\mathbf{y}}}\tilde{\bar{\mathbf{l}}}_{\mathbf{0}}^{-1}\bar{\mathbf{a}}$.

Definition 2.11. A clustered tensegrity structure is defined as equivalent to a classic tensegrity structure when the properties of the clustered structure are found from (7) to (10) and the clustering matrix is arbitrary.

Definition 2.12. A classic tensegrity structure is defined as equivalent to a clustered tensegrity structure when the properties of the classic structure are found by setting $\mathbf{S} \equiv \mathbf{I}$ in all of the property definitions of a clustered tensegrity structure.

Definition 2.13. The vector of force densities for a clustered structure is the vector of clustered forces in the elements divided by the vector of clustered lengths.

$$\bar{\boldsymbol{\lambda}} = \tilde{\bar{\mathbf{z}}}\tilde{\bar{\mathbf{k}}}\tilde{\bar{\mathbf{l}}}^{-1}(\bar{\mathbf{l}} - \bar{\mathbf{l}}_{\mathbf{0}}) = \tilde{\bar{\mathbf{z}}}\tilde{\bar{\mathbf{y}}}\tilde{\bar{\mathbf{a}}}\tilde{\bar{\mathbf{l}}}_{\mathbf{0}}^{-1}\tilde{\bar{\mathbf{l}}}^{-1}(\bar{\mathbf{l}} - \bar{\mathbf{l}}_{\mathbf{0}}) \tag{11}$$

Definition 2.14. The contribution of force density that an actuator supplies can be redefined in a similar way.

$$\bar{\boldsymbol{\lambda}}_{\mathbf{a}} = \tilde{\bar{\mathbf{z}}}\tilde{\bar{\mathbf{k}}}\tilde{\bar{\mathbf{l}}}^{-1}\tilde{\bar{\mathbf{l}}}_{\mathbf{0}}\bar{\epsilon}_{\mathbf{a}} = \tilde{\bar{\mathbf{z}}}\tilde{\bar{\mathbf{y}}}\tilde{\bar{\mathbf{a}}}\tilde{\bar{\mathbf{l}}}^{-1}\bar{\epsilon}_{\mathbf{a}}. \tag{12}$$

3. Energy approach

To develop the clustered tensegrity equilibrium equations there are many approaches that can be taken. The energy approach is utilized in this paper to derive these equations and to determine if a given equilibrium state is stable. In general, the total potential energy of a system can be defined for an infinitely small neighborhood around a given state of the system by a Taylor series expansion.

$$\begin{aligned}&\Pi_{sys}(\mathbf{p_1} + \boldsymbol{\delta}\mathbf{p_1}, \mathbf{p_2} + \boldsymbol{\delta}\mathbf{p_2}, \ldots, \mathbf{p_n} + \boldsymbol{\delta}\mathbf{p_n})\\ &\quad = \Pi_{sys}(\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}) + \delta\Pi_{sys} + \delta^2\Pi_{sys} + \cdots\end{aligned} \tag{13}$$

where the first variation of the potential energy is,

$$\delta\Pi_{sys} = \frac{1}{1!}\sum_{i=1}^{n}\frac{\partial\Pi_{sys}(\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n})}{\partial\mathbf{p_i}}\boldsymbol{\delta}\mathbf{p_i} \tag{14}$$

and the second variation is,

$$\delta^2\Pi_{sys} = \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\Pi_{sys}(\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n})}{\partial\mathbf{p_i}\partial\mathbf{p_j}}\boldsymbol{\delta}\mathbf{p_i}\boldsymbol{\delta}\mathbf{p_j} \tag{15}$$

An arbitrarily small virtual displacement, $\boldsymbol{\delta}\mathbf{p_i}$, is applied to the system and equilibrium occurs when the slope of the potential energy of the system with respect to the nodal positions is equal to zero (Bazant and Cedolin, 2003).

$$\delta\Pi_{sys} = 0 \text{ for any } \boldsymbol{\delta}\mathbf{p_i} \qquad \text{which gives,} \quad \frac{\partial\Pi_{sys}}{\partial\mathbf{p}} = \mathbf{0} \tag{16}$$

Also for tensegrity structures in equilibrium the cable elements must not go into compression. The stability of the equilibrium points can be determined by using the second variation of the potential energy. In general, the second variation of the potential energy must be positive definite for a structure to be stable.

$$\delta^2\Pi_{sys} > 0 \text{ for any } \boldsymbol{\delta}\mathbf{p_i} \qquad \text{which gives,} \quad \frac{\partial^2\Pi_{sys}}{\partial\mathbf{p}^2} > \mathbf{0} \tag{17}$$

The total potential energy of the entire system is the sum of the internal strain energy of the structure, U, the work done by active elements, W_a, and the external loads, W_e, (Bazant and Cedolin, 2003).

$$\Pi_{sys} = U - W_a - W_e \tag{18}$$

The potential energy of a discrete system (finite number of degrees of freedom) can be written as the sum of the energy contributions of each element or load. The ones vector, $\mathbf{1}$, replaces the summation to convert the statement for the potential energy into a full matrix representation.

$$\begin{aligned}U - W_a - W_e &= \sum\mathbf{U_{ele}} - \sum\mathbf{W_a} - \sum\mathbf{W_e}\\ &= \mathbf{1}^{\mathbf{T}}\mathbf{U_{ele}} - \mathbf{1}^{\mathbf{T}}\mathbf{W_a} - \mathbf{1}^{\mathbf{T}}\mathbf{W_e}\end{aligned} \tag{19}$$

The potential energy of each element is, in general, the summation of all of the differential contributions of the work done to move a point from a to b, where the differential of work done is the inner product of force and a differential displacement along the path from a to b.

$$\Pi_{sys} = \mathbf{1}^{\mathbf{T}}\int_{\mathbf{a}}^{\mathbf{b}}\tilde{\bar{\mathbf{f}}}_{\mathbf{ele}}\mathbf{du} - \mathbf{1}^{\mathbf{T}}\int_{\mathbf{a}}^{\mathbf{b}}\tilde{\bar{\mathbf{f}}}_{\mathbf{a}}\mathbf{du} - \mathbf{1}^{\mathbf{T}}\int_{\mathbf{a}}^{\mathbf{b}}\tilde{\bar{\mathbf{f}}}_{\mathbf{e}}\mathbf{du} \tag{20}$$

The vector, $\bar{\mathbf{f}}_{\mathbf{ele}}$, is composed of the element forces, $\bar{\mathbf{f}}_{\mathbf{a}}$ is a vector of the actuator forces and $\bar{\mathbf{f}}_{\mathbf{e}}$ is a vector of the external loads. These variables are defined for a tensegrity structure with clustered elements.

At this point, some assumptions can be made for tensegrity structures. The first assumption is that the structures described are pin-jointed structures such that bending stresses do not develop within the elements. This leads to the first two integrals being constrained to displacements only along the axis of the elements

(i.e. uniform element strains). The second assumption is that the elements of the structure exhibit a linear elastic constitutive law. Lastly, since the actuators are in series with structural elements, the actuator forces and element forces are equivalent. When these assumptions are applied, the integral equations are changed to the following.

$$\Pi_{sys} = \mathbf{1}^{\mathbf{T}} \int^{\bar{\epsilon}} \tilde{\tilde{\mathbf{k}}} \tilde{\mathbf{u}} \mathbf{du} - \mathbf{1}^{\mathbf{T}} \int_{\mathbf{a}}^{\bar{\epsilon}} \tilde{\tilde{\mathbf{k}}} \tilde{\mathbf{u}} \mathbf{du_a} - \int^{\mathbf{p}} \bar{\mathbf{f}}_{\mathbf{e}}^{\mathbf{T}} \mathbf{dp} \tag{21}$$

The vector, $\mathbf{u}$, is composed of the displacements of the elements acting along the element axis. In general, the actuator displacements are not equivalent to the element displacements due to the strain within the elements. The vector, $\bar{\epsilon}$, is composed of the element strains. Furthermore, $\mathbf{u} = \tilde{\tilde{\mathbf{l}}}_{\mathbf{0}} \bar{\epsilon}$ is substituted.

$$\Pi_{sys} = \mathbf{1}^{\mathbf{T}} \int^{\bar{\epsilon}} \tilde{\tilde{\mathbf{k}}} \tilde{\tilde{\mathbf{l}}}_{\mathbf{0}}^{\mathbf{2}} \tilde{\bar{\epsilon}} \mathbf{d}\bar{\epsilon} - \mathbf{1}^{\mathbf{T}} \int_{\mathbf{a}}^{\bar{\epsilon}} \tilde{\tilde{\mathbf{k}}} \tilde{\tilde{\mathbf{l}}}_{\mathbf{0}}^{\mathbf{2}} \tilde{\bar{\epsilon}} \mathbf{d}\bar{\epsilon}_{\mathbf{a}} - \int^{\mathbf{p}} \bar{\mathbf{f}}_{\mathbf{e}}^{\mathbf{T}} \mathbf{dp} \tag{22}$$

Evaluating the integrals and applying $\bar{\epsilon} = \tilde{\tilde{\mathbf{l}}}_{\mathbf{0}}^{-\mathbf{1}} (\bar{\mathbf{l}} - \bar{\mathbf{l}}_{\mathbf{0}})$ gives the following relations for the potential energy of the system, the strain energy of the clustered structure, the work done by the actuators and the work done by the external forces.

$$\Pi_{sys} = \frac{1}{2} (\bar{\mathbf{l}} - \bar{\mathbf{l}}_{\mathbf{0}})^{\mathbf{T}} \tilde{\tilde{\mathbf{k}}} (\bar{\mathbf{l}} - \bar{\mathbf{l}}_{\mathbf{0}}) - \bar{\epsilon}_{\mathbf{a}}^{\mathbf{T}} \tilde{\tilde{\mathbf{l}}}_{\mathbf{0}} \tilde{\tilde{\mathbf{k}}} (\bar{\mathbf{l}} - \bar{\mathbf{l}}_{\mathbf{0}}) - \int^{\mathbf{p}} \mathbf{f}_{\mathbf{e}}^{\mathbf{T}} \mathbf{dp} \tag{23}$$

Determining the clustered equilibrium equations from the equilibrium condition is shown in Section 4 and the stability condition is discussed in Section 6.

4. Clustered equilibrium with active elements

Setting the first variation of the potential energy to zero satisfies the equilibrium condition.

$$\frac{\partial \Pi_{sys}}{\partial \mathbf{p}} = (\bar{\mathbf{l}} - \bar{\mathbf{l}}_{\mathbf{0}})^{\mathbf{T}} \tilde{\tilde{\mathbf{k}}} \frac{\partial \bar{\mathbf{l}}}{\partial \mathbf{p}} - \bar{\epsilon}_{\mathbf{a}}^{\mathbf{T}} \tilde{\tilde{\mathbf{l}}}_{\mathbf{0}} \tilde{\tilde{\mathbf{k}}} \frac{\partial \bar{\mathbf{l}}}{\partial \mathbf{p}} - \mathbf{f}_{\mathbf{e}}^{\mathbf{T}}(\mathbf{p}) = \mathbf{0}^{\mathbf{T}} \tag{24}$$

where

$$\frac{\partial \bar{\mathbf{l}}}{\partial \mathbf{p}} = \mathbf{S} \frac{\partial \mathbf{l}}{\partial \mathbf{p}} = \mathbf{S} \tilde{\mathbf{l}}^{-\mathbf{1}} \tilde{\mathbf{g}}^{\mathbf{T}} \mathbf{M} \tag{25}$$

Replacing $\partial \bar{\mathbf{l}} / \partial \mathbf{p}$ in (24) gives the following.

$$(\bar{\mathbf{l}} - \bar{\mathbf{l}}_{\mathbf{0}})^{\mathbf{T}} \tilde{\tilde{\mathbf{k}}} \mathbf{S} \tilde{\mathbf{l}}^{-\mathbf{1}} \tilde{\mathbf{g}}^{\mathbf{T}} \mathbf{M} - \bar{\epsilon}_{\mathbf{a}}^{\mathbf{T}} \tilde{\tilde{\mathbf{l}}}_{\mathbf{0}} \tilde{\tilde{\mathbf{k}}} \mathbf{S} \tilde{\mathbf{l}}^{-\mathbf{1}} \tilde{\mathbf{g}}^{\mathbf{T}} \mathbf{M} - \mathbf{f}_{\mathbf{e}}^{\mathbf{T}}(\mathbf{p}) = \mathbf{0}^{\mathbf{T}} \tag{26}$$

Using the clustered force density definitions (11) and (12), the identities, $\tilde{\mathbf{z}} \tilde{\mathbf{g}}^{\mathbf{T}} = \tilde{\mathbf{g}}^{\mathbf{T}} \hat{\mathbf{z}}$, $\hat{\mathbf{z}} \mathbf{M} = \mathbf{C}^{\mathbf{T}}$ and $\tilde{\mathbf{z}} \mathbf{S} = \mathbf{S} \tilde{\mathbf{z}}$, the clustered equilibrium equations can be derived.

$$\mathbf{C} \tilde{\mathbf{g}} \tilde{\mathbf{l}}^{-\mathbf{1}} \mathbf{S}^{\mathbf{T}} \tilde{\bar{\mathbf{l}}} (\bar{\lambda} - \bar{\lambda}_{\mathbf{a}}) = \mathbf{f}_{\mathbf{e}}(\mathbf{p}) \tag{27}$$

The clustered equilibrium equations are a set of geometrically nonlinear equations that are valid for any tensegrity structure with or without clustered elements. The classic tensegrity equilibrium equations (no clustered elements) are a special case of the clustered tensegrity equilibrium equations. When the classic limit ($\mathbf{S} \equiv \mathbf{I}$) is applied, the barred vectors are reverted from their reduced form to their full form and the clustered equilibrium equations reduce to the following classic equilibrium equations.

$$\mathbf{C} \tilde{\mathbf{g}} (\lambda - \lambda_{\mathbf{a}}) = \mathbf{f}_{\mathbf{e}}(\mathbf{p}) \tag{28}$$

A less fundamental approach can be taken to derive the clustered equilibrium equations. The static condition (9) can be directly substituted into the classic equilibrium equations (28). First, the force densities in the classic equations are written out explicitly as the forces in the elements divided by the lengths of the elements.

$$\mathbf{C} \tilde{\mathbf{g}} \tilde{\mathbf{l}}^{-\mathbf{1}} (\mathbf{f} - \mathbf{f}_{\mathbf{a}}) = \mathbf{f}_{\mathbf{e}}(\mathbf{p}) \tag{29}$$

Substituting the static condition (9),

$$\mathbf{C} \tilde{\mathbf{g}} \tilde{\mathbf{l}}^{-\mathbf{1}} \mathbf{S}^{\mathbf{T}} (\bar{\mathbf{f}} - \bar{\mathbf{f}}_{\mathbf{a}}) = \mathbf{f}_{\mathbf{e}}(\mathbf{p}) \tag{30}$$

Replacing the clustered forces in the elements with the clustered force densities multiplied by the clustered lengths gives the same equilibrium equations as before.

$$\mathbf{C} \tilde{\mathbf{g}} \tilde{\mathbf{l}}^{-\mathbf{1}} \mathbf{S}^{\mathbf{T}} \tilde{\bar{\mathbf{l}}} (\bar{\lambda} - \bar{\lambda}_{\mathbf{a}}) = \mathbf{f}_{\mathbf{e}}(\mathbf{p}) \tag{31}$$

$$\text{To summarize :} \quad \mathbf{C} \tilde{\mathbf{g}} \lambda_{\mathbf{T}}^{*} = \mathbf{f}_{\mathbf{e}}(\mathbf{p}) \tag{32}$$

$$\text{where :} \quad \lambda_{\mathbf{T}}^{*} = \tilde{\mathbf{l}}^{-\mathbf{1}} \mathbf{S}^{\mathbf{T}} \tilde{\bar{\mathbf{l}}} \bar{\lambda}_{\mathbf{T}}$$

$$\bar{\lambda}_{\mathbf{T}} = \bar{\lambda} - \bar{\lambda}_{\mathbf{a}}$$

$$\text{such that :} \quad \bar{\lambda}_{cables} \geqslant \mathbf{0}$$

$$\|\bar{\lambda}\| > \mathbf{0}$$

The first constraint allows cables to only be in tension or slack, but restricts cables from being compressed. This is a necessary constraint since the tensegrity static model accepts cables carrying compressive loads, but realistically if a cable is compressed it buckles and carries no load instead of a compressive load. Therefore the static model must constrain the cables from being compressed, while allowing the struts to be either in compression or tension. The second constraint states that a tensegrity structure must have pre-stress in some of its elements in order to exist.

5. Pre-stress and mechanism analysis

5.1. Pre-stress analysis

To design an active tensegrity structure an initial state must be defined. From the initial state, subsequent shape changes or new equilibrium states can be determined quasi-statically or dynamically. To fully define the static state of a tensegrity structure the quadruple $\Gamma = \{\mathbf{p}, \mathbf{C}, \mathbf{S}, \lambda\}$ must be parameterized. It is assumed in this paper that the first three parameters, $\{\mathbf{p}, \mathbf{C}, \mathbf{S}\}$, are known and that the pre-stress states (equilibrium states) of the system are to be found under no external loads or actuation. The clustered tensegrity equilibrium equations are used to solve for the feasible pre-stress states.

$$\bar{\mathbf{A}} \bar{\lambda} = \mathbf{0} \tag{33}$$

The clustered equilibrium matrix, $\bar{\mathbf{A}}$, can be defined as $\bar{\mathbf{A}} = \mathbf{C} \tilde{\mathbf{g}} \tilde{\mathbf{l}}^{-\mathbf{1}} \mathbf{S}^{\mathbf{T}} \tilde{\bar{\mathbf{l}}}$. Since there are cable elements that cannot carry a compressive load the following constraints must be applied.

$$\bar{\lambda}_{cables} \geqslant \mathbf{0} \qquad \|\bar{\lambda}\| > \mathbf{0} \tag{34}$$

The solution space for the set of clustered pre-stress force densities can be found at the intersection of the null space solution $\Lambda = null(\bar{\mathbf{A}})$ and the constraint $\bar{\lambda}_{\mathbf{c}} \geqslant \mathbf{0}$ (Quirant et al., 2003; Masic, 2004; Quirant, 2007). The second constraint does not need to be explicitly applied since it is implicit in the null space solution. This solution space constitutes a convex polyhedral cone (Fig. 3) (Fukuda and Prodon, 1995). The edges of the polyhedral cone in Fig. 3 are known as extreme rays and constitute the basis that spans the solution space of the polyhedral cone, i.e. every solution can be decomposed into a linear combination of extreme rays. The extreme rays, $\Lambda^{\mathbf{E}}$, are also known as the extreme directions (Masic and Skelton, 2006) and denote the elementary pre-stress basis or elementary pre-stress modes of a tensegrity structure. For clarification, the extreme directions represent the most fundamental sub-units of a tensegrity structure that have their own state of

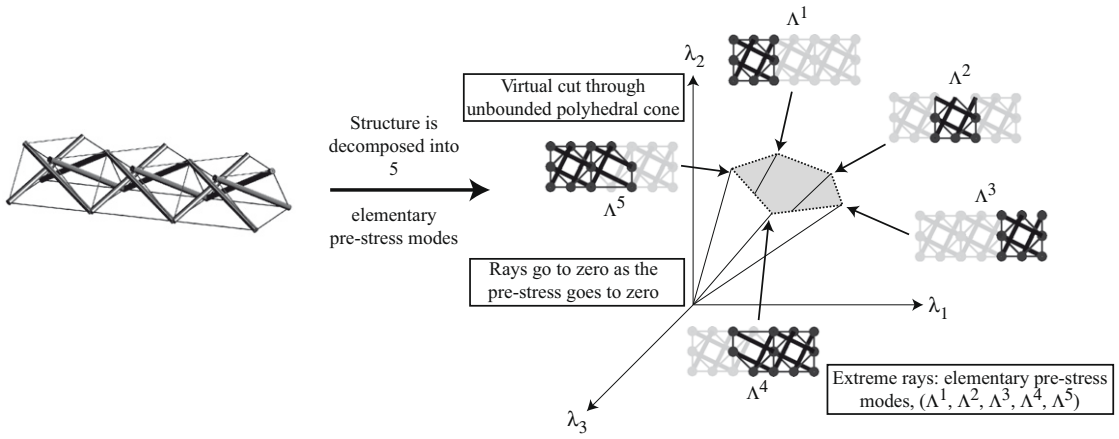

Fig. 3. Decomposition of a three cell beam structure into its five elementary pre-stress modes. The elementary pre-stress modes are the extreme rays of a polyhedral cone that goes to zero as the magnitude of the pre-stress goes to zero. Any feasible pre-stress mode is a linear superposition of the extreme rays. Each dimension of the space represents the force density in an element. This figure is a geometric interpretation and is not exact since the example structure has 46 elements, but the figure shows 3 axes.

self-stress (Fig. 3). Determining the null space solution alone is not sufficient to find the elementary pre-stress modes of a tensegrity structure, but it does give an initial linearly independent basis within the solution space that can be transformed into the extreme basis, $\mathbf{\Lambda^E}$. The algorithm used to determine the extreme basis is similar to the ones used by Wagner (2003) and Gagneur and Klamt (2004). In both papers the extreme directions of a polyhedral cone represented the elementary modes of chemical reactions from a set of stoichiometric equations and reaction flow constraints. The chemical/biomedical application of finding extreme directions of a polyhedral cone for chemical reaction systems is directly analogous to the structural application, where irreversible reaction constraints are synonymous with positive/negative force density constraints.

A similar algorithm was developed that uses the null space solution as the initial matrix solution for the elementary basis since it satisfies the systems of equations, however the constraints (34) must be used to transform the initial basis into the true fundamental basis. To transform the basis the principle of superposition is applied to the initial basis, i.e. the elementary modes can be determined by a linear superposition of the initial basis. However, there are only special superpositions that yield the elementary basis. First, the new basis vector must satisfy the force density constraints. Second, the new vector must be a minimal set meaning that it cannot be decomposed into two other vectors that satisfy the equilibrium equations and the constraints.

A combinatorial algorithm determines the set of all possible pre-stress states (up to an arbitrary scaling constant chosen by the designer) for a given parametric set, $\{\mathbf{p}, \mathbf{C}, \mathbf{S}\}$. When clustering is applied to a classic structure with a known pre-stress basis, the new pre-stress analysis will admit a number of clustered pre-stress states, $\bar{s}$, which is less than or equal to the number of classic pre-stress states for the structure ($\bar{s} \leqslant s$). Even if the number of clustered pre-stress states is equivalent to the number of classic pre-stress states the relative values of the clustered state can be different than the relative values of the classic state. Thus having an algorithm that finds all of the pre-stress states for a clustered structure is vital.

It *cannot* be assumed that an arbitrary clustering route produces a self-stressable tensegrity structure even if the classic version of the structure has a feasible self-stress state. Fig. 4 shows a planar tensegrity cross that classically has a well known self-stress state for any nodal configuration. Once cables 1 and 3 are clustered together the preceding pre-stress algorithm finds no feasible pre-stress basis. To show the validity of the pre-stress results an experiment was set up where a single cable is routed from node 1 to node 4. Another cable is added between node 1 and node 2 to create the classic tensegrity cross. When the cables are pre-stressed the structure has a well defined configuration (Fig. 4a). If the cable between node 1 and node 2 is removed then the structure cannot be pre-stressed in its initial configuration, i.e. there is no feasible self-stress state for the given parametric set (Fig. 4c). This loss of pre-stress can be simply explained by looking at a free-body diagram of node 2 (Fig. 4b). Since the tensions in cables 1 and 3 must equilibrate (consequence of clustering) they pair to form a force vector of arbitrary magnitude with a direction that always bisects the angle between the two cables. For the non-square configuration given in the example, the strut force vector can never fully

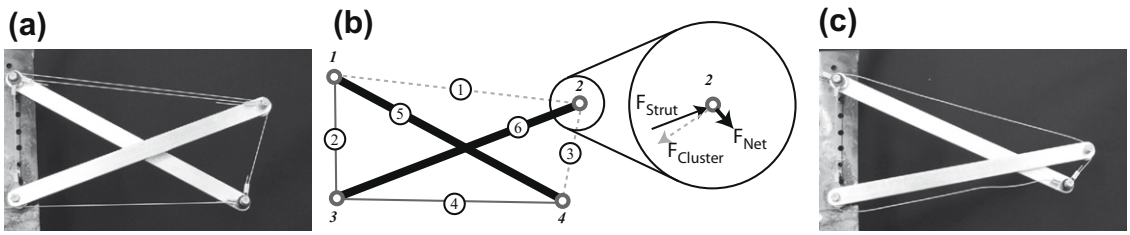

Fig. 4. (a) A classic planar tensegrity structure given a state of pre-stress. (b) The classic structure is converted to a clustered structure by cutting the bottom of the two top cables. The other top cable routes over a pulley on the upper right node and connects to the lower right node forming a cluster. The clustered structure does not have a pre-stress state for the same configuration which is shown in the free body diagram of the top right node.

balance the cluster force vector leaving a net unbalanced force vector that causes the structure to move away from its initial configuration. If the tensegrity cross is a perfect square then the force vectors will balance and there is a self-stress state (confirmed by the combinatorial algorithm).

5.2. Mechanism analysis

Tensegrity structures may also have internal mechanisms that need to be determined to understand the response and the stability of the structures. From the kinematic interpretation of clustering it is known that the kinematic constraints acting on the nodes are reduced which leads to the conclusion that the number of internal mechanisms may increase but not decrease. Thus it is seen that the number of clustered mechanisms in a structure, $\bar{m}$, is greater than or equal to the number of mechanisms for the equivalent classic structure ($\bar{m} \geqslant m$).

The approach for determining the mechanisms of a clustered structure is the same as the approach given by Pellegrino and Calladine (1986) other than the form of the clustered compatibility matrix, $\overline{\mathbf{B}}$, is different than the classic compatibility matrix. The clustered equivalent is the following.

$$\overline{\mathbf{B}} = \overline{\mathbf{A}}^{\mathbf{T}} = \tilde{\bar{\mathbf{l}}}\mathbf{S}\tilde{\bar{\mathbf{l}}}^{-1}\tilde{\mathbf{g}}^{\mathbf{T}}\mathbf{C}^{\mathbf{T}} \tag{35}$$

By solving for the kernel of the compatibility matrix a set of nodal displacements are determined that, to first order, cause no extension of the elements of the structure and are termed mechanisms. If the structure has no boundary constraints to reduce the size of the compatibility matrix then there are two types of mechanisms for a structure: rigid-body and internal mechanisms. When the mechanisms are found for a structure the internal mechanisms and rigid-body mechanisms will be mixed. Normally, only internal mechanisms are considered when determining if a structure is kinematically indeterminate since the rigid-body mechanisms are assumed. A procedure to determine the rigid-body mechanisms of a structure is presented in Pellegrino and Calladine (1986). Once the rigid-body mechanisms are found, $\mathbf{r} \in \mathbb{R}^{n\times 3(dim-1)}$, the full set of mechanisms, $\mathbf{d}_* \in \mathbb{R}^{n\times 3n-rank(\mathbf{A})}$ can be orthogonalized to the set of rigid-body motions by the Gram–Schmidt orthogonalization procedure to give the set of internal mechanisms, $\mathbf{d} \in \mathbb{R}^{n\times m}$.

$$\mathbf{d} = \mathbf{d}_* - \mathbf{r}\left[\mathbf{d}_*^{\mathbf{T}}\mathbf{r}(\mathbf{r}^{\mathbf{T}}\mathbf{r})^{-1}\right]^{\mathbf{T}} \tag{36}$$

If internal mechanisms are found in the structure then it would be considered unstable if the states of self-stress do not stabilize the mechanisms.

6. Stability

The pre-stress basis $\mathbf{\Lambda}^E$ of a tensegrity structure Γ constitutes a set of feasible force densities that satisfy equilibrium. These equilibrium solutions may be stable, neutrally stable, or unstable, making it important to classify the stability. Furthermore, even if a classical structure is stable it is not guaranteed that the equivalent clustered structure is stable since mechanisms may be produced by clustering. The condition for stability of a structure has to do with the positive definiteness of the second variation of the potential energy presented in Section 3. The condition on the second variation leads to the following inequality on the tangent stiffness matrix of the structure. Stability is ensured if,

$$\mathscr{K} = \frac{\partial^2 \Pi_{sys}}{\partial \mathbf{p}^2} > \mathbf{0} \tag{37}$$

The tangent stiffness matrix for a classic tensegrity structure is given by Masic et al. (2005) as the following.

$$\mathscr{K} = \mathbf{C}\tilde{\mathbf{g}}\tilde{\mathbf{y}}\tilde{\mathbf{a}}\tilde{\mathbf{l}}^{-3}\tilde{\mathbf{g}}^{\mathbf{T}}\mathbf{C}^{\mathbf{T}} - \mathbf{C}\hat{\lambda}\mathbf{M} \tag{38}$$

The first term reflects a material stiffness that arises from strain within the elements and the second term is a pre-stress stiffness (also called the stress matrix, $\mathbf{\Omega}$) (Connelly and Whiteley, 1996) that arises from a reorientation of the pre-stress within the elements. This tangent stiffness matrix is valid for all pin-jointed structures except when clustering is present. Thus, before the stability of a clustered tensegrity structure is classified the clustered stiffness matrix must be derived.

The stiffness matrix for a clustered tensegrity structure is derived by taking the second derivative of the potential energy or the first derivative of the equilibrium equations. Thus the stiffness matrix is the following

$$\mathscr{K} = \frac{\partial}{\partial \mathbf{p}}\left[\mathbf{C}\tilde{\mathbf{g}}\lambda_{\mathbf{T}}^* - \mathbf{f}_{\mathbf{e}}\right] \tag{39}$$

In this derivation, external forces and the actuation forces are allowed to contribute to the tangent stiffness matrix. The stiffness contribution from external loads can, for example, represent an elastic foundation. The stiffness contribution from actuation can be used to investigate the special case of an element being a zero rest length spring. This case is obtained by giving the zero rest length cable element an actuation strain of 1. The theory of using zero rest length springs is fully discussed by Schenk et al. (2007). Even by allowing these stiffness contributions, the stability of self-stress states is normally determined by setting both the actuation strain and the external loads to zero. Since the equilibrium equations are nonlinear in the nodal point positions the chain rule has to be applied while using the identity $\mathbf{C}\tilde{\mathbf{g}}\lambda_{\mathbf{T}}^* \equiv \mathbf{C}\hat{\lambda}_{\mathbf{T}}^*\mathbf{M}\mathbf{p}$.

$$\mathscr{K} = \mathbf{C}\hat{\lambda}_{\mathbf{T}}^*\mathbf{M} + \mathbf{C}\tilde{\mathbf{g}}\frac{\partial \lambda_{\mathbf{T}}^*}{\partial \mathbf{p}} - \frac{\partial \mathbf{f}_{\mathbf{e}}}{\partial \mathbf{p}} \tag{40}$$

$$\frac{\partial \lambda_{\mathbf{T}}^*}{\partial \mathbf{p}} = \tilde{\mathbf{l}}^{-1}\mathbf{S}^{\mathbf{T}}\tilde{\bar{\lambda}}_{\mathbf{T}}\mathbf{S}\frac{\partial \mathbf{l}}{\partial \mathbf{p}} - \tilde{\mathbf{l}}^{-1}\tilde{\lambda}_{\mathbf{T}}^*\frac{\partial \mathbf{l}}{\partial \mathbf{p}} + \tilde{\mathbf{l}}^{-1}\mathbf{S}^{\mathbf{T}}\tilde{\bar{\mathbf{l}}}\frac{\partial \bar{\lambda}_{\mathbf{T}}}{\partial \mathbf{p}} \tag{41}$$

$$\frac{\partial \mathbf{l}}{\partial \mathbf{p}} = \tilde{\mathbf{l}}^{-1}\tilde{\mathbf{g}}^{\mathbf{T}}\mathbf{M} \tag{42}$$

$$\frac{\partial \bar{\lambda}_{\mathbf{T}}}{\partial \mathbf{p}} = \tilde{\bar{\mathbf{l}}}^{-1}\tilde{\bar{\mathbf{z}}}\tilde{\bar{\mathbf{y}}}\tilde{\bar{\mathbf{a}}}\tilde{\bar{\mathbf{l}}}_{\mathbf{0}}^{-1}\frac{\partial \bar{\mathbf{l}}}{\partial \mathbf{p}} - \tilde{\bar{\mathbf{l}}}^{-1}\tilde{\bar{\lambda}}_{\mathbf{T}}\frac{\partial \bar{\mathbf{l}}}{\partial \mathbf{p}} \tag{43}$$

Substituting Eq. (25) into the previous equation gives the following.

$$\frac{\partial \bar{\lambda}_{\mathbf{T}}}{\partial \mathbf{p}} = (\mathbf{I} - \tilde{\bar{\epsilon}}_{\mathbf{a}})\tilde{\bar{\mathbf{z}}}\tilde{\bar{\mathbf{y}}}\tilde{\bar{\mathbf{a}}}\tilde{\bar{\mathbf{l}}}^{-2}\mathbf{S}\tilde{\mathbf{l}}^{-1}\tilde{\mathbf{g}}^{\mathbf{T}}\mathbf{M} \tag{44}$$

$$\frac{\partial \lambda_{\mathbf{T}}^*}{\partial \mathbf{p}} = \tilde{\mathbf{l}}^{-1}\mathbf{S}^{\mathbf{T}}(\mathbf{I} - \tilde{\bar{\epsilon}}_{\mathbf{a}})\tilde{\bar{\mathbf{z}}}\tilde{\bar{\mathbf{y}}}\tilde{\bar{\mathbf{a}}}\tilde{\bar{\mathbf{l}}}^{-1}\mathbf{S}\tilde{\mathbf{l}}^{-1}\tilde{\mathbf{g}}^{\mathbf{T}}\mathbf{M} - \tilde{\mathbf{l}}^{-1}\left(\tilde{\lambda}_{\mathbf{T}}^* - \mathbf{S}^{\mathbf{T}}\tilde{\bar{\lambda}}_{\mathbf{T}}\mathbf{S}\right)\tilde{\mathbf{l}}^{-1}\tilde{\mathbf{g}}^{\mathbf{T}}\mathbf{M} \tag{45}$$

$$\begin{aligned}\mathscr{K} = {} & \mathbf{C}\hat{\lambda}_{\mathbf{T}}^*\mathbf{M} + \mathbf{C}\tilde{\mathbf{g}}\tilde{\mathbf{l}}^{-1}\mathbf{S}^{\mathbf{T}}(\mathbf{I} - \tilde{\bar{\epsilon}}_{\mathbf{a}})\tilde{\bar{\mathbf{z}}}\tilde{\bar{\mathbf{y}}}\tilde{\bar{\mathbf{a}}}\tilde{\bar{\mathbf{l}}}^{-1}\mathbf{S}\tilde{\mathbf{l}}^{-1}\tilde{\mathbf{g}}^{\mathbf{T}}\mathbf{M} \\ & - \mathbf{C}\tilde{\mathbf{g}}\tilde{\mathbf{l}}^{-1}\left(\tilde{\lambda}_{\mathbf{T}}^* - \mathbf{S}^{\mathbf{T}}\tilde{\bar{\lambda}}_{\mathbf{T}}\mathbf{S}\right)\tilde{\mathbf{l}}^{-1}\tilde{\mathbf{g}}^{\mathbf{T}}\mathbf{M} - \frac{\partial \mathbf{f}_{\mathbf{e}}}{\partial \mathbf{p}}\end{aligned} \tag{46}$$

Using the identities, $\tilde{\mathbf{z}}\mathbf{S} \equiv \mathbf{S}\tilde{\mathbf{z}}, \tilde{\mathbf{z}}\tilde{\mathbf{g}}^{\mathbf{T}} \equiv \tilde{\mathbf{g}}^{\mathbf{T}}\tilde{\mathbf{z}}$ and $\hat{\mathbf{z}}\mathbf{M} \equiv \mathbf{C}^{\mathbf{T}}$ the stiffness matrix is obtained.

$$\begin{aligned}\mathscr{K} = {} & \mathbf{C}\tilde{\mathbf{g}}\tilde{\mathbf{l}}^{-1}\mathbf{S}^{\mathbf{T}}(\mathbf{I} - \tilde{\bar{\epsilon}}_{\mathbf{a}})\tilde{\bar{\mathbf{y}}}\tilde{\bar{\mathbf{a}}}\tilde{\bar{\mathbf{l}}}^{-1}\mathbf{S}\tilde{\mathbf{l}}^{-1}\tilde{\mathbf{g}}^{\mathbf{T}}\mathbf{C}^{\mathbf{T}} + \mathbf{C}\hat{\lambda}_{\mathbf{T}}^*\mathbf{M} \\ & - \mathbf{C}\tilde{\mathbf{g}}\tilde{\mathbf{l}}^{-1}\left(\tilde{\lambda}_{\mathbf{T}}^* - \mathbf{S}^{\mathbf{T}}\tilde{\bar{\lambda}}_{\mathbf{T}}\mathbf{S}\right)\tilde{\mathbf{l}}^{-1}\tilde{\mathbf{g}}^{\mathbf{T}}\mathbf{M} - \frac{\partial \mathbf{f}_{\mathbf{e}}}{\partial \mathbf{p}}\end{aligned} \tag{47}$$

In the classical limit $\mathbf{S} \equiv \mathbf{I}$ and $\tilde{\bar{\epsilon}}_{\mathbf{a}} \equiv \mathbf{0}$. Using these identities, $\tilde{\mathbf{y}} \to \tilde{\mathbf{y}}$, $\tilde{\mathbf{a}} \to \tilde{\mathbf{a}}$, $\tilde{\bar{\mathbf{l}}} \to \tilde{\mathbf{l}}$, $\bar{\lambda}_{\mathbf{T}} \to \lambda$ and $\lambda_{\mathbf{T}}^* \to \lambda$, causing the third term to vanish. If the stiffness contribution from external forces is set to zero then the clustered stiffness matrix in the classic limit matches

Table 1
Stability classification.

Stability	Classification	Sign of the stiffness matrix (without rigid-body motions)
Stable	First-order rigid	$\mathscr{K} > 0$ when $\bar{\lambda} = 0$
	Pre-stress stabilized	$\mathscr{K} > 0$ for some admissible $\bar{\lambda} \neq 0$ $\mathscr{K} \geq 0$ when $\bar{\lambda} = 0$
Neutrally stable	Finite mechanism	$\mathscr{K} \geqslant 0$ for any admissible $\bar{\lambda} \neq 0$
Unstable	–	$\mathscr{K} \leqslant 0$ or indefinite

the classic stiffness matrix (38) derived by Masic et al. (2005).[3] Deriving the clustered stiffness matrix from the clustered potential energy or the clustered equilibrium equations is necessary. One cannot simply substitute clustered variables into the classic stiffness matrix to obtain the clustered stiffness matrix. The stiffness matrix can be rewritten to reflect the contribution to the material stiffness, terms one and two, and the contribution to the pre-stress stiffness, terms three and four.

$$\mathscr{K} = \overline{\mathbf{A}}\left(\tilde{\mathbf{y}}\tilde{\mathbf{a}}\tilde{\mathbf{l}}_{\mathbf{0}}^{-1}\tilde{\mathbf{l}}^{-1} - 2\tilde{\mathbf{y}}\tilde{\mathbf{a}}\tilde{\mathbf{l}}^{-2}\tilde{\epsilon}_{\mathbf{a}}\right)\overline{\mathbf{A}}^{\mathbf{T}} + \mathbf{C}\left(\hat{\lambda}_{\mathbf{T}}^{*} - \tilde{\mathbf{g}}\tilde{\mathbf{l}}^{-1}\tilde{\lambda}_{\mathbf{T}}^{*}\tilde{\mathbf{l}}^{-1}\tilde{\mathbf{g}}^{\mathbf{T}}\right)\mathbf{M} - \frac{\partial \mathbf{f_e}}{\partial \mathbf{p}} \tag{48}$$

Connelly and Whiteley (1996) developed a rigorous mathematical approach to investigate the different classifications of the rigidity of tensegrity structures. The classifications are still relevant for clustered structures and they are compiled in Table 1.

First-order rigidity occurs when a structure has no internal mechanisms (i.e. the kernel of $\overline{\mathbf{A}}^{\mathbf{T}}$ is the rigid-body mechanisms alone). Thus any load on the structure will cause an extension of an element so that the first term of the stiffness matrix must be positive definite without the rigid-body motions. To determine positive definiteness the sign of the eigenvalues of a matrix may be investigated. For every rigid-body motion there is a zero eigenvalue which are overlooked when investigating the positive definiteness without the rigid-body motions.

Pre-stress stabilization occurs when a structure does have internal mechanisms but they are stabilized by a state of self-stress. Once the internal mechanisms for a structure are found from (36) they can be multiplied by the stiffness matrix to determine the force necessary to displace the structure in the direction of the mechanisms ($\mathbf{f} = \mathscr{K}\mathbf{d}$). If $\mathbf{f}^{\mathbf{T}}\mathbf{d} > 0$ the mechanisms are stabilized, if $\mathbf{f}^{\mathbf{T}}\mathbf{d} = 0$ they are neutrally stable and if $\mathbf{f}^{\mathbf{T}}\mathbf{d} < 0$ they are unstable.

Finite mechanisms in this context are internal mechanisms that are of a higher-order than first-order infinitesimal or have a truly finite range-of-motion. Finite mechanisms cannot be stabilized by a state of self-stress and thus the stiffness matrix is positive semi-definite for any admissible state of self-stress.

7. Results and discussion

An example of a tensegrity structure with clustered elements is shown in Fig. 5. In this example a classic tensegrity beam structure is composed of three four-strut prismatic structures. The modules are attached together in a way that no bars are connected to each other, making it a class 1 beam structure. Every top node of the four-strut unit cell has two cables that connect to the bottom nodes, which is necessary for there to be a feasible pre-stress basis when clustered elements are added to the structure. To be able to classically actuate this structure in a bending deformation, actuators can be embedded in 10 of the top elements and 10 of the bottom elements in the span direction. In this scenario antagonistic actuation can be utilized to minimize the addition of strain energy caused by actuation. Antagonistic actuation contracts the top elements while the bottom elements are simultaneously expanded.

Instead of using embedded actuation, clustered actuation can be applied to this structure. The clustering routes are chosen, at this point, by designer intuition to achieve a desired deformation of the structure when the clusters are actuated. An optimization algorithm could also be used such as the one proposed in Moored and Bart-Smith (2007) to determine the best clustering route for a target deformation field. For this example a bending deformation is desirable. To actuate in a bending mode four clustering routes are created. Each route runs along the top and bottom edges of the structure and is composed of 5 of the embedded active elements. Before clustering is applied there are 84 independent pre-stress modes, 17 internal mechanisms and 20 active elements. The internal mechanisms are stabilized by the states of self-stress, so the structure is pre-stress stable. After clustering is applied there is one pre-stress mode, 25 mechanisms and four active elements. The stiffness matrix is positive semi-definite when there is no pre-stress and positive definite (minus the rigid-body motions) when pre-stress is present, making the clustered structure pre-stress stabilized. The reduction in the number of pre-stress modes follows the outcome that the independent states of self-stress in a clustered structure are less than or equal to the number of states of self-stress in the equivalent classic structure ($\bar{s} \leqslant s$). The mechanisms also follow the outcome that the number of mechanisms in the clustered structure is greater than or equal to the number of mechanisms in the equivalent classic structure ($\bar{m} \geqslant m$).

In Fig. 5 three of the nodes are pinned and the top clusters have actuated with 10% contraction while the bottom clusters have been expanded by 10% causing 36% spanwise tip deflection (normalized by the span length) in the positive z-direction. There is also a 7% tip deflection (normalized by the span length) in the positive y-direction. This unwanted deflection in the y-direction could be minimized by intuitively varying the actuation amounts or an optimization strategy could be utilized to determine the necessary actuation strains to reach a target displacement field, similar to the one presented by Moored and Bart-Smith (2007).

This example also shows a radical reduction in the number of pre-stress states of the structure from 84 down to 1 global pre-stress state. Having only one state makes the application of pre-stress during the fabrication of a structure much more straightforward. Stress can be applied to a single element in order to pre-stress the entire structure, after all of the other elements have been constructed to their proper lengths, $\bar{\mathbf{l}}_{\mathbf{0}}$. This characteristic has been taken advantage of by Kwan et al. (1993) and Smaili and Motro (2005).

Clustered actuation can also reduce the power required to accelerate a tensegrity structure with embedded actuators. Assume that a tensegrity beam is being dynamically activated in a bending deformation with the tip oscillating through an amplitude, h, in a sinusoidal motion. Further assume that the structural mass and the actuator mass are evenly distributed throughout the structure. The total mass of the structure and actuators would be

$$M = m_s + Nm_a. \tag{49}$$

The mass of the structure is m_s, the mass of a single actuator is m_a and the number of actuators is N. The force require to accelerate the total mass through a vertical sinusoidal motion is $F = Ma$ where the maximum acceleration,

$$a_{max} \approx 2\pi^2 f^2 h, \tag{50}$$

is approximated at the center of mass where the amplitude is $h/2$. So the force required to accelerate the structure would be,

$$F \approx 2\pi^2 M f^2 h. \tag{51}$$

[3] The sign difference on the second term is due to a slightly different formulation than Masic et al. (2005) where $\hat{\mathbf{z}}\mathbf{M} \equiv \mathbf{C}^{\mathbf{T}}$ instead of $\hat{\mathbf{z}}\mathbf{M} \equiv -\mathbf{C}^{\mathbf{T}}$.

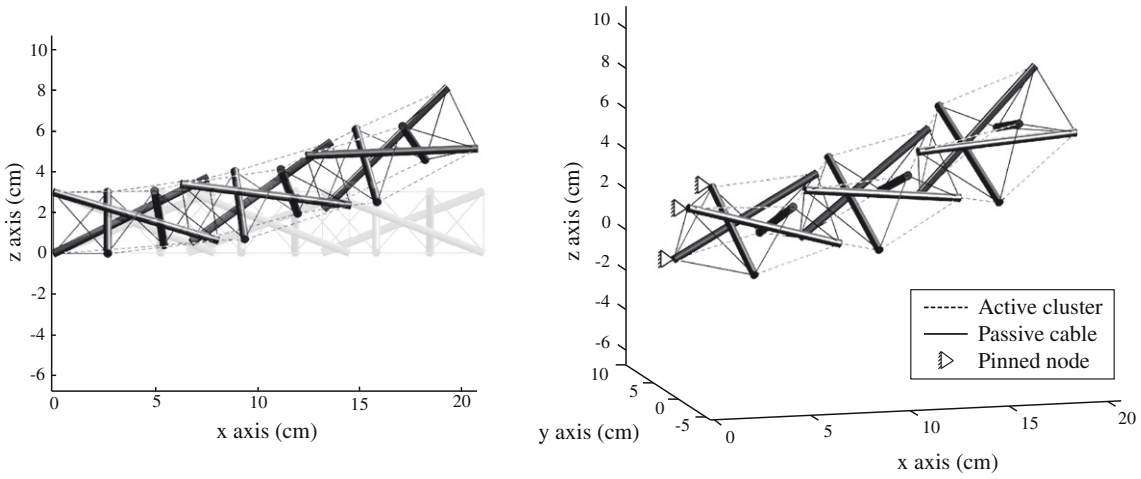

Fig. 5. A class 1 tensegrity beam composed of three unit cells with four clusters. There are about 84 pre-stress modes for the classic structure which is reduced to 1 global pre-stress mode for the clustered scenario. The top two clusters are contracted by 10% while the bottom two clusters are expanded by 10% to bend the structure in the spanwise direction.

The power required to accelerate the total mass would be,

$$P \approx 2\pi^3 m_s f^3 h^2 + 2\pi^3 N m_a f^3 h^2 \tag{52}$$

By applying clustered actuation, the actuators can be migrated out of the part of the structure that is accelerating to the base of the beam where there is no acceleration. This then reduces the required power for actuation.

$$P_{reduction} \approx 2\pi^3 N m_a f^3 h^2 \tag{53}$$

Thus the power ratio of a dynamic clustered structure can be compared to a dynamic classic structure with embedded actuators.

$$P_{ratio} \approx m_s / M \tag{54}$$

So if the mass of the actuators is approximately the same as the mass of the structure then there is a power savings of about 50%. This concept follows the idea that the actuators should not be supplying power to move themselves, but to only be accelerating the structure.

8. Conclusions

The potential energy for a clustered tensegrity structure has been developed. The equilibrium equations and stiffness matrix of a clustered tensegrity structure have been derived. Finding the pre-stress basis, the mechanisms, and the stability of a clustered structure has been discussed. The addition of the clustering matrix, **S**, to the classic set of parameters allows the classic tensegrity equations to be cast into a new form that includes clustered elements. The compact clustered equations are shown to be more general than the classic equations since they allow $\mathbf{S} \neq \mathbf{I}$. It is shown that clustering can offer a scalable actuation strategy, reduce the number of actuators, reduce the number of pre-stress modes to a single global mode, reduce the power requirements for accelerating a structure and to relieve element size limitations due to embedded actuators. This formulation of clustered actuation also uses a the binary clustering matrix which is well suited for future optimization of clustering routes.

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