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# Ordering the Zeroes of Legendre Functions $P_\nu^m(z_0)$ When Considered as a Function of $\nu$

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In the following, we will consider the problem of ordering the zeroes of the Legendre functions $P_\nu^m(z_0)$ when $m$ is a nonnegative integer and $0<z_0<1$. Let $\nu=\nu_j^m(z_0)$ denote the $j$th positive root of $P_\nu^m(z_0)=0$, where $j=1, 2, \ldots$. It is well known from the Sturm–Liouville theory that $\nu_j^m(z_0)<\nu_j^{m+1}(z_0)<\nu_{j+1}^m(z_0)$. We will show that $\nu_j^{m+2}(z_0)<\nu_{j+1}^m(z_0)$. Using these and several other inequalities, we will also show that $\nu_1^0<\nu_1^1<\nu_1^2<\nu_2^0<\nu_1^3<\nu_2^1<\nu_1^4<\nu_2^2<\nu_3^0<\nu_1^5$ for all $0<z_0<1$. Moreover, this is the unique ordering of the first ten $\nu_j^m(z_0)$'s for $0<z_0<1$. © 1990 Academic Press, Inc.

## 1. INTRODUCTION

Let $m$ denote a nonnegative integer. For fixed $z_0 \in [0, 1)$, we will denote the positive zeroes of the Legendre function $P_\nu^m(z_0)=0$ by $\nu=\nu_j^m(z_0)$, where $j=1, 2, \ldots$. In this paper, we will be concerned with questions that are related to ordering the $\nu_j^m(z_0)$'s. For example, given $\nu_j^m(z_0)$ and $\nu_i^n(z_0)$ with $0<z_0<1$:

[Q1] What conditions on $(m, j)$ and $(n, i)$ imply $\nu_j^m(z_0)<\nu_i^n(z_0)$?

[Q1] was motivated by questions about the order and multiplicity of the eigenvalues of the Laplacian on a spherical cap (see [1]). The Sturm–Liouville theory provides some partial answers to these questions. In particular (see [11, Chap. VII]),

$$\nu_j^m(z_0)<\nu_j^{m+1}(z_0)<\nu_{j+1}^m(z_0). \tag{1.1}$$

However, the above inequalities are not sufficient to order all the $\nu_j^m(z_0)$'s. To the best of the author's knowledge, it is not known whether or not

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[Q1] has been considered for the $\nu$-zeroes of the Legendre functions. On the other hand, the related problem of how a zero "$z_0$" behaves as the degree $\nu$ is varied has been studied via the Sturm comparison theorem (see [7–9] for a discussion of this and related results).

In this paper, we will present some results that are related to [Q1]. In general, the condition $\nu_j^m(z_0) = \nu_i^n(z_0)$ does not imply $(m, j) = (n, i)$ (see the Remark at the end of Section 2). However, in Section 4, we show that the set of $z_0$'s, for which $\nu_j^m(z_0) = \nu_i^n(z_0)$ and $(m, j) \neq (n, i)$, has measure 0. In Section 2, we list some properties that are related to Legendre functions. In addition, we show that $\nu = \nu_j^m(z_0)$ is an analytic function of $z_0$ for $z_0 \in (0, 1)$ (see Lemma 1).

The main result of this paper are contained in Section 3, where we show that

$$\nu_j^{m+2}(z_0) < \nu_{j+1}^m(z_0), \tag{1.2}$$

$$\nu_2^0(z_0) < \nu_1^3(z_0), \tag{1.3}$$

$$\nu_2^1(z_0) < \nu_2^0(z_0) + 1, \tag{1.4}$$

$$\nu_2^1(z_0) < \nu_1^4(z_0), \tag{1.5}$$

$$\nu_3^0(z_0) < \nu_1^5(z_0). \tag{1.6}$$

Although (1.1)–(1.6) are not sufficient to order all the $\nu_j^m(z_0)$'s, they do imply that the first ten zeroes are (in increasing order)

$$\nu_1^0 < \nu_1^1 < \nu_1^2 < \nu_2^0 < \nu_1^3 < \nu_2^1 < \nu_1^4 < \nu_2^2 < \nu_3^0 < \nu_1^5 \qquad \text{for all} \quad 0 < z_0 < 1. \tag{1.7}$$

## 2. The Legendre Functions and Related Properties

In the following section, we list some of the known properties related to the Legendre functions and their zeroes. Unless otherwise stated, throughout the remainder of this paper $m$ will always denote a nonnegative integer and $i, j, k, n$ will always denote positive integers.

A function $y = P_\nu^m(z)$ that satisfies

$$\frac{d}{dz}\left((1-z^2)\frac{d}{dz}y\right) + \left(\nu(\nu+1) - \frac{m^2}{1-z^2}\right)y = 0, \qquad -1 < z < 1, \tag{2.1a}$$

$$y(1) \text{ bounded}, \tag{2.1b}$$

is called a Legendre function of the first kind of degree $\nu$ and order $m$. If $-1 < z \leqslant 1$ and $\nu > 0$, then $P_\nu^m(z)$ can be expressed as (see [2, p. 148])

$$P_\nu^m(z) = \frac{(-1)^m \Gamma(\nu+m+1)}{2^m m!\, \Gamma(\nu-m+1)} (1-z^2)^{m/2}$$
$$\times \sum_{n=0}^{\infty} \frac{(1+m+\nu)_n (m-\nu)_n}{(m+1)_n\, n!\, 2^n} (1-z)^n, \tag{2.2}$$

where $\Gamma(z)$ is the gamma function and $(a)_n$ denotes the Pochhammer symbol,

$$(a)_0 = 1,$$
$$(a)_n = a(a+1) \cdots (a+n-1).$$

There are various identities which relate contiguous Legendre functions. We list a few that will be needed later (see [2, p. 161]):

$$P_\nu^{m+2}(z) + 2(m+1)\, z(1-z^2)^{-1/2} P_\nu^{m+1}(z) + (\nu-m)(\nu+m+1) P_\nu^m(z) = 0, \tag{2.3a}$$

$$P_{\nu-1}^m(z) - P_{\nu+1}^m(z) = (2\nu+1)(1-z^2)^{1/2} P_\nu^{m-1}(z), \tag{2.3b}$$

$$(\nu-m)(\nu-m+1) P_{\nu+1}^m(z) - (\nu+m)(\nu+m+1) P_{\nu-1}^m(z)$$
$$= (2\nu+1)(1-z^2)^{1/2} P_\nu^{m+1}(z). \tag{2.3c}$$

It is well known that $P_\nu^m(z) = P^m{}_{-\nu-1}(z)$ (see [2, p. 122]). We will consider only nonnegative values of $\nu$.

For a fixed $z_0 \in [0, 1)$, we will refer to the problem of finding a pair $(\nu(\nu+1), y)$ that satisfies (2.1) and the boundary condition

$$y(z_0) = 0, \tag{2.4}$$

as eigenvalue problem ( # ). For such a pair, $\nu(\nu+1)$ is called an eigenvalue and $y$ the corresponding eigenfunction. When $0 \leqslant z_0 < 1$, the solutions of ( # ) are pairs

$$(\nu_j^m(\nu_j^m+1), P_{\nu_j^m}^m(z)), \tag{2.5}$$

where $P_{\nu_j^m}^m(z_0) = 0$, $m = 0, 1, \ldots$, and $j = 1, 2, \ldots$.

For fixed $m$, $P_\nu^m(z)$ is an analytic function of $\nu$ in the neighborhood of any point $\nu_0$ at which the function $P_{\nu_0}^m(z)$ is finite [5, p. 189]. Moreover, the roots of

$$f(\nu) = P_\nu^m(z_0) = 0 \tag{2.6a}$$

are simple, i.e. (see [6, p. 241]),

$$f'(\nu_j^m) \neq 0. \tag{2.6b}$$

To emphasize that the $\nu_j^m$'s depend on $z_0$, we write $\nu_j^m = \nu_j^m(z_0)$. Questions on how the spectrum of an eigenvalue problem behaves as a parameter, such as $z_0$, is varied have been studied by a number of authors (see [11, p. 174] or [3, p. 409]). The problem of how a root of a Legendre function varies as a function of $\nu$ is considered in [7, 8]. We are interested in how a typical $\nu_j^m(z_0)$ behaves as a function of the parameter $z_0$ for $0 < z_0 < 1$. Each $\nu_j^m(z_0)$ is increasing in $z_0$ when $0 < z_0 < 1$ (see [4]). As the following result shows, each $\nu_j^m$ is analytic on (0, 1).

LEMMA 1. *For every nonnegative integer m and positive integer j, there is a unique function $\nu = \nu_j^m(\alpha)$ that is analytic for $\alpha \in (0, 1)$ and satisfies $P_\nu^m(\alpha) = 0$ for $\alpha \in (0, 1)$.*

*Proof.* Let $\nu_i = \nu_j^m(\alpha_i)$ denote the $j$th positive zero of (2.6a) for $z_0 = \alpha_i$, $i = 1, ..., N$, $\varepsilon = \alpha_1 < \alpha_2 < \cdots < \alpha_N = 1 - \varepsilon$ $(0 < \varepsilon \ll 1)$. From (2.6b), it follows that

$$\left.\frac{\partial P_\nu^m(\alpha_i)}{\partial \nu}\right|_{\nu = \nu_i} \neq 0.$$

By the implicit function theorem, there exist $g_i(\alpha)$ and $U_i = \{\alpha : |\alpha - \alpha_i| < \delta_i\}$ such that for $i = 1, ..., N$,

(i) $U_i \subset (0, 1)$,

(ii) $g_i(\alpha_i) = \nu_i$, $g_i(\alpha)$ is unique and analytic for $\alpha \in U_i$,

(iii) $P_{g_i(\alpha)}^m(\alpha) = 0$ for all $\alpha \in U_i$.

Since $[\varepsilon, 1 - \varepsilon]$ is compact, one can choose a finite number of $U_i$'s that cover $[\varepsilon, 1 - \varepsilon]$. Since the $g_i$'s are unique on $U_i$, it follows that $g_i(\alpha) = g_j(\alpha)$ whenever $\alpha \in U_i \cap U_j$. If we define $\nu_j^m(\alpha) = g_i(\alpha)$ for $\alpha \in U_i$, then it follows (since $\varepsilon$ can be chosen arbitrarily small) that we can denote the solution of $P_\nu^m(\alpha) = 0$ by $\nu = \nu_j^m(\alpha)$ for $0 < \alpha < 1$.

Suppose $m$ is a nonnegative integer and $k$ is a positive integer. From Rodrigues' formula [7, p. 246], it follows that $P_{m+2j-1}^m(0) = 0$. Hence, when $z_0 = 0$, the solutions of (#) are those pairs in the form (2.5) with

$$\nu_j^m(0) = m + 2j - 1. \tag{2.7}$$

Since each $\nu_j^m(z_0)$ is increasing as a function of $z_0$, it follows that

$$\nu_j^m(z_0) > m + 2j - 1 \qquad \text{for} \quad 0 < z_0 < 1. \tag{2.8}$$

In general, given an arbitrary $z_0 \in (0, 1)$, it is difficult to determine the $\nu_j^m(z_0)$'s without resorting to numerical methods. However, when $z_0$ is a root of a Legendre function with $\nu = n$ ($n$ a positive integer), one can easily determine certain of the $v_j^m(z_0)$'s. For $m \leqslant n$, $P_n^m(z)$ has $n-m$ zeroes on the interval $-1 < z < 1$ (see [7, p. 260]). These will be denoted by $z_{n,k}^m$ (i.e., $P_n^m(z_{n,k}^m) = 0$ and $k = 1, 2, ..., n-m$). By convention, $z_{n,k}^m > z_{n,k+1}^m$. Thus, if $z_0 = z_{n,k}^m$ for some integers $m$, $n$, and $k$, then it follows that

$$v_k^m(z_{n,k}^m) = n. \tag{2.9}$$

Bruns' inequalities give a way of determining intervals in which the $z_{n,k}$'s fall (when $m = 0$, we will drop the superscript and write $z_{n,k}$ for $z_{n,k}^0$). Let $\phi_{n,k} \in (0, \pi)$ be defined by the equation

$$\cos(\phi_{n,k}) = z_{n,k}.$$

Bruns' inequalities state that (see [7, p. 189])

$$\frac{k-\frac{1}{2}}{n+\frac{1}{2}}\pi < \phi_{n,k} < \frac{k}{n+\frac{1}{2}}\pi \qquad \text{for} \quad k = 1, 2, ..., n. \tag{2.10}$$

It follows from (2.10) that

$$\cos\left(\frac{k}{n+\frac{1}{2}}\pi\right) < z_{n,k} < \cos\left(\frac{k-\frac{1}{2}}{n+\frac{1}{2}}\pi\right) \qquad \text{for} \quad k = 1, 2, ..., n. \tag{2.11}$$

Although $\phi_{n,k}$ in (2.10) may lie in the interval $(0, \pi)$, in this paper we will be concerned with $\phi_{n,k} \in (0, \frac{1}{2}\pi]$. In [9], Szegö obtained a sharper estimate for an interval containing a $\phi_{n,k} \in (0, \frac{1}{2}\pi]$. In particular, he showed

$$\frac{j_k}{\sqrt{(n+\frac{1}{2})^2 + c}} < \phi_{n,k} < \frac{j_k}{n+\frac{1}{2}}, \qquad 0 < \phi_{n,k} \leqslant \tfrac{1}{2}\pi, \qquad k = 1, ..., n, \tag{2.12}$$

where $c = 1 - (2/\pi)^2$ and $j_k$ is the $k$th nonnegative zero of the Bessel function $J_0(z)$ (see [9, Sect. III; Eqs. (4) and (7)]). From (2.12), it follows that

$$\cos\left(\frac{j_k}{n+\frac{1}{2}}\right) < z_{n,k} < \cos\left(\frac{j_k}{\sqrt{(n+\frac{1}{2})^2 + c}}\right). \tag{2.13}$$

In the sections that follow, we will be interested in determining the order of the $v_j^m(z_0)$'s. Certain properties are well known. For example, from the general Sturm–Liouville theory, we have

$$v_j^m(z_0) < v_j^{m+1}(z_0) < v_{j+1}^m(z_0), \tag{2.14}$$

for every $0 < z_0 < 1$ (see [11, Chap. VII]). However, (2.14) does not give a complete ordering of the $v_j^m$'s.

*Remark.* In general, the $\nu_j^m$'s need not be distinct. For example, one can verify that $z_{14,3}^5 = z_{14,1}^{10} = 1/\sqrt{5}$. Hence, when $z_0 = 1/\sqrt{5}$, it follows that $\nu_3^5(z_0) = \nu_1^{10}(z_0) = 14$.

## 3. Ordering the Zeroes of $P_\nu^m(z_0)$

THEOREM 1. *If* $0 < z_0 < 1$, *then*

$$\nu_j^{m+1} < \nu_j^{m+2} < \nu_{j+1}^m < \nu_{j+1}^{m+1}. \tag{3.1}$$

*Proof.* The first and third inequalities of (3.1) follow from (2.14). That is,

$$\nu_{j+1}^m, \nu_j^{m+2} \in (\nu_j^{m+1}, \nu_{j+1}^{m+1}).$$

From (2.2), we see that

$$P_{m+1}^m(z_0) = \frac{(-1)^m \,\Gamma(2(m+1))}{2^m m!} (1 - z_0^2)^{m/2} z_0. \tag{3.2}$$

By (2.8), $\nu_1^{m+1} > m+1$. Moreover, since $P_{\nu_j^{m+1}}^m(z_0) \neq 0$, it follows from (3.2) that

$$\operatorname{sign}(P_{\nu_j^{m+1}}^m(z_0)) = (-1)^{m+j} \qquad \text{for} \quad j = 1, 2, .... \tag{3.3}$$

By applying (2.14), it follows that the sign of $P_\nu^{m+1}(z_0)$ must be constant for $\nu_j^{m+1} < \nu < \nu_{j+1}^{m+1}$, i.e.,

$$\operatorname{sign}(P_\nu^{m+1}(z_0)) = (-1)^{m+j+1} \qquad \text{for} \quad \nu_j^{m+1} < \nu < \nu_{j+1}^{m+1}. \tag{3.4}$$

From (2.3a) (with $\nu = \nu_j^{m+1}$ and $z = z_0$), it follows that the sign of $P_{\nu_j^{m+1}}^{m+2}(z_0)$ must be opposite the sign of $P_{\nu_j^{m+1}}^m(z_0)$, i.e.,

$$\operatorname{sign}(P_{\nu_j^{m+1}}^{m+2}(z_0)) = (-1)^{m+j+1}. \tag{3.5}$$

Both $P_\nu^m(z_0)$ and $P_\nu^{m+2}(z_0)$ change sign exactly once in the interval $(\nu_j^{m+1}, \nu_{j+1}^{m+1})$. Hence,

$$\operatorname{sign}(P_{\nu_{j+1}^{m+1}}^m(z_0)) = (-1)^{m+j+1}, \tag{3.6}$$

$$\operatorname{sign}(P_{\nu_{j+1}^{m+1}}^{m+2}(z_0)) = (-1)^{m+j+2}. \tag{3.7}$$

If we assume $\nu_{j+1}^m \leqslant \nu_j^{m+2}$, then it follows that

$$\begin{aligned}\operatorname{sign}(P_\nu^{m+2}(z_0)) &= \operatorname{sign}(P_\nu^{m+1}(z_0)) = \operatorname{sign}(P_\nu^m(z_0)) \\ &= (-1)^{m+j+1} \qquad \text{for} \quad \nu_{j+1}^m \leqslant \nu \leqslant \nu_j^{m+2}.\end{aligned} \tag{3.8}$$

Since (3.8) contradicts (2.3a), we conclude $v_j^{m+2} < v_{j+1}^m$ and the proof of Theorem 1 is complete.

*Remark* 1. From (2.7), it follows that $v_j^{m+2}(0) = v_{j+1}^m(0)$.

*Remark* 2. If $0 < z_0 < 1$ and $v_j^{m+n} = v_k^m$ for some nonnegative integers $m$, $n$, $j$, and $k$, it follows from Theorem 1 that $n \geqslant 3$.

From (2.7), we see that $v_2^1(0) = 4$ and $v_2^0(0) = 3$. The following result shows that the distance between $v_2^1(z_0)$ and $v_2^0(z_0)$ is strictly greater than 1 for all $0 < z_0 < 1$.

LEMMA 2. *If* $0 < z_0 < 1$, *then* $v_2^1(z_0) > v_2^0(z_0) + 1$.

*Proof.* The proof consists of two parts. First, we will show that $v_2^1(z_0) \neq v_0^2(z_0) + 1$ for any $z_0 \in (0, 1)$. Suppose there is a $z_0 \in (0, 1)$ for which $v_2^1(z_0) = v_2^0(z_0) + 1$. Let

$$v = v_2^0(z_0), \tag{3.9}$$

$$v + 1 = v_2^1(z_0). \tag{3.10}$$

From (2.3b) with $m = 1$ and $v$, $v + 1$ as above, we have

$$P_{v-1}^1(z_0) - P_{v+1}^1(z_0) = (2v+1)(1 - z_0^2)^{1/2} P_v(z_0). \tag{3.11}$$

However, from (3.9) and (3.10), we see that $v$ and $v + 1$ are zeroes of $P_v(z_0)$ and $P_v^1(z_0)$, respectively. From (3.11), it follows that

$$P_{v-1}^1(z_0) = 0. \tag{3.12}$$

From (3.12), we conclude that $v - 1 = v_j^1(z_0)$ for some $j$. However, $v_2^1(z_0) = v + 1$. Hence, $j = 1$, i.e.,

$$v_1^1(z_0) = v - 1. \tag{3.13}$$

Substituting (3.10) and (3.13) into (2.3c) with $m = 1$, we find

$$(2v+1)(1 - z_0^2)^{1/2} P_v^2(z_0) = 0. \tag{3.14}$$

Hence, $v = v_j^2(z_0)$ for some $j$. Moreover, from (3.9), we see that

$$v = v_2^0(z_0) = v_j^2(z_0) \qquad \text{for some} \quad z_0 \in (0, 1).$$

By Remark 2 following Theorem 1, this is impossible and it follows that

$$v_2^1(z_0) \neq v_2^0(z_0) + 1 \qquad \text{for all} \quad 0 < z_0 < 1. \tag{3.15}$$

Since $v_2^0(z_0)$, $v_2^1(z_0)$ are analytic for $0 < z_0 < 1$, it follows from (3.15) that either $\nu_2^1(z_0) > \nu_2^0(z_0) + 1$ or $\nu_2^1(z_0) < \nu_2^0(z_0) + 1$. In order to complete the proof of Lemma 2, we note that

$$z_{5,2}^1 = [(7 - 2\sqrt{7})/21]^{1/2} \doteq 0.2852, \quad z_{4,2} = [(15 - 2\sqrt{30})/35]^{1/2} \doteq 0.3399,$$

$$\nu_2^0(z_{5,2}^1) < \nu_2^0(z_{4,2}) = 4, \tag{3.16}$$

$$\nu_2^1(z_{5,2}^1) = 5. \tag{3.17}$$

Substracting (3.16) from (3.17), we obtain

$$\nu_2^1(z_{5,2}^1) > \nu_2^0(z_{5,2}^1) + 1. \tag{3.18}$$

By (3.18), we find that

$$\nu_2^1(z_0) > \nu_2^0(z_0) + 1 \qquad \text{for all} \quad 0 < z_0 < 1. \tag{3.19}$$

Before completing this section, we first note a few inequalities that will be needed later:

$$\frac{2\sqrt{2}}{\sqrt{n(n+1)+6}} < \frac{9\sqrt{3}}{2(2n+3)} < \sin\left(\frac{3\pi}{2n+3}\right) \qquad \text{for} \quad n \geqslant 3. \tag{3.20}$$

The first inequality in (3.20) is verified by a simple calculation. The second inequality in (3.20) follows from the observations that

$$\frac{3\sqrt{3}}{2\pi} x \leqslant \sin x \qquad \text{for} \quad 0 \leqslant x \leqslant \frac{\pi}{3},$$

$$\frac{3\pi}{2n+3} \leqslant \frac{\pi}{3} \qquad \text{when} \quad n \geqslant 3.$$

From (3.20) and (2.11) (with $n$ replaced by $n+1$ and $k=2$), we find that

$$\left(\frac{n(n+1)-2}{n(n+1)+6}\right)^{1/2} > \cos\left(\frac{3\pi}{2n+3}\right) > z_{n+1,2} \qquad \text{for} \quad n \geqslant 3. \tag{3.21}$$

Applying (2.13) and arguing in a similar fashion as above, one can also show that

$$\left(\frac{n(n+1)-6}{n(n+1)+18}\right)^{1/2} > \cos\left(\frac{j_2}{\sqrt{(n+\frac{1}{2})^2 + c}}\right) > z_{n,2} \qquad \text{for} \quad n \geqslant 5, \tag{3.22}$$

$$\left(\frac{n(n+1)-12}{n(n+1)+56}\right)^{1/2} > \cos\left(\frac{j_3}{\sqrt{(n+\frac{3}{2})^2 + c}}\right) > z_{n+1,3} \qquad \text{for} \quad n \geqslant 6. \tag{3.23}$$

THEOREM 2. *If* $0 \leqslant z_0 < 1$, *then* $\nu_2^0(z_0) < \nu_1^3(z_0)$.

*Proof.* Suppose there is a $z_0 \in [0, 1)$ such that

$$v = v_2^0(z_0) = v_1^3(z_0), \tag{A.1}$$

and $n$ is that integer for which

$$n \leqslant v < n+1. \tag{A.2}$$

From (2.7), it follows that $v_2^0(0) = 3$ and $v_1^3(0) = 4$. Moreover, $v_2^0(z_0)$, $v_1^3(z_0)$ are increasing in $z_0$. Hence, if (A.1) and (A.2) are true, then for such a $z_0$, it follows that $0 < z_0 < 1$ and $n \geqslant 4$. From (2.3a) with $m = 1$ and $m = 0$, we obtain

$$\begin{pmatrix} 4z_0(1-z_0^2)^{-1/2} & (v-1)(v+2) \\ 1 & 2z_0(1-z_0^2)^{-1/2} \end{pmatrix} \begin{pmatrix} P_v^2(z_0) \\ P_v^1(z_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.24}$$

By Remark 2 following Theorem 1, it follows that $P_v^2(z_0) \neq 0$ and $P_v^1(z_0) \neq 0$. From (3.24), it follows that

$$\det \begin{pmatrix} 4z_0(1-z_0^2)^{-1/2} & (v-1)(v+2) \\ 1 & 2z_0(1-z_0^2)^{-1/2} \end{pmatrix} = 0. \tag{3.25}$$

Solving for $z_0^2$ in (3.25), we obtain

$$z_0^2 = \frac{v(v+1)-2}{v(v+1)+6}. \tag{3.26}$$

For an $n$ satisfying (A.2), it follows from (3.21) and (3.26) that

$$z_0 = \left(\frac{v(v+1)-2}{v(v+1)+6}\right)^{1/2} \geqslant \left(\frac{n(n+1)-2}{n(n+1)+6}\right)^{1/2} > \cos\left(\frac{3\pi}{2n+3}\right) > z_{n+1,2}. \tag{3.27}$$

From (3.27), we conclude $z_0 > z_{n+1,2}$. Since $v_2^0(z_0)$ is increasing in $z_0$, it follows from (2.9) that

$$v_2^0(z_0) > v_2^0(z_{n+1,2}) = n+1. \tag{3.28}$$

However, (3.28) contradicts (A.2) and the proof of Theorem 2 is complete.

THEOREM 3. *If* $0 \leqslant z_0 < 1$, *then* $v_2^1(z_0) < v_1^4(z_0)$.

*Proof.* Suppose there is a $z_0 \in [0, 1)$ such that

$$v = v_2^1(z_0) = v_1^4(z_0), \tag{B.1}$$

and $n$ is that integer for which

$$n \leqslant v < n+1. \tag{B.2}$$

From (2.7), (B.1), and (B.2), we see that $n \geqslant 5$. From (2.3a) with $m=2$ and $m=1$, we obtain

$$\begin{pmatrix} 6z_0(1-z_0^2)^{-1/2} & (\nu-2)(\nu+3) \\ 1 & 4z_0(1-z_0^2)^{-1/2} \end{pmatrix} \begin{pmatrix} P_\nu^3(z_0) \\ P_\nu^2(z_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.29}$$

Since $P_\nu^3(z_0) \neq 0$ and $P_\nu^2(z_0) \neq 0$, from (3.29) it follows that

$$\det \begin{pmatrix} 6z_0(1-z_0^2)^{-1/2} & (\nu-2)(\nu+3) \\ 1 & 4z_0(1-z_0^2)^{-1/2} \end{pmatrix} = 0. \tag{3.30}$$

Solving for $z_0^2$ in (3.30), we obtain

$$z_0^2 = \frac{\nu(\nu+1)-6}{\nu(\nu+1)+18}. \tag{3.31}$$

For an $n$ satisfying (B.2), it follows from (3.22) and (3.31) that

$$z_0 = \left(\frac{\nu(\nu+1)-6}{\nu(\nu+1)+18}\right)^{1/2} \geqslant \left(\frac{n(n+1)-6}{n(n+1)+18}\right)^{1/2} > \cos\left(\frac{j_2}{\sqrt{(n+\frac{1}{2})^2+c}}\right) > z_{n,2}. \tag{3.32}$$

Thus,

$$\nu_2^0(z_0) > \nu_2^0(z_{n,2}) = n. \tag{3.33}$$

From Lemma 2 and (3.33), it follows that

$$\nu_2^1(z_0) > \nu_2^0(z_0) + 1 > n+1. \tag{3.34}$$

Equation (3.34) contradicts (B.2) and the proof of Theorem 3 is complete.

THEOREM 4. *If* $0 \leqslant z_0 < 1$, *then* $\nu_3^0(z_0) < \nu_1^5(z_0)$.

*Proof.* Suppose there is a $z_0 \in (0, 1)$ such that

$$\nu = \nu_3^0(z_0) = \nu_1^5(z_0), \tag{C.1}$$

and $n$ is that integer for which

$$n \leqslant \nu < n+1. \tag{C.2}$$

From (2.7), (C.1), and (C.2), we find $n \geqslant 6$. From (2.3a) with $m = 3, 2, 1, 0$, we obtain

$$\begin{pmatrix} 8\sigma & (\nu-3)(\nu+4) & 0 & 0 \\ 1 & 6\sigma & (\nu-2)(\nu+3) & 0 \\ 0 & 1 & 4\sigma & (\nu-1)(\nu+2) \\ 0 & 0 & 1 & 2\sigma \end{pmatrix} \begin{pmatrix} P_\nu^4(z_0) \\ P_\nu^3(z_0) \\ P_\nu^2(z_0) \\ P_\nu^1(z_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.35}$$

where $\sigma = z_0/(1-z_0^2)^{1/2}$. Since $P_\nu^4(z_0) \neq 0$ and $P_\nu^1(z_0) \neq 0$, from (3.35) it follows that

$$\det \begin{pmatrix} 8\sigma & (\nu-3)(\nu+4) & 0 & 0 \\ 1 & 6\sigma & (\nu-2)(\nu+3) & 0 \\ 0 & 1 & 4\sigma & (\nu-1)(\nu+2) \\ 0 & 0 & 1 & 2\sigma \end{pmatrix} = 0. \tag{3.36}$$

After substituting $\lambda = \nu(\nu+1)$, $s = \sigma^2$ into (3.36) and simplifying, we obtain

$$384s^2 - 72(\lambda-4)s + (\lambda-2)(\lambda-12) = 0. \tag{3.37}$$

We denote the two solutions of (3.37) by

$$s^\pm = \frac{9(\lambda-4) \pm \sqrt{3}\,(19\lambda^2 - 104\lambda + 240)^{1/2}}{96}. \tag{3.38}$$

Since $\nu \geqslant 6$, it follows that $\lambda \geqslant 42$. Since $19\lambda^2 - 104\lambda + 240 > 1856/19$ and $\lambda \geqslant 42$, we see that $s^+ > 0$. A straightforward calculation shows that $s^- \leqslant 0$ if and only if $2 \leqslant \lambda \leqslant 12$. Hence, when $\lambda \geqslant 42$, $s^+ > s^- > 0$. The particular "$z_0$" that satisfies (C.1) is related to either $s^+$ or $s^-$, i.e.,

$$z_0 \in \left\{ \left(\frac{s^+}{1+s^+}\right)^{1/2}, \left(\frac{s^-}{1+s^-}\right)^{1/2} \right\}. \tag{3.39}$$

Furthermore, for such a $z_0$ we have

$$\begin{aligned} z_0^2 &\geqslant \min\left\{\frac{s^+}{1+s^+}, \frac{s^-}{1+s^-}\right\} \\ &= \frac{s^-}{1+s^-} \\ &\geqslant \frac{9(\lambda-4) - \sqrt{3}\,(19\lambda^2 - 104\lambda + 240)^{1/2}}{9\lambda + 60 - \sqrt{3}\,(19\lambda^2 - 104\lambda + 240)^{1/2}} \\ &> \frac{\lambda-12}{\lambda+56} \\ &= \frac{\nu(\nu+1) - 12}{\nu(\nu+1) + 56}. \end{aligned} \tag{3.40}$$

Combining (3.40) with (3.23), we find

$$z_0 > \left(\frac{\nu(\nu+1)-12}{\nu(\nu+1)+56}\right)^{1/2} \geqslant \left(\frac{n(n+1)-12}{n(n+1)+56}\right)^{1/2}$$

$$> \cos\left(\frac{j_3}{\sqrt{(n+\frac{3}{2})^2+c}}\right) > z_{n+1,3}. \tag{3.41}$$

From (3.41), we find

$$\nu_3^0(z_0) > \nu_3^0(z_{n+1,3}) = n+1. \tag{3.42}$$

However, (3.42) contradicts (C.2) and the proof of Theorem 4 is complete.

## 4. Concluding Remarks

By combining the results of Sections 2 and 3, we conclude that

$$\nu_1^0 < \nu_1^1 < \nu_1^2 < \nu_2^0 < \nu_1^3 < \nu_2^1 < \nu_1^4 < \nu_2^2 < \nu_3^0 < \nu_1^5 \qquad \text{for all} \quad 0 < z_0 < 1. \tag{4.1}$$

By applying Theorems 1–4 and (2.14), it follows that for $0 < z_0 < 1$,

$$\nu_1^5 < \nu_2^3 < \nu_3^1,$$

$$\nu_1^5 < \nu_1^6,$$

$$\nu_{11} = \min\{\nu_2^3, \nu_1^6\} = \min\{\nu_j^m : \nu_j^m \neq \nu_1^0, \nu_1^1, \nu_1^2, \nu_2^0, \nu_1^3, \nu_1^4, \nu_2^1, \nu_2^2, \nu_3^0, \nu_1^5\}. \tag{4.2}$$

Hence, $\nu_1^5$ is distinct. Moreover, the order of the first ten $\nu_j^m(z_0)$'s is given by (4.1).

Numerical calculations indicate that $\nu_{11} = \nu_2^3$ and

$$\begin{aligned} \nu_3^1(z_0) &< \nu_1^6(z_0) && \text{for} \quad 0 \leqslant z_0 < 0.810843468, \\ \nu_1^6(z_0) = \nu_3^1(z_0) &= 15.78011308 && \text{for} \quad z_0 = 0.810843468, \\ \nu_3^1(z_0) &> \nu_1^6(z_0) && \text{for} \quad 0.810843468 < z_0 < 1. \end{aligned}$$

It is interesting to note that the Bessel functions, $J_m$ and $J_{m+n}$, have no common zeroes other than the origin (see [10, p. 484]). That is, if $j_k^m$ denotes the $k$th positive zero of $J_m(z)$, then $j_k^m = j_i^n$ if and only if $(m, k) = (n, i)$. By the remark at the end of Section 1, it is clear that the $\nu_j^m(z_0)$'s are not distinct for every $z_0 \in (0, 1)$. On the other hand, suppose $\nu_j^m(z_0) = \nu_i^n(z_0)$ with $(m, j) \neq (n, i)$ and $\nu_j^m(z) = \nu_i^n(z)$ for all $z$ in a dense set $\mathcal{N}_{z_0}$ such that $z_0 \in \mathcal{N}_{z_0} \subset (0, 1)$. Since $\nu_j^m, \nu_i^n$ are analytic, it follows

$v_j^m(z) \equiv v_i^n(z)$ for all $z \in (0, 1)$ (which is impossible). It follows that the measure of the set,

$$\mathscr{Z}_0 = \{z_0 \mid 0 < z_0 < 1, v_j^m(z_0) = v_i^n(z_0) \text{ and } (m, j) \neq (n, i)\},$$

is zero. Hence, in every neighborhood of a $z_0$ for which $v_j^m(z_0) = v_i^n(z_0)$ and $(m, j) \neq (n, i)$, there is a $z_0'$ such that $v_j^m(z_0') \neq v_i^n(z_0')$ and $|z_0 - z_0'|$ is arbitrarily small. In other words, for *almost every* $z_0 \in (0, 1)$, the condition $v_j^m(z_0) = v_i^n(z_0)$ implies that $(m, j) = (n, i)$.

All of the results that are presented here can be applied to ordering the eigenvalues of the Laplacian on a spherical cap with a half-angle opening of $\theta_0 \in (0, \frac{1}{2}\pi)$ (in the context of this paper, $z_0 = \cos \theta_0$). This is because $\mu$ is an eigenvalue of the Laplacian if and only if $\mu = \mu_j^m = v_j^m(v_j^m + 1)$ for some $(m, j)$ (see [1]). In particular, we can obtain the order of the first ten eigenvalues of the Laplacian from (4.1) by replacing the $v_j^m$'s with the corresponding $\mu_j^m$'s.

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