Topology and its Applications 156 (2009) 2642–2650

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Which infinite abelian groups admit an almost maximally almost-periodic group topology?

A.P. Nguyen [1]

Department of Mathematics, University of Manitoba, Winnipeg, MB, R3T 2N2, Canada

A R T I C L E   I N F O

Article history:
Received 1 December 2008
Received in revised form 23 February 2009
Accepted 3 March 2009

MSC:
primary 22A05
secondary 20K45, 54H11

Keywords:
Almost maximally almost-periodic
Prüfer group
von Neumann radical
T-sequence

A B S T R A C T

A topological group G is said to be *almost maximally almost-periodic* if its von Neumann radical $\mathbf{n}(G)$ is non-trivial, but finite. In this paper, we prove that every abelian group with an infinite torsion subgroup admits a (Hausdorff) almost maximally almost-periodic group topology. Some open problems are also formulated.

1. Introduction

Every topological group G admits a "largest" compact Hausdorff group bG and a continuous homomorphism $\rho_G : G \to bG$ such that every continuous homomorphism $\varphi : G \to K$ into a compact Hausdorff group K factors uniquely through ρ_G:

$$\begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[dr, "\rho_G"'] & & K \\ & bG \arrow[ur, dashed, "\exists!\tilde{\varphi}"'] & \end{tikzcd} \tag{1}$$

The group bG is called the *Bohr-compactification* of G, and the image $\rho_G(G)$ is dense in bG. The kernel of ρ_G is called the *von Neumann radical* of G, and is denoted by $\mathbf{n}(G)$. One says that G is *maximally almost-periodic* if $\mathbf{n}(G) = 1$, and *minimally almost-periodic* if $\mathbf{n}(G) = G$ (cf. [8]).

It is well known that the discrete topology is maximally almost-periodic on every abelian group (cf. [6, 4.23]). Ajtai, Havas, and Komlós, and independently, Zelenyuk and Protasov, showed that every infinite abelian group admits a (Hausdorff) group topology that is not maximally almost-periodic (cf. [1] and [13, Theorem 16]). While these results provide a group topology where the von Neumann radical is non-trivial, they remain silent about the size of the von Neumann radical of the group. In particular, they do not guarantee that the von Neumann radical is finite. Motivated by these observations, Lukács called a Hausdorff topological group G *almost maximally almost-periodic* if $\mathbf{n}(G)$ is non-trivial, but finite (cf. [7]). He

E-mail address: umnguyeb@cc.umanitoba.ca.

[1] I am grateful for the financial support of the Faculty of Science at the University of Manitoba that enabled me to carry out this research.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2009.04.005

proved, among other results, that for every prime $p \neq 2$, the Prüfer group $\mathbb{Z}(p^\infty)$ admits a (Hausdorff) almost maximally almost-periodic group topology (cf. [7, 4.4]).

The aim of this paper is to substantially extend the results of Lukács in several directions. The main results of the paper are as follows:

Theorem A. *Let A be an abelian group with an infinite torsion subgroup. Then A admits a (Hausdorff) almost maximally almost-periodic group topology.*

Theorem B. *Let p be a prime, and $x \in \mathbb{Z}(p^\infty)$ a non-zero element. Then there is a (Hausdorff) group topology τ on $\mathbb{Z}(p^\infty)$ such that $\mathbf{n}(\mathbb{Z}(p^\infty), \tau) = \langle x \rangle$.*

Most of the effort in this paper is put toward proving Theorem B, which implies Theorem A. Once Theorem B has been established, Theorem A follows from it and from another result of Lukács (cf. [7, 3.1]). Since Theorem B was proven by Lukács for all primes $p > 2$ (cf. [7, 4.4]), it remains to be shown that the statement also holds for $p = 2$.

The paper is structured as follows: In order to make the manuscript more self-contained, in Section 2, we have collected some preliminary results and techniques that will be used throughout the paper. Section 3 is a somewhat technical preparation for the proof of Theorem B, which is presented in Section 4 along with the proof of Theorem A. Finally, in Section 5, we formulate two open problems stemming from the results presented in this paper, and discuss what is known to us, at this point, about their solution.

2. Preliminaries

In this section, we have collected some preliminary results and techniques that are used throughout the paper. Thus, the experienced or expert reader may wish to skip this section.

In this paper, all groups are abelian, and all group topologies are Hausdorff, unless otherwise stated. For a topological group A, let $\hat{A} = \mathcal{H}(A, \mathbb{T})$ denote the Pontryagin dual of A—in other words, the group of *continuous characters* of A (i.e., continuous homomorphisms $\chi : A \to \mathbb{T}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$) equipped with the compact-open topology. It follows from the famous Peter–Weyl Theorem [10, Theorem 33] that the Bohr-compactification of A can be quite easily computed: $bA = \widehat{\hat{A}_d}$, where $\hat{A}_d$ stands for the group $\hat{A}$ with the discrete topology. Thus,

$$\mathbf{n}(A) = \bigcap_{\chi \in \hat{A}} \ker \chi. \tag{2}$$

The group $\mathbb{Z}(p^\infty)$ can be seen as the subgroup of $\mathbb{Q}/\mathbb{Z}$ generated by elements of p-power order, or as the group formed by all p^nth roots of unity in $\mathbb{C}$. Throughout this note, the additive notation provided by $\mathbb{Q}/\mathbb{Z}$ is used, and we set $e_n = \frac{1}{p^n} + \mathbb{Z}$. The Pontryagin dual $\widehat{\mathbb{Z}(p^\infty)}$ of $\mathbb{Z}(p^\infty)$ is the p-adic group $\mathbb{Z}_p$. We let χ_1 denote the natural embedding of $\mathbb{Z}(p^\infty)$ into $\mathbb{T}$. Lukács, who proved Theorem B for $p \neq 2$ (cf. [7, 4.4]), used so-called T-sequences as his main machinery to produce almost maximally almost-periodic group topologies on $\mathbb{Z}(p^\infty)$. While the outstanding case of $p = 2$ requires special attention, the techniques used in this paper are nevertheless similar.

A sequence $\{a_n\}$ in a group G is a *T-sequence* if there is a Hausdorff group topology τ on G such that $a_n \xrightarrow{\tau} e$. In this case, the group G equipped with the finest group topology with this property is denoted by $G\{a_n\}$. The notion of a T-sequence was introduced and extensively investigated by Zelenyuk and Protasov, who characterized T-sequences (and so-called T-filters), and studied the topological properties of $G\{a_n\}$, where $\{a_n\}$ is a T-sequence (cf. [13, Theorems 1–2] and [11, 2.1.3, 2.1.4, 3.1.4]). These two authors used the technique of T-sequences to prove the following results (some of which were also obtained independently by Ajtai, Havas, and Komlós [1]).

Theorem 2.1.

(a) *([1, §2], [13, Example 4]) $\mathbb{Z}$ admits a minimally almost-periodic group topology.*
(b) *([1, §4], [13, Example 6]) $\mathbb{Z}(p^\infty)$ admits a minimally almost-periodic group topology for every prime p.*
(c) *([13, Example 6], [2, 3.3]) Let $\chi \in \widehat{\mathbb{Z}(p^\infty)} = \mathbb{Z}_p$. One has $\chi(e_n) \longrightarrow 0$ if and only if there is $m \in \mathbb{Z}$ such that $\chi = m\chi_1$.*

Since $\mathbb{Z}(2^\infty)$ is an abelian group, we need only the abelian version of the Zelenyuk–Protasov criterion:

Theorem 2.2. *([11, 2.1.4], [13, Theorem 2]) Let $\underline{a} = \{a_k\}$ be a sequence in an abelian group A. For $l, m \in \mathbb{N}$, put*

$$A(l,m)_{\underline{a}} = \left\{ m_1 a_{k_1} + \cdots + m_h a_{k_h} \;\middle|\; m \leqslant k_1 < \cdots < k_h,\, m_i \in \mathbb{Z}\backslash\{0\},\, \sum |m_i| \leqslant l \right\}. \tag{3}$$

Then $\{a_k\}$ is a T-sequence if and only if for every $l \in \mathbb{N}$ and $g \neq 0$, there exists $m \in \mathbb{N}$ such that $g \notin A(l,m)_{\underline{a}}$.

For a group A, we put $A[n] = \{a \in A \mid na = 0\}$ for every $n \in \mathbb{N}$. The group A is *almost torsion-free* if $A[n]$ is finite for every $n \in \mathbb{N}$ (cf. [12]). Clearly, the Prüfer groups $\mathbb{Z}(p^\infty)$ are almost torsion-free. Lukács characterized T-sequences in almost torsion-free groups as follows.

Theorem 2.3. *([7, 2.2]) Let A be an almost torsion-free group, and let $\underline{a} = \{a_k\}$ be a sequence in A. The following statements are equivalent*:

(i) *For every $l, n \in \mathbb{N}$, there exists $m_0 \in \mathbb{N}$ such that $A[n] \cap A(l,m)_{\underline{a}} = \{0\}$ for every $m \geqslant m_0$.*
(ii) *$\{a_k\}$ is a T-sequence.*

Lukács also provided sufficient conditions for a sequence in $\mathbb{Z}(p^\infty)$ to be a T-sequence.

Lemma 2.4. *([7, 4.1]) Let $\{a_k\}$ be a sequence in $\mathbb{Z}(p^\infty)$ such that $o(a_k) = p^{n_k}$. If $n_{k+1} - n_k \longrightarrow \infty$, then $\{a_k\}$ is a T-sequence.*

It turns out that the class of abelian groups that admit an almost maximally almost-periodic group topology is upward closed in the sense that if a group A belongs there, then so does every abelian group B that contains A as a subgroup (cf. Theorem 4.2). Thus, in the proof of Theorem A, we restrict our attention to torsion groups. In particular, we rely on the following result on the structure of infinite abelian groups to confine our attention further to two special subgroups.

Lemma 2.5. *([3, Theorems 8.4, 23.1, 27.2]) Every infinite abelian group contains a subgroup that is isomorphic to $\mathbb{Z}$, or $\mathbb{Z}(p^\infty)$, or an infinite direct sum of non-trivial finite cyclic groups.*

It follows from the above lemma that in the proof of Theorem A, we can focus on the following two types of subgroups: Prüfer groups (which are taken care of by Theorem B), and direct sums of infinitely many finite groups, which are addressed by the following result.

Theorem 2.6. *([7, 3.1]) If A is a direct sum of infinitely many non-trivial finite abelian groups, then A admits an almost maximally almost-periodic group topology.*

3. The canonical form in $\mathbb{Z}(2^\infty)$

The aforementioned result of Lukács concerning $\mathbb{Z}(p^\infty)$ is based on a *canonical form*, which he introduced for writing each element of $\mathbb{Z}(p^\infty)$ uniquely as an integer combination of the elements $\{e_n\}_{n\in\mathbb{N}}$ with certain additional conditions on the coefficients (cf. [7, 4.6]). The canonical form of Lukács, however, requires the prime p to be odd, and thus fails in the case of $p = 2$. In this section, we remedy this, provide a unique canonical form for elements of $\mathbb{Z}(2^\infty)$, and establish some technical properties of the canonical form that are needed for the proof of Theorem B.

Each element $y \in \mathbb{Z}(2^\infty)$ admits many representations of the form $y = \sum \sigma_n e_n$, where $\sigma_n \in \mathbb{Z}$ with only finitely many of the σ_n being non-zero. In order to find a canonical form for these elements, we first eliminate the summands with odd indices.

Lemma 3.1. *Let $y = \sum \sigma_n e_n \in \mathbb{Z}(2^\infty)$. Then y can be represented in the form of $y = \sum \sigma'_{2n} e_{2n}$, where $\sigma'_{2n} \in \mathbb{N}$ and $\sum |\sigma'_{2n}| \leqslant 2 \sum |\sigma_n|$.*

Proof. Let K be the largest index such that $\sigma_K \neq 0$, and N the smallest integer that satisfies $K \leqslant 2N$. Since $e_{2n-1} = 2e_{2n}$ for every $n \in \mathbb{N}$, one has

$$y = \sum_{n=1}^{2N} \sigma_n e_n = \sum_{n=1}^{N} \sigma_{2n-1} e_{2n-1} + \sum_{n=1}^{N} \sigma_{2n} e_{2n} \tag{4}$$

$$= \sum_{n=1}^{N} 2\sigma_{2n-1} e_{2n} + \sum_{n=1}^{N} \sigma_{2n} e_{2n} = \sum_{n=1}^{N} (2\sigma_{2n-1} + \sigma_{2n}) e_{2n}. \tag{5}$$

Thus, by setting $\sigma'_{2n} = 2\sigma_{2n-1} + \sigma_{2n}$ for every $n \in \mathbb{N}$, one obtains $y = \sum \sigma'_{2n} e_{2n}$ and

$$\sum_{n=1}^{N} |\sigma'_{2n}| = \sum_{n=1}^{N} |2\sigma_{2n-1} + \sigma_{2n}| \leqslant \sum_{n=1}^{N} \big(|2\sigma_{2n-1}| + |\sigma_{2n}|\big) \tag{6}$$

$$\leqslant \sum_{n=1}^{N} \big(2|\sigma_{2n-1}| + 2|\sigma_{2n}|\big) = 2\sum_{n=1}^{2N} |\sigma_n|, \tag{7}$$

as desired. □

Definition 3.2. Let $y \in \mathbb{Z}(2^\infty)$. We say that $y = \sum \sigma_{2n} e_{2n}$ is the *canonical form* of the element y if $\sigma_{2n} \in \{-1, 0, 1, 2\}$ for every $n \in \mathbb{N}$ (and $\sigma_{2n} = 0$ for all but finitely many indices n); in this case, we put $\Lambda(y) = \{n \in \mathbb{N} \mid \sigma_{2n} \neq 0\}$ and $\lambda(y) = |\Lambda(y)|$.

In order for $\Lambda(y)$ and $\lambda(y)$ to be well-defined, we first show that each $y \in \mathbb{Z}(2^\infty)$ admits a unique canonical form. We put $\underline{e} = \{e_n\}_{n=1}^\infty$, and use the notation introduced in Theorem 2.2.

Theorem 3.3. *Let $y = \sum \sigma_{2n} e_{2n} \in \mathbb{Z}(2^\infty)$. Then,*

(a) *y admits a canonical form $y = \sum \sigma'_{2n} e_{2n}$ that satisfies*

$$\sum f\left(\sigma'_{2n}\right) \leqslant \sum f(\sigma_{2n}), \tag{8}$$

where $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \max\{-2x, x\}$;

(b) *the canonical form is unique*;

(c) *$\lambda(z) \leqslant 4l$ for every $z \in \mathbb{Z}(2^\infty)(l, 1)_{\underline{e}}$ and $l \in \mathbb{N}$.*

In order to make the proof of Theorem 3.3 more transparent, we summarize the properties of the function $f(x)$ in the following lemma, whose easy, but nevertheless technical, proof is omitted.

Lemma 3.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \max\{-2x, x\}$. Then, for every $a, b \in \mathbb{R}$:*

(a) *$f(a) \leqslant 2|a|$;*
(b) *$f(a) \leqslant |b|$ if and only if $-\frac{1}{2}|b| \leqslant a \leqslant |b|$;*
(c) *$f(a) \geqslant |b|$ if and only if $a \leqslant -\frac{1}{2}|b|$ or $a \geqslant |b|$;*
(d) *$f(a+b) \leqslant f(a) + f(b)$;*
(e) *$f(a) + f(b) \leqslant f(a+4b)$, provided that $a \in [-1, 2]$ and $|b| \geqslant 1$ or $b = 0$.*

In what follows, we also rely on the following well-known property of p-groups.

Remark 3.5. Let $o(x)$ denote the order of an element x in a group. If P is a p-group, $a, b \in P$, and $o(a) \neq o(b)$, then $o(a+b) = \max\{o(a), o(b)\}$.

Proof of Theorem 3.3. (a) Let $2N$ be the largest index such that $\sigma_{2N} \neq 0$. We proceed by induction on N. If $N = 1$, then $y = \sigma_2 e_2$, and one may write $\sigma_2 = \sigma'_2 + 4m$ with $\sigma'_2 \in \{-1, 0, 1, 2\}$ and $m \in \mathbb{Z}$. Since $4e_2 = 0$,

$$y = \left(\sigma'_2 + 4m\right)e_2 = \sigma'_2 e_2 + m(4e_2) = \sigma'_2 e_2, \tag{9}$$

and by Lemma 3.4(e), $f(\sigma'_2) + f(m) \leqslant f(\sigma_2)$. In particular, $f(\sigma'_2) \leqslant f(\sigma_2)$.

Suppose that the statement holds for all elements with a representation where the maximal non-zero index less than $2N$ and $N > 1$. Let $\sigma_{2N} = \sigma'_{2N} + 4m$ be a division of σ_{2N} by 4 with residue in $\mathbb{Z}$ such that $\sigma'_{2N} \in \{-1, 0, 1, 2\}$. Since $4e_{2N} = e_{2N-2}$, one obtains that

$$y = \sum_{n=1}^{N} \sigma_{2n} e_{2n} = \sum_{n=1}^{N-2} \sigma_{2n} e_{2n} + \sigma_{2N-2} e_{2N-2} + \sigma_{2N} e_{2N} \tag{10}$$

$$= \sum_{n=1}^{N-2} \sigma_{2n} e_{2n} + \sigma_{2N-2} e_{2N-2} + \left(\sigma'_{2N} + 4m\right)e_{2N} \tag{11}$$

$$= \sum_{n=1}^{N-2} \sigma_{2n} e_{2n} + (\sigma_{2N-2} + m)e_{2N-2} + \sigma'_{2N} e_{2N}. \tag{12}$$

The element $z = \sum_{n=1}^{N-2} \sigma_{2n} e_{2n} + (\sigma_{2N-2} + m)e_{2N-2}$ satisfies the inductive hypothesis, and so it can be represented in the canonical form $z = \sum_{n=1}^{N-1} \sigma'_{2n} e_{2n}$, where $\sigma'_n \in \{-1, 0, 1, 2\}$ and

$$\sum_{n=1}^{N-1} f\left(\sigma'_{2n}\right) \leqslant \sum_{n=1}^{N-2} f(\sigma_{2n}) + f(\sigma_{2N-2} + m). \tag{13}$$

Thus, one has

$$y = z + \sigma'_{2N} e_{2N} = \sum_{n=1}^{N} \sigma'_{2n} e_{2n}. \tag{14}$$

By Lemma 3.4(d),

$$f(\sigma_{2N-2}+m) \leqslant f(\sigma_{2N-2}) + f(m), \tag{15}$$

and by Lemma 3.4(e),

$$f(m) + f(\sigma'_{2N}) \leqslant f(\sigma_{2N}). \tag{16}$$

Therefore, one obtains that

$$\sum_{n=1}^{N} f(\sigma'_{2n}) \overset{(13)}{\leqslant} \sum_{n=1}^{N-1} f(\sigma'_{2n}) + f(\sigma'_{2N}) \leqslant \sum_{n=1}^{N-2} f(\sigma_{2n}) + f(\sigma_{2N-2}+m) + f(\sigma'_{2N}) \tag{17}$$

$$\overset{(15)}{\leqslant} \sum_{n=1}^{N-2} f(\sigma_{2n}) + f(\sigma_{2N-2}) + f(m) + f(\sigma'_{2N}) = \sum_{n=1}^{N-1} f(\sigma_{2n}) + f(m) + f(\sigma'_{2N}) \tag{18}$$

$$\overset{(16)}{\leqslant} \sum_{n=1}^{N-1} f(\sigma_{2n}) + f(\sigma_{2N}) = \sum_{n=1}^{N} f(\sigma_{2n}). \tag{19}$$

Hence, (8) holds for y, as desired.

(b) Suppose that $\sum \sigma_{2n}e_{2n} = \sum \nu_{2n}e_{2n}$ are two distinct canonical representations of the same element in $\mathbb{Z}(2^\infty)$. Then, $\sum(\sigma_{2n}-\nu_{2n})e_{2n} = 0$ and $|\sigma_{2n}-\nu_{2n}| \leqslant 3$. Let $2N$ be the largest index such that $\sigma_{2N} \neq \nu_{2N}$. (Since all coefficients are zero, except for a finite number of indices, such an N exists.) This means that $0 < |\sigma_{2N}-\nu_{2N}| \leqslant 3$, and so $2^{2N-1} \leqslant o((\sigma_{2N}-\nu_{2N})e_{2N})$. On the other hand,

$$o\left(\sum_{n<N}(\sigma_{2n}-\nu_{2n})e_{2n}\right) \leqslant \max_{n<N} o\big((\sigma_{2n}-\nu_{2n})e_{2n}\big) \leqslant 2^{2N-2} < o\big((\sigma_{2N}-\nu_{2N})e_{2N}\big). \tag{20}$$

Therefore, by Remark 3.5,

$$o\left(\sum(\sigma_{2n}-\nu_{2n})e_{2n}\right) = \max\left\{o\left(\sum_{n<N}(\sigma_{2n}-\nu_{2n})e_{2n}\right), o\big((\sigma_{2N}-\nu_{2N})e_{2N}\big)\right\} \geqslant 2^{2N-1}, \tag{21}$$

contrary to the assumption that $\sum(\sigma_{2n}-\nu_{2n})e_{2n} = 0$. Hence, $\sigma_{2n} = \nu_{2n}$ for every $n \in \mathbb{N}$.

(c) Let $z = \nu_1 e_{n_1} + \cdots + \nu_t e_{n_t}$, where $\sum|\nu_i| \leqslant l$ and $n_1 < \cdots < \nu_t$. By Lemma 3.1, z can be expressed as $z = \sum \sigma_{2n}e_{2n}$, such that $\sum|\sigma_{2n}| \leqslant 2\sum|\nu_n| \leqslant 2l$. By Lemma 3.4(a), $f(\sigma_{2n}) \leqslant 2|\sigma_{2n}|$, so one obtains that $\sum f(\sigma_{2n}) \leqslant 2\sum|\sigma_{2n}| \leqslant 4l$. By (a), z admits a canonical form $z = \sum \sigma'_{2n}e_{2n}$ where $\sum f(\sigma'_{2n}) \leqslant \sum f(\sigma_{2n}) \leqslant 4l$. Since $f(\sigma'_{2n}) \geqslant 0$, there can only be at most $4l$ indices with non-zero coefficients σ'_{2n}. □

Lemma 3.6. *Let $m \in \mathbb{Z}\backslash\{0\}$, and put $l = \lceil \log_4|m| \rceil$. If $n > l$, then $\Lambda(me_{2n}) \subseteq \{n-l,\dots,n-1,n\}$ and $1 \leqslant \lambda(me_{2n})$.*

Proof. Since $n > l$, we have that $2^{2n} > |m|$, and so $me_{2n} \neq 0$. Thus, $1 \leqslant \lambda(me_{2n})$. One may expand m in the form of $m = \mu_0 + \mu_2 2^2 + \cdots + \mu_{2l}2^{2l}$, where $\mu_i \in \{-1,0,1,2\}$. Therefore,

$$me_{2n} = \mu_0 e_{2n} + \mu_2 e_{2n-2} + \cdots + \mu_{2l}e_{2n-2l} \tag{22}$$

is in the canonical form. Hence, $\Lambda(me_{2n}) \subseteq \{n-l,\dots,n-1,n\}$, as desired. □

Lemma 3.7. *Let $y, z \in \mathbb{Z}(2^\infty)$ be such that $\lambda(y) > \lambda(z)$, and suppose that $\Lambda(y) = \{k_1,\dots,k_g\}$ where $k_1 < \cdots < k_g$ and $g = \lambda(y)$. Then, $o(y-z) > 4^{k_{g-\lambda(z)}-1}$.*

Proof. Let $y = \sum \nu_{2n}e_{2n}$ and $z = \sum \mu_{2n}e_{2n}$ be the canonical forms of y and z. Since Theorem 3.3(b) provides that the canonical form is unique, $\lambda(y) > \lambda(z)$ implies that $y \neq z$, and thus $y - z \neq 0$. Let N be the largest integer such that $\nu_{2N} - \mu_{2N} \neq 0$. Then, $\nu_{2n} = \mu_{2n}$ for every $n > N$. In particular, $\mu_{2k_i} \neq 0$ for every $k_i > N$. Therefore, there are at most $\lambda(z)$ many indices k_i that satisfy $k_i > N$. Hence, $N \geqslant k_{g-\lambda(z)}$. Since $0 < |\nu_{2N}-\mu_{2N}| \leqslant 3$, one has $2^{2N-1} \leqslant o((\nu_{2N}-\mu_{2N})e_{2N})$. On the other hand,

$$o\left(\sum_{n<N}(\nu_{2n}-\mu_{2n})e_{2n}\right) \leqslant \max_{n<N} o\big((\nu_{2n}-\mu_{2n})e_{2n}\big) \leqslant 2^{2N-2} < o\big((\nu_{2N}-\mu_{2N})e_{2N}\big). \tag{23}$$

Consequently, by Remark 3.5,

$$o\left(\sum(\nu_{2n}-\mu_{2n})e_{2n}\right) = \max\left\{o\left(\sum_{n<N}(\nu_{2n}-\mu_{2n})e_{2n}\right), o\big((\nu_{2N}-\mu_{2N})e_{2N}\big)\right\} \geqslant 2^{2N-1} > 4^{N-1}. \tag{24}$$

Hence, $o(y-z) = o(\sum(\nu_{2n}-\mu_{2n})e_{2n}) > 4^{k_{g-\lambda(z)}-1}$, as desired. □

Remark 3.8. If $y_1, y_2 \in \mathbb{Z}(2^\infty)$ and $\Lambda(y_1) \cap \Lambda(y_2) = \emptyset$, then $\Lambda(y_1 + y_2) = \Lambda(y_1) \cup \Lambda(y_2)$ and $\lambda(y_1 + y_2) = \lambda(y_1) + \lambda(y_2)$.

Proposition 3.9. *Let $y = \nu_1 e_{2n_1} + \cdots + \nu_t e_{2n_t}$, where $\nu_i \neq 0$ and $0 < n_1 < \cdots < n_t$ are integers. Put $l_i = \lceil \log_4 |\nu_i| \rceil$, and suppose that $n_i < n_{i+1} - l_{i+1}$ for each $1 \leqslant i < t$. Then,*

(a) $t \leqslant \lambda(y)$;
(b) *if $z \in \mathbb{Z}(2^\infty)$ is such that $\lambda(z) < \lambda(y)$, then $o(y - z) > 4^{n_{t-\lambda(z)} - l_{t-\lambda(z)} - 1}$.*

Proof. (a) By Lemma 3.6,

$$\Lambda(\nu_i e_{2n_i}) \subseteq \{n_i - l_i, \ldots, n_i\} \tag{25}$$

for each $1 \leqslant i \leqslant t$. Thus, the sets $\Lambda(\nu_i e_{2n_i})$ are pairwise disjoint, because $n_i < n_{i+1} - l_{i+1}$. Therefore, by Remark 3.8, one obtains that $\lambda(y) = \lambda(\nu_1 e_{2n_1}) + \cdots + \lambda(\nu_t e_{2n_t}) \geqslant t$, and

$$\Lambda(y) = \bigcup_{i=1}^{t} \Lambda(\nu_i e_{2n_i}) \subseteq \bigcup_{i=1}^{t} \{n_i - l_i, \ldots, n_i\}. \tag{26}$$

(b) Suppose that $\Lambda(y) = \{k_1, \ldots, k_g\}$ (increasingly ordered). For any i such that $t - i \leqslant 0$, define $n_{t-i} = n_1$ and $l_{t-i} = l_1$. We proceed by induction on i to show that $k_{g-i} \geqslant n_{t-i} - l_{t-i}$ for all $0 \leqslant i \leqslant g - 1$.

For $i = 0$, Lemma 3.6 yields that $\Lambda(y) \cap \Lambda(\nu_t e_{2n_t}) = \Lambda(\nu_t e_{2n_t}) \neq \emptyset$. Since $n_i < n_{i+1} - l_{i+1}$ for each $0 \leqslant i < t$, $\Lambda(\nu_t e_{2n_t})$ contains the largest elements of $\bigcup_{i=1}^{t} \{n_i - l_i, \ldots, n_i\}$, and thus contains the largest value in $\Lambda(y)$, namely k_g. Hence, $k_g \geqslant n_t - l_t$. Suppose that the statement holds for all integers $i < N$. For $i = N$, if $k_{g-N} < n_{t-N} - l_{t-N}$, then for all $N \leqslant i \leqslant g - 1$, $k_{g-i} \leqslant k_{g-N} < n_{t-N} - l_{t-N}$. Moreover, by the inductive hypothesis, for all $0 \leqslant i < N$, one has that $k_{g-i} \geqslant n_{t-i} - l_{t-i} > n_{t-N}$, because $n_i < n_{i+1} - l_{i+1}$. So,

$$\Lambda(\nu_{t-N} e_{2n_{t-N}}) \cap \{n_{t-N} - l_{t-N}, \ldots, n_{t-N}\} \subseteq \Lambda(y) \cap \{n_{t-N} - l_{t-N}, \ldots, n_{t-N}\} = \emptyset. \tag{27}$$

Thus, by (25), $\Lambda(\nu_{t-N} e_{2n_{t-N}}) = \emptyset$, which contradicts $1 \leqslant \lambda(\nu_{t-N} e_{2n_{t-N}})$ from Lemma 3.6. Hence, one has that $k_{g-N} \geqslant n_{t-N} - l_{t-N}$ for all N. It follows from (26) that $k_i \geqslant n_1 - l_1$ for all i. So, $k_{g-i} \geqslant n_{t-i} - l_{t-i}$ holds even for $t - i \leqslant 0$. Thus, for $i = \lambda(z)$, $k_{g-\lambda(z)} \geqslant n_{t-\lambda(z)} - l_{t-\lambda(z)}$. By Lemma 3.7, $o(y - z) > 4^{k_{\lambda(y)-\lambda(z)} - 1} \geqslant 4^{n_{t-\lambda(z)} - l_{t-\lambda(z)} - 1}$, as desired. □

Corollary 3.10. *Let $l \in \mathbb{N}$, $z \in \mathbb{Z}(2^\infty)(l, 1)_{\underline{e}}$, and $y = e_{2n_1} + \cdots + e_{2n_t}$ such that $n_1 < \cdots < n_t$, $4l < t$, and $n_i < n_{i+1} - l$. Then, $o(\mu y + z) > 4^{n_{t-4l} - l - 1} \geqslant 4^{n_1 - l - 1}$ for every $\mu \in \mathbb{Z}$ such that $0 \leqslant |\mu| \leqslant l$.*

Proof. Since $|\mu| \leqslant l$, one has that $\log_4 |\mu| < l$ and $\lceil \log_4 |\mu| \rceil \leqslant l$. So, $\mu y = \mu e_{2n_1} + \cdots + \mu e_{2n_t}$ satisfies the conditions of Proposition 3.9. By Proposition 3.9(a), one obtains that $4l < t \leqslant \lambda(\mu y)$. Moreover, if $z = \nu_1 e_{n_1} + \cdots + \nu_s e_{n_s} \in \mathbb{Z}(2^\infty)(l, 1)_{\underline{e}}$, where $n_1 < \cdots < n_s$ and $\sum_{i=1}^{s} |\nu_i| \leqslant l$, then

$$-z = (-\nu_1) e_{n_1} + \cdots + (-\nu_s) e_{n_s} \in \mathbb{Z}\big(2^\infty\big)(l, 1)_{\underline{e}}, \tag{28}$$

since $\sum_{i=1}^{s} |-\nu_i| = \sum_{i=1}^{s} |\nu_i| \leqslant l$. By Theorem 3.3(c), $\lambda(-z) \leqslant 4l < \lambda(\mu y)$. Thus, μy and $-z$ satisfy the conditions of Proposition 3.9(b), and so one has that

$$o(\mu y + z) = o\big(\mu y - (-z)\big) > 4^{n_{t-\lambda(z)} - \lceil \log_4 |\mu| \rceil - 1} \geqslant 4^{n_{t-\lambda(z)} - l - 1} > 4^{n_{t-4l} - l - 1} \geqslant 4^{n_1 - l - 1}, \tag{29}$$

as desired. □

4. Proofs of Theorems A and B

In this section, we first prove Theorem B, and then we apply this result to prove Theorem A. For Theorem B, Lukács has already established this result for all primes but $p = 2$ (cf. [7, 4.4]). Thus, we confine our attention to the Prüfer group $\mathbb{Z}(2^\infty)$. Clearly, Theorem 4.1 below, together with the result of Lukács, implies Theorem B. In Theorem 4.1, we construct a T-sequence in $\mathbb{Z}(2^\infty)$, and show that its von Neumann radical $\mathbf{n}(G)$ is a prefixed cyclic subgroup.

Theorem 4.1. *For $x \in \mathbb{Z}(2^\infty) \backslash \{0\}$ such that $o(x) = 2^{k_0}$, put*

$$b_k = -x + e_{2(k^3 - k^2)} + \cdots + e_{2(k^3 - 2k)} + e_{2(k^3 - k)} + e_{2k^3}. \tag{30}$$

Consider the sequence $\{d_k\}$, defined as $b_1, e_1, b_2, e_2, b_3, e_3, \ldots$. Then,

(a) *$\{d_k\}$ is a T-sequence in $\mathbb{Z}(2^\infty)$;*
(b) *the underlying group of $\widehat{\mathbb{Z}(2^\infty)\{d_k\}}$ is $\langle 2^{k_0} \chi_1 \rangle$;*
(c) *$\mathbf{n}(\mathbb{Z}(2^\infty)\{d_k\}) = \langle x \rangle$.*

Proof. (a) To shorten the notation, we denote $A = \mathbb{Z}(2^\infty)$.

In order to show that $\{d_k\}$ is a T-sequence, we prove that it satisfies statement (i) of Theorem 2.3. To that end, let $l, n \in \mathbb{N}$. For every $k \geqslant k_0$, we have that

$$o(e_{2(k^3-k^2)} + \cdots + e_{2(k^3-k)} + e_{2k^3}) = 2^{2k^3} > 2^{k_0} = o(-x), \tag{31}$$

and so by Remark 3.5,

$$o(b_k) = \max\{o(-x), o(e_{2(k^3-k^2)} + \cdots + e_{2(k^3-k)} + e_{2k^3})\} = 2^{2k^3}. \tag{32}$$

Since $e_k \longrightarrow 0$, in the subgroup topology inherited from $\mathbb{Q}/\mathbb{Z}$, $\{e_k\}$ is a T-sequence. Thus, by Theorem 2.3, there exists M_1 such that for every $m \geqslant M_1$, $A[n] \cap A(l,m)_{\underline{e}} = \{0\}$. On the other hand, since $o(b_k) = 2^{2k^3}$ for every $k \geqslant k_0$, and $2(k+1)^3 - 2k^3 \longrightarrow \infty$, by Lemma 2.4, $\{b_k\}$ is also a T-sequence. So, by Theorem 2.3, there exists M_2 such that for any $m \geqslant M_2$, $A[n] \cap A(l,m)_{\underline{b}} = \{0\}$.

Put $m_0 = \max\{M_1, M_2, 4l + n + k_0\}$. For any $m \geqslant m_0$, one has that

$$A[n] \cap A(l,m)_{\underline{e}} = A[n] \cap A(l,m)_{\underline{b}} = \{0\}. \tag{33}$$

Since

$$A(l,2m)_{\underline{d}} \subseteq A(l,m)_{\underline{e}} \cup A(l,m)_{\underline{b}} \cup \big(A(l,m)_{\underline{e}} \backslash \{0\} + A(l,m)_{\underline{b}} \backslash \{0\}\big), \tag{34}$$

it suffices to show that for every $m \geqslant m_0$,

$$\big(A(l,m)_{\underline{e}} \backslash \{0\} + A(l,m)_{\underline{b}} \backslash \{0\}\big) \cap A[n] = \emptyset. \tag{35}$$

Let $z \in A(l,m)_{\underline{e}} \backslash \{0\}$ and $w = m_1 b_{k_1} + \cdots + m_h b_{k_h} \in A(l,m)_{\underline{b}} \backslash \{0\}$ where $0 < \sum |m_i| \leqslant l$ and $m \leqslant k_1 < \cdots < k_h$. There are $k_h + 1$ summands in $y = e_{2(k_h^3 - k_h^2)} + \cdots + e_{2(k_h^3 - k_h)} + e_{2k_h^3}$. Moreover, the indices of every two consecutive summands differ by k_h, and $k_h + 1 > k_h > m \geqslant m_0 > 4l$. Since $|m_h| \leqslant \sum |m_i| \leqslant l$, y and z satisfy the hypothesis of Corollary 3.10, and one obtains that $o(m_h y + z) > 4^{k_h^3 - k_h^2 - l - 1}$. Since $k_h > l \geqslant 1$,

$$k_h^3 - k_h^2 - l - 1 > k_h^3 - k_h^2 - k_h - 1 \geqslant k_h^3 - 3k_h^2 + 3k_h - 1 = (k_h - 1)^3. \tag{36}$$

So, $o(m_h y + z) > 4^{(k_h^3 - k_h^2 - l - 1)} > 4^{(k_h - 1)^3}$. Moreover, one has that $k_0 < m_0 - 1 < k_h - 1$, because $m_0 = \max\{M_1, M_2, 4l + n + k_0\}$. Thus,

$$o(-m_h x) \leqslant 2^{k_0} < 4^{(k_h - 1)^3} < o(m_h y + z). \tag{37}$$

Thus, $o(-m_h x) \neq o(m_h y + z)$, and by Remark 3.5,

$$o(m_h b_{k_h} + z) = o\big((-m_h x) + (m_h y + z)\big) \tag{38}$$

$$= \max\{o(-m_h x), o(m_h y + z)\} = o(m_h y + z) > 4^{(k_h - 1)^3}. \tag{39}$$

On the other hand,

$$o(w - m_h b_{k_h}) \leqslant o(b_{k_{h-1}}) = 4^{k_{h-1}^3} \leqslant 4^{(k_h - 1)^3} < o(m_h b_{k_h} + z). \tag{40}$$

By Remark 3.5,

$$o(w + z) = o\big((w - m_h b_{k_h}) + (m_h b_{k_h} + z)\big) \tag{41}$$

$$= \max\{o(w - m_h b_{k_h}), o(m_h b_{k_h} + z)\} > 4^{(k_h - 1)^3} > 4^{(m_0 - 1)^3} > n. \tag{42}$$

Therefore, $A[n] \cap A(l,2m)_{\underline{d}} = \{0\}$ for every $m \geqslant m_0$. Hence, by Theorem 2.3, $\{d_k\}$ is a T-sequence.

(b) Since every continuous character of $\mathbb{Z}(2^\infty)\{d_k\}$ is also a continuous character of $\mathbb{Z}(2^\infty)$, $\widehat{\mathbb{Z}(2^\infty)\{d_k\}}$ is contained in $\mathbb{Z}_2 = \widehat{\mathbb{Z}(2^\infty)}$. By the universal property of $\mathbb{Z}(2^\infty)\{d_k\}$, $\chi \in \widehat{\mathbb{Z}(2^\infty)}$ is a continuous character of $\mathbb{Z}(2^\infty)\{d_k\}$ if and only if $\chi(d_k) \longrightarrow 0$, that is, $\chi(b_k) \longrightarrow 0$ and $\chi(e_k) \longrightarrow 0$. By Theorem 2.1(c), $\chi(e_k) \longrightarrow 0$ holds if and only if $\chi = m\chi_1$ for some $m \in \mathbb{Z}$. On the other hand, since

$$0 \leqslant \frac{1}{2^{2(k^3-k^2)}} + \cdots + \frac{1}{2^{2(k^3-2k)}} + \frac{1}{2^{2(k^3-k)}} + \frac{1}{2^{2k^3}} \leqslant \frac{k+1}{2^{2(k^3-k^2)}} \longrightarrow 0, \tag{43}$$

one has that $\chi_1(b_k) \longrightarrow -x$ in $\mathbb{T}$. Thus, $\chi(b_k) = m\chi_1(b_k) \longrightarrow 0$ if and only if $-mx = 0$, which means that $x \in \ker \chi$, and $o(x) = 2^{k_0} \mid m$. So, $\chi \in \widehat{\mathbb{Z}(2^\infty)}$ is a continuous character of $\mathbb{Z}(2^\infty)\{d_k\}$ if and only if $\chi = m\chi_1$ for $m \in \mathbb{Z}$ and $2^{k_0} \mid m$. Therefore, the underlying group of $\widehat{\mathbb{Z}(2^\infty)\{d_k\}}$ is $2^{k_0}\mathbb{Z}$.

(c) It follows from the argument in part (b) that $x \in \ker\chi$ for every continuous character χ of $\mathbb{Z}(2^\infty)\{d_k\}$, and so $x \in \bigcap \ker\chi = \mathbf{n}(\mathbb{Z}(2^\infty)\{d_k\})$. Thus, $\langle x\rangle \subseteq \mathbf{n}(\mathbb{Z}(2^\infty)\{d_k\})$. On the other hand, since $2^{k_0}\chi_1$ is a continuous character of $\mathbb{Z}(2^\infty)\{d_k\}$, one has that

$$\mathbf{n}\big(\mathbb{Z}\big(2^\infty\big)\{d_k\}\big) = \bigcap \ker\chi \subseteq \ker 2^{k_0}\chi_1 = \langle x\rangle. \tag{44}$$

Therefore, $\mathbf{n}(\mathbb{Z}(2^\infty)\{d_k\}) = \langle x\rangle$, as desired. □

We proceed by introducing some notations prior to the proof of Theorem A. We denote by Ab and Ab(Haus) the categories of abelian groups and abelian Hausdorff topological groups, respectively (with the usual morphisms). To abbreviate the notations, we introduce the following class of abelian groups:

$$\mathcal{A} = \big\{A \in \mathsf{Ab} \;\big|\; (\exists\tau)\big((A,\tau) \in \mathsf{Ab}(\mathsf{Haus}) \wedge (A,\tau) \text{ is almost maximally almost-periodic}\big)\big\}. \tag{45}$$

As indicated in Section 2, the proof of Theorem A is based on establishing certain algebraic properties of the class $\mathcal{A}$.

Theorem 4.2. *Let $B \in \mathsf{Ab}$, and let A be a subgroup of B. If $A \in \mathcal{A}$, then $B \in \mathcal{A}$.*

Proof. Let $\pi : B \to B/A$ denote the canonical projection, and let τ_A be a Hausdorff group topology on A such that $\mathbf{n}(A,\tau_A)$ is non-trivial and finite. Then, τ_A can be extended to a Hausdorff group topology τ_B on B by taking the neighborhoods of 0 with respect to τ_A as a base for the neighborhoods of 0 with respect to τ_B. It remains to be seen that $\mathbf{n}(A,\tau_A) = \mathbf{n}(B,\tau_B)$. Let $x \in \mathbf{n}(A,\tau_A)$. Since every continuous character $\chi \in \widehat{(B,\tau_B)}$ restricted to A is a continuous character of A, one has that $\chi(x) = \chi_{|A}(x) = 0$. Thus, $x \in \mathbf{n}(B,\tau_B)$. This shows that $\mathbf{n}(A,\tau_A) \subseteq \mathbf{n}(B,\tau_B)$. Conversely, let $x \in \mathbf{n}(B,\tau_B)$. Assume that $x \notin A$. Then, $\pi(x) \neq 0$. Since A is an open subgroup of B, the quotient B/A is a discrete abelian group. Consequently, there exists a continuous character $\psi : B/A \to \mathbb{T}$ such that $\psi(\pi(x)) \neq 0$ (cf. [10, Theorem 39]). So, $\psi\pi \in \widehat{(B,\tau_B)}$ and $\psi\pi(x) \neq 0$, which contradicts the assumption that $x \in \mathbf{n}(B,\tau_B)$. This shows that $x \in A$. To conclude, we prove that $x \in \mathbf{n}(A,\tau_A)$. Every continuous character $\psi \in \widehat{(A,\tau_A)}$ can be extended to a character χ on B (because $\mathbb{T}$ is divisible). Since $\chi_{|A} = \psi$ is continuous on A, which is an open subgroup of B, χ is continuous on B. Thus, $\chi \in \widehat{(B,\tau_B)}$, and it follows that $\psi(x) = \chi(x) = 0$. Therefore, $x \in \mathbf{n}(A,\tau_A)$. Hence, $\mathbf{n}(B,\tau_B) = \mathbf{n}(A,\tau_A)$, which is non-trivial and finite. □

Proof of Theorem A. Let B be an abelian group with an infinite torsion subgroup A. Since A is infinite and torsion, by Lemma 2.5, A contains a subgroup that is isomorphic to either $\mathbb{Z}(p^\infty)$ or $\bigoplus_{i=1}^\infty C_i$, where each C_i is a non-trivial finite cyclic group. Moreover, one has that $\mathbb{Z}(p^\infty) \in \mathcal{A}$ by Theorem B, and $\bigoplus_{i=1}^\infty C_i \in \mathcal{A}$ by Theorem 2.6. Therefore, it follows from Theorem 4.2 that $A \in \mathcal{A}$, and hence one has that $B \in \mathcal{A}$, as desired. □

Remark 4.3. In the original version of this manuscript, it was only shown that Theorem A holds for abelian torsion groups G where $|G| = \aleph_0$ or $|G| > \mathfrak{c}$. I am grateful to the anonymous referee for suggesting Theorem 4.2, which led to an improvement in the statement of Theorem A, as well as great simplification of its proof.

5. Two open problems

Theorem A naturally leads to the following problem.

Problem I. Is there an infinite abelian group E with a non-trivial torsion subgroup that does not admit an almost maximally almost-periodic group topology?

Discussion 5.1. Theorem A implies that if such an infinite abelian group E exists, then its torsion subgroup must be finite. At the time this manuscript is being revised, a negative answer to this problem was conjectured by Gabriyelyan (cf. [4, Theorem 5]), but his proof dated February 4, 2009, available on the ArXiv preprint server, is incomplete. In Gabriyelyan's manuscript, a variation of Theorem 4.2 was also presented.

Theorem A also raises another non-trivial question.

Problem II.

(a) Which abelian topological groups occur as the von Neumann radical of a (Hausdorff) abelian topological group?
(b) Which abelian groups occur (algebraically) as the von Neumann radical of a (Hausdorff) abelian topological group?

Discussion 5.2. Due to the algebraic nature of this discussion, we are more interested in part (b) of this problem. We put $\cong_a$ to denote an isomorphism in Ab. To abbreviate the notations, we introduce the following class of abelian groups:

$$\mathcal{B} = \left\{ A \in \mathsf{Ab} \;\middle|\; \exists G \in \mathsf{Ab}(\mathsf{Haus}),\, \mathbf{n}(G) \cong_a A \right\}. \tag{46}$$

(a) By Theorem 2.1(a) and (b), $\mathbb{Z} \in \mathcal{B}$ and $\mathbb{Z}(p^\infty) \in \mathcal{B}$ for every prime p.
(b) One has that $\mathbb{R} \in \mathcal{B}$, because $\mathbb{R}$ admits a minimally almost periodic Hausdorff group topology coarser than the Euclidean topology of $\mathbb{R}$ (cf. [9, Theorem]).
(c) Since every continuous character of $\mathbb{Q}$ can be extended uniquely to a continuous character of $\mathbb{R}$, and every continuous character of $\mathbb{R}$ restricted to $\mathbb{Q}$ is a continuous character of $\mathbb{Q}$, $\mathbf{n}(\mathbb{R}, \tau) \cap \mathbb{Q} = \mathbf{n}(\mathbb{Q}, \tau_{|\mathbb{Q}})$. Thus, $\mathbb{Q} \in \mathcal{B}$.
(d) If $\{A_\alpha\}_{\alpha \in I}$ is an arbitrary family in $\mathcal{B}$, then $G \in \mathcal{B}$ for every abelian group G such that $\bigoplus_{\alpha \in I} A_\alpha \subseteq G \subseteq \prod_{\alpha \in I} A_\alpha$; in particular, $\bigoplus_{\alpha \in I} A_\alpha \in \mathcal{B}$, and $\prod_{\alpha \in I} A_\alpha \in \mathcal{B}$.
(e) Due to the algebraic structures of abelian groups, it follows from the foregoing observations that the class $\mathcal{B}$ includes the following types of abelian groups: free abelian groups, divisible groups, direct sums of cyclic groups, bounded torsion groups, and finitely generated groups.
(f) We feel that we have not exploited the full strength of (d). Therefore, this task is left to another day.

Remark 5.3. (Added on *March 16, 2009.*) After this paper was accepted for publication, the following noteworthy details came to the author's attention:

(a) On March 8, 2009, Gabriyelyan posted a complete solution of Problem I on the ArXiv preprint server (cf. [5]).
(b) It appears that Gabriyelyan is claiming credit for Theorem 4.2. Although, as noted in Discussion 5.1, a variant of this result appeared in Gabriyelyan's February 4, 2009 preprint (cf. [4]), Theorem 4.2 was suggested to the author by the referee for this paper (see Remark 4.3). The principle behind Theorem 4.2 is not a novel one, though; it was already known to Ajtai, Havas, and Komlós (see paragraph below Theorem′ in [1, p. 22]) and to Zelenyuk and Protasov (see proof of Theorem 16 in [13, top of p. 459]). The author of this paper did not claim credit for Theorem 4.2. Thus, as a courtesy to Gabriyelyan, the author hereby explicitly disclaims all credit for Theorem 4.2 (which the author considers folklore).

Acknowledgements

This paper stems from my undergraduate summer research project in 2008, under the supervision of Dr. Gábor Lukács. I wish to express my gratitude to Dr. Lukács for his patience and guidance throughout the project, and for his extensive help with the preparation of this manuscript.

I am thankful to Dr. Brendan Goldsmith for the helpful correspondence concerning the structure of reduced p-groups.

I am grateful to Karen Kipper for her kind help in proof-reading this paper for grammar and punctuation.

Last but not least, I would like to thank the anonymous referee for her/his helpful comments and suggestions, which led to significant improvements of the manuscript, not only in form, but also in substance.

References

[1] M. Ajtai, I. Havas, J. Komlós, Every group admits a bad topology, in: Studies in Pure Mathematics, Birkhäuser, Basel, 1983, pp. 21–34.
[2] D. Dikranjan, C. Milan, A. Tonolo, A characterization of the maximally almost periodic abelian groups, J. Pure Appl. Algebra 197 (1–3) (2005) 23–41.
[3] L. Fuchs, Infinite Abelian Groups, vol. I, Pure Appl. Math., vol. 36, Academic Press, New York, 1970.
[4] S. Gabriyelyan, On T-sequences and characterized subgroups, preprint, March 8, 2009, arXiv:0902.0723v2.
[5] S. Gabriyelyan, Characterization of almost maximally almost-periodic groups, preprint, March 8, 2009, arXiv:0903.1425v1.
[6] E. Hewitt, K.A. Ross, Abstract Harmonic Analysis, Academic Press Inc., Publishers, New York, 1963.
[7] G. Lukács, Almost maximally almost-periodic group topologies determined by T-sequences, Topology Appl. 153 (15) (2006) 2922–2932.
[8] J. von Neumann, E.P. Wigner, Minimally almost periodic groups, Ann. of Math. (2) 41 (1940) 746–750.
[9] J.W. Nienhuys, A solenoidal monothetic minimally almost periodic group, Fund. Math. 73 (2) (1971/1972) 167–169.
[10] L.S. Pontryagin, Selected Works, vol. 2, third ed., Gordon & Breach Science Publishers, New York, 1986. Topological groups, Edited and with a preface by R.V. Gamkrelidze, Translated from the Russian and with a preface by Arlen Brown, With additional material translated by P.S.V. Naidu.
[11] I. Protasov, E. Zelenyuk, Topologies on Groups Determined by Sequences, Math. Stud. Monogr. Ser., vol. 4, VNTL Publishers, L'viv, 1999.
[12] M. Tkachenko, I. Yaschenko, Independent group topologies on abelian groups, Topology Appl. 122 (1–2) (2002) 425–451.
[13] E.G. Zelenyuk, I.V. Protasov, Topologies on abelian groups, Izv. Akad. Nauk SSSR Ser. Mat. 54 (5) (1990) 1090–1107; English translation: Math. USSR-Izv. 37 (2) (1991) 445–460.