# Stability and convergence of the two parameter cubic spline collocation method for delay differential equations 

Hong Su ${ }^{\text {a,b,d }}$, Shui-Ping Yang ${ }^{\text {a,b,c,** }, ~ L i-P i n g ~ W e n ~}{ }^{\text {a,b }}$<br>${ }^{\text {a }}$ School of Mathematics and Computational Science, Xiangtan University, Hunan, 411105, China<br>${ }^{\mathrm{b}}$ Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Xiangtan University, Hunan, 411105, China<br>${ }^{\text {c }}$ Department of Mathematics, Huizhou University, Guangdong, 516007, China<br>${ }^{\text {d }}$ Huigang Middle School, Huizhou, Guangdong, 516007, China

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#### Abstract

In this paper, we propose the cubic spline collocation method with two parameters for solving delay differential equations (DDEs). Some results of the local truncation error and the convergence of the spline collocation method are given. We also obtain some results of the linear stability and the nonlinear stability of the method for DDEs. In particular, we design an algorithm to obtain the ranges of the two parameters $\alpha, \beta$ which are necessary for the P-stability of the collocation method. Some illustrative examples successfully verify our theoretical results.


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## 1. Introduction

DDEs arise in the models of many real life applications, for example, control theory, medical science, biology, lasers, and population growth. In the past decades, there has been increasing interest in the numerical solution of the initial value problem of DDEs. Some numerical methods which have successfully solved ordinary differential equations (ODEs) have been adopted to solve DDEs [1-11]. In recent years, spline collocation methods as numerical methods have been used to solve many differential equations such as ODEs, DDEs and partial differential equations (PDEs). Blaga et al. have successfully applied the spline function of even degree to solve DDEs in [9]. Ibrahim et al. have discussed the 2 h -step spline method for DDEs in [7], but the results of the stability and convergence have not been given. Engelborghs et al. have proposed collocation methods for periodic solutions of DDEs in [4]. Moreover, in [1], El-Hawary et al. have successfully obtained the numerical solutions of DDEs by means of the four point spline collocation method. They have also studied the convergence and stability of collocation methods. However, to the best of our knowledge, the nonlinear stability and the convergence of the cubic spline method for ODEs cannot be obtained by using their proposed methods, so we introduce different ways to obtain the convergence and the nonlinear stability of the cubic spline collocation method for solving DDEs. Moreover, in this paper, we design an algorithm for obtaining the value range of the two parameters when the method is P-stable.

This paper is organized as follows. In Section 2, we present the description of the method. Some theorems of the local truncation error, the convergence and the nonlinear stability of the cubic spline collocation method for solving DDEs are given in Section 3. In Section 4, we study the linear stability of the method. Finally, some numerical examples are provided.

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## 2. Description of the method for solving DDEs

Consider the initial value problem of DDEs

$$
\left\{\begin{array}{l}
y^{\prime}(t)=f(t, y(t), y(t-\tau)), \quad 0 \leq t \leq T  \tag{1}\\
y(t)=\rho(t), \quad-\tau \leq t \leq 0
\end{array}\right.
$$

where $\tau>0, \rho:[-\tau, 0] \rightarrow R$ is a continuous differentiable mapping, $f:[0, T] \times R \times R \rightarrow R$ is a given mapping, satisfies the following Lipschitz condition

$$
\begin{equation*}
\left|f(t, y, z)-f\left(t, y^{\prime}, z^{\prime}\right)\right| \leq L_{1}\left|y-y^{\prime}\right|+L_{2}\left|z-z^{\prime}\right|, \quad \forall t \in[0, T], y, z, y^{\prime}, z^{\prime} \in R \tag{2}
\end{equation*}
$$

Define a uniform partition on $I=[0, T]$ into $N$ equal parts and denote

$$
h=\frac{T}{N}, \quad t_{i}=i h, \quad I_{i}=\left[t_{i-1}, t_{i}\right], \quad i=1,2, \ldots, N
$$

and $\tau=m h, t_{i}-\tau \in I_{i-m}=\left[t_{i-1-m}, t_{i-m}\right]$. Then, the cubic $C^{1}$-spline function $S(t)$ can be written on each subinterval $I_{i}$ by

$$
S(t)=\left\{\begin{array}{l}
\bar{\xi}^{2}(2 \xi+1) S_{i-1}^{(0)}+\bar{\xi}^{2} \xi S_{i-1}^{(1)}+\xi^{2}(2 \bar{\xi}+1) S_{i}^{(0)}-\xi^{2} \bar{\xi} S_{i}^{(1)}, \quad \forall t \in I_{i}  \tag{3}\\
\rho(t), \quad \tau \leq t \leq 0
\end{array}\right.
$$

where $\xi=\frac{t-t_{i-1}}{h} \in[0,1], \bar{\xi}=1-\xi, S_{i}^{(0)}=S\left(t_{i}\right), S_{i}^{(1)}=h S^{\prime}\left(t_{i}\right), i=0, \ldots, N$. Substituting $t_{i-1+\alpha}$ and $t_{i-1+\beta}$ into (3), respectively, we get

$$
\begin{align*}
{\left[\begin{array}{l}
S\left(t_{i-1}+h \alpha\right) \\
S\left(t_{i-1}+h \beta\right)
\end{array}\right] } & =\left[\begin{array}{ll}
\bar{\alpha}^{2}(2 \alpha+1) & \bar{\alpha}^{2} \alpha \\
\bar{\beta}^{2}(2 \beta+1) & \bar{\beta}^{2} \beta
\end{array}\right]\left[\begin{array}{l}
S_{i-1}^{(0)} \\
S_{i-1}^{(1)}
\end{array}\right]+\left[\begin{array}{ll}
\alpha^{2}(2 \bar{\alpha}+1) & -\alpha^{2} \bar{\alpha} \\
\beta^{2}(2 \bar{\beta}+1) & -\beta^{2} \bar{\beta}
\end{array}\right]\left[\begin{array}{l}
S_{i}^{(0)} \\
S_{i}^{(1)}
\end{array}\right] \\
& =C(\alpha, \beta)\left[\begin{array}{l}
S_{i-1}^{(0)} \\
S_{i-1}^{(1)}
\end{array}\right]+D(\alpha, \beta)\left[\begin{array}{l}
S_{i}^{(0)} \\
S_{i}^{(1)}
\end{array}\right], \quad \forall \alpha, \beta \in[0,1] \tag{4}
\end{align*}
$$

where $\bar{\alpha}=1-\alpha, \bar{\beta}=1-\beta$. Next, differentiating Eq. (3) on both the two sides, we can obtain

$$
\begin{equation*}
h S^{\prime}(t)=\bar{\xi}(-6 \xi) S_{i-1}^{(0)}+\bar{\xi}(1-3 \xi) S_{i-1}^{(1)}+\xi(6 \bar{\xi}) S_{i}^{(0)}+\xi(1-3 \bar{\xi}) S_{i}^{(1)} \tag{5}
\end{equation*}
$$

By using the collocation conditions

$$
\begin{equation*}
S^{\prime}\left(t_{i-1+\phi}\right)=f\left(t_{i-1+\phi}, S\left(t_{i-1+\phi}\right), S\left(t_{i-1+\phi}-\tau\right)\right), \quad \phi \in\{\alpha, \beta\} \tag{6}
\end{equation*}
$$

where $\alpha, \beta$ are collocation points, and $0<\alpha<\beta \leq 1$, denote $\bar{\phi}=1-\phi$, then (6) can be written as follows

$$
\begin{equation*}
\phi(6 \bar{\phi}) S_{i}^{(0)}+\phi(1-3 \bar{\phi}) S_{i}^{(1)}=\bar{\phi}(6 \phi) S_{i-1}^{(0)}-\bar{\phi}(1-3 \phi) S_{i-1}^{(1)}+h f\left(t_{i-1+\phi}, S\left(t_{i-1+\phi}\right), S\left(t_{i-1+\phi}-\tau\right)\right) \tag{7}
\end{equation*}
$$

Let $f_{i-1+\phi}=f\left(t_{i-1+\phi}, S\left(t_{i_{-1+\phi}}\right), S\left(t_{i-1+\phi}-\tau\right)\right)$, we can obtain the following equivalent recurrent formula

$$
\begin{equation*}
\underline{S}_{i}=A \underline{S}_{i-1}+h B \underline{\underline{l}}_{i} \tag{8}
\end{equation*}
$$

where

$$
\begin{aligned}
& A=\tilde{A}^{-1} H, \quad B=\tilde{A}^{-1}, \quad \underline{S}_{i}=\left(S_{i}^{(0)}, S_{i}^{(1)}\right)^{T}, \\
& \underline{f}_{i}=\left(f_{i-1+\alpha}, f_{i-1+\beta}\right)^{T}=\binom{f\left(t_{i-1+\alpha}, S\left(t_{i-1+\alpha}\right), S\left(t_{i-1+\alpha}-\tau\right)\right)}{f\left(t_{i-1+\beta}, S\left(t_{i-1+\beta}\right), S\left(t_{i-1+\beta}-\tau\right)\right)}
\end{aligned}
$$

and

$$
\widetilde{A}^{-1}=\left[\begin{array}{ll}
6 \alpha \bar{\alpha} & \alpha(1-3 \bar{\alpha}) \\
6 \beta \bar{\beta} & \beta(1-3 \bar{\beta})
\end{array}\right]^{-1}, \quad H=\left[\begin{array}{cc}
6 \alpha \bar{\alpha} & \bar{\alpha}(1-3 \alpha) \\
6 \beta \bar{\beta} & \bar{\beta}(1-3 \beta)
\end{array}\right]
$$

So we can get the following scheme

$$
\left[\begin{array}{l}
S_{i}^{(0)}  \tag{9}\\
S_{i}^{(1)}
\end{array}\right]=\left[\begin{array}{cc}
1 & \frac{6 \alpha \beta-3 \alpha-3 \beta+2}{6 \alpha \beta} \\
0 & \frac{(1-\alpha)(1-\beta)}{\alpha \beta}
\end{array}\right]\left[\begin{array}{l}
S_{i-1}^{(0)} \\
S_{i-1}^{(1)}
\end{array}\right]+h\left[\begin{array}{cc}
\frac{2-3 \beta}{6 \alpha(\alpha-\beta)} & \frac{3 \alpha-2}{6 \beta(\alpha-\beta)} \\
\frac{1-\beta}{\alpha(\alpha-\beta)} & \frac{\alpha-1}{\beta(\alpha-\beta)}
\end{array}\right]\left[\begin{array}{l}
f_{i-1+\alpha} \\
f_{i-1+\beta}
\end{array}\right]
$$

## 3. Theoretical analysis

In this section, we give some theoretical results of the cubic spline collocation method for DDEs.
Theorem 3.1 (Consistence). For numerical method (8), the local truncation error is $O\left(h^{4}\right)$.
Proof. Substituting true solution to (8), then we have the following local truncation error

$$
e_{i}^{1}=\left[\begin{array}{c}
y\left(t_{i}\right) \\
h y^{\prime}\left(t_{i}\right)
\end{array}\right]-A\left[\begin{array}{c}
y\left(t_{i-1}\right) \\
h y^{\prime}\left(t_{i-1}\right)
\end{array}\right]-h B\left[\begin{array}{l}
f\left(t_{i-1+\alpha}, S\left(t_{i-1+\alpha}\right), S\left(t_{i-1+\alpha}-\tau\right)\right) \\
f\left(t_{i-1+\beta}, S\left(t_{i-1+\beta}\right), S\left(t_{i-1+\beta}-\tau\right)\right)
\end{array}\right],
$$

where

$$
S(t)=\bar{\xi}^{2}\left[(2 \xi+1) y\left(t_{i-1}\right)+\left(\xi y^{\prime}\right)\left(t_{i-1}\right) h\right]+\xi^{2}\left[(2 \bar{\xi}+1) y\left(t_{i}\right)-\bar{\xi} y^{\prime}\left(t_{i}\right) h\right], \quad t \in\left[t_{i-1}, t_{i}\right]
$$

is a cubic Hermite interplot of $y(t)$ on subinterval $t \in\left[t_{i-1}, t_{i}\right]$. Applying Taylor's expansions

$$
y(t)=S(t)+O\left(h^{4}\right), \quad t \in\left[t_{i-1}, t_{i}\right], y \in C^{1}[a, b]
$$

then

$$
|S(t)-y(t)| \leq C h^{4}, \quad t \in\left[t_{i-1}, t_{i}\right]
$$

where $C$ is an appropriate constant. Combine (8) with Lipschitz condition (2), then we obtain

$$
\left|e_{i}^{1}-\mathrm{e}_{i}^{2}\right|=O\left(h^{4}\right), \quad i=1, \ldots, N
$$

where

$$
\mathrm{e}_{i}^{2}=\left[\begin{array}{c}
y\left(t_{i}\right) \\
h y^{\prime}\left(t_{i}\right)
\end{array}\right]-A\left[\begin{array}{c}
y\left(t_{i-1}\right) \\
h y^{\prime}\left(t_{i-1}\right)
\end{array}\right]-h B\left[\begin{array}{c}
y^{\prime}\left(t_{i-1+\beta}\right) \\
y^{\prime}\left(t_{i-1+\beta}\right)
\end{array}\right] .
$$

Now using the Taylor expansion

$$
y(x)=q_{3}(x)+O\left(h^{4}\right)
$$

and noting that the methods are accurate for polynomials of degree $\leq 3$, namely $e_{i}^{2}=0$, hence

$$
\left|\mathrm{e}_{i}^{1}\right|=O\left(h^{4}\right) .
$$

Theorem 3.2 (Convergence). If $\|A\| \leq 1$, numerical method (8) for solving problem (1) is convergence.
Proof. First, we denote

$$
\begin{aligned}
& \epsilon_{i}=\binom{\left|S_{i}^{(0)}-y\left(t_{i}\right)\right|}{\left|S_{i}^{(1)}-h y^{\prime}\left(t_{i}\right)\right|}, \quad \epsilon_{i}^{\prime}=\binom{\left|S\left(t_{i+\alpha}\right)-y\left(t_{i+\alpha}\right)\right|}{\left|S\left(t_{i+\beta}\right)-y\left(t_{i+\beta}\right)\right|}, \\
& \tilde{\epsilon}_{i}=\binom{\left|S\left(t_{i-1+\alpha}-\tau\right)-y\left(t_{i-1+\alpha}-\tau\right)\right|}{\left|S\left(t_{i-1+\beta}-\tau\right)-y\left(t_{i-1+\beta}-\tau\right)\right|}, \quad X_{i}=\max _{0 \leq \leq \leq i}\left\{\left\|\epsilon_{s}\right\|\right\},
\end{aligned}
$$

where

$$
\begin{align*}
{\left[\begin{array}{l}
y\left(t_{i-1}+h \alpha\right) \\
y\left(t_{i-1}+h \beta\right)
\end{array}\right] } & =\left[\begin{array}{ll}
\bar{\alpha}^{2}(2 \alpha+1) & \bar{\alpha}^{2} \alpha \\
\bar{\beta}^{2}(2 \beta+1) & \bar{\beta}^{2} \beta
\end{array}\right]\left[\begin{array}{c}
y_{i-1} \\
h y_{i-1}^{\prime}
\end{array}\right]+\left[\begin{array}{ll}
\alpha^{2}(2 \bar{\alpha}+1) & -\alpha^{2} \bar{\alpha} \\
\beta^{2}(2 \bar{\beta}+1) & -\beta^{2} \bar{\beta}
\end{array}\right]\left[\begin{array}{c}
y_{i} \\
h y_{i}^{\prime}
\end{array}\right] \\
& =C(\alpha, \beta)\left[\begin{array}{c}
y_{i-1} \\
h y_{i-1}^{\prime}
\end{array}\right]+D(\alpha, \beta)\left[\begin{array}{c}
y_{i} \\
h y_{i}^{\prime}
\end{array}\right]+O\left(h^{4}\right), \quad \forall \alpha, \beta \in[0,1] . \tag{10}
\end{align*}
$$

The cubic spline collocation function $S(t)$, with $C(\alpha, \beta)$ and $D(\alpha, \beta)$ in Eq. (10) as the functions with respect to $\alpha$ and $\beta$, is continuous on [0, 1], so

$$
\begin{align*}
{\left[\begin{array}{l}
S\left(t_{i-1}+h \alpha\right)-y\left(t_{i-1}+h \alpha\right) \\
S\left(t_{i-1}+h \beta\right)-y\left(t_{i-1}+h \beta\right)
\end{array}\right] } & =\left[\begin{array}{ll}
\bar{\alpha}^{2}(2 \alpha+1) & \bar{\alpha}^{2} \alpha \\
\bar{\beta}^{2}(2 \beta+1) & \bar{\beta}^{2} \beta
\end{array}\right]\left[\begin{array}{c}
S_{i-1}^{(0)}-y_{i-1} \\
S_{i-1}^{(1)}-h y_{i-1}^{\prime}
\end{array}\right]+\left[\begin{array}{ll}
\alpha^{2}(2 \bar{\alpha}+1) & -\alpha^{2} \bar{\alpha} \\
\beta^{2}(2 \bar{\beta}+1) & -\beta^{2} \bar{\beta}
\end{array}\right]\left[\begin{array}{c}
S_{i}^{(0)}-y_{i} \\
S_{i}^{(1)}-h y_{i}^{\prime}
\end{array}\right] \\
& =C(\alpha, \beta)\left[\begin{array}{c}
S_{i-1}^{(0)}-y_{i-1} \\
S_{i-1}^{(1)}-h y_{i-1}^{\prime}
\end{array}\right]+D(\alpha, \beta)\left[\begin{array}{c}
S_{i}^{(0)}-y_{i} \\
S_{i}^{(1)}-h y_{i}^{\prime}
\end{array}\right]+O\left(h^{4}\right), \tag{11}
\end{align*}
$$

$\forall \alpha, \beta \in[0,1]$, then there must exist constants $\lambda_{1}>0, \lambda_{2}>0$ such that

$$
\begin{align*}
& \left\|\epsilon_{i-1}^{\prime}\right\| \leq \lambda_{1}\left\|\epsilon_{i-1}\right\|+\lambda_{2}\left\|\epsilon_{i}\right\|+C h^{4} \leq \lambda X_{i}+C h^{4}, \\
& \left\|\tilde{\epsilon}_{i}\right\| \leq \max \left\{\lambda X_{i-m}, \lambda X_{i-m+1}\right\}+C h^{4} \leq \lambda X_{i}+C h^{4}, \quad \forall i \geq 1, \tag{12}
\end{align*}
$$

where $C$ is an appropriate constant. Combine Theorem 3.1 and Eq. (8) with the following formula

$$
\begin{align*}
y(t)= & \left(1+2 \frac{t-t_{i-1}}{h}\right)\left(\frac{t_{i}-t}{h}\right)^{2} y\left(t_{i-1}\right)+\frac{\left(t-t_{i-1}\right)\left(t_{i}-t\right)^{2}}{h^{2}} y^{\prime}\left(t_{i-1}\right) \\
& +\left(1+2 \frac{t_{i}-t}{h}\right)\left(\frac{t-t_{i-1}}{h}\right)^{2} y\left(t_{i}\right)-\frac{\left(t-t_{i-1}\right)^{2}\left(t_{i}-t\right)}{h^{2}} y^{\prime}\left(t_{i-1}\right)+O\left(h^{4}\right), \quad \forall t \in\left[t_{i-1}, t_{i}\right], \tag{13}
\end{align*}
$$

we have

$$
\left[\begin{array}{c}
S\left(t_{i}\right)-y\left(t_{i}\right) \\
h S^{\prime}\left(t_{i}\right)-h y^{\prime}\left(t_{i}\right)
\end{array}\right]=A\left[\begin{array}{c}
S\left(t_{i-1}\right)-y\left(t_{i-1}\right) \\
h S^{\prime}\left(t_{i-1}\right)-h y^{\prime}\left(t_{i-1}\right)
\end{array}\right]+h B\left[\begin{array}{c}
f_{i-1+\alpha}-f\left(t_{i-1+\alpha}, y\left(t_{i-1+\alpha}\right), y\left(t_{i-1+\alpha}-\tau\right)\right) \\
f_{i-1+\beta}-f\left(t_{i-1+\beta}, y\left(t_{i-1+\beta}\right), y\left(t_{i-1+\beta}-\tau\right)\right)
\end{array}\right]+O\left(h^{4}\right) .
$$

By using Lipschitz condition (2), we obtain

$$
\begin{align*}
\left\|\epsilon_{i}\right\| & \leq\|A\|\left\|\epsilon_{i-1}\right\|+h\|B\|\left[L_{1}\left(\lambda_{1}\left\|\epsilon_{i-1}\right\|+\lambda_{2}\left\|\epsilon_{i}\right\|\right)+L_{2} \tilde{\epsilon}_{i}\right]+C h^{4}, \\
& \leq\|A\|\left\|\epsilon_{i-1}\right\|+h\|B\| \lambda\left(L_{1}+L_{2}\right) X_{i}+C h^{4} \\
& \leq\|A\|\left\|\epsilon_{i-1}\right\|+h\|B\| \lambda\left(L_{1}+L_{2}\right)\left\|\epsilon_{i}\right\|+C h^{4}+h\|B\| \lambda\left(L_{1}+L_{2}\right) X_{i-1}, \tag{14}
\end{align*}
$$

and

$$
\begin{equation*}
\left[1-h\|B\| \lambda\left(L_{1}+L_{2}\right)\right]\left\|\epsilon_{i}\right\| \leq\left[\|A\|+h\|B\| \lambda\left(L_{1}+L_{2}\right)\right] X_{i-1}+C h^{4} \tag{15}
\end{equation*}
$$

Noting that $\|A\| \leq 1$, from the above equation we obtain

$$
\begin{equation*}
\left\|\epsilon_{i}\right\| \leq \frac{\|A\|+h\|B\| \lambda\left(L_{1}+L_{2}\right)}{1-h\|B\| \lambda\left(L_{1}+L_{2}\right)} X_{i-1}+C h^{4} \leq \frac{1+h\|B\| \lambda\left(L_{1}+L_{2}\right)}{1-h\|B\| \lambda\left(L_{1}+L_{2}\right)} X_{i-1}+C h^{4} \tag{16}
\end{equation*}
$$

Since $\|B\| \lambda\left(L_{1}+L_{2}\right)>0$, for any $c_{0} \in(0,1)$, we select $h$ such that $h\|B\| \lambda\left(L_{1}+L_{2}\right) \leq c_{0}$, then

$$
\begin{align*}
\left\|\epsilon_{i}\right\| \leq X_{i} & \leq\left(1+c_{1} h\right) X_{i-1}+C h^{4} \\
& \leq\left(1+c_{1} h\right)^{i} X_{0}+C h^{4} \sum_{j=0}^{i-1}\left(1+h c_{1}\right)^{j} \\
& \leq \exp \left(i h c_{1}\right) X_{0}+C h^{4} N \exp \left(i h c_{1}\right) \\
& \leq \exp \left(T c_{1}\right) X_{0}+C h^{4} N \exp \left(T c_{1}\right) \tag{17}
\end{align*}
$$

where $c_{1}=\frac{2 L\|B\| \lambda\left(L_{1}+L_{2}\right)}{1-c_{0}}$. From Theorem 3.1 and the definition of $\left\|\epsilon_{i}\right\|$, it always gives

$$
\left\|\epsilon_{i}\right\| \rightarrow 0 \quad \text { as } h \rightarrow 0
$$

when the initial error $\left\|\varepsilon_{0}\right\|=0$. Consequently, numerical method (8) is convergence.
Theorem 3.3 (Nonlinear Stability). If $\|A\| \leq 1$, numerical method (8) is stable, i.e. there exists a constant $C$, for any perturbation $\epsilon_{i}, i=0,1,2, \ldots$, such that

$$
\left\|\varepsilon_{i}\right\| \leq C\left\|\varepsilon_{0}\right\|
$$

where

$$
\epsilon_{i}=\binom{\left|z_{i}-y_{i}\right|}{\left|h z_{i}^{\prime}-h y_{i}^{\prime}\right|} .
$$

Proof. Consider the initial value problem

$$
\left\{\begin{array}{l}
y^{\prime}(t)=f(t, y(t), y(t-\tau)), \quad 0 \leq t \leq T  \tag{18}\\
y(t)=\varphi(t), \quad-\tau \leq t \leq 0
\end{array}\right.
$$

and the perturbation problem

$$
\left\{\begin{array}{l}
z^{\prime}(t)=f(t, z(t), z(t-\tau)), \quad 0 \leq t \leq T,  \tag{19}\\
z(t)=\psi(t), \quad-\tau \leq t \leq 0,
\end{array}\right.
$$

where $f$ satisfies Lipschitz condition (2).

Denoting that

$$
\begin{aligned}
\epsilon_{i} & =\binom{\left|z_{i}-y_{i}\right|}{\left|h z_{i}^{\prime}-h y_{i}^{\prime}\right|}, \quad \epsilon_{i}^{\prime}=\binom{\left|S_{z}\left(t_{i+\alpha}\right)-S_{y}\left(t_{i+\alpha}\right)\right|}{\left|S_{z}\left(t_{i+\beta}\right)-S_{y}\left(t_{i+\beta}\right)\right|}, \\
\tilde{\epsilon}_{i} & =\binom{\left|S_{z}\left(t_{i-1+\alpha}-\tau\right)-S_{y}\left(t_{i-1+\alpha}-\tau\right)\right|}{\left|S_{z}\left(t_{i-1+\beta}-\tau\right)-S_{y}\left(t_{i-1+\beta}-\tau\right)\right|}, \\
X_{i} & =\max _{0 \leq s \leq i}\left\{\left\|\epsilon_{s}\right\|, \max _{-\tau \leq t \leq 0}\|\psi(t)-\varphi(t)\|\right\}, \quad X_{0}=\max _{-\tau \leq t \leq 0}\|\psi(t)-\varphi(t)\|,
\end{aligned}
$$

where $S_{y}(\cdot), S_{z}(\cdot)$ are the spline collocation functions of (18) and (19), respectively.
Noting that the definitions of the collocation functions $S_{y}(t), S_{z}(t)$, there must exist constants $\lambda_{1}>0, \lambda_{2}>0, \lambda>0$ which satisfy

$$
\begin{align*}
& \left\|\epsilon_{i-1}^{\prime}\right\| \leq \lambda_{1}\left\|\epsilon_{i-1}\right\|+\lambda_{2}\left\|\epsilon_{i}\right\| \leq \lambda X_{i} \\
& \left\|\tilde{\epsilon}_{i}\right\| \leq \max \left\{\lambda X_{i-m}, \lambda X_{i-m+1}\right\} \leq \lambda X_{i}, \quad \forall i \geq 1 \tag{20}
\end{align*}
$$

By means of Theorem 3.1 and (8), using Lipschitz condition (2), we have

$$
\begin{align*}
\left\|\epsilon_{i}\right\| & \leq\|A\|\left\|\epsilon_{i-1}\right\|+h\|B\|\left[L_{1}\left(\lambda_{1}\left\|\epsilon_{i-1}\right\|+\lambda_{2}\left\|\epsilon_{i}\right\|\right)+L_{2} \tilde{\epsilon}_{i}\right] \\
& \leq\|A\|\left\|\epsilon_{i-1}\right\|+h\|B\| \lambda\left(L_{1}+L_{2}\right) X_{i} \\
& \leq\|A\|\left\|\epsilon_{i-1}\right\|+h\|B\| \lambda\left(L_{1}+L_{2}\right)\left\|\epsilon_{i}\right\|+h\|B\| \lambda\left(L_{1}+L_{2}\right) X_{i-1}, \tag{21}
\end{align*}
$$

and

$$
\begin{equation*}
\left[1-h\|B\| \lambda\left(L_{1}+L_{2}\right)\right]\left\|\epsilon_{i}\right\| \leq\left[\|A\|+h\|B\| \lambda\left(L_{1}+L_{2}\right)\right] X_{i-1} . \tag{22}
\end{equation*}
$$

Since $\|A\| \leq 1$, then from the above equation we have

$$
\begin{equation*}
\left\|\epsilon_{i}\right\| \leq \frac{\|A\|+h\|B\| \lambda\left(L_{1}+L_{2}\right)}{1-h\|B\| \lambda\left(L_{1}+L_{2}\right)} X_{i-1} \leq \frac{1+h\|B\| \lambda\left(L_{1}+L_{2}\right)}{1-h\|B\| \lambda\left(L_{1}+L_{2}\right)} X_{i-1} \tag{23}
\end{equation*}
$$

Noting that $\|B\| \lambda\left(L_{1}+L_{2}\right)>0$, for any $c_{0} \in(0,1)$, we select $h$ such that $h\|B\| \lambda\left(L_{1}+L_{2}\right) \leq c_{0}$, then

$$
\begin{align*}
\left\|\epsilon_{i}\right\| \leq X_{i} & \leq\left(1+c_{1} h\right) X_{i-1} \leq\left(1+c_{1} h\right)^{i} X_{0} \\
& \leq \exp \left(i h c_{1}\right) X_{0} \leq \exp \left(T c_{1}\right) X_{0} \tag{24}
\end{align*}
$$

where $c_{1}=\frac{2 L\|B\| \lambda\left(L_{1}+L 2\right)}{1-c_{0}}$. Let $C=\exp \left(T c_{1}\right)$, it shows that the numerical method (8) is stable.

## 4. Linear stability analysis

Consider the initial value problem of DDE

$$
\left\{\begin{array}{l}
y^{\prime}(t)=\lambda y(t)+q y(t-\tau), \quad t \geq 0, \tau>0  \tag{25}\\
y(t)=\varphi(t), \quad t \leq 0
\end{array}\right.
$$

where $\lambda, q \in \mathbb{C}$ are any given parameters, $\tau$ is a constant delay. For the purpose of study the linear stability of the method for solving problem (25), we introduce the following definition and theorems.

Theorem 4.1 ([3]). If the coefficients $\lambda, q$ of Eq. (25) satisfy

$$
\begin{equation*}
\operatorname{Re}(\lambda)<0, \quad|q|+\operatorname{Re}(\lambda)<0, \tag{26}
\end{equation*}
$$

then for any initial function $\varphi \in C(-\infty, 0]$, the solution of problem (25) $y(t)$ satisfies

$$
\lim _{t \rightarrow+\infty} y(t)=0
$$

Definition 4.1 ([3] (P-Stability)). If the coefficients $\lambda$, $q$ of Eq. (25) satisfy (26), and the numerical solutions $y_{n}$ of the method for solving problem (25) satisfies

$$
\lim _{n \rightarrow \infty} y_{n}=0
$$

then the numerical method is $P$-stable, where $y_{n} \approx y(n h), h>0$ is stepsize, $m h=\tau, m$ is a positive integer.

Now, applying the cubic spline collocation method to solve problem (25), we have

$$
\begin{equation*}
S^{\prime}\left(t_{i-1+\phi}\right)=\lambda S\left(t_{i-1+\phi}\right)+q S\left(t_{i-1-m+\phi}\right), \quad \phi \in\{\alpha, \beta\}, i=1, \ldots, N, \tag{27}
\end{equation*}
$$

where $h=\frac{\tau}{m}, m \geq 1$, and $m$ is an integer, $S\left(t_{i-1-m+\phi}\right)=S\left(t_{i-1+\phi}-m h\right)$.
Let $z=h \lambda, v=h q$, then from the above equation, we have

$$
\begin{align*}
& \bar{\phi}(-6 \phi) S_{i-1}^{(0)}+\bar{\phi}(1-3 \phi) S_{i-1}^{(1)}+\phi(6 \bar{\phi}) S_{i}^{(0)}+\phi\left(1-\phi^{\prime}\right) \\
&= z\left[\bar{\phi}^{2}(2 \phi+1) S_{i-1}^{(0)}+\bar{\phi}^{2} \phi S_{i-1}^{(1)}+\phi^{2}(2 \bar{\phi}+1) S_{i}^{(0)}-\phi^{2} \bar{\phi} S_{i}^{(1)}\right] \\
&+v\left[\bar{\phi}^{2}(2 \phi+1) S_{i-m-1}^{(0)}+\bar{\phi}^{2} \phi S_{i-m-1}^{(1)}+\phi^{2}(2 \bar{\phi}+1) S_{i-m}^{(0)}-\phi^{2} \bar{\phi} S_{i-m}^{(1)}\right] \tag{28}
\end{align*}
$$

and

$$
A_{1} \underline{S}_{i}+B_{1} \underline{S}_{i-m}=A_{2} \underline{S}_{i-1}+B_{2} \underline{S}_{i-m-1},
$$

where $\underline{S}_{i}=\left(S_{i}^{(0)}, S_{i}^{(1)}\right)^{T}, \underline{S}_{i-m}=\left(S_{i-m}^{(0)}, S_{i-m}^{(1)}\right)^{T}$,

$$
\begin{array}{ll}
A_{1}=\left(\begin{array}{lll}
6 \alpha \bar{\alpha}-z \alpha^{2}(1+2 \bar{\alpha}) & \alpha(1-3 \bar{\alpha})+z \alpha^{2} \bar{\alpha} \\
6 \beta \bar{\beta}-z \beta^{2}(1+2 \bar{\beta}) & \beta(1-3 \bar{\beta})+z \beta^{2} \bar{\beta}
\end{array}\right), & B_{1}=\left(\begin{array}{ll}
-v \alpha^{2}(1+2 \bar{\alpha}) & v \alpha^{2} \bar{\alpha} \\
-v \beta^{2}(1+2 \bar{\beta}) & v \beta^{2} \bar{\beta}
\end{array}\right), \\
A_{2}=\left(\begin{array}{lll}
6 \alpha \bar{\alpha}+z \bar{\alpha}^{2}(1+2 \alpha) & \bar{\alpha}(3 \alpha-1)+z \bar{\alpha}^{2} \alpha \\
6 \beta \bar{\beta}+z \bar{\beta}^{2}(1+2 \beta) & \bar{\beta}(3 \beta-1)+z \bar{\beta}^{2} \beta
\end{array}\right), & B_{2}=\left(\begin{array}{ll}
v \bar{\alpha}^{2}(1+2 \alpha) & v \bar{\alpha}^{2} \alpha \\
v \bar{\beta}^{2}(1+2 \beta) & v \bar{\beta}^{2} \beta
\end{array}\right) .
\end{array}
$$

Hence, we can obtain

$$
\left(A_{1}, B_{1}\right)\binom{\underline{S}_{i}}{\underline{S}_{i-m}}=\left(A_{2}, B_{2}\right)\binom{\underline{S}_{i-1}}{\underline{S}_{i-m-1}} .
$$

Denoting $C(z, v)=\left(A_{1}, B_{1}\right), D(z, v)=\left(A_{2}, B_{2}\right), M_{i}=\left(\underline{S}_{i}, \underline{S}_{i-m}\right)^{T}, M_{i-1}=\left(\underline{S}_{i-1}, \underline{S}_{i-m-1}\right)^{T}$, and using Definition 4.1 and Theorem 4.1, we obtain the following theorem.

Theorem 4.2 ([1]). The numerical method (27) is P-stable, if the coefficients $\lambda, q$ (or $z=\lambda h, v=q h$ ) of Eq. (25) satisfy (26), and the eigenvalues $\mu_{j}(z, v)$ of the following problem

$$
\begin{equation*}
\mu C(z, v) \cdot x=D(z, v) \cdot x, \quad x \neq \mathbf{0} \tag{29}
\end{equation*}
$$

are in the unit disk, i.e.

$$
\begin{equation*}
\left|\mu_{j}(z, v)\right|<1, j=1,2 \tag{30}
\end{equation*}
$$

Next, we investigate the value range of $\alpha$ and $\beta$ when the numerical method is $P$-stable and design algorithm as follows

## Algorithm 1.

```
For \(0 \leq \theta \leq 2 \pi\)
    \(z=a e^{\overline{i \theta}} ; v=b e^{i \theta}\);
    if \(\operatorname{Re}(z)+|v|<0\)
    For \(0 \leq \alpha \leq 1\)
        For \(\alpha<\beta \leq 1\)
            \(\operatorname{root}\left(\operatorname{det}\left(\zeta C(z, v) C(z, v)^{T}-D(z, v) D(z, v)^{T}\right)=0\right)\)
                if \(\left|\left\{\operatorname{root}\left(\operatorname{det}\left(\zeta C(z, v) C(z, v)^{T}-D(z, v) D(z, v)^{T}\right)=0\right)\right\}^{\frac{1}{2}}\right|<1\)
                        record \(\alpha, \beta\);
            end
        end
        end
        end
    end
```

Using Algorithm 1, we can obtain the results of $\alpha, \beta$ as in Fig. 1, from which we can see that the ranges of $\alpha, \beta$ satisfying $P$ stability depend on the coefficients $z, v$ of problem (25).

## 5. Illustrative examples

In order to verify our theoretical results, we present some numerical examples in this section.


Fig. 1a. The ranges of $\alpha, \beta$ satisfying $P$-stability, $a=-2, b=1$.


Fig. 1b. The ranges of $\alpha, \beta$ satisfying $P$-stability, $a=-5, b=1$.


Fig. 1c. The ranges of $\alpha, \beta$ satisfying $P$-stability, $a=-10, b=1$.
Example 5.1. Consider the initial value problem as follows

$$
\left\{\begin{array}{l}
y^{\prime}(t)=A y(t)+y\left(t-\frac{3 \pi}{2}\right)-A \sin (t), \quad 0 \leq t \leq T  \tag{31}\\
y(t)=\mathrm{e}^{-p t}+\sin (t), \quad-\frac{3 \pi}{2} \leq t \leq 0
\end{array}\right.
$$

where $A=-p-\mathrm{e}^{-\frac{3 p \pi}{2}}, T=10, p=1.0$. The exact solution is $y(t)=\mathrm{e}^{-p t}+\sin (t)$. If we select two collocation points $\alpha=0.5, \beta=1.0$ and apply the cubic spline collocation method (8) to solve problem (31), then the results of error estimation and error order can be obtained as shown in Table 1 . When $h=0.1$ and $h=0.01$, the error of the numerical solutions and the exact solution are shown in Fig. 2. From the numerical results we find that the error between the numerical solutions and the exact solution is very small. The order of error is very high. In particular, we select $T=100, h=0.1$ and obtain the numerical result as in Fig. 3. From this we can see that the error is also very small and almost tends to $10^{-7}$, which show that the convergence of the cubic spline method for solving ODEs can be verified and the method is robust.


Fig. 1d. The ranges of $\alpha, \beta$ satisfying $P$-stability, $a=-100, b=1$.

Table 1
The error of numerical solution and exact solution of problem (31) when $\alpha=0.5, \beta=1.0$.

| $h$ | $\\|$ error $1 \\|_{L_{1}}$ | Order | $\\|$ error $\\|_{L_{2}}$ | Order |
| :--- | :--- | :--- | :--- | :--- |
| $\frac{10}{50}$ | $1.845504 \times 10^{-6}$ | - | $2.066386 \times 10^{-6}$ | - |
| $\frac{10}{100}$ | $1.215364 \times 10^{-7}$ | 3.9246 | $1.359508 \times 10^{-7}$ | 3.9260 |
| $\frac{10}{200}$ | $7.234833 \times 10^{-9}$ | 4.0478 | $8.108477 \times 10^{-9}$ | 4.0675 |
| $\frac{10}{400}$ | $7.587401 \times 10^{-11}$ | 6.5983 | $1.440994 \times 10^{-10}$ | 5.8143 |



Fig. 2a. The error of numerical solution and true solution of problem (31) with $h=0.1$.

Example 5.2. Consider the following initial value problem

$$
\left\{\begin{array}{l}
y^{\prime}(t)=\frac{1}{2} \mathrm{e}^{\frac{t}{2}} y\left(\frac{t}{2}\right)+\frac{1}{2} y(t), \quad 0 \leq t \leq 10  \tag{32}\\
y(0)=1
\end{array}\right.
$$

the exact solution of which is $y(t)=e^{t}$. In our theoretical analysis, we have not involved the cubic spline collocation method to solve proportion (or variable) DDEs. In fact, the method for solving this problem is also very effective. First, let $\alpha=0.5, \beta=1.0, h=0.1$ and $\alpha=0.5, \beta=1.0, h=0.01$, and using the cubic spline collocation method to solve (32), respectively, we can obtain the relative error of the numerical solution and the exact solution as shown in Fig. 4, from which we know that the accuracy of the method is still very high and the numerical solution is very reliable.

Example 5.3. Consider the initial value problem

$$
\left\{\begin{array}{l}
y^{\prime}(t)=(-2+0.5 \cos (t)) y(t)+y(t-2), \quad 0 \leq t \leq T,  \tag{33}\\
y(3)=\sin (t), \quad t \leq 0,
\end{array}\right.
$$



Fig. 2b. The error of numerical solution and true solution of problem (31) with $h=0.01$.


Fig. 3. The error of numerical solution and true solution of problem (31), $h=0.1, T=100$.


Fig. 4a. The error of numerical solution and true solution of (32), $h=0.1$.
and the perturbation problem

$$
\left\{\begin{array}{l}
z^{\prime}(t)=(-2+0.5 \cos (t)) z(t)+z(t-2), \quad 0 \leq t \leq T,  \tag{34}\\
z(3)=\cos (t), \quad t \leq 0 .
\end{array}\right.
$$

We use the cubic spline collocation method to solve problems (33) and (34). Selecting $\alpha=0.5, \beta=1.0, T=25$ and $h=0.1, h=0.01$, we obtain the errors of numerical solutions as in Fig. 5. From Fig. 5 we can see it clearly that as $t$


Fig. 4b. The error of numerical solution and true solution of (32), $h=0.01$.


Fig. 5a. The error of numerical solutions of (33) and (34), $h=0.1$.


Fig. 5b. The error of numerical solutions of (33) and (34), $h=0.01$.
increases the absolute error of numerical solutions of problem (33) and (34) constantly decreases, and finally tends to 0 . Thus, we can draw the conclusion that cubic spline collocation method (8) for DDEs is stable, which precisely verifies our theoretical analysis.

## 6. Conclusion

In this paper, we have successfully applied the cubic spline collocation method to solve DDEs, and obtained some theorems of local truncation error and convergence. The analysis of nonlinear stability and linear stability of the method has also been given. Moreover, we have designed an algorithm for solving the ranges of the two parameters $\alpha, \beta$ when the method is $P$-stable. In particular, we have successfully obtained the numerical solution of proportion delay differential equations (PDDEs) by using the cubic spline collocation method. In fact, this method is efficient for solving some other differential equations such as neutral delay differential equations (NDDEs). Finally, the numerical results have shown that the cubic spline collocation method for solving DDEs is very robust and efficient.

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[^0]:    * Corresponding author at: School of Mathematics and Computational Science, Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Xiangtan University, Hunan, 411105, China.

    E-mail addresses: lucretine@126.com (H. Su), yang52053052@yahoo.com.cn (S.-P. Yang), lpwen@xtu.edu.cn (L.-P. Wen).

