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SOME EXAMPLES OF PRECIPITOUS IDEALS

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0. Introduction

Generic ultrapowers were first introduced by Solovay [14] in his study of saturated ideals. Let I be an ideal on a regular, uncountable cardinal κ (in this paper, any ideal on κ will be κ-complete and nontrivial, that is, $\kappa \notin I$ and $\{\alpha\} \in I$ for any $\alpha \in \kappa$). Let I^+ be the collection of all *sets of positive measure*, i.e. $I^+ = \{S \subseteq \kappa : S \notin I\}$. Let R_I be the notion of forcing

$$R_I = (I^+, \subseteq).$$

Forcing conditions are sets of positive measure and S_1 is stronger than S_2 iff $S_1 \subseteq S_2$. If $W = W(I) \subseteq I^+$ is R_I-generic, then W is called an *I-generic ultrafilter*. The ultrapower of the ground model by W is called the *generic ultrapower*, written Ult_W. W will not be countably complete and hence Ult_W may not be well founded. The ideal I is said to be *precipitous* if every condition in R_I forces that Ult_W is well founded. These ideals were introduced by the first author and Prikry in [5] and [6].

The norm induced by I on functions f from κ into the ordinals was first defined by Galvin and Hajnal in [2], as follows: say $f < g$ iff $f(\alpha) < g(\alpha)$ for almost all α (mod I) (a property P holds for *almost all* α (mod I) if $\{\alpha \in \kappa : P \text{ fails}\} \in I$). The partial ordering $<$ of ordinal functions is well founded; let $\|f\|$, the *norm* of f, be the rank of f in the ordering $<$. Similarly, if S is a set of positive measure, we let

$$f <_S g \quad \text{iff } f(\alpha) < g(\alpha) \text{ for a.a. } \alpha \in S \text{ (mod } I),$$

and denote by $\|f\|_S$ the *S-norm* of f.

A function f is said to be an αth *function* if $\|f\|_S = \alpha$ for all conditions S in R_I or, equivalently, if for every generic W, α is the ordinal $[f]_W$ represented by f in the generic ultrapower Ult_W. The ideal I is *uniformly normed* if for every ordinal α, the αth function exists.

Let f be an ordinal function on κ. The *degree* of f, $\deg(f)$, is the least ordinal α (if it exists) such that some condition in R_I forces that f represents α in the generic ultrapower (otherwise we let $\deg(f) = \infty$). For every f we have $\|f\| \leqslant \deg(f)$, The ideal I is *normed* if $\deg(f) = \|f\|$ for all ordinal functions f on κ.

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The concept of a uniform norm was introduced (before precipitous ideals) by the first author in a talk at the M.I.T. Set Theory Conference in March 1975. In [4] he further studied uniform norms and introduced the degree of a function and the concept of a normed ideal. Proofs of otherwise unattributed assertions in this introduction may be found in [4]. Normed and uniformly normed ideals have also been studied (with different terminology) by Levinski in [11].

If I is any uniformly normed ideal, then I is normed, and if I is normed, then I is precipitous. In [4] it was asked whether either of the converses hold. The main result of this paper is that neither does:

Main theorem. *Suppose κ is a measurable cardinal. Then there exists a generic extension in which $\kappa = \aleph_1$ and there exist κ-complete ideals I_1, I_2 and I_3 such that*
(i) *I_1 is uniformly normed,*
(ii) *I_2 is normed, but not uniformly normed,*
(iii) *I_3 is precipitous, but not normed.*

The main theorem is proved in two stages. In the first stage (Section 1) we use a technique of Kunen and Paris from [10] to construct a model in which κ is still measurable such that by combining a large number of ultrafilters we can find ideals having the properties of the theorem. In the second stage we use a Levy collapse to make $\kappa = \aleph_1$ and show that the ideals still have the desired properties. This technique was used by the second author [7] to show that the ideal generated by a measure on κ after a Lévy collapse is precipitous. Several people, including the present authors, noticed that his method of proof applies to any κ-chain condition forcing and to the more general case when κ carries a precipitous ideal in the ground model (cf. [8, 11, 12]). Levinski also observed that an αth function on the measurable cardinal is the αth function in the extension (thus the precipitous ideal constructed in [7] is uniformly normed). We state a general theorem in Section 2 that includes these results. The second stage of the proof of the main theorem is given in Section 3, using results from Section 2.

The proof in Section 3 heavily uses the explicit definition of the ideals from Section 1. It is not known whether a Lévy collapse (or more generally, any κ-cc notion of forcing) always preserves these properties.

K. Kunen has pointed out that if κ is supercompact and $2^\kappa = \kappa^+$, then it can be proved outright that there is a precipitous, but unnormed, ideal.

1. Examples of ideals: κ measurable

In this section the main result is

Theorem 1.1. *If κ is measurable, then there is a generic extension in which κ is still* *measurable and there are ideals J_1, J_2 and J_3 such that*

(i) *J_1 is uniformly normed,*

(ii) *J_2 is normed, but not uniformly normed,*

(iii) *J_3 is precipitous but not normed.*

We will begin by describing a way of combining ideals and we will then define the generic extension. After these preliminaries we will describe the ideals and prove that they have the required properties.

Let

$$\{I_a : a \in A\} \tag{1}$$

be a collection of κ-complete ideals over κ. Then $I = \bigcap\{I_a : a \in A\}$ is a κ-complete ideal over κ. In general, I does not inherit any special properties of the I_a's. Let us call the collection (1) *separated* if there exist sets

$$\{E_a : a \in A\}$$

such that for each $a \in A$, E_a belongs to the filter $F(I_a)$, the dual of I_a, and for all $b \neq a$, $E_b \in I_a$. If this is the case, we call I the *sum* of the I_a:

$$\sum_a I_a = \bigcap\{I_a : a \in A\}.$$

Let us recall some basic terminology from [6] and [4]: Let I be a κ-complete ideal over κ. An *I-partition* of a set $S \in I^+$ is a collection P of subsets of S such that

(i) $P \subset I^+$,

(ii) if $X, Y \in P$ and $X \neq Y$, then $X \cap Y \in I$,

(iii) P is maximal: if $X \subseteq S$ has positive measure, then $X \cap Y \in I^+$ for some $Y \in P$.

A partition P_1 is a *refinement* of P_2, $P_1 \leqslant P_2$, if for every $X \in P_2$ there is $Y \in P_1$ such that $Y \subseteq X$. An *I-function* is a function whose domain is a set $S \in I^+$. A *functional* on a set $S \in I^+$ is a collection F of ordinal I-functions such that

(i) $P_F = \{\mathrm{dom}(f) : f \in F\}$ is an I-partition of S;

(ii) if $f, g \in F$ and $f \neq g$, then $\mathrm{dom}(f) \neq \mathrm{dom}(g)$.

We call two functionals F and G on S *equivalent* (mod I)

$$F = G \ (\mathrm{mod}\ I)$$

if there exists an I-partition P of S that refines P_F and P_G and such that for every $X \in P$ the respective functions $f \in F$ and $g \in G$ agree everywhere on X. Similarly,

$$F < G$$

means that P_F refines P_G and that each $f \in F$ is less than the respective $g \in G$ everywhere on dom(f). The ideal I is precipitious if and only if for every $S \in I^+$, the partial ordering $<$ of functionals on S is well founded. If F is a functional on S, then the *degree* of F,

$$\mathrm{deg}_S(F)$$

is the length of $<$ below F, if $<$ is well founded below F. If f is an ordinal I-function, then $\deg_S(f) = \deg_S(\{f\})$. If F is a functional on κ, then we drop the subscript S. Functionals correspond to Boolean valued names for ordinal functions in the generic ultrapower. We recall Theorem 1.4 of [4]:

The degree of a functional F is the least ordinal α such that for some $f \in F$ and some set $S \subseteq \mathrm{dom}(f)$ of postive measure,

$$S \Vdash f \text{ represents } \alpha \text{ in the generic ultrapower } \mathrm{Ult}_{W(I)}.$$

The following lemma is proved by an easy verification:

Lemma 1.2. *Let $\{I_a : a \in A\}$ be a family of κ-complete ideals over κ and assume that the I_a are separated by a family $\{E_a : a \in A\}$. Let $I = \sum_a I_a$. Then*

(i) *$\{E_a : a \in A\}$ is an I-partition of κ;*

(ii) *for every $a \in A$, every I_a-partition of E_a is an I-partition of E_a and vice versa; every I_a-functional on E_a is an I-functional on E_a and vice versa, and its I_a-degree is equal to its I-degree;*

(iii) *for every functional F,*

$$\deg(F) = \min\{\deg_{E_a}(F) : a \in A\};$$

(iv) *I precipitous if and only if each I_a is precipitous;*

(v) *I is normal if and only if each I_a is normal.*

In this section we shall construct two examples of precipitous ideals on a measurable cardinal κ. The idea is to let $I = \sum_a I_a$ where for each $a \in A$, I_a is a normal κ-complete prime ideal, and $|A| = \kappa^+$. Clearly, every κ-complete prime ideal is precipitous (and the generic ultrapower is just the ordinary ultrapower), and in fact, it is uniformly normed (any function that represents α is the αth function).

In order to obtain a family of κ^+ separated normal measure on κ, we employ the technique of Kunen and Paris from [10].

Thus let $\mathcal{M}$ denote the ground model, a model of ZFC+"there exists a measurable cardinal κ". We can assume that the GCH is true in $\mathcal{M}$, as otherwise we could make a preliminary generic extension to make it true. We shall extend $\mathcal{M}$ generically to a model $\mathcal{M}[G]$ in which κ is still a measurable cardinal, and in which we can find families $\{I_a : a \in A\}$ of normal κ-complete prime ideals such that $\sum_a I_a$ is respectively a precipitous ideal that is not normed, and a normed ideal that is not uniformly normed.

Let us work in $\mathcal{M}$, and let us construct a notion of forcing that yields $\mathcal{M}[G]$. We follow closely [10, Section 3].

Let κ be a measurable cardinal and let U be a normal measure on κ. Following [9], let U_2 be the ultrafilter on $[\kappa]^2 = \{(\alpha, \beta) : \alpha < \beta < \kappa\}$ defined by

$$X \in U_2 \leftrightarrow \{\alpha : \{\beta : (\alpha, \beta) \in X\} \in U\} \in U.$$

Let $\mathcal{N}_1$ and $\mathcal{N}_2$ denote the ultrapowers of $\mathcal{M}$ by U and U_2 respectively, and let $i = i_{01}$ and $j = i_{02}$ be the corresponding elementary embeddings,

$$i : \mathcal{M} \to \mathcal{N}_1, \qquad j : \mathcal{M} \to \mathcal{N}_2.$$

Let $\kappa_1 = i(\kappa)$ and $\kappa_2 = j(\kappa)$.

We shall now review some basic facts about the ultrapowers $\mathcal{N}_1$ and $\mathcal{N}_2$ that we might need later on:

If $X \subseteq \kappa$, then

$$X = i(X) \cap \kappa = j(X) \cap \kappa, \quad \text{and} \quad i(X) = j(X) \cap \kappa_1.$$

If $X \subseteq \kappa$ and $Y \subseteq [\kappa]^2$, then

$$X \in U \leftrightarrow \kappa \in i(X) \leftrightarrow \kappa \in j(X), \qquad Y \in U_2 \leftrightarrow (\kappa, \kappa_1) \in j(X).$$

If f is any function on κ and g is any function on $[\kappa]^2$, let $[f]_U$ and $[g]_{U_2}$ denote the elements of $\mathcal{N}_1$ and $\mathcal{N}_2$ represented respectively by f and g. We have

$$[f]_U = (i(f))(\kappa), \qquad [g]_{U_2} = (j(g))(\kappa, \kappa_1).$$

Let d, π_0 and π_1 be functions defined as follows:

$$d(\alpha) = \alpha, \qquad \pi_0(\alpha, \beta) = \alpha, \qquad \pi_1(\alpha, \beta) = \beta.$$

Then

$$[d]_U = \kappa, \qquad [\pi_0]_{U_2} = \kappa, \qquad [\pi_1]_{U_2} = \kappa_1.$$

Finally, let g be such that $g(\alpha, \beta) < \beta$ for every $\alpha, \beta \in \kappa$. Then there is a function $f : \kappa \to \kappa$ such that $g(\alpha, \beta) = f(\alpha)$ for almost all (α, β) mod U_2 (this follows from normality).

We shall now define a notion of forcing P. The intention is to add generically a function G from (a subset of) κ_1 into κ_1 such that

(a) $G(\alpha) \geq \alpha$,

(b) if $\alpha_1, \alpha_2 \in \mathrm{dom}(G)$ and $\alpha_1 < \alpha_2$, then $G(\alpha_1) < \alpha_2$.

(The property (b) will prevent the collapse of cardinals.)

First we define P_κ:

$p \in P_k$ iff (i) p *is* a function and $\mathrm{dom}(p) \subseteq \kappa$, $\mathrm{ran}(p) \subseteq \kappa$:
(ii) $p(\alpha) \geq \alpha$ for all α;
(iii) if $\alpha_1, \alpha_2 \in \mathrm{dom}(p)$ and $\alpha_1 < \alpha_2$, then $p(\alpha_1) < \alpha_2$;
(iv) for every regular cardinal λ, $|\mathrm{dom}(p) \cap \lambda| < \lambda$. (2)

(The last property is the well-known Easton's device.)

Now let

$$P = j(P_\kappa) \quad \text{and} \quad p \leq q \quad \text{iff} \quad p \supseteq q.$$

(the fact that P is defined in $\mathcal{N}_2 = j(\mathcal{M})$ is the essence of the Kunen–Paris method: it enables us to extend the ultrafilter U_2 in $\mathcal{M}[G]$.)

For every η let

$$P_\eta = \{p \in P \colon p \subseteq \eta \times \eta\}, \qquad P^\eta = \{p \in P \colon p \subseteq (\kappa_2 - \eta) \times (\kappa_2 - \eta)\} \tag{3}$$

and note that since $\mathcal{N}_2$ is closed under κ-sequences, P_κ as defined in (3) is the same as the P_κ defined in (2).

Lemma 1.3. *Forcing with P preserves all regular cardinals.*[1]

Proof. We want to show that every regular cardinal in $\mathcal{M}$ remains a regular cardinal in the extension. Since $|P_\kappa| = \kappa$ and $|\kappa_2| = \kappa^+$, it is easy to see that $|P| = \kappa^+$. Thus it suffices to show that regular cardinals $\leqslant \kappa^+$ are preserved. Let G be a generic set of conditions. Let λ and η be regular cardinals in $\mathcal{M}$, $\lambda < \eta \leqslant \kappa^+$, and assume that η has a cofinal λ-sequence in $\mathcal{M}[G]$.

Let α and β be such that $\alpha \leqslant \lambda \leqslant \beta$ and $\{(\alpha, \beta)\} \in G$. It follows from (2(iii)) that $\mathcal{M}[G] = \mathcal{M}[G \cap P_\alpha][G \cap P^{\beta+1}]$. Since $|P_\alpha| \leqslant \lambda$ and $P^{\beta+1}$ is λ-closed, Easton's theorem applies (cf. [1]) and every λ-sequence in $\mathcal{M}[G]$ is in $\mathcal{M}[G \cap P_\alpha]$. But η has no cofinal λ-sequence in $\mathcal{M}[G \cap P_\alpha]$. $\square$

Lemma 1.4. *If G is P-generic over $\mathcal{M}$ and $\kappa \in \mathrm{dom}(G)$, then there is an elementary embedding $j^G : \mathcal{M}[G \cap P_k] \to \mathcal{N}_2[G]$ extending the embedding $j : \mathcal{M} \to \mathcal{N}_2$.*

Proof. j^G is defined in the obvious way:

$$j^G(\mathrm{int}_{G \cap P_\kappa}(x)) = \mathrm{int}_G(j(x))$$

where int denotes the interpretation of a Boolean-valued name by the generic filter. G is generic over $\mathcal{N}_2$ and since $G \supset G \cap P_\kappa = j[G \cap P_\kappa]$, j^G is well defined and elementary. $\square$

Now we define a normal measure U^G on κ:

$$X \in U^G \quad \text{iff} \quad \kappa \in j^G(X).$$

Each U^G is a measure since every $X \subseteq \kappa$ in $\mathcal{M}[G]$ belongs to $\mathcal{M}[G \cap P_\kappa]$. Since j^G extends j, it is clear that $U^G \supset U$.

Now for the rest of this section let G be a fixed generic function from a subset of κ_2 into κ_2 such that

$$G(\kappa) = \kappa, \qquad G(\kappa^+) = \kappa^+, \quad \text{and} \quad G(\kappa_1) = \kappa_1.$$

Proof of Theorem 1.1(i). *$\mathcal{M}[G]$ has a uniformly normed ideal J_1.*

Let J_1 be the dual of the ultrafilter U^G. Since J_1 is maximal it is uniformly normed. $\square$

Now if $a \in \mathrm{dom}(G)$ and $b \geqslant a$ is such that $b < \kappa_2$, then we say that G' is *the*

[1] We use this opportunity to correct a mistake in [10]. In Lemma 3.6 ii it is claimed incorrectly that (what we call) P_κ has the κ-chain condition. Since the only implicit use of the κ-c.c. is the preservation of cardinals, our Lemma 1.3 will do instead.

same as G except that $G'(a)=b$ if

$$\operatorname{dom}(G')=(\operatorname{dom}(G)-[a,b])\cup\{a,b+1\},$$
$$G'(\nu)=G(\nu)\quad\text{if } \nu\in\operatorname{dom}(G) \text{ and } \nu<a \text{ or } \nu>b,$$
$$G'(a)=b,$$
$$G'(b+1)=(\text{least member of } \operatorname{ran}(G)-(b+1)).$$

Lemma 1.5. *Suppose* $a\in\operatorname{dom}(G)$, $\kappa\leqslant a\leqslant b$, *and G' is the same as G except that* $G'(a)=b$. *Then G' is P-generic over $\mathcal{M}$ and* $P(\kappa)\cap\mathcal{M}[G']=P(\kappa)\cap\mathcal{M}[G]$.

Proof. That G' is P-generic is straightforward. Note that $a\in\operatorname{dom}(G)$ is needed here. If a is a cardinal and $a\notin\operatorname{dom}(G)$, then there will not be any generic G' in $\mathcal{M}[G]$ such that $a\in\operatorname{dom}(G')$.

That $P(\kappa)\cap\mathcal{M}[G']=P(\kappa)\cap\mathcal{M}[G]$ follows immediately from the fact that $G'\cap P_\kappa=G\cap P_\kappa$. □

Lemma 1.5 is still true if we change G at finitely many points of its domain instead of only one. Any such modified function G' will give rise by Lemma 1.4 to a map $j^{G'}:\mathcal{M}[G\cap P_\kappa]\to\mathcal{N}_2[G']$, and hence to a ultrafilter $U^{G'}$ on κ in $\mathcal{M}[G]$. We will construct J_2 and J_3 by taking sums of the ideals dual to ultrafilters so obtained. Since J_3 is simpler we will take it first:

Proof of Theorem 1.1(iii). *In $\mathcal{M}[G]$ there is a precipitous ideal J_3 on κ which is not normed.*

We will in fact construct J_3 so that the constant function on κ has degree κ_2, but every function $f:\kappa\to\kappa$ has degree less than κ_1. Then $\|f\|\leqslant\deg(f)<\kappa_1$, for all $f:\kappa\to\kappa$ and the norm of the constant function is at most (and, in fact, is exactly) κ_1. Since its norm is less than its degree, J_3 is not normed.

In $\mathcal{M}$, the cofinality of κ_1 is κ^+. Thus let us consider, in $\mathcal{M}$, a fixed increasing sequence

$$\{b_a: a<\kappa^+\}$$

cofinal in κ_1. Let (in $\mathcal{M}[G]$)

$$A=\{a: \kappa\leqslant a<\kappa_1,\ a\in\operatorname{ran}(G) \text{ and } b_a\in\operatorname{dom}(G)\}.$$

For every $a\in A$, let G_a be the set of all (α,β) such that

$$\begin{aligned}
&\text{either} && \alpha<\kappa \quad\text{and}\quad (\alpha,\beta)\in G,\\
&\text{or} && (\alpha,\beta)=(\kappa,a),\\
&\text{or} && a<\alpha<b_a \quad\text{and}\quad (\alpha,\beta)\in G,\\
&\text{or} && (\alpha,\beta)=(b_a,\kappa_1),\\
&\text{or} && b_a<\alpha \quad\text{and}\quad (\alpha,\beta)\in G.
\end{aligned}$$

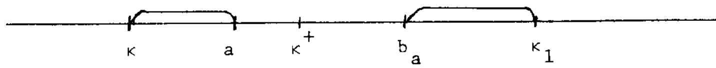

Fig. 1.

Thus G_a is the same as G except that $G_a(\kappa) = a$ and $G_a(b_a) = \kappa_1$. Let $j_a = j^{G_a}$ and $U_a = U^{G_a}$ be the embeddings and ultrafilters given by Lemma 1.5. Let I_a be the dual ideal to U_a and set

$$J_3 = I = \sum_{a \in A} I_a.$$

After some preliminaries we will show that I has the required properties.

Let p be a condition and assume that $p(\kappa) = \kappa$ and $p(\kappa_1) = \kappa_1$. Let $a < \kappa^+$ and assume that p forces that $a \in A$; i.e. $a \in \operatorname{ran}(p)$ and $b_a \in \operatorname{dom}(p)$. We denote by

$$p/a$$

the condition obtained from p in the same way as G_a is obtained from G: $(\alpha, \beta) \in p/a$ iff

$$\begin{aligned} \text{either} \quad & \alpha < \kappa \quad \text{and} \quad p(\alpha) = \beta, \\ \text{or} \quad & (\alpha, \beta) = (\kappa, a), \\ \text{or} \quad & a < \alpha < b_a \quad \text{and} \quad p(\alpha) = \beta, \\ \text{or} \quad & (\alpha, \beta) = (b_a, \kappa_1), \\ \text{or} \quad & b_a < \alpha \quad \text{and} \quad p(\alpha) = \beta. \end{aligned}$$

Then we have

$$p \Vdash X \in U_a \quad \text{iff} \quad p/a \Vdash \kappa \in j(X).$$

We shall now show that the measures U_a, $a \in A$, are separated:

Lemma 1.6. *There exists a collection $\{E_a : a \in A\}$ of subsets of κ such that $E_a \in U_a$ for each $a \in A$, and if $a' \neq a$, then $E_a \notin U_{a'}$.*

Proof. For each ordinal a let f_a be a function that represents a in the ultrapower $\mathcal{N}_1$:

$$[f_a]_U = a.$$

If $a \in A$, we define

$$E_a = \{\alpha < \kappa : G(\alpha) = f_a(\alpha)\}.$$

Now if a, a' are in A, we have

$$\begin{aligned} E_a \in U_{a'} &\leftrightarrow \kappa \in j_{a'}(E_a) \\ &\leftrightarrow (j_{a'}(G))(\kappa) = (j_{a'}(f_a))(\kappa) \\ &\leftrightarrow G_{a'}(\kappa) = (jf_a)(\kappa) \\ &\leftrightarrow a' = a. \quad \square \end{aligned}$$

In order to get more information on the ultrapowers $\mathrm{Ult}_{U_a}(\mathcal{M}[G])$ we shall now use the embeddings j_a to define certain ultrafilters over $[\kappa]^2$.

For each $a \in A$, let U_a^* be the following ultrafilter over $[\kappa]^2$: For $X \subseteq [\kappa]^2$, let

$$X \in U_a^* \quad \text{iff} \quad (\kappa, \kappa_1) \in j_a(X).$$

It is clear that each U_a^* extends the measure U_2 on $[\kappa]^2$ in $\mathcal{M}$. Let $\mathrm{Ult}_{U_a^*}(\mathcal{M}[G])$ be the ultrapower of $\mathcal{M}[G]$ by U_a^*, and let $i_a : \mathcal{M}[G] \to \mathrm{Ult}_{U_a^*}(\mathcal{M}[G])$ be the associated elementary embedding. As usual, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{M}[G] \arrow[r, "j_a"] \arrow[d, "i_a"] & \mathcal{N}_2[G_a] \\
\mathrm{Ult}_{U_a^*}(\mathcal{M}[G]) \arrow[ur, "k_a"'] &
\end{tikzcd}$$

Fig. 2.

where embedding k_a is defined as follows:

$$k_a([f]_{U_a^*}) = (j_a(f))(\kappa, \kappa_1).$$

Here $f \in \mathcal{M}[G]$ is a function on $[\kappa]^2$ and $[f]_{U_a^*}$ is the element of $\mathrm{Ult}_{U_a^*}(\mathcal{M}[G])$ represented by f.

If f is an ordinal function on $[\kappa]^2$ in $\mathcal{M}$ and $[f]_{U_2}$ the element of $\mathcal{N}_2 = \mathrm{Ult}_{U_2}(\mathcal{M})$ represented by f, then

$$[f]_{U_2} \leq [f]_{U_a^*}$$

since U_a^* extends U_2. On the other hand, since

$$[f]_{U_2} = (j(f))(\kappa, \kappa_1).$$

we have

$$k_a([f]_{U_a^*}) = [f]_{U_2}$$

and it follows that

$$[f]_{U_a^*} = [f]_{U_2}.$$

[We could now go on and show that $i_a = j_a$ and that k_a is the identity mapping.] Thus we have:

Lemma 1.7. *For every $a \in A$ and every function $f \in \mathcal{M}$ on $[\kappa]^2$, $[f]_{U_a^*} = [f]_{U_2}$. In particular,*

$$[\mathrm{const}(\kappa)]_{U_a^*} = \kappa_2$$

(where $\mathrm{const}(\kappa)$ *denotes the constant function with value* κ*).*

Lemma 1.8. *For every $a \in A$, the ultrafilters U_a and U_a^* are equivalent, that is, there is a 1–1 function $\pi : \kappa^2 \to \kappa$ such that $U_a = \{\pi[x] : x \in U_a^*\}$.*

Corollary 1.9. *For every $a \in A$,*

$$[\mathrm{const}(\kappa)]_{U_a} = \kappa_2.$$

Proof. Let $a \in A$. Let π_0 be the projection of $[\kappa]^2$ to κ:

$$\pi_0(\alpha, \beta) = \alpha.$$

It is clear from the definition of U_a and U_a^* that for every $X \subseteq \kappa$,

$$\begin{aligned} X \in U_a &\leftrightarrow \kappa \in j_a(X) \\ &\leftrightarrow (\kappa, \kappa_1) \in \pi_0^{-1}(j_a(X)) \\ &\leftrightarrow \pi_0^{-1}(X) \in U_a^*. \end{aligned}$$

Thus is suffices to show that π_0 is 1–1 on a set in U_a^*. We recall that f_{b_a} is the function on κ that represents b_a in $\mathrm{Ult}_U(\mathcal{M})$. Let

$$Z_a = \{(\alpha, \beta) \in [\kappa]^2 : G(f_{b_a}(\alpha)) = \beta\}.$$

It is clear that π_0 is 1–1 on Z_a. Since $(j(f_{b_a}))(\kappa) = b_a$ and $(j_a(G))(b_a) = G_a(b_a) = \kappa_1$, we have $(\kappa, \kappa_1) \in j_a(Z_a)$, and so $Z_a \in U_a^*$. $\square$

Lemma 1.10. *For every $f : \kappa \to \kappa$ there exists $a \in A$ such that $[f]_{U_a} < \kappa_1$.*

Proof. In view of the commutative diagram (Fig. 3) where $k([f]_{U_a}) = (j_a(f))(\kappa)$, we have $[f]_{U_a} \leqslant (j_a(f))(\kappa)$ and so it suffices to show that for every $f : \kappa \to \kappa$ there is $a \in A$ such that $(j_a(f))(\kappa) < \kappa_1$.

$$\begin{tikzcd} \mathcal{M}[G] \arrow[r, "j_a"] \arrow[d] & N_2[G_a] \\ \mathrm{Ult}_{U_a}(\mathcal{M}[G]) \arrow[ur, "k"'] & \end{tikzcd}$$

Fig. 3.

Thus let f be a P_κ-name for a function from κ to κ, and let p be a forcing condition in G. We may assume that $p(\kappa) = \kappa$, $p(\kappa^+) = \kappa^+$ and $p(\kappa_1) = \kappa_1$. It suffices to find a stronger condition $q \supseteq p$ and $a < \kappa^+$ such that $a \in \mathrm{ran}(q)$, $b_a \in \mathrm{dom}(q)$, and that

$$q/a \Vdash (jf)(\kappa) < \kappa_1.$$

Consider the unique decomposition

$$p = r \cup s \cup t \cup \{(\kappa_1, \kappa_1)\} \cup u.$$

where

$$\begin{aligned} r &\in P_\kappa, & s(\kappa) &= \kappa, \\ s &\in P^\kappa - P_{\kappa^+}, & t(\kappa^+) &= \kappa^+, \\ t &\in P^{\kappa^+} - P_{\kappa_1}, & u &\in P^{\kappa_1 + 1}. \end{aligned} \tag{4}$$

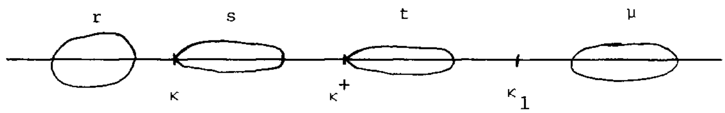

Fig. 4.

Since $p \in \mathcal{N}_2$, there is a function $\langle p_{\alpha\beta} : (\alpha, \beta) \in [\kappa]^2 \rangle$ that represents p in $\mathcal{N}_2 = \mathrm{Ult}_{U_2}(\mathcal{M})$. In view of (4) there exist conditions r, s_α $(\alpha < \kappa)$, t_α $(\alpha < \kappa)$ and $u_{\alpha\beta}$ $(\alpha, \beta < \kappa)$ in P_κ such that for almost all α, β (mod U_2).

$$p_{\alpha\beta} = r \cup s_\alpha \cup t_\alpha \cup \{(\beta, \beta)\} \cup u_{\alpha\beta}. \tag{5}$$

We may assume that (5) holds for all $\alpha > \sup(r)$ and all $\beta > \sup(t_\alpha)$, and also that

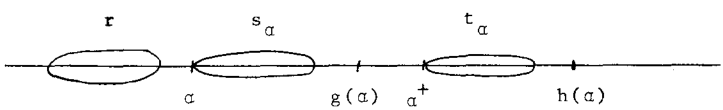

Fig. 5.

$s_\alpha(\alpha) = \alpha$, $\sup(s_\alpha) < \alpha^+$, $t_\alpha(\alpha^+) = \alpha^+$ and $\min(u_{\alpha\beta}) > \beta$. Let us fix an ordinal $\alpha > \sup(r)$. Let $g(\alpha) < \alpha^+$ be some ordinal such that $g(\alpha) > \sup(s_\alpha)$. For every ξ such that $g(\alpha) \leq \xi < \alpha^+$, let η_α^ξ be an ordinal and $r_\alpha^\xi \supseteq r$ and $t_\alpha^\xi \supseteq t_\alpha$ be conditions such that

$$\sup(r_\alpha^\xi) < \alpha, \qquad \xi < \min(t_\alpha^\xi), \tag{6}$$

and that

$$r_\alpha^\xi \cup \{(\alpha, \xi)\} \cup t_\alpha^\xi \Vdash f(\alpha) = \eta_\alpha^\xi.$$

Let $h(\alpha)$ be an ordinal such that

$$\sup(t_\alpha^\xi) < h(\alpha) \tag{7}$$

for all $\xi < \alpha^+$, and also $h(\alpha) > \eta_\alpha^\xi$ for all $\xi < \alpha^+$. Hence for all $\xi < \alpha^+$,

$$r_a^\xi \cup \{(\alpha, \xi)\} \cup t_\alpha^\xi \Vdash f(\alpha) < h(\alpha). \tag{8}$$

Now let $a < \kappa^+$ be such that $[g]_U < a$ and $[h]_U < b_a$. Let $\langle a(\alpha) : \alpha < \kappa \rangle$ and $\langle b(\alpha) : \alpha < \kappa \rangle$ represent a and b_a respectively; we may assume that $g(\alpha) < a(\alpha) < \alpha^+$ and $h(\alpha) < b(\alpha)$ for all α. From (8) it follows that for all $\alpha > \sup(r)$ and all $\beta > b(\alpha)$ (hence for almost all α, β mod U_2)

$$r_\alpha^{a(\alpha)} \cup \{(\alpha, a(\alpha))\} \cup t_\alpha^{a(\alpha)} \cup \{(b(\alpha), \beta)\} \cup u_{\alpha\beta} \Vdash f(\alpha) < \beta. \tag{9}$$

Let r^a and t^a be the elements of P_{κ_1} represented by $\langle r_\alpha^{a(\alpha)} : \alpha < \kappa \rangle$ and $\langle t_\alpha^{a(\alpha)} : \alpha < \kappa \rangle$. Clearly $r^a \supseteq r$ and $t^a \supseteq t$, and by (6) and (7), we have $\sup(r^a) < \kappa$, $a < \min(t)$ and $\sup(t) < b_a$. Let

$$\bar{q} = r^a \cup \{(\kappa, a)\} \cup \{(b_a, \kappa_1)\} \cup u$$

and

$$q = r^a \cup s \cup \{(a, a)\} \cup t^a \cup \{(\kappa_1, \kappa_1)\} \cup u.$$

Now it is clear $q \supseteq p$, that $q/a = \bar{q}$, and that (by (9))

$$\bar{q} \Vdash (jf)(\kappa) < \kappa_1. \quad \square$$

End of proof of Theorem 1.1(iii). Recall that

$$J_3 = I = \sum_{a \in A} I_a$$

where I_a is the dual ideal of U_a. The ideal I is a normal precipitous ideal on κ, by Lemma 1.2(iv) and (v). By Corollary 1.9 and Lemma 1.2(iii) we have

$$\deg(\mathrm{const}(\kappa)) = \min_a [\mathrm{const}(\kappa)]_{U_a} = \kappa_2.$$

If $f : \kappa \to \kappa$, then by Lemma 1.10 there is $a \in A$ such that

$$\deg(f) \leqslant \deg_{E_a}(f) = [f]_{U_a} < \kappa_1. \quad \square$$

Thus we have found in $\mathcal{M}[G]$ a precipitous ideal on κ that is not normed. Next we shall find a normed ideal that is not uniformly normed. This ideal will be constructed using the ideals I_α and J_α from the following lemma.

Lemma 1.11. *There exists a separated family of normal ideals*

$$\{I_a, J_a : a \in A\}$$

such that

(i) *For each ordinal α there are functions f_α and g_α such that*

(a) *for each $a \in A$, f_α is the* αth *function for I_a and g_α is the* αth *function for J_a,*

(b) *for each $a \in A$ and $\alpha < \beta$, $f_\alpha(\xi) < f_\beta(\xi)$ almost everywhere* mod J_a;

and

(ii) *The ideal $I = \sum_{a \in A} (I_a, J_a)$ is not uniformly normed.*

Proof of Theorem 1.1(ii) from Lemma 1.11. *There is an ideal J_2 in $\mathcal{M}[G]$ which is normed but not uniformly normed.*

We take J_2 equal to $I = (\sum_{a \in A} I_a) + (\sum_{a \in A} J_a)$ from Lemma 1.11. I is not uniformly normed by Lemma 1.11(ii) so we need to show that I is normed. Suppose $f : \kappa \to \mathrm{ON}$ and let γ be the I-degree of f. We need to show that γ is also the I-norm of f. Let f_α and g_α be as given in Lemma 1.11(i). By Lemma 1.2(iii), if $a \in A$, then $f(\xi) \leqslant f_\gamma(\xi)$ for almost all $\xi \bmod I_a$ and $f(\xi) \geqslant g_\gamma(\xi)$ for almost all $\xi \bmod J_a$. Let

$$S = \{\xi : f(\xi) < f_\gamma(\xi)\}.$$

Note that $S \in I_a$ for all $a \in A$. We define, for all $\alpha < \gamma$,

$$h_\alpha(\xi) = \begin{cases} f_\alpha(\xi) & \text{if } \xi \notin S, \\ g_\alpha(\xi) & \text{if } \xi \in S. \end{cases}$$

If $\alpha<\beta<\gamma$, and $a\in A$, then

$$f_\alpha(\xi)<f_\beta(\xi)<f_\gamma(\xi)\leqslant f(\xi) \tag{10}$$

for almost all $\xi\notin S \bmod I_a$, and by Lemma 1.11(i(b)) also for almost all $\xi\notin S \bmod J_a$. Also,

$$g_\alpha(\xi)<g_\beta(\xi)<g_\gamma(\xi)\leqslant f(\xi) \tag{11}$$

for almost all $\xi\notin S \bmod J_a$, and since $S\in I_a$, we conclude from (10) and (11) that

$$h_\alpha(\xi)<h_\beta(\xi)<f(\xi) \tag{12}$$

holds for almost all $\xi \bmod I$. From (12) it follows that $\|f\|\geqslant\gamma=\deg(f)$, and hence $\|f\|=\gamma$. □

Proof of Lemma 1.11. As in the proof of Theorem 1.1(iii), the ideals are duals of normal measures in $\mathcal{M}[G]$ defined using Lemmas 1.4 and 1.5. Let us consider, in $\mathcal{M}$, a fixed mapping

$$a\mapsto b_a \quad (\kappa\leqslant a<\kappa^+)$$

of $\kappa^+-\kappa$ onto $\kappa_1-\kappa^+$, such that every $b\in\kappa_1-\kappa^+$ is $=b_a$ for κ^+ many a's. Let (in $\mathcal{M}[G]$)

$$A=\{a\colon a\in\operatorname{ran}(G) \text{ and } b_a\in\operatorname{ran}(G)\}.$$

For every $a\in A$, let

$$\begin{aligned}(\alpha,\beta)\in G_a \quad &\text{iff either} \quad (\alpha<\kappa \text{ or } a<\alpha<\kappa^+ \text{ or } b_a<\alpha) \text{ and } (\alpha,\beta)\in G,\\ &\qquad\quad \text{or} \quad (\alpha,\beta)=(\kappa,a),\\ &\qquad\quad \text{or} \quad (\alpha,\beta)=(\kappa^+,b_a);\\ (\alpha,\beta)\in H_a \quad &\text{iff either} \quad (\alpha<\kappa \text{ or } a<\alpha<\kappa^+ \text{ or } \kappa_1<\alpha) \text{ and } (\alpha,\beta)\in G,\\ &\qquad\quad \text{or} \quad (\alpha,\beta)=(\kappa,a),\\ &\qquad\quad \text{or} \quad (\alpha,\beta)=(\kappa^+,\kappa_1).\end{aligned}$$

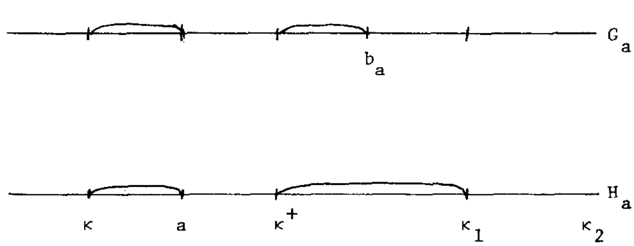

Fig. 6.

Thus G_a is the same as G except that $G_a(\kappa)=a$ and $G_a(\kappa^+)=b_a$ and H_a is the same as G except that $H_a(\kappa)=a$ and $H_a(\kappa^+)=\kappa_1$. By Lemma 1.5, G_a and H_a are both P-generic over $\mathcal{M}$. Let $i_a:\mathcal{M}[G\cap P_\kappa]\to\mathcal{N}_2[G_a]$ and $j_a:\mathcal{M}[G\cap P_\kappa]\to\mathcal{N}_2[H_a]$ be as given by Lemma 1.4 and let U_a and V_a be the normal measures defined by

i_a and j_a respectively. Thus we have $U_a \supset U$ and $V_a \supset U$. Let p be a condition such that $p(\kappa)=\kappa$, $p(\kappa^+)=\kappa^+$, $p(\kappa_1)=\kappa_1$ and a, $b_a \in \operatorname{ran}(p)$. We denote by

$$p/a \quad \text{and} \quad p//a$$

the modifications of p corresponding to G_a and H_a:

$$\begin{aligned}(\alpha,\beta)\in p/a \quad \text{iff either} \quad & (\alpha<\kappa \text{ or } a<\alpha<\kappa^+ \text{ or } b_a<\alpha) \text{ and } p(\alpha)=\beta,\\ \text{or} \quad & (\alpha,\beta)=(\kappa,a),\\ \text{or} \quad & (\alpha,\beta)=(\kappa^+,b_a);\end{aligned}$$

$$\begin{aligned}(\alpha,\beta)\in p//a \quad \text{iff either} \quad & (\alpha<\kappa \text{ or } a<\alpha<\kappa^+ \text{ or } b_a<\alpha) \text{ and } p(\alpha)=\beta,\\ \text{or} \quad & (\alpha,\beta)=(\kappa,a),\\ \text{or} \quad & (\alpha,\beta)=(\kappa^+,\kappa_1).\end{aligned}$$

We have

$$\begin{aligned}p \Vdash X\in U_a \quad &\text{iff} \quad p/a \Vdash \kappa\in j(X),\\ p \Vdash X\in V_a \quad &\text{iff} \quad p//a \Vdash \kappa\in j(X).\end{aligned}$$

The measures U_a, V_a $(a\in A)$ are separated:

Lemma 1.12. *There are sets E_a and F_a $(a\in A)$ such that for all $a\in A$, $E_a\in U_a$ and $F_a\in V_a$, and $E_a\notin V_{a'}$, $F_a\notin U_{a'}$ for all a', and $E_a\notin U_{a'}$, $F_a\notin V_{a'}$ for all $a'\neq a$.*

Proof. For each ordinal α, let f_α be the αth function for U:

$$[f_\alpha]_U=\alpha.$$

For $a\in A$ we define

$$\begin{aligned}E_a&=\{\alpha<\kappa\colon G(\alpha)=f_a(\alpha) \text{ and } G(\alpha^+)=f_{b_a}(\alpha)\},\\ F_a&=\{\alpha<\kappa\colon G(\alpha)=f_a(\alpha) \text{ and } G(\alpha^+)\neq f_{b_a}(\alpha)\}.\end{aligned}$$

If a and a' are in A, then a computation like the one in Lemma 1.6 shows that

$$\begin{aligned}E_a\in U_{a'} &\leftrightarrow G_{a'}(\kappa)=a \quad \text{and} \quad G_{a'}(\kappa^+)=b_a \leftrightarrow a=a',\\ F_a\in U_{a'} &\leftrightarrow G_{a'}(\kappa)=a \quad \text{and} \quad G_{a'}(\kappa^+)\neq b_a \leftrightarrow \text{never},\\ E_a\in V_{a'} &\leftrightarrow H_{a'}(\kappa)=a \quad \text{and} \quad H_{a'}(\kappa^+)=b_a \leftrightarrow \text{never},\\ F_a\in V_{a'} &\leftrightarrow H_{a'}(\kappa)=a \quad \text{and} \quad H_{a'}(\kappa^+)\neq b_a \leftrightarrow a=a'. \quad \Box\end{aligned}$$

Lemma 1.13. *For every $a\in A$ and every function $f\in\mathcal{M}$ on κ, $[f]_{U_a}=[f]_U$. In particular $[f_\alpha]_{U_a}=\alpha$ for every ordinal α where f_α is the αth function in $\mathcal{M}$ for U.*

Proof. Since $G_a(\kappa_1)=\kappa_1$, it is easy to see that $G_a\cap P_{\kappa_1}$ is P_{κ_1}-generic over $\mathcal{M}$. Thus we can define an elementary embedding

$$i_a^*\colon \mathcal{M}[G]\to\mathcal{N}_1[G_a\cap P_{\kappa_1}]$$

$$\begin{tikzcd} \mathcal{M}[G] \arrow[r, "i_a^*"] \arrow[d] & N_1[G_a \cap P_{\kappa_1}] \\ \mathrm{Ult}_{U_a}(\mathcal{M}[G]) \arrow[ur, "k_a"'] & \end{tikzcd}$$

Fig. 7.

extending $i:\mathcal{M} \to \mathcal{N}_1$. Since every subset of κ in $\mathcal{M}[G]$ has a P-name and since

$$i(X) = j(X) \cap \kappa_1$$

or every $X \subseteq \kappa$ in $\mathcal{M}$, we have

$$i_a^*(X) = i_a(X) \cap \kappa_1$$

for all $X \subseteq \kappa$ in $\mathcal{M}[G]$. In particular,

$$X \in U_a \leftrightarrow \kappa \in i_a^*(X).$$

Now consider the commutative diagram (Fig. 7) where

$$k_a([f]_{U_a}) = (i_a^*(f))(\kappa). \tag{13}$$

If f is an ordinal function on κ in $\mathcal{M}$, then (13) yields

$$k_a([f]_{U_a}) = (i(f))(\kappa) = [f]_U$$

and so $[f]_{U_a} = [f]_U$. □

For each $a \in A$, let

$$V_a^* = \{X \subseteq [\kappa]^2 : (\kappa, \kappa_1) \in j_a(X)\}$$

V_a^* is an ultrafilter over $[\kappa]^2$ extending U_2.

Lemma 1.14. *For every $a \in A$ and every function $f \in \mathcal{M}$ on $[\kappa]^2$, $[f]_{V_a^*} = [f]_{U_2}$.*

Proof. Consider the triangle as shown in Fig. 8 where

$$k_a([f]_{V_a^*}) = (j_a(f))(\kappa, \kappa_1).$$

If $f \in \mathcal{M}$, then

$$k_a([f]_{V_a^*}) = (j(f))(\kappa, \kappa_1) = [f]_{U_2}$$

and the lemma follows. □

$$\begin{tikzcd} \mathcal{M}[G] \arrow[r, "j_a"] \arrow[d] & N_2[H_a] \\ \mathrm{Ult}_{V_a^*}(\mathcal{M}[G]) \arrow[ur, "k_a"'] & \end{tikzcd}$$

Fig. 8.

Lemma 1.15. *For every* $a \in A$,

$$X \in V_a \leftrightarrow \pi_0^{-1}(X) \in V_a^*$$

and π_0 *is* 1–1 *on some* $Z \in V_a^*$.

Proof. The first statement is clear. Let

$$Z = \{(\alpha, \beta) \in [\kappa]^2 \colon G(\alpha^+) = \beta\}.$$

Since $H_a(\kappa^+) = \kappa_1$, we have $(\kappa, \kappa_1) \in j_a(Z)$ and so $Z \in V_a^*$ for all $a \in A$. $\square$

Lemma 1.16. *For every ordinal* γ *there exists a function* g_γ *such that*

$$[g_\gamma]_{V_a} = \gamma$$

for all $a \in A$.

Proof. Let $h_\gamma \in \mathcal{M}$ be a function on $[\kappa]^2$ such that $[h_\gamma]_{U_2} = \gamma$. Let

$$g_\gamma(\alpha) = h_\gamma(\alpha, G(\alpha^+)) \tag{14}$$

for all $\alpha < \kappa$. Since $Z \in V_a^*$, $h_\gamma(\alpha, \beta) = g_\gamma(\pi_0(\alpha, \beta))$ for almost all α, β mod V_a^*, and so by Lemma 1.15 and 1.14

$$[g_\gamma]_{V_a} = [g_\gamma \circ \pi_0]_{V_a^*} = [h_\gamma]_{V_a^*} = [h_\gamma]_{U_2} = \gamma. \quad \square$$

Lemma 1.17. *There is no* $X \subseteq \kappa$ *such that for every* $a \in A$, $X \in V_a$ *but* $X \notin U_a$.

Proof. Let X be a P_κ-name for a subset of κ, and let $p \in G$ be such that $p(\kappa) = \kappa$, $p(\kappa^+) = \kappa^+$ and $p(\kappa_1) = \kappa_1$. Let us assume that

$$p \Vdash (\forall\, a \in A)\, X \in V_a. \tag{15}$$

We shall find a stronger condition $q \supseteq p$ and $a < \kappa^+$ such that $a, b_a \in \operatorname{ran}(q)$ and that

$$q \Vdash X \in U_a.$$

Let

$$p = r \cup s \cup t \cup \{(\kappa_1, \kappa_1)\} \cup u$$

be the decomposition of p as in the proof of Lemma 1.10 (see Fig. 4). Let s_α, t_α and $u_{\alpha\beta}$ be as in Lemma 1.10. From (15) it follows that whenever a is such that

$$\sup(s) < a < \kappa^+ \quad \text{and} \quad \sup(t) < b_a < \kappa_1,$$

then

$$r \cup \{(\kappa, a)\} \cup \{(\kappa^+, \kappa_1)\} \cup u \Vdash \kappa \in j(X). \tag{16}$$

Note that for almost all α, for all ξ such that $\sup(s_\alpha) < \xi < \alpha^+$, for almost all β.

$$r \cup \{(\alpha, \xi)\} \cup \{(\alpha^+, \beta)\} \cup u_{\alpha\beta} \Vdash \alpha \in X. \tag{17}$$

Otherwise, for almost all α pick $\xi = g(\alpha)$ such that for almost all β (17) fails, and then if $a=[g]_U$, we have

$$r\cup\{(\kappa,a)\}\cup\{(\kappa^+,\kappa_1)\}\cup u \Vdash \kappa\notin j(X)$$

contrary to (16). Thus for almost all α there is $B_\alpha\in U$ such that for all $\beta\in B_\alpha$ and all ξ with $\sup(s_\alpha)<\xi<\alpha^+$, (17) holds. Let B be the diagonal intersection of the B_α's. So for almost all α, all ξ such that $\sup(s_\alpha)<\xi<\alpha^+$ and all $\beta\in B$ with $\beta>\alpha^+$, (17) holds. Since the set $i(B)$ is unbounded in κ_1, there exists $b\in i(B)$ such that $b>\sup(t)$. By the assumption on the mapping $a\mapsto b_a$ there are arbitrarily large $a<\kappa^+$ such that $b_a=b$. Fix such an a larger than $\sup(s)$. Let $\langle a(\alpha)\colon \alpha<\kappa\rangle$ and $\langle b(\alpha)\colon \alpha<\kappa\rangle$ represent a and b.

For almost all $\alpha\in B$ we have $\sup(s_\alpha)<a(\alpha)<\alpha^+$ and $\alpha^+<b(\alpha)\in B$, and so (17) holds for $\xi=a(\alpha)$ and $\beta=b(\alpha)$, in other words

$$r\cup\{(\alpha,a(\alpha))\}\cup\{(\alpha^+,b(\alpha))\}\cup u_{\alpha,b(\alpha)}\Vdash\alpha\in X. \tag{18}$$

Let $v\in P_{\kappa_1}$ be represented by $\langle U_{\alpha,b(\alpha)}\colon \alpha<\kappa\rangle$. From (18) we get

$$r\cup\{(\kappa,a)\}\cup\{(\kappa^+,b_a)\}\cup v\Vdash\kappa\in j(X). \tag{19}$$

Now let

$$\bar q=r\cup\{(\kappa,a)\}\cup\{(\kappa^+,b_a)\}\cup v\cup\{(\kappa_1,\kappa_1)\}\cup u$$

and

$$q=r\cup s\cup\{(a,a)\}\cup t\cup\{(b_a,b_a)\}\cup v\cup\{(\kappa_1,\kappa_1)\}\cup u.$$

We have $q\supseteq p$ and $a,b\in\operatorname{ran}(q)$, and $q/a=\bar q$. From (19) we get

$$q/a\Vdash\kappa\in j(X)$$

and so

$$q\Vdash X\in U_a.\quad\square$$

End of proof of Lemma 1.11. Let I_a and J_a be respectively the duals of U_a and V_a. The ideals I_a and J_a are separated by Lemma 1.12. For each α, let f be the αth function (in $\mathcal{M}$) for the measure U. By Lemma 1.13, each f_α is the αth function for each I_a. Since each V_a extends U, we have $f_\alpha<f_\beta$ almost everywhere mod J_a whenever $\alpha<\beta$, for each $a\in A$. For each α, let g_α be the function from Lemma 1.16 so that g_α is the αth function for each J_a. This completes the proof of clause (i) of Lemma 1.11 and it remains to show that the ideal I is not uniformly normed. Let us show that the κ_1st function for I does not exist. Assume to the contrary that f is an ordinal function on κ such that for all $a\in A$

$$[f]_{U_a}=[f]_{V_a}=\kappa_1$$

and let

$$X=\{\alpha\colon f(\alpha)<\kappa\}.$$

By Lemma 1.13, κ_1 is represented in each U_a by the constant function $\operatorname{const}(\kappa)$ with value κ, while by Lemmas 1.14 and 1.15, $\operatorname{const}(\kappa)$ represents κ_2 in each V_a (in fact κ_1 is represented in each V_a by the function $g(\alpha)=G(\alpha^+)$). It follows that for all $a\in A$, $X\in V_a$ and $X\notin U_a$, contrary to Lemma 1.17. $\square$

2. Properties of ideals preserved by κ-c.c. forcing

The consistency of the existence of a precipitous ideal on ω_1 was established in [7] by collapsing a measurable cardinal κ and showing that the ideal on κ generated by a κ-complete prime ideal in the ground model is precipitous. The method employed in [7] admits a generalization: it was observed by several people that the method works if one starts with only a precipitous ideal (instead of a prime ideal) in the ground model, and uses any κ-c.c. notion of forcing instead of the Lévy collapse. We shall now state a general theorem on preservation of various properties of ideals under κ-chain condition forcing. Some of the results have been known to others.

Let I be a κ-complete ideal over an uncountable regular cardinal (in the ground model $\mathcal{M}$). Let $\mathcal{P}$ be a notion of forcing that satisfies the κ-chain condition, and let $\mathcal{M}[G]$ be a generic extension of $\mathcal{M}$ by a $\mathcal{P}$-generic set of conditions G. In $\mathcal{M}[G]$, let $\bar{I}$ be the ideal over κ generated by I. In the following theorem, the statements about I are taken in the ground model while the statements about $\bar{I}$ are meant in $\mathcal{M}[G]$.

Theorem 2.1. (a) *If I is κ-complete, then $\bar{I}$ is κ-complete.*

(b) *If I is normal, then $\bar{I}$ is normal.*

(c) *If P is an I-partition, then P is an $\bar{I}$-partition.*

(d) *If F is an I-functional, then F is an $\bar{I}$-functional.*

(e) *If I is not precipitous, then $\bar{I}$ is not precipitous.*

(f) *If I is precipitous, then $\bar{I}$ is precipitous.*

(g) *If the I-degree of F is α, then the $\bar{I}$-degree of F is α.*

(h) *If I is uniformly normed, then $\bar{I}$ is uniformly normed.*

(i) *If I is not uniformly normed, then $\bar{I}$ is not uniformly normed.*

(j) *If $I=\sum_{a\in A} I_a$, then $\bar{I}=\sum_{a\in A}\bar{I}_a$.*

Proof. (a) This has been known for years; cf. [13] or [14]. Let $\lambda<\kappa$ and let $p\Vdash\{X_\alpha:\alpha<\lambda\}\subseteq\bar{I}$. Then for each $\alpha<\lambda$, $p\Vdash(\exists\, Y\in I)\, X_\alpha\subseteq Y$, and by κ-c.c. and the κ-completeness of I there exists $Y_\alpha\in I$ such that $p\Vdash X_\alpha\subseteq Y_\alpha$. It follows that $p\Vdash\bigcup_\alpha X_\alpha\subseteq\bigcup_\alpha Y_\alpha\in I$.

(b) Well known and easy.

(c) Cf. [9] or [13]. Let P be an I-partition, and let $\mathbf{A}$ and $p\in\mathcal{P}$ be such that $p\Vdash\mathbf{A}\in\bar{I}^+$. We shall find an $X\in P$ such that $p\nVdash\mathbf{A}\cap X\in\bar{I}$. (Hence some $q\leqslant p$ forces $\mathbf{A}\cap X\in\bar{I}^+$ and so P is an $\bar{I}$-partition.) Let

$$S=\{\alpha: p\nVdash\alpha\notin\mathbf{A}\}. \tag{20}$$

Then $S\in I^+$ and so there is an $X\in P$ such that $S\cap X\in I^+$. We claim that $p\nVdash\mathbf{A}\cap X\in\bar{I}$. If $p\Vdash\mathbf{A}\cap X\in\bar{I}$, then using the κ-chain condition we find $N\in I$ such that

$$p\Vdash\mathbf{A}\cap X\subseteq N. \tag{21}$$

Let $\alpha \in (S \cap X) - N$. Then because $\alpha \in X - N$, (21) implies that $p \Vdash \alpha \in \mathbf{A}$, and so $\alpha \notin S$, a contradiction.

(d) Follows from (c).

(e) [8]. A descending sequence of I-functionals is a descending sequence of $\bar{I}$-functionals.

(f) The special case is in [7]; for the general case, see [8] or [12]. The present proof follows the arguments from [7], [3] and [8].

Let $\bar{W}$ be an $R_{\bar{I}}$-generic ultrafilter over $\mathcal{M}[G]$. We shall show that the generic ultrapower $\mathrm{Ult}_W(\mathcal{M}[G])$ is well founded. Let

$$W = \bar{W} \cap \mathcal{M}.$$

Using (c), one can easily see that W is an R_I-generic ultrafilter over $\mathcal{M}$. Let

$$j: \mathcal{M} \to \mathrm{Ult}_W(\mathcal{M}) \quad \text{and} \quad \bar{j}: \mathcal{M}[G] \to \mathrm{Ult}_{\bar{W}}(\mathcal{M}[G])$$

be the embeddings associated with the respective ultrapowers by W and $\bar{W}$. Since I is precipitous, $\mathrm{Ult}_W(\mathcal{M})$ is well founded.

Let us consider the notion of forcing $j(\mathcal{P}) \in \mathrm{Ult}_W(\mathcal{M})$. Every $p \in j(\mathcal{P})$ is represented by some

$$\langle p_\alpha : \alpha < \kappa \rangle \in \mathcal{M}$$

where $p_\alpha \in \mathcal{P}$ for all $\alpha < \kappa$. For each $\bar{p} = \langle p_\alpha : \alpha < \kappa \rangle$ let

$$T_{\bar{p}} = \{\alpha < \kappa : p_\alpha \in G\}.$$

If $\bar{p}$ and $\bar{q}$ represent the same $p \in j(\mathcal{P})$, and since $W \subseteq \bar{W}$, we have

$$T_{\bar{p}} \in \bar{W} \leftrightarrow T_{\bar{q}} \in \bar{W}.$$

Thus the set $H \subseteq j(\mathcal{P})$ is well defined by

$$H = \{p \in j(\mathcal{P}) : T_{\bar{p}} \in \bar{W} \text{ for } [\bar{p}]_W = p\}.$$

The crucial fact about H is:

$$H \text{ is } j(\mathcal{P})\text{-generic over } \mathrm{Ult}_W(\mathcal{M}) \text{ and } j[G] \subseteq H. \tag{22}$$

Since the proof of (22) appears, explicitly or implicitly, in the references cited above, we omit it.

Now (22) permits us to extend $j: \mathcal{M} \to \mathrm{Ult}_W(\mathcal{M})$ to an elementary embedding $\tilde{j}: \mathcal{M}[G] \to (\mathrm{Ult}_W(\mathcal{M}))[H]$ by defining

$$\tilde{j}(\mathrm{int}_G(x)) = \mathrm{int}_H(j(x))$$

where int_G and int_H denote the interpretations by G and H respectively. Let

$$\vartheta = [d]_W$$

be the ordinal represented in $\mathrm{Ult}_W(\mathcal{M})$ by the function $d(\alpha) = \alpha$. It is easy to verify that

$$\bar{W} = \{X \subseteq \kappa : \vartheta \in \tilde{j}(X)\}.$$

$$\begin{tikzcd} \mathcal{M}[G] \arrow[r, "\tilde{j}"] \arrow[d, "\bar{j}"'] & (\mathrm{Ult}_W(\mathcal{M}))[H] \\ \mathrm{Ult}_{\bar{W}}(\mathcal{M}[G]) \arrow[ur, "k"'] & \end{tikzcd}$$

Fig. 9.

Thus consider the commutative diagram (Fig. 9) where k is defined by

$$k([f]_{\bar{W}}) = (\tilde{j}(f))(\vartheta).$$

A standard argument shows that k is well defined and elementary and so $\mathrm{Ult}_{\bar{W}}(\mathcal{M}[G])$ is well founded, as desired.

The diagram in Fig. 9 provides additional information. Namely, k is an isomorphism (and so $\mathrm{Ult}_{\bar{W}}(\mathcal{M}[G]) = (\mathrm{Ult}_W(\mathcal{M}))[H]$ and $\bar{j} = \tilde{j}$). To prove this, it suffices to show that k is onto. In turn, it suffices to show that H is in the range of k, and so is every $x \in \mathrm{Ult}_W(\mathcal{M})$. As for H, we have

$$k(\bar{j}(G)) = k([\mathrm{const}(G)]_{\bar{W}}) = (\tilde{j}(\mathrm{const}(G)))(\vartheta) = \tilde{j}(G) = H.$$

Let $x \in \mathrm{Ult}_W(\mathcal{M})$. Then $x = [f]_W$ for some $f \in \mathcal{M}$, and

$$k([f]_{\bar{W}}) = (\tilde{j}(f))(\vartheta) = (j(f))(\vartheta) = [f]_W = x. \tag{23}$$

(g) Since clearly $\deg_I(F) \leqslant \deg_{\bar{I}}(F)$, it suffices to show that whenever $S \in I^+$ and $f \in \mathcal{M}$ is a function on S such that $S \Vdash f$ represents α, then for any $R_{\bar{I}}$-generic ultrafilter $\bar{W} \ni S$ over $\mathcal{M}[G]$, $[f]_{\bar{W}} = \alpha$. Thus let $\bar{W}$ be R_I-generic over $\mathcal{M}[G]$ such that $S \in \bar{W}$. Using the diagrm in Fig. 9 and the equalities in (23), we have

$$[f]_{\bar{W}} = [f]_W$$

for some R_I-generic W containing S. Hence $[f]_{\bar{W}} = \alpha$.

(h) For the special case when I is prime, see [11]. For every ordinal α, let $f_\alpha \in \mathcal{M}$ be the αth function for I. It follows from (g) that each f_α is the αth function for $\bar{I}$ (in $\mathcal{M}[G]$).

(i) Let us assume that $g \in \mathcal{M}[G]$ is the αth function for $\bar{I}$; we shall find $h \in \mathcal{M}$ such that $\deg_S(h) = \alpha$ for all $S \in I^+$. Let $F \in \mathcal{M}$ be the αth functional for I. By (g), F is the αth functional in $\mathcal{M}[G]$, and so for every $f \in F$, $f(\xi) = g(\xi)$ almost everywhere (mod $\bar{I}$) on the domain of f. Thus for every $f \in F$ there exists $X \subseteq \mathrm{dom}(f)$ such that $X \in \mathcal{M}$, $\mathrm{dom}(f) - X \in I$, and $f(\xi) = g(\xi)$ for all $\xi \in X$.

Let all this be forced by some condition $p \in G$. Using the κ-chain condition, we can find (in $\mathcal{M}$), for every $f \in F$ a set $X_f \subseteq \mathrm{dom}(f)$ such that $\mathrm{dom}(f) - X_f \in I$ and that p forces that $f(\xi) = g(\xi)$ for all $\xi \in X_f$.

It follows that if $X_{f_1} \cap X_{f_2} \neq \emptyset$, then f_1 and f_2 agree on $X_{f_1} \cap X_{f_2}$. So if we let $h = \bigcup \{f \upharpoonright X_f : f \in F\}$, then h is a function. And the preceding arguments make it clear that h is the αth function for I.

(j) If the I_a's are separated then the $\bar{I}_a$'s are also separated (by the same sets) and it is immediate that $\bar{I} = \sum_a \bar{I}_a$.

3. Proof of main theorem

Let κ be a measurable cardinal and let U be a normal measure on κ. First we extend the ground model $\mathcal{M} = L[U]$ to the model $\mathcal{M}[G]$ presented in Section 1. In $\mathcal{M}[G]$, consider the normal precipitous ideals J_1, J_2 and J_3 on κ given by Theorem 1.1. J_1 is uniformly normed. J_2 is the sum $\sum_a (I_a, J_a)$ where the I_a and J_a satisfy Lemma 1.11(i), and is not uniformly normed. J_3 is not normed because the constant function on κ has degree κ_2 and norm κ_1.

We further extend $\mathcal{M}[G]$ by Lévy-collapsing κ to $\aleph_1$. The notion of forcing involved has the κ-chain condition. Let

$$I_1 = \bar{J}_1, \qquad I_2 = \bar{J}_2, \qquad I_3 = \bar{J}_3$$

be the ideals generated by J_1, J_2, J_3. By Theorem 2.1, I_1, I_2 and I_3 are normal precipitous ideals on $\aleph_1$.

I_1 is uniformly normed by Theorem 2.1(h).

I_2 is not uniformly normed, by Theorem 2.1(i). By Theorem 2.1(j), $I_2 = \sum_a (\bar{I}_a, \bar{J}_a)$, and by (g) and (h), the ideals $\bar{I}_a$ and $\bar{J}_a$ also satisfy Lemma 1.11(i), with the same functions f_α and g_α. Thus I_2 is normed for the same reason J_2 is.

It remains to prove that I_3 is not normed. By Theorem 2.1(g), the I_3-degree of const(κ) is κ_2. Since the final model is a generic extension of the intermediate model by a c.c.c. extension, for every function $f: \kappa \to \kappa$ (in the final model) there is a function $h: \kappa \to \kappa$ in the intermediate model such that $f(\alpha) < h(\alpha)$ for all α. Then $\|f\| \leq \deg(f) \leq \deg(h)$. But $\deg_{I_3}(h) = \deg_{J_3}(h)$ by Theorem 2.19 and $\deg_{J_3}(h) < \kappa_1$. Thus $\|f\| < \kappa_1$ and it follows that $\|\mathrm{const}(\kappa)\| = \kappa_1$.

References

[1] W.B. Easton, Powers of regular cardinals, Ann. Math. Logic 1 (1970) 139–178.
[2] F. Galvin and A. Hajnal, Inequalities for cardinal powers, Ann. Math. 101 (1975) 491–498.
[3] T. Jech, Precipitous ideals, in: R. Gandy and M. Hyland, eds., Logic Colloquim 76 (North-Holland, Amsterdam, 1977) pp. 521–530.
[4] T. Jech, Some properties of κ-complete ideals defined in terms of infinite games, Ann. Pure Appl. Logic to appear.
[5] T. Jech and K. Prikry, Ideals of sets and the power set operation, Bull. AMS 82 (1976) 593–595.
[6] T. Jech and K. Prikry, Ideal over uncountable sets: Application of almost disjoint functions and generic ultrapowers, Memoirs AMS vol. 18, No. 214. (AMS, Providence, RI 1979).
[7] T. Jech, M. Magidor, W. Mitchell and K. Prikry, Precipitous ideals, J. Symbolic Logic 45 (1980) 1–8.
[8] Y. Kakuda, On a condition for Cohen extensions which preserve precipitous ideals, J. Symbolic Logic, to appear.
[9] K. Kunen, Some applications of iterated ultrapowers in set theory, Ann. Math. Logic 1 (1970) 179–227.
[10] K. Kunen and J.B. Paris, Boolean extensions and measurable cardinals, Ann. Math. Logic 2 (1971) 359–377.
[11] J.-P. Levinski, Etude combinatoire des filtres en relation avec le problème des cardinaux singuliers, Thèse de 3 ème Cycle, Université Paris 7 (1980).
[12] M. Magidor, Precipitous ideals and Σ^1_4 sets, Israel J. Math., to appear.
[13] K. Prikry, Changing measurable into accessible cardinals, Dissertationes Math. 68 (1970) 5–52.
[14] R. Solovay, Real-values measurable cardinals, in: D. Scott, ed., Axiomatic Set Theory, Proc. Symp. Pure Math. 13, I (AMS, Providence, RI, 1971) pp. 397–428.