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On sign-changing solutions for nonlinear operator equations ☆

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Abstract

In this paper, the existence of sign-changing solutions for nonlinear operator equations is discussed by using the topological degree and fixed point index theory. The main theorems are some new three-solution theorems which are different from the famous Amann's and Leggett-Williams' three-solution theorems as well as the results in [F. Li, G. Han, Generalization for Amann's and Leggett–Williams' three-solution theorems and applications, J. Math. Anal. Appl. 298 (2004) 638–654]. These three solutions are all nonzero. One of them is positive, another is negative, and the third one is a sign-changing solution. Furthermore, the theoretical results are successfully applied to both integral and differential equations.

Keywords: Cone; Fixed point index; The index of isolated zero point; *e*-Continuous; Completely continuous operator

1. Introduction and preliminaries

The purpose of this paper is mainly to establish some theoretical existence results of sign-changing solutions for nonlinear operator equations. It is well known that the study of existence of sign-changing solutions is very useful and interesting both in theory and in applications. Much attention has been attached to this problem. For example, in a recent paper [24], Xu and Sun have

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discussed the existence of sign-changing solutions to the following second order three-point boundary value problem:

$$\begin{cases} -u''(t) = f(u(t)), & t \in [0,1], \\ u(0) = 0, & \alpha u(\eta) = u(1), \end{cases} \tag{1.1}$$

where $\alpha, \eta \in (0,1)$, $f \in C(\mathbb{R}^1, \mathbb{R}^1)$. They have proved that the three-point boundary value problem (1.1) has at least one sign-changing solution under conditions (A_1), (A_2) and (A_3), which are listed as follows:

(A_1) $f : \mathbb{R}^1 \to \mathbb{R}^1$ is continuous and strictly increasing, $f(0) = 0$;

(A_2) there exists a positive integer n_0 such that

$$\lambda_{2n_0} < \beta_0 < \lambda_{2n_0+1},$$

where $\lim_{x \to 0} f(x)/x = \beta_0$, and $\{\lambda_n\}$ with

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \cdots$$

is the sequence of all positive solutions of the equation $\sin\sqrt{x} = \alpha \sin \eta\sqrt{x}$;

(A_3) $\lim_{x \to \infty} f(x)/x < 2(1 - \alpha\eta)$.

In this paper, we abstract some more general conditions from (A_1), (A_2) and (A_3) and obtain some existence results of sign-changing solution for increasing operators. The detail can be seen in the second section of this paper. In Section 3, by using topological degree and fixed point index theory, we continue to study the existence of sign-changing solutions for general operators. The main results are more useful and important in theory. As applications, the theoretical results are applied to both integral and differential equations, in Section 4. As you will see, the main theorems are actually some new three-solution theorems which are different from the famous Amann's and Leggett–Williams' three-solution theorems as well as the results in [17] since our main results present the existence of sign-changing solutions. In the nonlinear functional analysis, the existence of multiple solutions to nonlinear operator equations is also important in theory and applications. Many authors devote themselves to this problem. For instance, see [1–6,8–12,14–19,21]. However, to the authors' knowledge, few papers have considered the multiplicity results of one positive, one negative and one sign-changing solution for operator equations.

For the discussion of the following sections, we state here some definitions, notations and known results. For convenience of readers, we suggest that one refer to [7,10,15] for details.

Let E be a real Banach space. A nonempty closed convex subset P is called a cone in E if it satisfies the following conditions:

(i) $x \in P, \lambda \geqslant 0$ imply $\lambda x \in P$;

(ii) $x \in P$ and $-x \in P$ imply $x = 0$, where 0 denotes the zero element of E.

Define a partial ordering with respect to cone P by $x \leqslant y$ iff $y - x \in P$ [6,10,11]. Sometimes we shall write $x < y$ to indicate that $x \leqslant y$ but $x \neq y$. Let D be a nonempty subset of E. An operator $A : D \to E$ is said to be increasing if $Ax \leqslant Ay$ for all $x, y \in D$ with $x \leqslant y$. A fixed point u of operator A is said to be a sign-changing fixed point if $u \notin P \cup (-P)$. An order interval $[u, v]$ is defined as $[u, v] = \{x \in E: u \leqslant x \leqslant v\}$. It is obvious that any order interval is a closed convex subset of E. Then it is a retract of E by Dugundji theorem [7]. An operator $Q : E \to E$ is called a positive operator if $Q(P) \subset P$. An element $x \in E$ is called a positive element if $x \in P \setminus \{0\}$.

Theorem *(Leray–Schauder).* [25, Proposition 14.5, p. 619] *Let $A : E \to E$ be completely continuous, $A0 = 0$, and A be Fréchet differentiable at* 0. *Assume that* 1 *is not an eigenvalue of the Fréchet derivative $A'(0)$. Let $F = \{x \in E \setminus \{0\}\colon\ Ax = x\}$. Then there exists $\tau > 0$ such that $F \cap B_\tau = \emptyset$, where $B_\tau = \{x \in E\colon\ \|x\| < \tau\}$. That is,* 0 *is an isolated zero point of the completely continuous vector field $I - A$ and*

$$\operatorname{ind}(I - A, 0) = \operatorname{ind}\big(I - A'(0), 0\big) = (-1)^k,$$

where k is the sum of the algebraic multiplicities of the real eigenvalues of $A'(0)$ in $(1, +\infty)$.

Theorem *(Krein–Rutman).* [13,25] *Let E be a Banach space, $P \subset E$ a total cone and K a linear compact positive operator with $r(K) > 0$, where $r(K)$ denotes the spectral radius of K. Then $r(K)$ is an eigenvalue of K with a positive eigenvector. Meanwhile, $r(K)$ is an eigenvalue of K^*, the dual operator of K, with positive eigenvector in P^*, where P^* is the dual cone of P.*

Definition 1.1. [15] Let $A : D \to E$ be an operator, $e \in P \setminus \{0\}$, and $x_0 \in D$. If for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$-\varepsilon e \leqslant Ax - Ax_0 \leqslant \varepsilon e$$

for all $x \in D$ with $\|x - x_0\| < \delta$, then A is called e-continuous at x_0. If A is e-continuous at each point $x \in D$, then A is called e-continuous on D.

It is easy to see that if $A : D \to E$ is a linear operator, then A is e-continuous on D iff A is e-continuous at 0.

2. Three-solution theorems for increasing operators

In this section, we mainly consider the existence of sign-changing solutions for nonlinear increasing operators.

Lemma 2.1. *Let P be a cone in E. Assume that $A : E \to E$ is a completely continuous operator, $A0 = 0$, and A is Fréchet differentiable at* 0. *If the following conditions*:

(i) 1 *is not an eigenvalue of the Fréchet derivative $A'(0)$*;
(ii) *there exists $h \in P \setminus \{0\}$ such that $\|x\|h \leqslant x$ for all $x \in P$ with $Ax = x$, and $x \leqslant -\|x\|h$ for all $x \in (-P)$ with $Ax = x$*

are satisfied, then there exists $\alpha > 0$ such that $\alpha h \leqslant x$ for all $x \in F^+$ and $x \leqslant -\alpha h$ for all $x \in F^-$, where $F^+ = \{x \in P \setminus \{0\}\colon\ Ax = x\}$ and $F^- = \{x \in (-P) \setminus \{0\}\colon\ Ax = x\}$.

Proof. It follows from condition (i) and Leray–Schauder theorem that there exists $\tau > 0$ such that $F \cap B_\tau = \emptyset$, where $F = \{x \in E \setminus \{0\}\colon\ Ax = x\}$. This implies that $\|x\| \geqslant \tau$ for all $x \in F$. Then it follows from condition (ii) that $x \geqslant \|x\|h \geqslant \tau h$ for all $x \in F^+$ and that $x \leqslant -\|x\|h \leqslant -\tau h$ for all $x \in F^-$. Therefore, for $\alpha = \tau$, the conclusion of Lemma 2.1 holds. The proof is completed. □

Lemma 2.2. *Let P be a normal cone in E, $A : E \to E$ be a completely continuous increasing operator, and A be e-continuous on E. Suppose that*

(i) *there exist* $u_0 \in (-P) \setminus \{0\}$ *and* $v_0 \in P \setminus \{0\}$ *such that* $u_0 \leqslant Au_0$ *and* $Av_0 \leqslant v_0$;
(ii) *there exist* $u_1 \in [0, v_0] \setminus \{0, v_0\}$, $v_1 \in [u_0, 0] \setminus \{0, u_0\}$ *and* $\delta > 0$ *such that* $u_1 + \delta e \leqslant Au_1$ *and* $Av_1 \leqslant v_1 - \delta e$.

Let $X = [u_0, v_0]$, $\Omega_1 = \{x \in X$: *there exists* $\lambda > 0$ *such that* $u_1 + \lambda e \leqslant Ax\}$ *and* $\Omega_2 = \{x \in X$: *there exists* $\lambda > 0$ *such that* $Ax \leqslant v_1 - \lambda e\}$. *Then* Ω_1 *and* Ω_2 *are disjoint nonempty open subsets of* X, *and the fixed point index* $i(A, \Omega_1, X) = 1$, $i(A, \Omega_2, X) = 1$.

Proof. Since P is a normal cone in E, $X = [u_0, v_0]$ is a nonempty bounded closed convex subset of E. Then X is a retract of E. Moreover, by condition (i) and the increasing property of operator A, we have that $A(X) \subset X$. It is obvious that Ω_1 is a nonempty subset of X since $u_1 \in \Omega_1$. It follows from the e-continuity of operator A on E that Ω_1 is a nonempty open subset of X. We now prove that $x \neq (1-t)Ax + tu_1$ for all $x \in \partial\Omega_1$ and $t \in [0, 1]$, where $\partial\Omega_1$ is the boundary of Ω_1 with respect to X. In fact, if there exist $x_0 \in \partial\Omega_1$ and $t_0 \in [0, 1]$ such that $x_0 = (1-t_0)Ax_0 + t_0u_1$, then $u_1 \leqslant Ax_0$ since $x_0 \in \partial\Omega_1$. So $u_1 = (1-t_0)u_1 + t_0u_1 \leqslant (1-t_0)Ax_0 + t_0u_1 = x_0$. It follows from condition (ii) and the increasing property of operator A that $u_1 + \delta e \leqslant Au_1 \leqslant Ax_0$. This implies that $x_0 \in \Omega_1$. This is a contradiction with $x_0 \in \partial\Omega_1$. By the homotopy invariance and normalization of the fixed point index, we have that $i(A, \Omega_1, X) = i(u_1, \Omega_1, X) = 1$.

Similarly, we can prove that Ω_2 is also a nonempty open subset of X and $i(A, \Omega_2, X) = 1$. It is obvious that Ω_1 and Ω_2 are disjoint since $v_1 < 0 < u_1$. The proof is completed. □

Lemma 2.3. *Let* P *be a cone in* E, A: $E \to E$ *be a completely continuous increasing operator,* $A0 = 0$, *and* A *be* e*-continuous at* 0. *Suppose that*

(i) *there exist* $u_0 \in (-P) \setminus \{0\}$ *and* $v_0 \in P \setminus \{0\}$ *such that* $u_0 \leqslant Au_0$ *and* $Av_0 \leqslant v_0$. *And there exists* $\beta > 0$ *such that* $u_0 \leqslant -\beta e$ *and* $\beta e \leqslant v_0$;
(ii) 0 *is an isolated zero point of* $I - A$ *and the index of isolated zero point* $\mathrm{ind}(I - A, 0) = 1$.

Then $i(A, B_\rho \cap X, X) = 1$ *for sufficiently small* $\rho > 0$, *where* $X = [u_0, v_0]$.

Proof. Since 0 is an isolated zero point of the completely continuous vector field $I - A$, there exists $\rho_0 > 0$ such that 0 is a unique zero point of $I - A$ in B_{ρ_0}, and

$$\mathrm{ind}(I - A, 0) = \deg(I - A, B_\rho, 0), \quad \rho \in (0, \rho_0). \tag{2.1}$$

Notice $A0 = 0$. For the positive number β in condition (i), it follows from the e-continuity of operator A at 0 that there exists $\rho_1 \in (0, \rho_0]$ such that $-\beta e \leqslant Ax \leqslant \beta e$ for all $x \in E$ with $\|x\| \leqslant \rho_1$. By condition (i), it is easy to show that $A(\bar{B}_\rho) \subset X$ for all $\rho \in (0, \rho_1]$, where $\bar{B}_\rho = \{x \in E\colon \|x\| \leqslant \rho\}$.

Let r be a retraction from E into X. Using the definition of fixed point index [11, (2.3.4), p. 84], for every given $\rho \in (0, \rho_1)$, we have

$$i(A, B_\rho \cap X, X) = \deg\bigl(I - A \circ r, B_R \cap r^{-1}(B_\rho \cap X), 0\bigr), \tag{2.2}$$

where $B_R \supset \overline{B_\rho \cap X}$. We claim that 0 is a unique fixed point of $A \circ r$ in $B_R \cap r^{-1}(B_\rho \cap X)$. In fact, suppose that x^* is a fixed point of $A \circ r$ in $B_R \cap r^{-1}(B_\rho \cap X)$, then $x^* = A \circ r(x^*)$. Since $r(E) \subset X$ and $A(X) \subset X$, $x^* = A \circ r(x^*) \in X$. Then $x^* = r(x^*) \in B_\rho \cap X$ and $x^* = Ax^*$. Since 0 is a unique zero point of $I - A$ in B_{ρ_0}, $x^* = 0$. Noticing that $B_R \cap r^{-1}(B_\rho \cap X)$ is an open subset of E and $0 \in B_R \cap r^{-1}(B_\rho \cap X)$, we have that there exists $\rho_2 \in (0, \rho)$ such that $B_{\rho_2} \subset B_R \cap r^{-1}(B_\rho \cap X)$. Therefore it follows from the excision property of the Leray–Schauder degree that

$$\deg\bigl(I - A \circ r, B_R \cap r^{-1}(B_\rho \cap X), 0\bigr) = \deg(I - A \circ r, B_{\rho_2}, 0). \tag{2.3}$$

If there exist $x_0 \in \partial B_{\rho_2}$ and $t_0 \in [0, 1]$ such that

$$x_0 = (1 - t_0)Ax_0 + t_0 A \circ r(x_0),$$

then it follows from $A(\bar{B}_{\rho_2}) \subset X, r(E) \subset X$ and $A(X) \subset X$ that $x_0 \in X$. Then $x_0 = Ax_0$. This is a contradiction with the fact that 0 is a unique zero point of $I - A$ in B_{ρ_0}. Therefore we have $x \neq (1 - t)Ax + tA \circ r(x)$ for all $x \in \partial B_{\rho_2}$ and $t \in [0, 1]$. Hence the homotopy invariance of Leray–Schauder degree implies that

$$\deg(I - A \circ r, B_{\rho_2}, 0) = \deg(I - A, B_{\rho_2}, 0). \tag{2.4}$$

It follows from (2.1)–(2.4) and condition (ii) that $i(A, B_\rho \cap X, X) = \operatorname{ind}(I - A, 0) = 1$ for all $\rho \in (0, \rho_1)$. The proof is completed. □

Lemma 2.4. *Let $A = KF$, where $K : E \to E$ is a bounded linear operator and is e-continuous at 0, $F0 = 0$ and $F : E \to E$ is Fréchet differentiable at 0. If $A'(0) = KF'(0)$ has an eigenvalue $\lambda_0 > 1$ with eigenvector v satisfying $\mu e \leqslant v$, where $\mu > 0$, then there exists $\gamma > 0$ such that $tv + \delta e \leqslant A(tv)$ and $A(-tv) \leqslant -tv - \delta e$ for all $t \in (0, \gamma)$, where $\delta = \delta(t) = t(\lambda_0 - 1)\mu/2 > 0$.*

Proof. Since the operator K is e-continuous at 0, for given $\varepsilon = (\lambda_0 - 1)\mu/2 > 0$, there exists $\eta > 0$ such that $-\varepsilon e \leqslant Kx \leqslant \varepsilon e$ for all $x \in E$ with $\|x\| < \eta$. It follows from the differentiability of operator F at 0 that $F(tv) - F'(0)(tv) = g(t) = o(t)$ as $t \to 0$. Then there exists $\gamma > 0$ such that $\|g(t)/t\| < \eta$ for all $|t| \in (0, \gamma)$. Therefore for each $t \in (0, \gamma)$, we have

$$A(tv) - t\lambda_0 v = A(tv) - A'(0)(tv) = K\bigl[F(tv) - F'(0)(tv)\bigr] = tK\bigl(g(t)/t\bigr) \geqslant -t\varepsilon e.$$

Hence,

$$A(tv) \geqslant tv + t\bigl[(\lambda_0 - 1)v - \varepsilon e\bigr] \geqslant tv + t\bigl[(\lambda_0 - 1)\mu - \varepsilon\bigr]e = tv + \delta e,$$

where $\delta = t(\lambda_0 - 1)\mu/2 > 0$.

In the same way, for each $t \in (0, \gamma)$, we have

$$\begin{aligned} A(-tv) + t\lambda_0 v &= A(-tv) - A'(0)(-tv) = K\bigl[F(-tv) - F'(0)(-tv)\bigr] \\ &= -tK\bigl[g(-t)/(-t)\bigr] \leqslant t\varepsilon e. \end{aligned}$$

So

$$A(-tv) \leqslant -tv - t\bigl[(\lambda_0 - 1)v - \varepsilon e\bigr] \leqslant -tv - t\bigl[(\lambda_0 - 1)\mu - \varepsilon\bigr]e = -tv - \delta e,$$

where $\delta = t(\lambda_0 - 1)\mu/2 > 0$ is the same as above. The proof is completed. □

Theorem 2.1. *Let P be a normal cone in E, $A : E \to E$ be a completely continuous increasing operator and e-continuous on E. Suppose that*

(i) *$A0 = 0$, A is Fréchet differentiable at 0, 1 is not an eigenvalue of the Fréchet derivative $A'(0)$, and the index of isolated zero point* $\operatorname{ind}(I - A'(0), 0) = 1$;

(ii) *there exist* $u_0 \in (-P) \setminus \{0\}$ *and* $v_0 \in P \setminus \{0\}$ *such that* $u_0 \leqslant Au_0$ *and* $Av_0 \leqslant v_0$. *And there exists* $\beta > 0$ *such that* $u_0 \leqslant -\beta e$ *and* $\beta e \leqslant v_0$;

(iii) *there exist* $u_1 \neq v_0$, $v_1 \neq u_0$ *and* $\sigma > 0$ *such that* $\sigma e \leqslant u_1$ *and* $v_1 \leqslant -\sigma e$. *And there exists* $\delta > 0$ *such that* $u_1 + \delta e \leqslant Au_1$ *and* $Av_1 \leqslant v_1 - \delta e$, *furthermore* $u_1 \leqslant x$ *for all* $x \in F^+$ *and* $x \leqslant v_1$ *for all* $x \in F^-$, *where* F^+ *and* F^- *are defined as in Lemma* 2.1.

Then A has at least three fixed points, one of which is positive, another is negative, and the third one is a sign-changing fixed point.

Proof. Let $X = [u_0, v_0]$. Then X is a bounded closed convex subset of E and $A(X) \subset X$. Thus, the fixed point index

$$i(A, X, X) = 1. \tag{2.5}$$

Let $\Omega_1 = \{x \in X\colon$ there exists $\lambda > 0$ such that $u_1 + \lambda e \leqslant Ax\}$ and $\Omega_2 = \{x \in X\colon$ there exists $\lambda > 0$ such that $Ax \leqslant v_1 - \lambda e\}$. Then it follows from condition (iii) and Lemma 2.2 that Ω_1 and Ω_2 are disjoint nonempty open subsets of X, and the fixed point index

$$i(A, \Omega_1, X) = 1, \qquad i(A, \Omega_2, X) = 1. \tag{2.6}$$

By condition (i) and Leray–Schauder theorem, we get that the index of isolated zero point $\operatorname{ind}(I - A, 0) = \operatorname{ind}(I - A'(0), 0) = 1$. This equality together with condition (ii) and Lemma 2.3 implies that

$$i(A, B_\rho \cap X, X) = 1 \tag{2.7}$$

for sufficiently small $\rho > 0$.

For the positive number σ in condition (iii), since A is e-continuous at 0, there exists $\rho > 0$ sufficiently small such that $-(\sigma/2)e \leqslant Ax \leqslant (\sigma/2)e$ for all $x \in B_\rho$. If there exists $x^* \in B_\rho \cap \Omega_1$, then $Ax^* \leqslant (\sigma/2)e$ and $\sigma e \leqslant u_1 \leqslant u_1 + \lambda^* e \leqslant Ax^*$ by condition (iii) and $x^* \in \Omega_1$, where $\lambda^* > 0$ is some number. This is a contradiction. So $B_\rho \cap \Omega_1 = \emptyset$. Similarly, we have that $B_\rho \cap \Omega_2 = \emptyset$.

It follows from (2.5)–(2.7) and the additivity of fixed point index that

$$\begin{aligned} &i\big(A, X \setminus (\overline{\Omega_1 \cup \Omega_2 \cup B_\rho}), X\big) \\ &\quad = i(A, X, X) - i(A, \Omega_1, X) - i(A, \Omega_2, X) - i(A, B_\rho \cap X, X) \\ &\quad = 1 - 1 - 1 - 1 = -2. \end{aligned} \tag{2.8}$$

Hence, A has at least three fixed points x_1, x_2, x_3 such that $x_1 \in \Omega_1$, $x_2 \in \Omega_2$ and $x_3 \in X \setminus (\overline{\Omega_1 \cup \Omega_2 \cup B_\rho})$. It is obvious that x_1 is positive and x_2 is negative. Since $x_3 \in X \setminus (\overline{\Omega_1 \cup \Omega_2 \cup B_\rho})$, $x_3 \neq 0$. If $x_3 \in F^+$, then it follows from condition (iii) that $u_1 \leqslant x_3$. Then it follows from condition (iii) and the increasing property of operator A that $u_1 + \delta e \leqslant Au_1 \leqslant Ax_3$. This implies that $x_3 \in \Omega_1$. This is a contradiction with $x_3 \in X \setminus (\overline{\Omega_1 \cup \Omega_2 \cup B_\rho})$. Thus $x_3 \notin F^+$. Similarly, we can prove that $x_3 \notin F^-$. Therefore x_3 is a sign-changing fixed point. The proof is completed. □

Remark 2.1. By the condition (iii) of Theorem 2.1, we have that $u_1 < Au_1$ and $Av_1 < v_1$. So when P is a solid cone and A is strongly increasing, Amann's theorem holds. Therefore, Theorem 2.1 generalizes the results of Amann's theorem.

Theorem 2.2. *Let P be a normal cone in E, $A = KF$, where $F : E \to E$ is a continuous and bounded increasing operator, $K : E \to E$ is a positive linear completely continuous operator and is also e-continuous on E. Suppose that*

(i) $F0 = 0$, *F is Fréchet differentiable at* 0, *and $KF'(0)$ has an eigenvalue $\lambda_0 > 1$ with eigenvector v satisfying $\mu e \leqslant v \leqslant \lambda e$, where $\mu > 0$ and $\lambda > 0$;*
(ii) 1 *is not an eigenvalue of the operator $KF'(0)$, and the index of isolated zero point* $\mathrm{ind}(I - KF'(0), 0) = 1$;
(iii) *there exist $u_0 \in (-P) \setminus \{0\}$ and $v_0 \in P \setminus \{0\}$ such that $u_0 \leqslant Au_0$ and $Av_0 \leqslant v_0$, and there also exists $\beta > 0$ such that $u_0 \leqslant -\beta e$ and $\beta e \leqslant v_0$;*
(iv) *there exists $h \geqslant \nu e$ with $\nu > 0$ such that $\|x\|h \leqslant x$ for all $x \in P$ with $Ax = x$, and $x \leqslant -\|x\|h$ for all $x \in (-P)$ with $Ax = x$.*

Then A has at least one sign-changing fixed point, one positive fixed point and one negative fixed point.

Proof. We only need to verify all the conditions of Theorem 2.1. By the chain rule for derivatives of composite operator [25], we have that $A'(0) = KF'(0)$. So the condition (i) of Theorem 2.1 holds. The condition (ii) of Theorem 2.1 is obviously satisfied since it is an assumption. Furthermore, we will verify the condition (iii) of Theorem 2.1. It follows from condition (ii), (iv) and Lemma 2.1 that there exists $\alpha > 0$ such that $x \geqslant \alpha h \geqslant \alpha\nu e$ for all $x \in F^+$ and $x \leqslant -\alpha h \leqslant -\alpha\nu e$ for all $x \in F^-$. It follows from condition (i) and Lemma 2.4 that there exists $\gamma > 0$ such that $tv + \delta e \leqslant A(tv)$ and $A(-tv) \leqslant -tv - \delta e$ for all $t \in (0, \gamma)$, where $\delta = \delta(t) = t(\lambda_0 - 1)\mu/2 > 0$. Let $\gamma_0 = \min\{\gamma, \alpha\nu/\lambda, \beta/(2\lambda)\}$. Then we get that $x \geqslant \alpha\nu e \geqslant (\alpha\nu/\lambda)v \geqslant tv$ for all $x \in F^+$, $t \in (0, \gamma_0)$ and $x \leqslant -\alpha\nu e \leqslant -(\alpha\nu/\lambda)v \leqslant -tv$ for all $x \in F^-$, $t \in (0, \gamma_0)$, and that $tv \leqslant \beta/(2\lambda) \cdot \lambda e = (\beta/2)e \leqslant v_0/2 < v_0$ and $u_0 < u_0/2 \leqslant -(\beta/2)e = -\beta/(2\lambda) \cdot \lambda e \leqslant -tv$, and that $t\mu e \leqslant tv$ and $-tv \leqslant -t\mu e$. Hence, for $u_1 = tv$ and $v_1 = -tv$ with $t \in (0, \gamma_0)$, the condition (iii) of Theorem 2.1 holds obviously. The proof is completed. □

3. Three-solution theorems for general operators

Lemma 3.1. *Let P be a cone in E and h a bounded linear functional defined on E such that $h(x) > 0$ for all $x \in P \setminus \{0\}$. Let $A : E \to E$ be a completely continuous operator, Ω a nonempty open convex subset of E and $\Omega \subset P$. Suppose that*

(i) *there exist $R > 0$ and $\rho > 0$ such that $A(\bar{B}_R) \subset B_R$ and $\Omega \cap (B_R \setminus \bar{B}_\rho) \neq \emptyset$. In addition, there exists $z \in \Omega \cap (B_R \setminus \bar{B}_\rho)$ such that $h(x) \leqslant h(z)$ for all $x \in \Omega$ with $\|x\| \leqslant \rho$;*
(ii) *$Ax \in \Omega$ for all $x \in \partial\Omega \setminus \{0\}$;*
(iii) *$h(Ax) > h(x)$ for all $x \in \Omega$ with $0 < \|x\| \leqslant \rho$.*

Then the Leray–Schauder degree $\deg(I - A, \Omega \cap (B_R \setminus \bar{B}_\rho), 0) = 1$.

Proof. Let $U = \Omega \cap (B_R \setminus \bar{B}_\rho)$. Then it follows from condition (i) that U is a nonempty bounded open subset of E. For z in condition (i), if there exist $x_1 \in \partial U$ and $t_1 \in [0, 1]$ such that

$$x_1 = (1 - t_1)Ax_1 + t_1 z, \tag{3.1}$$

then $x_1 \in \partial\Omega$ and $\|x_1\| \in [\rho, R]$, or $x_1 \in \Omega$ and $\|x_1\| = \rho$, or $x \in \Omega$ and $\|x_1\| = R$. For the case $x_1 \in \partial\Omega$ and $\|x_1\| \in [\rho, R]$, it follows from condition (ii) that $Ax_1 \in \Omega$. Since Ω is an open convex subset of E, (3.1) and $z \in \Omega$ deduce that $x_1 \in \Omega$. This is a contradiction with $x_1 \in \partial\Omega$. For the case $x_1 \in \Omega$ and $\|x_1\| = \rho$, if $t_1 = 1$, then it follows from (3.1) that $x_1 = z \in \Omega \cap (B_R \setminus \bar{B}_\rho)$. So $\|x_1\| > \rho$. This is a contradiction with $\|x_1\| = \rho$. If $t_1 \in [0, 1)$, then it follows from conditions (iii) and (i) that $h(Ax_1) > h(x_1)$ and $h(z) \geqslant h(x_1)$. Then from (3.1) we have that

$$h(x_1) = (1 - t_1)h(Ax_1) + t_1 h(z) > (1 - t_1)h(x_1) + t_1 h(x_1) = h(x_1).$$

This is a contradiction. For the case $x_1 \in \Omega$ and $\|x_1\| = R$, if $t_1 = 1$, then it follows from (3.1) that $x_1 = z \in \Omega \cap (B_R \setminus \bar{B}_\rho)$. So $\|x_1\| < R$. This is a contradiction with $\|x_1\| = R$. If $t_1 \in [0, 1)$, then it follows from (3.1) that

$$\big\|(1 - t_1)Ax_1\big\| = \|x_1 - t_1 z\| \geqslant \|x_1\| - t_1\|z\| \geqslant (1 - t_1)R,$$

i.e., $\|Ax_1\| \geqslant R$, which is a contradiction with $A(\bar{B}_R) \subset B_R$. Hence, we have proved that $x \neq (1 - t)Ax + tz$ for all $x \in \partial U$ and $t \in [0, 1]$. According to the homotopy invariance and normalization of the Leray–Schauder degree,

$$\deg(I - A, U, 0) = \deg(I - z, U, 0) = \deg(I, U, z) = 1.$$

The proof is completed. □

Lemma 3.2. *Let P be a cone in E, h a bounded linear functional defined on E with $h(x) > 0$ for all $x \in P \setminus \{0\}$. Let $A : E \to E$ be a completely continuous operator, Ω a nonempty open convex subset of E and $\Omega \subset (-P)$. Suppose that*

- (i) *there exist $R > 0$ and $\rho > 0$ such that $A(\bar{B}_R) \subset B_R$ and $\Omega \cap (B_R \setminus \bar{B}_\rho) \neq \emptyset$. In addition, there exists $z \in \Omega \cap (B_R \setminus \bar{B}_\rho)$ such that $h(x) \geqslant h(z)$ for all $x \in \Omega$ with $\|x\| \leqslant \rho$;*
- (ii) *$Ax \in \Omega$ for all $x \in \partial\Omega \setminus \{0\}$;*
- (iii) *$h(Ax) < h(x)$ for all $x \in \Omega$ with $0 < \|x\| \leqslant \rho$.*

Then the Leray–Schauder degree $\deg(I - A, \Omega \cap (B_R \setminus \bar{B}_\rho), 0) = 1$.

The proof is similar to that of Lemma 3.1. For simplicity, we omit the proof.

Theorem 3.1. *Let P be a cone in E, h a bounded linear functional defined on E with $h(x) > 0$ for all $x \in P \setminus \{0\}$. Let $A : E \to E$ be a completely continuous operator, Ω_1 and Ω_2 both nonempty open convex subsets of E with $\Omega_1 \subset P$ and $\Omega_2 \subset (-P)$. Suppose that*

- (i) *$A0 = 0$, A is Fréchet differentiable at 0, 1 is not an eigenvalue of the Fréchet derivative $A'(0)$, and the index of isolated zero point* $\mathrm{ind}(I - A'(0), 0) = 1$;
- (ii) *there exist $R > 0$ and $\rho > 0$ such that $\Omega_1 \cap (B_R \setminus \bar{B}_\rho) \neq \emptyset$ and $\Omega_2 \cap (B_R \setminus \bar{B}_\rho) \neq \emptyset$. Moreover, $A(\bar{B}_R) \subset B_R$;*
- (iii) *there exists $z_1 \in \Omega_1 \cap (B_R \setminus \bar{B}_\rho)$ such that $h(x) \leqslant h(z_1)$ for all $x \in \Omega_1$ with $\|x\| \leqslant \rho$; there exists $z_2 \in \Omega_2 \cap (B_R \setminus \bar{B}_\rho)$ such that $h(x) \geqslant h(z_2)$ for all $x \in \Omega_2$ with $\|x\| \leqslant \rho$;*
- (iv) *$Ax \in \Omega_1$ for all $x \in P \setminus \{0\}$ and $Ax \in \Omega_2$ for all $x \in (-P) \setminus \{0\}$;*
- (v) *$h(Ax) > h(x)$ for all $x \in \Omega_1$ with $0 < \|x\| \leqslant \rho$;*
- (vi) *$h(Ax) < h(x)$ for all $x \in \Omega_2$ with $0 < \|x\| \leqslant \rho$.*

Then A has at least three nonzero fixed points, one of which is positive, another is negative, and the third one is a sign-changing fixed point.

Proof. Since $A(\bar{B}_R) \subset B_R$, we have that

$$\deg(I - A, B_R, 0) = 1. \tag{3.2}$$

It follows from condition (i) and the Leray–Schauder theorem that the index of isolated zero point $\operatorname{ind}(I - A, 0) = \operatorname{ind}(I - A'(0), 0) = 1$. So there exists $r \in (0, \rho)$ such that

$$\deg(I - A, B_r, 0) = 1. \tag{3.3}$$

It follows from conditions (ii)–(vi) and Lemmas 3.1 and 3.2 that

$$\deg\big(I - A, \Omega_1 \cap (B_R \setminus \bar{B}_r), 0\big) = 1, \qquad \deg\big(I - A, \Omega_2 \cap (B_R \setminus \bar{B}_r), 0\big) = 1. \tag{3.4}$$

It is obvious that B_r, $\Omega_1 \cap (B_R \setminus \bar{B}_r)$ and $\Omega_2 \cap (B_R \setminus \bar{B}_r)$ are three disjoint open subsets of B_R. Using (3.2)–(3.4) and the additivity of Leray–Schauder degree, we have that

$$\begin{aligned}
&\deg\big(I - A, B_R \setminus (\overline{\Omega_1 \cup \Omega_2 \cup B_r}\,), 0\big) \\
&\quad = \deg\big(I - A, B_R \setminus \big(\big[\,\overline{\Omega_1 \cap (B_R \setminus \bar{B}_r)}\,\big] \cup \big[\,\overline{\Omega_2 \cap (B_R \setminus \bar{B}_r)}\,\big] \cup \bar{B}_r\big), 0\big) \\
&\quad = \deg(I - A, B_R, 0) - \deg\big(I - A, \Omega_1 \cap (B_R \setminus \bar{B}_r), 0\big) \\
&\qquad - \deg\big(I - A, \Omega_2 \cap (B_R \setminus \bar{B}_r), 0\big) - \deg(I - A, B_r, 0) \\
&\quad = 1 - 1 - 1 - 1 = -2.
\end{aligned} \tag{3.5}$$

Thus, A has at least three fixed points x_1, x_2, x_3 such that $x_1 \in \Omega_1 \cap (B_R \setminus \bar{B}_r)$, $x_2 \in \Omega_2 \cap (B_R \setminus \bar{B}_r)$ and $x_3 \in B_R \setminus (\overline{\Omega_1 \cup \Omega_2 \cup B_r}\,)$. It is obvious that x_1 is positive and x_2 is negative. Since $x_3 \in B_R \setminus (\overline{\Omega_1 \cup \Omega_2 \cup B_r}\,)$, $\|x_3\| > r$. If $x_3 \in F^+$, then it follows from condition (iv) that $x_3 = Ax_3 \in \Omega_1$. This implies that $x_3 \in \Omega_1 \setminus \bar{B}_r$. This is a contradiction with $x_3 \in B_R \setminus (\overline{\Omega_1 \cup \Omega_2 \cup B_r}\,)$. Thus we obtain that $x_3 \notin F^+$. Similarly, we can prove that $x_3 \notin F^-$. Hence, x_3 is a sign-changing fixed point. The proof is completed. □

In the sequel of this section, we consider the existence of sign-changing fixed points for the composite operator $A = KF$. For convenience, we first list some rudimental conditions.

(B_1) $K : E \to E$ is a positive linear completely continuous operator with respect to a total cone P in E and its spectral radius $r = r(K) > 0$;
(B_2) $F : E \to E$ is a continuous and bounded operator. Moreover, $F(P \setminus \{0\}) \subset P \setminus \{0\}$ and $F((-P) \setminus \{0\}) \subset (-P) \setminus \{0\}$.

Noticing condition (B_1), by Krein–Rutman theorem, we have that there exists $h \in P^* \setminus \{0\}$ such that $K^*h = rh$, where P^* is the dual cone of P and K^* is the dual operator of K.

Theorem 3.2. *Let $A = KF$ and conditions* (B_1) *and* (B_2) *hold. Assume that Ω_1 and Ω_2 are both nonempty open convex subsets of E with $\Omega_1 \subset P$ and $\Omega_2 \subset (-P)$. Suppose that*

(i) *$F0 = 0$, F is Fréchet differentiable at 0, 1 is not an eigenvalue of the Fréchet derivative $A'(0)$, and the index of isolated zero point* $\operatorname{ind}(I - A'(0), 0) = 1$;

(ii) *there exist* $R > 0$ *and* $\rho > 0$ *such that* $\Omega_1 \cap (B_R \setminus \bar{B}_\rho) \neq \emptyset$ *and* $\Omega_2 \cap (B_R \setminus \bar{B}_\rho) \neq \emptyset$. *Moreover,* $A(\bar{B}_R) \subset B_R$;

(iii) *there exists* $z_1 \in \Omega_1 \cap (B_R \setminus \bar{B}_\rho)$ *such that* $h(x) \leqslant h(z_1)$ *for all* $x \in \Omega_1$ *with* $\|x\| \leqslant \rho$; *there exists* $z_2 \in \Omega_2 \cap (B_R \setminus \bar{B}_\rho)$ *such that* $h(x) \geqslant h(z_2)$ *for all* $x \in \Omega_2$ *with* $\|x\| \leqslant \rho$;

(iv) $Kx \in \Omega_1$ *for all* $x \in P \setminus \{0\}$ *and* $Kx \in \Omega_2$ *for all* $x \in (-P) \setminus \{0\}$;

(v) $h(Fx) > r^{-1}h(x)$ *for all* $x \in P$ *with* $0 < \|x\| \leqslant \rho$;

(vi) $h(Fx) < r^{-1}h(x)$ *for all* $x \in (-P)$ *with* $0 < \|x\| \leqslant \rho$,

where h *is mentioned before the theorem. Then* A *has at least a sign-changing fixed point, and also has at least one positive fixed point and one negative fixed point.*

Proof. We only need to verify all the conditions of Theorem 3.1. Since $K^*h = rh$, $h(Ax) = h(KFx) = (K^*h)(Fx) = rh(Fx)$ for all $x \in E$. Noticing assumption (B_2) and condition (iv), we easily see that the condition (iv) of Theorem 3.1 is satisfied. Other conditions are actually those of Theorem 3.1. The proof is completed. □

4. Applications

In this section, we apply our main results to both integral and differential equations. Firstly, we consider the integral equation

$$u(x) = \int_G k(x, y) f\big(y, u(y)\big)\, dy, \quad x \in G, \tag{4.1}$$

where G is a bounded closed domain of $\mathbb{R}^N$, $k : G \times G \to \mathbb{R}^1$ is nonnegative continuous and $k \not\equiv 0$ on $G \times G$, $f : G \times \mathbb{R}^1 \to \mathbb{R}^1$ is continuous.

Let $E = C(G)$ denote the space consisting of all continuous functions on G. Then E is a real Banach space with the norm $\|u\| = \max_{x \in G} |u(x)|$ for all $u \in E$. And let $P = \{u \in E\colon u(x) \geqslant 0, x \in G\}$. Then P is a normal and total cone in E. Let $e(x) = \int_G k(x, y)\, dy$, $x \in G$. Then $e > 0$.

Now we define operators $F, K, A : E \to E$ respectively by

$$(Fu)(x) = f\big(x, u(x)\big), \quad x \in G,\ \forall u \in E,$$

$$(Ku)(x) = \int_G k(x, y) u(y)\, dy, \quad x \in G,\ \forall u \in E,$$

and $A = KF$. It is obvious that $F : E \to E$ is a continuous and bounded operator. Since $k : G \times G \to \mathbb{R}^1$ is nonnegative continuous, $K : E \to E$ is a linear completely continuous operator and $K(P) \subset P$. So $A : E \to E$ is also completely continuous on E. By Riesz–Schauder theorem, we can suppose that the sequence of all positive eigenvalues of K is $\{\lambda_n\}$ and

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots.$$

Lemma 4.1. *Operators* $K, A : E \to E$ *are e-continuous on* E.

Proof. For any given $u_0 \in E$,

$$\begin{aligned}\left|Ku(x) - Ku_0(x)\right| &\leqslant \int_G k(x,y)\left|u(y) - u_0(y)\right| dy \\ &\leqslant \|u - u_0\| \int_G k(x,y)\,dy = \|u - u_0\| e(x), \quad x \in G,\ \forall u \in E.\end{aligned}$$

So K is e-continuous at u_0 and it follows from $F: E \to E$ is continuous that $A = KF$ is also e-continuous at u_0. The proof is completed. □

Lemma 4.2. *Suppose that*

(C_1) $f(\cdot, 0) = 0$ *on G, and for each $x \in G$, $f(x,u)$ is nondecreasing in u*;
(C_2) *there exists h with $\mu e \leqslant h$, where μ is a positive number, such that*

$$k(x,y) \geqslant h(x)k(z,y), \quad x, y, z \in G.$$

Then

$$\|Ku\|h \leqslant Ku, \quad u \in P, \qquad Ku \leqslant -\|Ku\|h, \quad u \in (-P),$$

and

$$\|Au\|h \leqslant Au, \quad u \in P, \qquad Au \leqslant -\|Au\|h, \quad u \in (-P).$$

Proof. For any given $u \in P$, it follows from the definition of K and condition (C_2) that

$$(Ku)(x) = \int_G k(x,y)u(y)\,dy \geqslant h(x) \int_G k(z,y)u(y)\,dy = h(x)(Ku)(z), \quad x, z \in G.$$

Then $Ku \geqslant \|Ku\|h$. Similarly, we can obtain that $Ku \leqslant -\|Ku\|h, u \in (-P)$. It follows from condition (C_1) that $F(P) \subset P$ and $F(-P) \subset (-P)$. So $Au = KFu \geqslant \|KFu\|h = \|Au\|h$, $u \in P$, and $Au = KFu \leqslant -\|KFu\|h = -\|Au\|h$, $u \in (-P)$. The proof is completed. □

Lemma 4.3. *Assume that $f(\cdot, 0) = 0$ on G, and $\lim_{u \to 0} f(x,u)/u = a$ uniformly for $x \in G$. Then the operator A is Fréchet differentiable at 0 and $A'(0) = aK$.*

Proof. By $\lim_{u \to 0} f(x,u)/u = a$ uniformly for $x \in G$, for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x,u)/u - a| < \varepsilon$ for all $x \in G$ and $|u| \in (0, \delta)$. So we have $\|Fu - au\| \leqslant \varepsilon \|u\|$ for all $u \in E$ with $\|u\| < \delta$. Consequently,

$$\lim_{\|u\| \to 0} \frac{\|Fu - F0 - au\|}{\|u\|} = 0.$$

This implies that the operator F is Fréchet differentiable at 0 and $F'(0) = aI$. It follows from the definition of A and the chain rule for derivatives of composite operator [25] that $A'(0) = KF'(0) = aK$. The proof is completed. □

Theorem 4.1. *Suppose that conditions* (C_1) *and* (C_2) *hold. Assume that*

(C_3) $\lim_{u\to 0} f(x,u)/u = a$ *uniformly for* $x \in G$,

$$1/\lambda_{2n_0} < a < 1/\lambda_{2n_0+1},$$

and the sum of the algebraic multiplicities of the eigenvalues λ_i *for all* $1 \leqslant i \leqslant 2n_0$ *is even*;

(C_4) $\lim_{u\to\infty} f(x,u)/u = f_\infty$ *uniformly for* $x \in G$, *and* $f_\infty < 1/\|e\|$.

Then the integral equation (4.1) *has at least three nontrivial solutions, one of which is positive, another is negative, and the third solution is sign-changing.*

Proof. It suffices to verify that all the conditions of Theorem 2.2 are satisfied. It follows from (C_3) and Lemma 4.3 that the eigenvalues of the operator aK in $(1,+\infty)$ are $a\lambda_1$, $a\lambda_2$, $\dots$, $a\lambda_{2n_0}$, and 1 is not an eigenvalue of aK. According to condition (C_3) and Leray–Schauder theorem, we can deduce that the index of isolated zero point $\operatorname{ind}(I - A'(0), 0) = 1$. That is the condition (ii) of Theorem 2.2.

Since P is a total cone in E, $K : E \to E$ is a completely continuous positive linear operator and the spectral radius $r(K) = \lambda_1 > 0$, it follows from Krein–Rutman theorem that there exists $v \in P \setminus \{0\}$ such that $Kv = \lambda_1 v$. Noticing that $\mu e \leqslant h$, $\mu > 0$ and according to Lemma 4.2, we have

$$\mu\lambda_1\|v\|e \leqslant \|\lambda_1 v\|h = \|Kv\|h \leqslant \lambda_1 v = Kv \leqslant \|v\|e.$$

So $\mu\|v\|e \leqslant v \leqslant \lambda_1^{-1}\|v\|e$. The condition (i) of Theorem 2.2 holds.

According to Lemma 4.2, we have

$$u = Au \geqslant \|Au\|h = \|u\|h \quad \text{for all } u \in P \text{ and } Au = u;$$
$$u = Au \leqslant -\|Au\|h = -\|u\|h \quad \text{for all } u \in (-P) \text{ and } Au = u.$$

It is easy to see that the condition (iv) of Theorem 2.2 is satisfied.

By condition (C_4), for some large $R > 0$, we have

$$f(x,R)/R < 1/\|e\|, \qquad f(x,-R)/(-R) < 1/\|e\|, \quad x \in G.$$

Let $u_0 = -R$, $v_0 = R$. Then $u_0 = -R \leqslant -R\|e\|^{-1}e$, $R\|e\|^{-1}e \leqslant R = v_0$. It follows that

$$\begin{aligned}
(Au_0)(x) &= \int_G k(x,y) f(y,-R)\,dy \geqslant -R\|e\|^{-1}\int_G k(x,y)\,dy = -R\|e\|^{-1}e(x)\\
&\geqslant -R = u_0(x), \quad x \in G;\\
(Av_0)(x) &= \int_G k(x,y) f(y,R)\,dy \leqslant R\|e\|^{-1}\int_G k(x,y)\,dy = R\|e\|^{-1}e(x)\\
&\leqslant R = v_0(x), \quad x \in G.
\end{aligned}$$

So $u_0 \leqslant Au_0$ and $Av_0 \leqslant v_0$. This implies that the condition (iii) of Theorem 2.2 holds. The proof is completed. □

Corollary 4.1. [24] *Suppose that conditions* (A_1)–(A_3) *mentioned in the first section are satisfied. Then the three-point boundary value problem* (1.1) *has at least one sign-changing solution. Moreover, the three-point boundary value problem* (1.1) *also has at least one positive and one negative solutions.*

Proof. Condition (C_1) obviously holds. Let

$$k(x,y)=\begin{cases}\begin{cases}(1-\frac{1-\alpha}{1-\alpha\eta}y)x, & 0\leqslant x\leqslant y\leqslant 1,\\ (1-\frac{1-\alpha}{1-\alpha\eta}x)y, & 0\leqslant y\leqslant x\leqslant 1,\end{cases} & y\leqslant \eta,\\ \begin{cases}\frac{1-y}{1-\alpha\eta}x, & 0\leqslant x\leqslant y\leqslant 1,\\ y-\frac{y-\alpha\eta}{1-\alpha\eta}x, & 0\leqslant y\leqslant x\leqslant 1,\end{cases} & \eta\leqslant y.\end{cases}$$

Let $h(x)=\sin\sqrt{\lambda_1}x$, $x\in[0,1]$. By direct calculation, it is easy to see that conditions (C_2)–(C_4) of Theorem 4.1 hold. The proof is completed. □

Now consider the following fourth-order nonlinear two-point boundary value problem (BVP):

$$\begin{cases}u^{(4)}(t)+\beta u''(t)-\alpha u(t)=f(t,u(t)), & t\in[0,1],\\ u(0)=u(1)=0,\\ u''(0)=u''(1)=0.\end{cases}\tag{4.2}$$

We assume the following rudimental conditions:

(D_1) $f:[0,1]\times\mathbb{R}^1\to\mathbb{R}^1$ is continuous, $f(\cdot,0)=0$ on $[0,1]$, and $f(\cdot,u)u\geqslant 0$ for all $u\in\mathbb{R}^1$;
(D_2) $\alpha,\beta\in\mathbb{R}^1$ satisfying $\beta<2\pi^2$, $\alpha\geqslant-\beta^2/4$, and $\alpha/\pi^4+\beta/\pi^2<1$.

Let λ_1,λ_2 be the roots of the polynomial $P(\lambda)=\lambda^2+\beta\lambda-\alpha$, namely

$$\lambda_1=:\bigl(-\beta+\sqrt{\beta^2+4\alpha}\bigr)/2,\qquad \lambda_2=:\bigl(-\beta-\sqrt{\beta^2+4\alpha}\bigr)/2.$$

Then we have that

$$\begin{aligned}&u^{(4)}(t)+\beta u''(t)-\alpha u(t)\\ &\quad=\left(-\frac{d^2}{dt^2}+\lambda_1\right)\left(-\frac{d^2}{dt^2}+\lambda_2\right)u(t)\\ &\quad=\left(-\frac{d^2}{dt^2}+\lambda_2\right)\left(-\frac{d^2}{dt^2}+\lambda_1\right)u(t),\quad t\in[0,1],\ \forall u\in C^4[0,1].\end{aligned}\tag{4.3}$$

By condition (D_2), it is easy to see that $\lambda_1\geqslant\lambda_2>-\pi^2$.

Let $G_i:[0,1]\times[0,1]\to\mathbb{R}^1$, $i=1,2$, be the Green's function of the linear boundary value problem

$$\begin{cases}-u''(t)+\lambda_i u(t)=0, & t\in[0,1],\\ u(0)=u(1)=0.\end{cases}$$

Set $\omega_i=\sqrt{|\lambda_i|}$. If $\lambda_i>0$, then G_i is explicitly given by

$$G_i(t,s)=\frac{1}{\omega_i\sinh\omega_i}\begin{cases}\sinh\omega_i t\cdot\sinh\omega_i(1-s), & 0\leqslant t\leqslant s\leqslant 1,\\ \sinh\omega_i s\cdot\sinh\omega_i(1-t), & 0\leqslant s\leqslant t\leqslant 1.\end{cases}$$

If $\lambda_i=0$, then G_i is expressed by

$$G_i(t,s)=\begin{cases}t(1-s), & 0\leqslant t\leqslant s\leqslant 1,\\ s(1-t), & 0\leqslant s\leqslant t\leqslant 1.\end{cases}$$

If $\lambda_i \in (-\pi^2, 0)$, then G_i is expressed by

$$G_i(t,s) = \frac{1}{\omega_i \sin\omega_i} \begin{cases} \sin\omega_i t \cdot \sin\omega_i(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \\ \sin\omega_i s \cdot \sin\omega_i(1-t), & 0 \leqslant s \leqslant t \leqslant 1. \end{cases}$$

See, for example, [20, Lemma 2.1].

Remark 4.1. It follows from the expression of G_i and (4.3) that

(i) $G_i : [0,1] \times [0,1] \to \mathbb{R}^1$ is continuous;
(ii) $G_i(t,s) > 0$, $t, s \in (0,1)$;
(iii) $G_i(t,s) = G_i(s,t)$ for all $t, s \in [0,1]$, $i = 1, 2$;
(iv) $\int_0^1 G_1(t,s)G_2(s,\tau)\,ds = \int_0^1 G_2(t,s)G_1(s,\tau)\,ds$, $t, \tau \in [0,1]$.

Let

$$C_0 = \max_{t\in[0,1]} \int_0^1\!\!\int_0^1 G_1(t,s)G_2(s,\tau)\,d\tau\,ds,$$

$$C_1 = \max_{t\in[0,1]} \int_0^1\!\!\int_0^1 \left|\frac{\partial G_1(t,s)}{\partial t} G_2(s,\tau)\right| d\tau\,ds.$$

Then $C_0 > 0$ and $C_1 > 0$.

Remark 4.2. It is easy to verify that the sequence of all eigenvalues of the following linear BVP:

$$\begin{cases} u^{(4)}(t) + \beta u''(t) - \alpha u(t) = \eta u(t), & t \in [0,1], \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0 \end{cases}$$

is $\{\eta_k\}_{k=1}^{\infty} = \{(k\pi)^4 - \beta(k\pi)^2 - \alpha\}_{k=1}^{\infty}$, the corresponding eigenfunction is $\phi_k = \sin k\pi t$, $t \in [0,1]$, and the algebraic multiplicity of η_k is 1 for all $k \in \mathbb{N} = \{1, 2, \ldots\}$.

Let $C[0,1]$ denote the space consisting of all continuous functions on $[0,1]$ with the norm $\|u\|_C = \max_{t\in[0,1]} |u(t)|$ for all $u \in C[0,1]$. Then $E = C[0,1]$ is a real Banach space. By $C^1[0,1]$ denote the space consisting of all continuously differentiable functions on $[0,1]$. Let $E_1 = \{u \in C^1[0,1]\colon u(0) = u(1) = 0\}$ with the norm $\|u\| = \max\{\|u\|_C, \|u'\|_C\}$ for all $u \in E_1$. Then E_1 is also a real Banach space. Let $P = \{u \in E\colon u(t) \geqslant 0,\ t \in [0,1]\}$ and $P_1 = \{u \in E_1\colon u(t) \geqslant 0,\ t \in [0,1]\}$. Then P is a total cone in E and P_1 is a cone in E_1.

It is well known that any solution of BVP (4.2) in $C^4[0,1]$ is equivalent to a solution of the following integral equation in E:

$$u(t) = \int_0^1 G(t,\tau) f\big(\tau, u(\tau)\big)\,d\tau, \quad t \in [0,1], \tag{4.4}$$

where $G(t,\tau)=\int_0^1 G_1(t,s)G_2(s,\tau)\,ds$, $t,\tau\in[0,1]$. And by (iii) and (iv) of Remark 4.1, we have

$$G(t,\tau)=\int_0^1 G_1(\tau,s)G_2(s,t)\,ds=\int_0^1 G_2(t,s)G_1(s,\tau)\,ds=G(\tau,t).$$

That is $G(t,\tau)=G(\tau,t)$ for all $t,\tau\in[0,1]$.

Define operators $K, F, A: E\to E$, respectively, by

$$(Ku)(t)=\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)u(\tau)\,ds\,d\tau,\quad t\in[0,1],\ \forall u\in E,$$

$$(Fu)(t)=f\bigl(t,u(t)\bigr),\quad t\in[0,1],\ \forall u\in E,$$

and $A=KF$. Then $F:E\to E$ is a continuous and bounded operator and $K:E\to E$ is linear continuous. And it follows from Remark 4.1(i) that $A,K:E\to E$ are completely continuous and $A,K:E_1\to E_1$ are also completely continuous.

Lemma 4.4. *All the positive eigenvalues of operator K are*

$$1/\eta_1, 1/\eta_2, \ldots, 1/\eta_n, \ldots,$$

and the algebraic multiplicity of each positive eigenvalue $1/\eta_n$ of K is 1.

Proof. Let λ be an eigenvalue of the linear operator K, and $u\in E\setminus\{0\}$ be an eigenvector corresponding to the eigenvalue λ. Then according to the definition of K, we have

$$\begin{cases} u^{(4)}(t)+\beta u''(t)-\alpha u(t)=\frac{1}{\lambda}u(t), & t\in[0,1],\\ u(0)=u(1)=0, \\ u''(0)=u''(1)=0.\end{cases}$$

Thus, according to Remark 4.2, λ is one of all the values

$$1/\eta_1, 1/\eta_2, \ldots, 1/\eta_n, \ldots,$$

and the algebraic multiplicity of $\lambda=1/\eta_n$ is 1 for all $n\in\mathbb{N}$. $\square$

Remark 4.3. By Lemma 4.4, the spectral radius of $K:E\to E$, $r=r(K)=1/\eta_1>0$. Since $K:E\to E$ is a linear completely continuous operator and P is a total cone in E, by Krein–Rutman theorem, there exist $\phi\in P\setminus\{0\}$ and $h\in P^*\setminus\{0\}$ such that $K\phi=r(K)\phi$ and $K^*h=r(K)h$, where K^* is the dual operator of K and P^* is the dual cone of P. It follows from Remark 4.2 that $\phi(t)=\sin\pi t$, $t\in[0,1]$. We now claim that $h\in P^*\setminus\{0\}$ can be taken in the following form:

$$h(u)=\int_0^1 \phi(t)u(t)\,dt,\quad u\in E. \tag{4.5}$$

In fact, for any given $u\in E$, it follows from the symmetry of G that

$$
\begin{aligned}
r(K)h(u) &= \int_0^1 r(K)\phi(t)u(t)\,dt = \int_0^1 (K\phi)(t)u(t)\,dt \\
&= \int_0^1 u(t)\,dt \int_0^1 G(t,\tau)\phi(\tau)\,d\tau = \int_0^1 \phi(\tau)\,d\tau \int_0^1 G(\tau,t)u(t)\,dt \\
&= \int_0^1 \phi(\tau)(Ku)(\tau)\,d\tau = h(Ku) = (K^*h)(u).
\end{aligned}
$$

So (4.5) holds.

Lemma 4.5. *Suppose that conditions* (D_1) *and* (D_2) *hold. Assume that* $\lim_{u\to 0} f(t,u)/u = f_0$ *uniformly for* $t \in [0,1]$. *Then the operator* $A: E_1 \to E_1$ *is Fréchet differentiable at* 0 *and* $A'(0) = f_0 K$.

Proof. By $\lim_{u\to 0} f(t,u)/u = f_0$ uniformly for $t \in [0,1]$, for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| f(t,u)/u - f_0 \right| < \varepsilon, \quad t \in [0,1],\ |u| \in (0,\delta].$$

This implies that

$$\left| f(t,u) - f_0 u \right| \leqslant \varepsilon |u|, \quad t \in [0,1],\ |u| \in [0,\delta].$$

Hence, for any $u \in E$ with $\|u\| \leqslant \delta$, we have

$$\|Au - A0 - f_0 Ku\|_C = \|KFu - f_0 Ku\|_C \leqslant C_0 \|Fu - f_0 u\|_C \leqslant \varepsilon C_0 \|u\|_C \leqslant \varepsilon C_0 \|u\|,$$

and

$$
\begin{aligned}
\left\|(Au - A0 - f_0 Ku)'\right\|_C &= \max_{t\in[0,1]} \left|(KFu - f_0 Ku)'(t)\right| \leqslant C_1 \|Fu - f_0 u\|_C \\
&\leqslant \varepsilon C_1 \|u\|_C \leqslant \varepsilon C_1 \|u\|.
\end{aligned}
$$

So we have

$$\lim_{\|u\|\to 0} \|Au - A0 - f_0 Ku\| / \|u\| = 0.$$

This implies that the operator A is Fréchet differentiable at 0 and $A'(0) = f_0 K$. The proof is completed. □

Theorem 4.2. *Suppose that conditions* (D_1) *and* (D_2) *hold. Assume that*

(D_3) $\lim_{u\to 0} f(t,u)/u = f_0$ *uniformly for* $t \in [0,1]$, *and* $f_0 \in (\eta_{2n_0}, \eta_{2n_0+1})$;
(D_4) $\lim_{u\to \infty} f(t,u)/u = f_\infty$ *uniformly for* $t \in [0,1]$, *and* $f_\infty \in (0, \min\{1/C_0, 1/C_1\})$.

Then BVP (4.2) *has at least one sign-changing solution. Moreover, the problem also has at least one positive solution and one negative solution.*

Proof. We will verify all the conditions of Theorem 3.1. Let

$$\Omega_1 = \{u \in E_1\colon\ u(t) > 0,\ t \in (0,1)\},$$
$$\Omega_2 = \{u \in E_1\colon\ u(t) < 0,\ t \in (0,1)\}.$$

Then Ω_1 and Ω_2 are disjoint open subsets of E_1. See [22,23]. It is obvious that Ω_1 and Ω_2 are both nonempty convex subsets of E_1 with $\Omega_1 \subset P_1$ and $\Omega_2 \subset (-P_1)$. It follows from conditions (D_1) and (D_3), Lemmas 4.4 and 4.5 that $A0 = 0$, $A: E_1 \to E_1$ is Fréchet differentiable at 0, $A'(0) = f_0 K$, 1 is not an eigenvalue of $A'(0)$ and the eigenvalues of $A'(0)$ in $(1, +\infty)$ are $f_0/\eta_1, f_0/\eta_2, \ldots, f_0/\eta_{2n_0}$. According to Leray–Schauder theorem, there exists $\delta > 0$ such that 0 is a unique zero point of $I - A$ in $B_\delta = \{u \in E_1\colon\ \|u\| < \delta\}$ and the index of isolated zero point $\operatorname{ind}(I - A'(0), 0) = 1$. So the condition (i) of Theorem 3.1 holds. By condition (D_3) and Remark 4.3, we have $\lim_{u\to 0} f(t,u)/u = f_0 > \eta_1 = 1/r$ uniformly for $t \in [0,1]$. Then there exists $\rho \in (0, \delta/2)$ such that

$$\big|f(t,u)\big| > r^{-1}|u|, \quad t \in [0,1],\ |u| \in (0,\rho]. \tag{4.6}$$

By condition (D_4), we can choose $\varepsilon_0 > 0$ such that $f_\infty(1+\varepsilon_0) < \min\{1/C_0, 1/C_1\}$. Then there exists $R_0 > \delta$ such that $|f(t,u)| < f_\infty(1+\varepsilon_0)|u|$ for all $t \in [0,1]$, $|u| \geqslant R_0$. Moreover, $|f(t,u)| - f_\infty(1+\varepsilon_0)|u|$ is obviously continuous on $[0,1] \times [-R_0, R_0]$. So there exists $C > 0$ such that $|f(t,u)| - f_\infty(1+\varepsilon_0)|u| \leqslant C$ for all $t \in [0,1]$, $|u| \leqslant R_0$. This implies that $|f(t,u)| \leqslant f_\infty(1+\varepsilon_0)|u| + C$ for all $t \in [0,1]$, $u \in \mathbb{R}^1$. Now we choose $R > \max\{R_0, C_0C/(1 - C_0 f_\infty(1+\varepsilon_0)), C_1C/(1 - C_1 f_\infty(1+\varepsilon_0))\}$. Then for each $u \in \bar{B}_R$, we have that

$$\begin{aligned}
\big|(Au)(t)\big| &\leqslant \int_0^1 G(t,\tau)\big|f\big(\tau, u(\tau)\big)\big|\,d\tau \\
&\leqslant \int_0^1 G(t,\tau)\big(f_\infty(1+\varepsilon_0)\big|u(\tau)\big| + C\big)\,d\tau \\
&\leqslant C_0 f_\infty(1+\varepsilon_0)R + C_0C < R, \quad t \in [0,1],
\end{aligned}$$

and

$$\begin{aligned}
\big|(Au)'(t)\big| &\leqslant \int_0^1\!\!\int_0^1 \left|\frac{\partial G_1(t,s)}{\partial t} G_2(s,\tau) f\big(\tau, u(\tau)\big)\right| d\tau\,ds \\
&\leqslant \int_0^1\!\!\int_0^1 \left|\frac{\partial G_1(t,s)}{\partial t} G_2(s,\tau)\right| \big(f_\infty(1+\varepsilon_0)\big|u(\tau)\big| + C\big)\,d\tau\,ds \\
&\leqslant C_1 f_\infty(1+\varepsilon_0)R + C_1C < R, \quad t \in [0,1].
\end{aligned}$$

This implies that $A(\bar{B}_R) \subset B_R$. Let $z_1(t) = 2\rho t(1-t)$, $t \in [0,1]$. It is easy to see that $z_1 \in \Omega_1 \cap (B_R \setminus \bar{B}_\rho)$ and $z_2 = -z_1 \in \Omega_2 \cap (B_R \setminus \bar{B}_\rho)$. So the condition (ii) of Theorem 3.1 holds. By Remark 4.3, we may choose

$$h(u) = \int_0^1 \phi(t)u(t)\,dt, \quad u \in E_1 \subset E.$$

It is obvious that h is a bounded linear functional defined on E_1 with $h(u) > 0$ for all $u \in P_1 \setminus \{0\}$. Now we verify the condition (iii) of Theorem 3.1. For $u \in \Omega_1$ with $\|u\| \leqslant \rho$, we have $u(t) \leqslant z_1(t)$, $t \in [0, 1]$. So

$$h(u) = \int_0^1 \phi(t)u(t)\,dt \leqslant \int_0^1 \phi(t)2\rho t(1-t)\,dt = h(z_1).$$

In a similar way, we have $h(u) \geqslant h(z_2)$ for all $u \in \Omega_2$ with $\|u\| \leqslant \rho$. For the condition (iv) of Theorem 3.1, it obviously holds by the definition of K and Remark 4.1. In the following, we prove that conditions (v) and (vi) of Theorem 3.1 are satisfied. For all $u \in \Omega_1 \subset P_1 \subset P$ with $0 < \|u\| \leqslant \rho$, by (4.6), we have

$$h(Au) = h(KFu) = (K^*h)(Fu) = rh(Fu) = r\int_0^1 \phi(t) f\big(t, u(t)\big)\,dt > r \cdot \frac{1}{r}\int_0^1 \phi(t)u(t)\,dt$$
$$= h(u).$$

Similarly, $h(Au) < h(u)$ for all $u \in \Omega_2$ with $0 < \|u\| \leqslant \rho$. The proof is completed. □

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