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Exponential iterated integrals and the relative solvable completion of the fundamental group of a manifold☆

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Abstract

We develop a class of integrals on a manifold M called *exponential iterated integrals*, an extension of K.T. Chen's iterated integrals. It is shown that the matrix entries of any upper triangular representation of $\pi_1(M, x)$ can be expressed via these new integrals. The ring of exponential iterated integrals contains the coordinate rings for a class of universal representations, called the *relative solvable completions* of $\pi_1(M, x)$. We consider exponential iterated integrals in the particular case of fibered knot complements, where the fundamental group always has a faithful relative solvable completion.

Keywords: Iterated integrals; Algebraic completions; Fundamental groups

1. Introduction

We are concerned with using integrals to determine the fundamental group of a smooth manifold M. Let PM denote the space of piecewise differentiable paths $\lambda: [0, 1] \to M$. A 1-form $\omega \in E^1(M; \mathbb{C})$ provides a $\mathbb{C}$-valued function on PM via integration:

$$\int \omega: PM \to \mathbb{C},$$

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$$\lambda \mapsto \int_\lambda \omega.$$

And $\int \omega$ induces a map on the fundamental group $\pi_1(M, x)$ if and only if ω is closed:

$$\int \omega: \pi_1(M, x) \to \mathbb{C}.$$

By the de Rham theorem, closed line integrals can distinguish two elements of $\pi_1(M, x)$ if and only if they are different in $H_1(M; \mathbb{C})$.

K.-T. Chen improved this approach with *iterated integrals* of 1-forms (see [2]). For $\mathbb{C}$-valued 1-forms $\omega_1, \omega_2, \ldots, \omega_n$,

$$\int_\lambda \omega_1\omega_2 \ldots \omega_n := \int_{0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant 1} f_1(t_1) f_2(t_2) \ldots f_n(t_n)\, \mathrm{d}t_1\, \mathrm{d}t_2 \ldots \mathrm{d}t_n,$$

where $f_i(t)\, \mathrm{d}t$ is the pullback of ω_i to $E^1([0, 1]; \mathbb{C})$ along $\lambda: [0, 1] \to M$. An iterated integral is a finite sum of these expressions, with a constant term, regarded as a function from *PM* into $\mathbb{C}$. A *closed iterated integral* is one that is constant on homotopy classes of paths $\lambda: [0, 1] \to M$ relative to $\{0, 1\}$, and thus induces a $\mathbb{C}$-valued function on $\pi_1(M, x)$. The vector space of iterated integrals on *PM* is denoted by $B(M)$ and the vector space of closed iterated integrals on the space of loops based at $x \in M$ is denoted by $H^0(B(M, x))$. Both are commutative Hopf algebras.

This larger class of integrals can be used to detect more structure in $\pi_1(M, x)$ than is detected by ordinary line integrals. Chen proved that integration induces a Hopf algebra isomorphism,

$$H^0(B(M, x)) \cong \mathcal{O}(\mathcal{U}(\pi_1(M, x))), \tag{1}$$

where $\mathcal{O}(\mathcal{U}(\pi_1(M, x)))$ denotes the coordinate ring of the unipotent completion of $\pi_1(M, x)$. Thus when the representation $\pi_1(M, x) \to \mathcal{U}(\pi_1(M, x))$ is faithful (this occurs, for example, when $\pi_1(M, x)$ is free), closed iterated integrals separate the elements of $\pi_1(M, x)$.

But there are important cases where $\pi_1(M, x) \to \mathcal{U}(\pi_1(M, x))$ is far from being faithful. Indeed, if $H_1(M; \mathbb{Z}) \cong \mathbb{Z}$ (as in the case of knot groups), $\mathcal{U}(\pi_1(M, x))$ is the additive group $\mathbb{G}_a$ and the kernel of the representation is the commutator subgroup of $\pi_1(M, x)$. Thus closed iterated integrals vanish on every element of the commutator subgroup, and they provide no advantage over ordinary line integrals in this case.

We shall overcome this limitation by considering a larger class of integrals, called *exponential iterated integrals*. The goal is to detect a larger quotient of $\pi_1(M, x)$ than is obtained by ordinary iterated integrals.

Exponential iterated integrals are certain infinite sums of ordinary iterated integrals, and their properties are similar. An exponential iterated integral is written as

$$\sum \int \mathrm{e}^{\delta_1}\omega_{12}\mathrm{e}^{\delta_2}\omega_{23}\mathrm{e}^{\delta_3} \cdots \omega_{(n-1)n}\mathrm{e}^{\delta_n},$$

where δ_i and $\omega_{i(i+1)}$ are 1-forms. If $L \subset E^1(M; \mathbb{C})$ is a $\mathbb{Z}$-module of closed 1-forms, we denote by $EB(M)^L$ the vector space of exponential iterated integrals whose exponents δ_i are in L, and by $H^0(EB(M, x)^L)$ the space of exponential iterated integrals on the loop space at x that are constant on homotopy classes of paths relative to endpoints. Each integral in $H^0(EB(M, x)^L)$ gives a map $\pi_1(M, x) \to \mathbb{C}$. We will show that these integrals are matrix entries of solvable representations of $\pi_1(M, x)$.

We then develop the notion of *relative solvable completion*, which is a special case of Deligne's relative unipotent completion (see [4]). Given a homomorphism $\rho: G \to T$ of an abstract group G into a diagonalizable algebraic group T with Zariski dense image, the *solvable completion of G relative to* ρ, denoted by $\mathscr{S}_\rho(G)$, is the inverse limit of algebraic representations $\phi: G \to S$ that fit into a commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & G & & & & \\
 & & & & \downarrow{\scriptstyle\phi} & \searrow^{\rho} & & & \\
1 & \longrightarrow & U & \longrightarrow & S & \longrightarrow & T & \longrightarrow & 1
\end{array}
$$

where the bottom row is exact, ϕ has Zariski dense image, and U is a unipotent group.

The essential link between exponential iterated integrals and relative solvable completion is given by the following theorem, a more general version of which is proved in Section 6.

Theorem 1.1. *Suppose* $\rho: \pi_1(M, x) \to T \subseteq (\mathbb{C}^*)^n$ *is a diagonal algebraic representation with Zariski dense image, and that* $\delta_1, \ldots, \delta_n$ *are closed* 1-*forms such that*

$$\rho(\lambda) = (e^{\int_\lambda \delta_1}, \ldots, e^{\int_\lambda \delta_n}) \in (\mathbb{C}^*)^n.$$

Let L denote the $\mathbb{Z}$-*submodule of* $E^1(M; \mathbb{C})$ *generated by* $\delta_1, \ldots, \delta_n$. *Then integration induces a Hopf algebra isomorphism*

$$H^0(EB(M, x)^L) \cong \mathcal{O}(\mathscr{S}_\rho(\pi_1(M, x))).$$

In Section 7 we consider the relationship between unipotent completion and relative solvable completion, and prove a result of which the following is a special case:

Theorem 1.2. *If G is a group such that* $G/[G, G]$ *is finitely generated and* $H_1([G, G]; \mathbb{C})$ *is finite dimensional, then there exists a diagonalizable algebraic representation* $\rho: G \to T$ *such that* $\mathscr{U}([G, G])$ *injects into* $\mathscr{S}_\rho(G)$.

All knot groups $G = \pi_1(S^3 \backslash K, x)$ of tame knots K satisfy the conditions of this theorem. The representation $\rho: G \to T$ can be obtained from the Alexander module of K. When K is a fibered knot, $[G, G]$ is free and thus it injects into its unipotent completion. Hence:

Corollary 1.3. *If* $K \subset S^3$ *is a fibered knot, there exists a diagonalizable algebraic representation* $\rho: \pi_1(S^3 \backslash K, x) \to T$ *such that the representation* $\pi_1(S^3 \backslash K, x) \to \mathscr{S}_\rho(\pi_1(S^3 \backslash K, x))$ *is injective.*

Combining Corollary 1.3 with Theorem 1.1 shows that exponential iterated integrals separate the elements of the group of a fibered knot. In Section 8 we consider the example of the trefoil knot (a fibered knot), providing an explicit description for the vector space of closed exponential iterated integrals on its complement in S^3.

2. Notation and conventions

Throughout, M is a C^∞-manifold with base point x. PM is the space of piecewise differentiable paths $\lambda\colon[0,1]\to M$, and $P_{x,x}M$ is the loop space at $x\in M$. $E^1(M;\mathbb{C})$ is the space of $\mathbb{C}$-valued 1-forms, and $B^1(M;\mathbb{C})$ is the space of closed $\mathbb{C}$-valued 1-forms. When I is an integral and $\lambda\in PM$ is a path, let $\langle I,\lambda\rangle$ denote the integral of I over λ. For $y\in M$, let $\mathbf{1}_y$ denote the constant loop at y. If δ is a path or 1-form, we write $[\delta]$ to mean the homotopy, homology, or cohomology class of δ, depending on the context.

By "algebraic group" we will always mean linear algebraic group over $\mathbb{C}$. We say that an algebraic group is *diagonalizable* if it is isomorphic to a closed subgroup of $(\mathbb{C}^*)^n$ for some n. If $\mathbf{G}$ is an algebraic group then $\mathbf{G}_u$ denotes the unipotent radical of $\mathbf{G}$. Let $\mathbf{M}_n(\mathbb{C})$, $\mathbf{B}_n(\mathbb{C})$, $\mathbf{U}_n(\mathbb{C})$, and $\mathbf{D}_n(\mathbb{C})$ denote, respectively, the $n\times n$ matrix ring, upper triangular matrix group, unipotent matrix group, and diagonal matrix group over $\mathbb{C}$.

3. Exponential iterated integrals

In this section we define exponential iterated integrals and show how they appear as matrix entries for the transport functions of certain trivialized vector bundles. Using this relationship we then prove several formal properties that will be used in later proofs.

Throughout this section let $\lambda\colon[0,1]\to M$ be a path. We begin with the definition of ordinary iterated integrals. Note that the definition given in the introduction extends easily to iterated integrals of 1-forms taking values in any $\mathbb{C}$-algebra.

Definition 3.1. Suppose $\omega_1,\ldots,\omega_n$ are 1-forms on M taking values in an associative $\mathbb{C}$-algebra A. Let

$$\int_\lambda \omega_1\omega_2\ldots\omega_n=\int_{0\leqslant t_1\leqslant t_2\leqslant\cdots\leqslant t_n\leqslant 1} F_1(t_1)F_2(t_2)\ldots F_n(t_n)\,\mathrm{d}t_1\,\mathrm{d}t_2\ldots\mathrm{d}t_n,$$

where $F_i(t)\,\mathrm{d}t=\lambda^*\omega_i\in E^1([0,1];\mathbb{C})\otimes A$.

The expression $\int\omega_1\ldots\omega_n$ denotes a map from PM to A. An A-valued iterated integral is a finite sum of these expressions, possibly including a constant term. The following theorem of Chen's provides the initial motivation for this definition [3, p. 253]:

Theorem 3.2. *Given a trivial vector bundle $\mathbb{C}^n\times M\to M$ with connection $\nabla=d-\omega$,*[1] *$\omega\in E^1(M;\mathbb{C})\otimes \mathbf{M}_n(\mathbb{C})$, let $T\colon PM\to GL(n,\mathbb{C})$ denote the transport function. For any $\lambda\in PM$, the sum*

$$I+\int_\lambda\omega+\int_\lambda\omega\omega+\int_\lambda\omega\omega\omega+\cdots$$

[1] This means that for a section $f\colon M\to\mathbb{C}^n$, $\nabla f=\mathrm{d}f-f\omega$.

converges absolutely, and

$$T(\lambda) = I + \int_\lambda \omega + \int_\lambda \omega\omega + \int_\lambda \omega\omega\omega + \cdots \qquad \square \tag{2}$$

When the matrix of 1-forms ω is strictly upper triangular, the series above is finite, and the transport function is given by a matrix of ($\mathbb{C}$-valued) iterated integrals. For example, if

$$\omega = \begin{bmatrix} 0 & \omega_{12} & 0 & 0 & \dots & 0 \\ 0 & 0 & \omega_{23} & 0 & \dots & 0 \\ 0 & 0 & 0 & \omega_{34} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \omega_{(n-1)n} \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{3}$$

computing the series yields

$$T = \begin{bmatrix} 1 & \int \omega_{12} & \int \omega_{12}\omega_{23} & \int \omega_{12}\omega_{23}\omega_{34} & \dots & \int \omega_{12}\omega_{23}\dots\omega_{(n-1)n} \\ 0 & 1 & \int \omega_{23} & \int \omega_{23}\omega_{34} & \dots & \int \omega_{23}\omega_{34}\dots\omega_{(n-1)n} \\ 0 & 0 & 1 & \int \omega_{34} & \dots & \int \omega_{34}\omega_{45}\dots\omega_{(n-1)n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \int \omega_{(n-1)n} \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}. \tag{4}$$

We write $B(M, x)$ for the vector space of functions $P_{x,x}M \to \mathbb{C}$ that are given by iterated integrals. The subspace of functions that are constant on homotopy classes is denoted by $H^0(B(M, x))$.[2]

Now we can define exponential iterated integrals:

Definition 3.3. For $n \geqslant 0$ and $\delta_1, \delta_2, \dots, \delta_n, \omega_{12}, \dots, \omega_{(n-1)n} \in E^1(M; \mathbb{C})$,

$$\int_\lambda e^{\delta_1}\omega_{12}e^{\delta_2}\omega_{23}e^{\delta_3}\dots e^{\delta_{n-1}}\omega_{(n-1)n}e^{\delta_n}$$
$$:= \sum_{m_1,\dots,m_n \geqslant 0} \int_\lambda \underbrace{\delta_1\delta_1\dots\delta_1}_{m_1 \text{ terms}} \omega_{12} \underbrace{\delta_2\delta_2\dots\delta_2}_{m_2 \text{ terms}} \omega_{23}\dots\omega_{(n-1)n} \underbrace{\delta_n\delta_n\dots\delta_n}_{m_n \text{ terms}}. \tag{5}$$

An exponential iterated integral is a finite sum of these expressions, regarded as a function from *PM* to $\mathbb{C}$. By the *length* of an exponential iterated integral we mean the number of linear 1-forms in its longest term. The integral above has length $n - 1$.[3]

---

[2] The reason for this notation is that $H^0(B(M, x))$ is the first cohomology group of the complex of higher iterated integrals (see [2, Section 1.5]).

[3] There is a possible ambiguity because two different integral expressions can compute the same map $PM \to \mathbb{C}$. So we will say that the length of an exponential iterated integral is the minimum length of its literal expressions.

To see where Definition 3.3 comes from, suppose $E=\mathbb{C}^n \times M \to M$ is a trivial bundle with connection $\nabla = d - \omega$ where ω is a superdiagonal matrix:

$$\omega = \begin{bmatrix} \delta_1 & \omega_{12} & 0 & 0 & \cdots & 0 \\ 0 & \delta_2 & \omega_{23} & 0 & \cdots & 0 \\ 0 & 0 & \delta_3 & \omega_{34} & \cdots & 0 \\ 0 & 0 & 0 & \delta_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \delta_n \end{bmatrix}. \tag{6}$$

Computing the matrix entries of the series (2) yields

$$T = \begin{bmatrix} \int e^{\delta_1} & \int e^{\delta_1}\omega_{12}e^{\delta_2} & \int e^{\delta_1}\omega_{12}e^{\delta_2}\omega_{23}e^{\delta_3} & \cdots & \int e^{\delta_1}\omega_{12}\dots\omega_{(n-1)n}e^{\delta_n} \\ 0 & \int e^{\delta_2} & \int e^{\delta_2}\omega_{23}e^{\delta_3} & \cdots & \int e^{\delta_2}\omega_{23}\dots\omega_{(n-1)n}e^{\delta_n} \\ 0 & 0 & \int e^{\delta_3} & \cdots & \int e^{\delta_3}\omega_{34}\dots\omega_{(n-1)n}e^{\delta_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \int e^{\delta_n} \end{bmatrix}. \tag{7}$$

Right away, then, we know from Theorem 3.2 that exponential iterated integrals are well-defined.

Proposition 3.4. *For any* 1-*forms* $\{\delta_j\}_j, \{\omega_{j(j+1)}\}_j \subseteq E^1(M;\mathbb{C})$, *the sum* (5) *converges absolutely*.

Also, since transport is invariant under reparametrization of paths, we have

Proposition 3.5. *For any* 1-*forms* $\delta_1,\dots,\delta_n,\omega_{12},\dots,\omega_{(n-1)n}$, *the integral*

$$\int_\lambda e^{\delta_1}\omega_{12}\dots\omega_{(n-1)n}e^{\delta_n}$$

is independent of the parametrization of λ.

And if $\lambda(0)=y$, then $T(\lambda\lambda^{-1}) = T(\mathbf{1}_y)$; thus,

Proposition 3.6. *For any* 1-*forms* $\delta_1,\dots,\delta_n,\omega_{12},\dots,\omega_{(n-1)n}$,

$$\int_{\lambda\lambda^{-1}} e^{\delta_1}\omega_{12}\dots\omega_{(n-1)n}e^{\delta_n} = \int_{\mathbf{1}_{\lambda(0)}} e^{\delta_1}\omega_{12}\dots\omega_{(n-1)n}e^{\delta_n}.$$

A formula for the integral $\int e^{\delta_1}\omega_{12}\dots\omega_{(n-1)n}e^{\delta_n}$ over λ^{-1} is easily verified from the definition:

Proposition 3.7. *For any* 1-*forms* $\delta_1,\dots,\delta_n,\omega_{12},\dots,\omega_{(n-1)n} \in E^1(M;\mathbb{C})$,

$$\int_{\lambda^{-1}} e^{\delta_1}\omega_{12}\dots\omega_{(n-1)n}e^{\delta_n} = \int_\lambda e^{-\delta_n}(-\omega_{(n-1)n})\dots(-\omega_{12})e^{-\delta_1}.$$

For the remaining propositions let

$$I = \int e^{\delta_1}\omega_{12}\dots\omega_{(n-1)n}e^{\delta_n}.$$

The expression (7) for parallel transport in E can be applied to prove a formula for the integral of I over a concatenation of paths:

Proposition 3.8. *For any paths $\alpha, \beta \in PM$ with $\alpha(1) = \beta(0)$,*

$$\langle I, \alpha\beta\rangle = \sum_{k=1}^{n} \int_\alpha e^{\delta_1}\omega_{12}\ldots\omega_{(k-1)k}e^{\delta_k} \int_\beta e^{\delta_k}\omega_{k(k+1)}\ldots\omega_{(n-1)n}e^{\delta_n}.$$

Proof. The transport map satisfies $T(\alpha\beta) = T(\alpha)T(\beta)$. The above formula comes from comparing the upper-right-hand matrix entries of $T(\alpha\beta)$ and $T(\alpha)T(\beta)$. □

Given $s, t \in [0, 1]$, let λ_s^t denote the subpath of λ from $\lambda(s)$ to $\lambda(t)$, defined by

$$\lambda_s^t(u) = \lambda(s + (t - s)u).$$

The concatenation of λ_0^t and λ_t^1 is equal to λ after reparametrization, thus

Corollary 3.9. *For any $t_0 \in [0, 1]$,*

$$\langle I, \lambda\rangle = \sum_{i=1}^{n} \int_{\lambda_0^{t_0}} e^{\delta_1}\omega_{12}\ldots\omega_{(i-1)i}e^{\delta_i} \int_{\lambda_{t_0}^1} e^{\delta_i}\omega_{i(i+1)}\ldots\omega_{(n-1)n}e^{\delta_n}.$$

A similar proposition (useful in doing induction on these integrals by length) follows immediately from the definition:

Proposition 3.10. *Let $f_{(i-1)i}(t)\,dt = \lambda^*\omega_{(i-1)i}$.*

$$\langle I, \lambda\rangle = \int_0^1 \left(\int_{\lambda_0^t} e^{\delta_1}\omega_{12}\ldots e^{\delta_{i-1}}\right) f_{(i-1)i}(t) \left(\int_{\lambda_t^1} e^{\delta_i}\ldots\omega_{(s-1)s}e^{\delta_s}\right) dt.$$

Note that

$$\begin{aligned}\int_\lambda e^\delta &= \sum_{n\in\mathbb{N}} \int_{0\leqslant t_1\leqslant t_2\leqslant\cdots\leqslant t_n\leqslant 1} f(t_1)f(t_2)\ldots f(t_n)\,dt_1\,dt_2\ldots dt_n \\ &= \sum_{n\in\mathbb{N}} \frac{1}{n!}\int_{t_1,t_2,\ldots,t_n\in[0,1]} f(t_1)f(t_2)\ldots f(t_n)\,dt_1\,dt_2\ldots dt_n \\ &= \sum_{n\in\mathbb{N}} \frac{1}{n!}\left(\int_{0\leqslant t\leqslant 1} f(t)\,dt\right)^n \\ &= e^{\int_\lambda \delta}\end{aligned}$$

(where $f(t)\,dt = \lambda^*\delta$), hence our notation.

Proposition 3.11. *For any $\delta \in E^1(M;\mathbb{C})$, $\lambda \in PM$,*

$$\int_\lambda e^\delta = e^{\int_\lambda \delta}.$$

Using this fact we can simplify integrals that have an exact exponent:

Proposition 3.12. *Suppose* $g: M \to \mathbb{C}$ *is a* C^∞ *function. Then*

$$\int e^{\delta_1}\omega_{12}\ldots\omega_{(i-1)i}e^{dg}\omega_{i(i+1)}\ldots\omega_{(n-1)n}e^{\delta_n}$$
$$= \int e^{\delta_1}\omega_{12}\ldots(e^{-g}\omega_{(i-1)i})(e^{g}\omega_{i(i+1)})\ldots\omega_{(n-1)n}e^{\delta_n}.$$

Proof. Again let $f_{(i-1)i}(t)\,\mathrm{d}t = \lambda^*\omega_{(i-1)i}$, $f_{i(i+1)}(t)\,\mathrm{d}t = \lambda^*\omega_{i(i+1)}$. Applying Proposition 3.10 twice,

$$\int_\lambda e^{\delta_1}\omega_{12}\ldots\omega_{(i-1)i}e^{\mathrm{d}g}\omega_{i(i+1)}\ldots\omega_{(n-1)n}e^{\delta_n}$$
$$= \int_{0\leqslant t\leqslant t'\leqslant 1}\left(\int_{\lambda_0^t} e^{\delta_1}\omega_{12}\ldots e^{\delta_{i-1}}\right) f_{(i-1)i}(t)\,\mathrm{d}t$$
$$\times \left(\int_{\lambda_t^{t'}} e^{\mathrm{d}f}\right) f_{i(i+1)}(t')\,\mathrm{d}t' \left(\int_{\lambda_{t'}^1} e^{\delta_{i+1}}\ldots\omega_{(n-1)n}e^{\delta_n}\right)$$
$$= \int_{0\leqslant t\leqslant t'\leqslant 1}\left(\int_{\lambda_0^t} e^{\delta_1}\omega_{12}\ldots e^{\delta_{i-1}}\right) f_{(i-1)i}(t)\,\mathrm{d}t$$
$$\times\, e^{-g(t)+g(t')} f_{i(i+1)}(t')\,\mathrm{d}t' \left(\int_{\lambda_{t'}^1} e^{\delta_{i+1}}\ldots\omega_{(n-1)n}e^{\delta_n}\right)$$
$$= \int_\lambda e^{\delta_1}\omega_{12}\ldots(e^{-g}\omega_{(i-1)i})(e^{g}\omega_{i(i+1)})\ldots\omega_{(n-1)n}e^{\delta_n},$$

as desired. □

Lastly, we show how the computation (7) of the transport function in terms of exponential iterated integrals can be generalized. Suppose $\nabla = d - \omega$ is a connection on the trivial bundle $\mathbb{C}^n \times M \to M$, where ω is an upper triangular matrix of 1-forms:

$$\omega = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} & \cdots & \omega_{1n} \\ 0 & \omega_{22} & \omega_{23} & \cdots & \omega_{2n} \\ 0 & 0 & \omega_{33} & \cdots & \omega_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega_{nn} \end{bmatrix}.$$

Matrix multiplication shows that the (i, j)th entry of

$$\int \omega^m := \int \underbrace{\omega\omega\ldots\omega}_{m \text{ terms}}$$

is

$$\sum_{i=k_1\leqslant\cdots\leqslant k_{m+1}=j} \int \omega_{k_1k_2}\omega_{k_2k_3}\ldots\omega_{k_mk_{m+1}}.$$

(We take the sum to be zero if $i > j$.) Thus by Theorem 3.2,

$$T = I + \left(\sum_{m>0, i=k_1 \leqslant \cdots \leqslant k_{m+1}=j} \int \omega_{k_1k_2}\omega_{k_2k_3} \ldots \omega_{k_mk_{m+1}} \right)_{1 \leqslant i,j \leqslant n}.$$

Grouping repeated terms,

$$\begin{aligned} T &= \left(\sum_{p>0, i=k_1 < \cdots < k_p = j, r_1, \ldots, r_p \geqslant 0} \int \omega_{k_1k_1}^{r_1}\omega_{k_1k_2}\omega_{k_2k_2}^{r_2}\omega_{k_2k_3} \ldots \omega_{k_{p-1}k_p}\omega_{k_pk_p}^{r_p} \right)_{1 \leqslant i,j \leqslant n} \\ &= \left(\sum_{p>0, i=k_1 < \cdots < k_p = j} \int \mathrm{e}^{\omega_{k_1k_1}}\omega_{k_1k_2}\mathrm{e}^{\omega_{k_2k_2}}\omega_{k_2k_3} \ldots \omega_{k_{p-1}k_p}\mathrm{e}^{\omega_{k_pk_p}} \right)_{1 \leqslant i,j \leqslant n}. \end{aligned}$$

The sums in this last expression are finite, so we have expressed T as a matrix of exponential iterated integrals.

We state this result for future reference:

Proposition 3.13. *Suppose $\nabla = d - \omega$ is a connection on the trivial bundle $\mathbb{C}^n \times M \to M$, where $\omega = (\omega_{ij})_{1 \leqslant i,j \leqslant n}$ is an upper triangular matrix of 1-forms. Then the transport function $T: PM \to \mathbf{GL}_n(\mathbb{C})$ is equal to an upper triangular matrix of exponential iterated integrals whose exponents are from the set $\{\omega_{11}, \omega_{22}, \ldots, \omega_{nn}\}$.*

4. Relative solvable representations

In order to describe the set of maps $\pi_1(M, x) \to \mathbb{C}$ that are given by exponential iterated integrals, we need first to explore the properties of a particular class of algebraic representations associated to a discrete group.

Suppose G is a group and $\rho: G \to (\mathbb{C}^*)^n$ is a diagonal representation. Let $T \subseteq (\mathbb{C}^*)^n$ denote the Zariski closure of $\rho(G)$. By a *solvable representation relative to* ρ we mean an algebraic representation $G \to S$ that fits into a commutative diagram

$$\begin{array}{ccccccccc} & & & & G & & & & \\ & & & & \downarrow & \searrow & & & \\ 1 & \longrightarrow & U & \longrightarrow & S & \longrightarrow & T & \longrightarrow & 1 \end{array} \tag{8}$$

where the bottom row is exact, U is a unipotent group, and the image of G in S is Zariski dense.

The canonical example of a relative solvable representation occurs when G has an upper triangular action on $\mathbb{C}^n$. The quotient of $\mathbf{B}_n(\mathbb{C})$ by $\mathbf{U}_n(\mathbb{C})$ is isomorphic to $\mathbf{D}_n(\mathbb{C})$, hence any representation

$G \to \mathbf{B}_n(\mathbb{C})$ fits into a diagram

$$\begin{array}{ccccccccc} & & & & G & & & & \\ & & & & \downarrow & \searrow & & & \\ 1 & \longrightarrow & \mathbf{U}_n(\mathbb{C}) & \longrightarrow & \mathbf{B}_n(\mathbb{C}) & \longrightarrow & \mathbf{D}_n(\mathbb{C}) & \longrightarrow & 1. \end{array}$$

The Zariski closure of G in $\mathbf{B}_n(\mathbb{C})$ is a relative solvable representation. In fact every relative solvable representation can be obtained in this manner, as the next proposition and corollary show.

Proposition 4.1. *Suppose* $1 \to U \to S \to T \to 1$ *is an exact sequence of algebraic groups with U unipotent and T diagonalizable. Then any linear representation* $S \to \boldsymbol{GL}(V)$ *has a common eigenvector*.

Proof. Suppose $S \to \mathbf{GL}(V)$ is a representation. Since U is unipotent we may find a nonzero vector $v \in V$ fixed by U. Let $W \subseteq V$ be the subspace spanned by $\{xv \mid x \in S\}$. Note that U fixes W, since for any $x \in S$, $y \in U$,

$$yxv = x(x^{-1}yx)v = xv.$$

So the action of S on W factors through T; and since T is diagonalizable it has a common eigenvector in W. □

Corollary 4.2. *Suppose*

$$\begin{array}{ccccccccc} & & & & G & & & & \\ & & & & \downarrow & \searrow & & & \\ 1 & \longrightarrow & U & \longrightarrow & S & \longrightarrow & T & \longrightarrow & 1 \end{array} \qquad (9)$$

is a relative solvable representation. Then there exists an embedding $S \hookrightarrow \boldsymbol{B}_n(\mathbb{C})$ *for some n, under which* $S \cap \boldsymbol{U}_n(\mathbb{C}) = U$.

Proof. Let $S \hookrightarrow \mathbf{GL}(V)$ be a faithful representation. Proceeding by induction with the proposition we may find a basis $\{v_1, \ldots, v_n\} \subseteq V$ that makes the action of S upper triangular. Thus we obtain the embedding $S \hookrightarrow \mathbf{B}_n(\mathbb{C})$. The intersection of S with $\mathbf{U}_n(\mathbb{C})$ is the unipotent radical of S, which is precisely U. □

We note in passing that the discussion above also explains the term "relative solvable." Since every connected solvable group is isomorphic to a closed subgroup of $\mathbf{B}_n(\mathbb{C})$ (see [6, Section 19]), every connected solvable representation with Zariski dense image may be expressed as a relative solvable representation. (The converse is not true, since a relative solvable representation need not be connected.)

By analogy we use the term *relative prosolvable representation* to mean a commutative diagram of the form of (8) with U prounipotent and T diagonalizable.

Proposition 4.3. *Suppose* $1 \to \mathscr{U} \to \mathscr{S} \to T \to 1$ *is an exact sequence with* $\mathscr{U}$ *prounipotent and T diagonalizable. Then the sequence splits, and any two splittings are conjugate by an element of* $\mathscr{U}$.

Proof. The assertion is well-known when $\mathscr{U}$ is unipotent (see [1, Proposition 5.1]). Choose compatible splittings for the unipotent quotients of $\mathscr{U}$ and take the inverse limit. □

Suppose $\rho: G \to T \subseteq (\mathbb{C}^*)^n$ is a diagonalizable representation as before. There is a unique relative prosolvable representation $G \to \mathscr{S}_\rho(G) \to T$ satisfying the following universal mapping property: if $\mathscr{S}$ is any prosolvable representation relative to ρ, then there is a homomorphism $\mathscr{S}_\rho(G) \to \mathscr{S}$ of G-representations that makes the following diagram commute:

$$\begin{array}{ccc} \mathscr{S}_\rho(G) & & \\ \downarrow & \searrow & \\ \mathscr{S} & \longrightarrow & T. \end{array}$$

$\mathscr{S}_\rho(G)$ is the inverse limit of all solvable representations relative to ρ. This is a special case of *relative Malcev completion*, a notion of Deligne's; see [4], Section 2 for a full development.

As an initial example of relative prosolvable completion, suppose G is finitely generated and abelian. Then $\mathscr{S}_\rho(G)$ must be abelian, so the splitting of Proposition 4.3 is a direct product: $\mathscr{S}_\rho(G) \cong \mathscr{U} \times T$ where $\mathscr{U}$ is prounipotent. $\mathscr{U}$ is abelian and therefore additive, and its dimension cannot exceed the free rank of G. This proves the next proposition.

Proposition 4.4. *If $\rho: G \to T$ is a homomorphism from a finitely generated abelian group into a diagonalizable group with Zariski dense image, then $\mathscr{S}_\rho(G) \cong \mathbb{G}_a^m \times T$ where m is the free rank of G.*

5. Algebraic properties of exponential iterated integrals

Suppose $L \subset E^1(M;\mathbb{C})$ is a $\mathbb{Z}$-module of 1-forms. Let $EB(M)^L$ denote the vector space of exponential iterated integrals with exponents from L. A closed exponential iterated integral is one that is constant on homotopy classes of paths $\lambda: [0,1] \to M$ relative to $\{0,1\}$. Let $H^0(EB(M)^L)$ denote the subspace of closed iterated integrals in $EB(M)^L$. In this section we show that $EB(M)^L$ and $H^0(EB(M)^L)$ are Hopf algebras, and that the groups Spec $EB(M)^L$ and Spec $H^0(EB(M)^L)$ are each an extension of a prodiagonalizable group by a prounipotent group.

We also let $EB(M,x)^L$ denote the space of functions on the loop space $P_{x,x}M$ given by exponential iterated integrals with exponents from L. ($EB(M,x)^L$ is the quotient of $EB(M)^L$ by integrals such as $\int \mathrm{e}^{\mathrm{d}f}$ that vanish on every loop at x. We will also refer to the elements of $EB(M,x)^L$ as exponential iterated integrals.) Write $H^0(EB(M,x)^L)$ for the subspace of functions constant on homotopy classes in $P_{x,x}M$. For convenience we will omit basepoints in the rest of this section, but everything we say for $EB(M)^L$ and $H^0(EB(M)^L)$ applies as well to $EB(M,x)^L$ and $H^0(EB(M,x)^L)$.

Lemma 5.1. *For any $\mathbb{Z}$-module $L \subseteq E^1(M;\mathbb{C})$, $EB(M)^L$ is closed under (pointwise) multiplication.*

Proof. It is sufficient to show that for any $\delta_i, \delta_i' \in L$, $\omega_{j(j+1)}, \omega'_{j(j+1)} \in E^1(M;\mathbb{C})$,

$$\int \mathrm{e}^{\delta_1}\omega_{12}\ldots\omega_{(n-1)n}\mathrm{e}^{\delta_n} \int \mathrm{e}^{\delta_1'}\omega_{12}'\ldots\omega'_{(n'-1)s'}\mathrm{e}^{\delta'_{n'}} \in EB(M)^L.$$

A formula for this product can be written down, but it is rather cumbersome and unnecessary for our purposes. We prove the result by induction on $n+n'$. Let $\lambda\colon [0,1] \to M$ be a path.

Note first that

$$\int_\lambda \mathrm{e}^{\delta} \int_\lambda \mathrm{e}^{\delta'} = \mathrm{e}^{\int_\lambda \delta} \mathrm{e}^{\int_\lambda \delta'} = \mathrm{e}^{\int_\lambda \delta+\delta'} = \int_\lambda \mathrm{e}^{\delta+\delta'}.$$

This proves the base case. For $n+n' \geqslant 1$ we apply Corollary 3.9 and Proposition 3.10 to split the product into a sum of products of smaller-length integrals. Assume without loss of generality that $n \geqslant 1$. Let $f_{(n-1)n}(t)\,\mathrm{d}t = \lambda^* \omega_{(n-1)n}$.

$$\begin{aligned}
&\int_\lambda \mathrm{e}^{\delta_1} \omega_{12} \ldots \omega_{(n-1)n} \mathrm{e}^{\delta_n} \int_\lambda \mathrm{e}^{\delta'_1} \omega'_{12} \ldots \omega'_{(n'-1)n'} \mathrm{e}^{\delta'_{n'}} \\
&\quad = \int_{0 \leqslant t \leqslant 1} \left(\int_{\lambda_0^t} \mathrm{e}^{\delta_1} \omega_{12} \ldots \mathrm{e}^{\delta_{n-1}} \right) f_{(n-1)n}(t)\,\mathrm{d}t \left(\int_{\lambda_t^1} \mathrm{e}^{\delta_n} \right) \\
&\qquad \times \left(\sum_{i=1}^{n'} \int_{\lambda_0^t} \mathrm{e}^{\delta'_1} \omega'_{12} \ldots \mathrm{e}^{\delta'_i} \int_{\lambda_t^1} \mathrm{e}^{\delta'_i} \omega'_{i(i+1)} \ldots \mathrm{e}^{\delta'_{n'}} \right) \\
&\quad = \sum_{i=1}^{n'} \int_{0 \leqslant t \leqslant 1} \left(\int_{\lambda_0^t} \mathrm{e}^{\delta_1} \omega_{12} \ldots \mathrm{e}^{\delta_{n-1}} \right) \left(\int_{\lambda_0^t} \mathrm{e}^{\delta'_1} \omega'_{12} \ldots \mathrm{e}^{\delta'_i} \right) \\
&\qquad \times f_{(n-1)n}(t)\,\mathrm{d}t \left(\int_{\lambda_t^1} \mathrm{e}^{\delta_n} \right) \left(\int_{\lambda_t^1} \mathrm{e}^{\delta'_i} \omega'_{i(i+1)} \ldots \mathrm{e}^{\delta'_{n'}} \right)
\end{aligned}$$

By inductive assumption both of

$$\int \mathrm{e}^{\delta_1} \omega_{12} \ldots \mathrm{e}^{\delta_{n-1}} \int \mathrm{e}^{\delta'_1} \omega'_{12} \ldots \mathrm{e}^{\delta'_i} \quad \text{and} \quad \int \mathrm{e}^{\delta_n} \int \mathrm{e}^{\delta'_i} \omega'_{i(i+1)} \ldots \mathrm{e}^{\delta'_{n'}}$$

may be expressed as exponential iterated integrals from $EB(M)^L$. Applying Proposition 3.10 then transforms each summand in the resulting expression into an exponential iterated integral. □

Therefore $EB(M)^L$ is a $\mathbb{C}$-algebra. (A homomorphism $\mathbb{C} \to EB(M)^L$ is given by $z \mapsto z \int \mathrm{e}^0$.) Clearly the product of two closed exponential iterated integrals is closed, thus $H^0(EB(M)^L)$ is likewise a $\mathbb{C}$-algebra.

The definition for comultiplication $\Delta\colon EB(M)^L \to EB(M)^L \otimes EB(M)^L$ comes from Proposition 3.8:

$$\begin{aligned}
&\Delta \int \mathrm{e}^{\delta_1} \omega_{12} \ldots \omega_{(n-1)n} \mathrm{e}^{\delta_n} \\
&\quad := \sum_{i=1}^{n} \int \mathrm{e}^{\delta_1} \omega_{12} \ldots \omega_{(i-1)i} \mathrm{e}^{\delta_i} \otimes \int \mathrm{e}^{\delta_i} \omega_{i(i+1)} \ldots \omega_{(n-1)n} \mathrm{e}^{\delta_n}. \qquad (10)
\end{aligned}$$

For $I \in EB(M)^L$ and $\alpha, \beta, \gamma \in PM$, we have

$$\langle \Delta I, (\alpha, \beta) \rangle = \langle I, \alpha\beta \rangle.$$

The coassociative property of Δ follows:

$$\langle(\Delta\otimes 1)\Delta I, (\alpha,\beta,\gamma)\rangle = \langle I, \alpha\beta\gamma\rangle = \langle(1\otimes\Delta)\Delta I, (\alpha,\beta,\gamma)\rangle.$$

And if I is constant on homotopy classes in PM, the function

$$(\alpha,\beta)\mapsto\langle\Delta I, (\alpha,\beta)\rangle = \langle I, \alpha\beta\rangle$$

is constant on homotopy classes in $PM\times PM$; thus Δ restricts to a comultiplication

$$\Delta\colon H^0(EB(M)^L)\to H^0(EB(M)^L)\otimes H^0(EB(M)^L).$$

The constant map on $EB(M)^L$ and $H^0(EB(M)^L)$ is given by $I\mapsto\langle I, \mathbf{1}_x\rangle$. (Note that $\langle I, \mathbf{1}_y\rangle = \langle I, \mathbf{1}_x\rangle$ for any $y\in M$.) Proposition 3.7 gives an antipode map on $EB(M)^L$:

$$i\colon \int \mathrm{e}^{\delta_1}\omega_{12}\ldots\omega_{(n-1)n}\mathrm{e}^{\delta_n} = (-1)^{n-1}\int \mathrm{e}^{-\delta_n}\omega_{(n-1)n}\ldots\omega_{12}\mathrm{e}^{-\delta_1}.$$

By Proposition 3.6,

$$\langle(1\otimes i)\Delta I, \lambda\rangle = \langle I, \lambda\lambda^{-1}\rangle = \langle I, \mathbf{1}_x\rangle.$$

And as with Δ, i restricts to an antipode map on $H^0(EB(M)^L)$. We have thus proven:

Proposition 5.2. *For any $\mathbb{Z}$-module of 1-forms $L\subseteq E^1(M;\mathbb{C})$,*

$$(EB(M)^L, \Delta, i) \quad\text{and}\quad (H^0(EB(M)^L), \Delta, i)$$

are Hopf algebras.

To demonstrate the structure of the group Spec $EB(M)^L$ we need a few definitions. Let $\mathscr{E}(M)^L\subseteq EB(M)^L$ denote the $\mathbb{C}$-algebra generated by integrals of the form $\int \mathrm{e}^\delta$ with $\delta\in L$. Since $\Delta\int \mathrm{e}^\delta = \int \mathrm{e}^\delta\otimes\int \mathrm{e}^\delta$, $\mathscr{E}(M)^L$ is the coordinate ring of a prodiagonalizable group. The inclusion $i\colon\mathscr{E}(M)^L\to EB(M)^L$ gives a surjective homomorphism

$$i^*\colon \mathrm{Spec}\, EB(M)^L\to\mathrm{Spec}\,\mathscr{E}(M)^L.$$

We show that the kernel of this homomorphism is a prounipotent group. The coordinate ring of the kernel is $EB(M)^L/I_0$, where I_0 is the ideal generated by integrals of the form $\int \mathrm{e}^\delta - 1$, $\delta\in L$. Consider the filtration of ideals

$$I_0\subseteq I_1\subseteq I_2\subseteq\cdots,$$

where

$$I_j = \langle I - \langle I, \mathbf{1}_x\rangle \mid I\in EB(M)^L \text{ of length } \leqslant j\rangle$$

This filtration corresponds to a filtration of subgroups in Spec $EB(M)^L/I_0$. For $j\geqslant 1$, the action of Δ on I_j/I_{j-1} is given by

$$\Delta\left(\int \mathrm{e}^{\delta_1}\omega_{12}\ldots\mathrm{e}^{\delta_{j+1}} + I_{j-1}\right)$$
$$= \int \mathrm{e}^{\delta_1}\omega_{12}\ldots\mathrm{e}^{\delta_{j+1}}\otimes 1 + 1\otimes\int \mathrm{e}^{\delta_1}\omega_{12}\ldots\mathrm{e}^{\delta_{j+1}} + (I_{j-1}\otimes I_j + I_j\otimes I_{j-1}).$$

Thus $EB(M)^L/I_0$ is indeed a prounipotent group. We have constructed an exact sequence of groups

$$1 \to \operatorname{Spec} EB(M)^L/I_0 \to \operatorname{Spec} EB(M)^L \to \operatorname{Spec} \mathscr{E}(M)^L \to 1,$$

where $\operatorname{Spec} EB(M)^L/I_0$ is prounipotent and $\operatorname{Spec} \mathscr{E}(M)^L$ is prodiagonalizable.

Now if we assume that L consists of closed 1-forms, the algebra $\mathscr{E}(M)^L$ is contained in $H^0(EB(M)^L)$, and we obtain a similar exact sequence for $\operatorname{Spec} H^0(EB(M)^L)$. Let $I_0' = I_0 \cap H^0(EB(M)^L)$. The sequence

$$1 \to \operatorname{Spec} H^0(EB(M)^L)/I_0' \to \operatorname{Spec} H^0(EB(M)^L) \to \operatorname{Spec} \mathscr{E}(M)^L \to 1$$

is a quotient of the one above by a prounipotent subgroup.

We summarize this discussion.

Theorem 5.3. *For any $\mathbb{Z}$-module $L \subseteq E^1(M; \mathbb{C})$, the group* $\operatorname{Spec} \mathscr{E}(M)^L$ *is prodiagonalizable, and there is an epimorphism*

$$\operatorname{Spec} EB(M)^L \to \operatorname{Spec} \mathscr{E}(M)^L$$

whose kernel is prounipotent. If L consists of closed 1*-forms, there is an epimorphism*

$$\operatorname{Spec} H^0(EB(M)^L) \to \operatorname{Spec} \mathscr{E}(M)^L$$

whose kernel is prounipotent.

6. The solvable de Rham theorem

Suppose that L is a $\mathbb{Z}$-module of closed 1-forms. Continuing the notation of Section 5, let $\mathscr{E}(M,x)^L \subseteq H^0(EB(M,x)^L)$ denote the subalgebra generated by the integrals $\int \mathrm{e}^{\delta}$, $\delta \in L$. Consider the homomorphism

$$\rho\colon \pi_1(M,x) \to \operatorname{Spec} \mathscr{E}(M,x)^L,$$

where $[\lambda]$ maps to the ideal of integrals that vanish on λ. This representation is prodiagonalizable and has Zariski dense image. If there is a finite set $\{\delta_1, \ldots, \delta_n\} \subseteq L$ that spans the image of L in $H^1(M; \mathbb{C})$ then ρ is an algebraic representation, and it may be expressed as $\rho\colon \pi_1(M,x) \to T \subseteq (\mathbb{C}^*)^n$,

$$\rho(\lambda) = \left(\int_\lambda \mathrm{e}^{\delta_1}, \ldots, \int_\lambda \mathrm{e}^{\delta_n} \right).$$

We shall call ρ "the representation defined by L."

As noted in the introduction, Chen proved that the Hopf algebra of closed iterated integrals on $P_{x,x}M$ is the coordinate ring of the unipotent completion of $\pi_1(M,x)$. Previous work has been done on extending this isomorphism: Hain in [5] constructed a class of integrals that compute the coordinate ring of the Malcev completion of $\pi_1(M,x)$ relative to any algebraic representation $\rho\colon \pi_1(M,x) \to S$. These integrals are written in the form

$$\int (\omega_1 \omega_2 \ldots \omega_r \mid \phi),$$

where the ω_i are 1-forms on a principal S-bundle over M, and ϕ is a matrix entry of S.

The solvable de Rham theorem shows that the coordinate ring of the Malcev completion may be computed using the more geometric (and more manageable) class of exponential iterated integrals, in the case where ρ is a diagonalizable representation defined by some $\mathbb{Z}$-module $L \subseteq E^1(M;\mathbb{C})$.

Theorem 6.1 (*The π_1 solvable de Rham theorem*). *Suppose L is a $\mathbb{Z}$-module of closed* 1*-forms whose image in $H^1(M;\mathbb{C})$ is finitely generated, and that $\rho\colon \pi_1(M,x) \to T \subseteq (\mathbb{C}^*)^n$ is the representation defined by L. Then integration induces a Hopf algebra isomorphism*

$$H^0(EB(M,x)^L) \cong \mathcal{O}(\mathcal{S}_\rho(\pi_1(M,x))).$$

Proof. The representation

$$\pi_1(M,x) \to \operatorname{Spec} H^0(EB(M,x)^L)$$

has Zariski dense image, and by Theorem 5.3 it is prosolvable relative to the diagonal representation ρ. Thus by the universal mapping property for $\mathcal{S}_\rho(\pi_1(M,x))$ (see Section 4) there exists an injection

$$H^0(EB(M,x)^L) \hookrightarrow \mathcal{O}(\mathcal{S}_\rho(\pi_1(M,x))).$$

To show that this map is surjective it will suffice to show that exponential iterated integrals compute the coordinate ring of any solvable representation of $\pi_1(M,x)$ relative to ρ. Our method is similar to that in Hain's proof of the π_1 de Rham theorem for ordinary iterated integrals in [3]. Henceforth let $\pi=\pi_1(M,x)$. Suppose $\psi\colon \pi \to S$ is a solvable representation relative to ρ. By Corollary 4.2, we may assume that S is a group of upper triangular $n \times n$ matrices whose diagonal entries are from $\mathcal{O}(T)$, meaning that they may be written as $\int \mathrm{e}^{\delta_k}$ with $\delta_k \in L$, $k=1,\dots,n$.

Let $\mathbb{C}^n = V^n \supset V^{n-1} \supset \cdots \supset V^0 = \{0\}$ denote the standard filtration,

$$V^k = \{(z_1,\dots,z_k,0,\dots,0) \mid z_1,\dots,z_k \in \mathbb{C}\},$$

which is stabilized by S.

Let $\widetilde{M}$ denote the universal cover of M. From the trivial flat bundle $\mathbb{C}^n \times \widetilde{M}$ we can obtain a bundle over M with monodromy ψ: let

$$E = \pi\backslash(\mathbb{C}^n \times \widetilde{M})$$

where π acts via $g \cdot (v,m) \mapsto (\psi(g)(v), mg^{-1})$. The filtration $V^n \supseteq V^{n-1} \supseteq \cdots \supseteq V^0$ induces a filtration of bundles

$$E = E^n \supseteq E^{n-1} \supseteq \cdots \supseteq E^0 = 0,$$

where the monodromy of the line bundle E^k/E^{k-1} is given by $\int \mathrm{e}^{\delta_k}$. We obtain for each k a trivialization

$$\mathbb{C} \times M \to E^k/E^{k-1}$$

via transport from x with respect to the connection $d-\delta_k$. Composing these maps with splittings $E^k/E^{k-1} \to E^k$, yields maps $\mathbb{C} \times M \to E^k$ for $k=1,\dots,n$. Adding these maps together we obtain an isomorphism

$$\mathbb{C}^n \times M \to E.$$

The induced connection form on $\mathbb{C}^n \times M$ is an upper triangular matrix with diagonal entries $\delta_1, \ldots, \delta_n$. By Proposition 3.13, the monodromy representation $\rho: \pi \to S$ is equal to a matrix of exponential iterated integrals with exponents from $\{\delta_1, \ldots, \delta_n\} \subseteq L$. These matrix entries generate the ring $\mathcal{O}(S)$ and this completes the proof. $\square$

It is natural now to ask which diagonalizable representations of $\pi_1(M, x)$ are defined by a module of closed 1-forms.

Proposition 6.2. *If $\rho: \pi_1(M, x) \to T \subseteq (\mathbb{C}^*)^n$ is a diagonalizable representation, then there exists a defining $\mathbb{Z}$-module $L \subseteq E^1(M; \mathbb{C})$ for ρ if and only if the induced map $H_1(M) \to T$ is trivial on* $\mathrm{Tor}(H_1(M))$.

Proof. If δ is a closed 1-form then the additive homomorphism

$$\int \delta: H^1(M) \to \mathbb{C}$$

kills $\mathrm{Tor}\, H^1(M)$, and the same is true for $\int \mathrm{e}^{\delta}: H^1(M) \to \mathbb{C}^*$. So any diagonalizable representation defined by 1-forms kills $\mathrm{Tor}\, H^1(M)$.

For the converse, suppose $\rho: \pi \to \mathbb{C}^*$ is a homomorphism that kills $\mathrm{Tor}\, H^1(M)$. Then ρ induces a map $\rho': H_1(M; \mathbb{C}) \to \mathbb{C}^*$. Choose a basis $\{z_i\}_i$ for $H_1(M; \mathbb{C})$, and choose elements $f_i \in \mathbb{C}$ such that $\mathrm{e}^{f_i} = \rho'(z_i)$. By the ordinary de Rham theorem we may find a closed 1-form δ such that $\int \delta$ takes z_i to f_i. Thus $\rho \cong \int \mathrm{e}^{\delta}$. This method extends easily to define arbitrary diagonalizable representations in terms of 1-forms. $\square$

Combining this result with the de Rham theorem gives a description for

$$H^0(EB(M, x)^{B^1(M;\mathbb{C})}), \tag{11}$$

the ring of all closed exponential iterated integrals with closed exponents. If $\pi \to \mathbf{G}$ is an algebraic representation with Zariski dense image, then $\mathcal{O}(\mathbf{G})$ is computed by exponential iterated integrals with closed exponents if and only if the reductive quotient $\mathbf{G}/\mathbf{G}_u$ is a diagonalizable representation that kills $\mathrm{Tor}\, H^1(M)$. The ring (11) is the direct limit of $\mathcal{O}(\mathbf{G})$ over all such representations.

7. The unipotent radical of the relative solvable completion

Recall that $H \to \mathcal{U}(H)$ denotes the unipotent completion of the group H, or equivalently, the solvable completion of H relative to the trivial representation $H \to \{1\}$. The functor $\mathcal{S}_\rho$ is right exact in the sense that

$$\mathcal{U}(\ker \rho) \to \mathcal{S}_\rho(G) \to \mathcal{S}_\rho(\mathrm{im}\, \rho) \to 1$$

is exact for any group G and diagonalizable representation $\rho: G \to T$. This is easily seen from the universal mapping property.

The main theorem of this section asserts conditions under which $\mathcal{U}(\ker \rho) \to \mathcal{S}_\rho(G)$ is an injection. The proof will require an understanding of how G acts by conjugation on the solvable completions of its subgroups. We begin with a discussion of the automorphism groups of prounipotent and relative prosolvable groups.

We will make free use of the equivalence of categories,

$$\{\text{prounipotent algebraic groups}/\mathbb{C}\} \longleftrightarrow \{\text{pronilpotent Lie algebras}/\mathbb{C}\}$$

provided by the **exp** and **log** maps.

Suppose $\mathscr{U}$ is a prounipotent group, and $\mathscr{U} = \mathscr{U}_0 \supseteq \mathscr{U}_1 \supseteq \mathscr{U}_2 \supseteq \cdots$ is a filtration by closed normal subgroups such that $\bigcap_{i=0}^{\infty} \mathscr{U}_i = \{1\}$. Let $\mathrm{Aut}_{\{\mathscr{U}_i\}}\mathscr{U}$ denote the group of automorphisms of $\mathscr{U}$ that stabilize $\{\mathscr{U}_i\}$.[4] Consider the graded vector space obtained from the Lie algebra $\mathfrak{u}$ of $\mathscr{U}$ with the induced filtration $\{\mathfrak{u}_i\}$:

$$\mathbf{gr}_{\{\mathfrak{u}_i\}}\,\mathfrak{u} = \mathfrak{u}/\mathfrak{u}_1 \oplus \mathfrak{u}_1/\mathfrak{u}_2 \oplus \mathfrak{u}_2/\mathfrak{u}_3 \oplus \cdots$$

The kernel of the morphism

$$\phi\colon \mathrm{Aut}_{\{\mathfrak{u}_i\}}\mathfrak{u} \to \mathrm{Aut}\,\mathbf{gr}_{\{\mathfrak{u}_i\}}\mathfrak{u} \tag{12}$$

is prounipotent, and the same is true of the equivalent morphism

$$\Phi\colon \mathrm{Aut}_{\{\mathscr{U}_i\}}\,\mathscr{U} \to \mathrm{Aut}\,\mathbf{gr}_{\{\mathfrak{u}_i\}}\,\mathfrak{u}\,. \tag{13}$$

If the chosen filtration is the central series $\{\mathscr{U}^{(1)}, \mathscr{U}^{(2)}, \ldots\}$ of $\mathscr{U}$, $\mathbf{gr}_{\{\mathfrak{u}^{(i)}\}}\mathfrak{u}$ has the additional structure of a graded Lie algebra generated by $\mathfrak{u}/\mathfrak{u}^{(2)}$. So an automorphism of $\mathbf{gr}_{\{\mathfrak{u}^{(i)}\}}\mathfrak{u}$ is determined by its action on $\mathfrak{u}/\mathfrak{u}^{(2)}$:

$$\Phi\colon \mathrm{Aut}_{\{\mathscr{U}^{(i)}\}}\,\mathscr{U} \to \mathrm{Aut}\,\mathbf{gr}_{\{\mathfrak{u}^{(i)}\}}\mathfrak{u} \hookrightarrow \mathbf{GL}(\mathfrak{u}/\mathfrak{u}^{(2)}). \tag{14}$$

Now suppose that

$$1 \longrightarrow \mathscr{U} \longrightarrow \mathscr{S} \xrightarrow{\rho} T \longrightarrow 1,$$

is an exact sequence where T is diagonalizable. Let $\mathrm{Aut}_{\rho,\{\mathscr{U}_i\}}\mathscr{S}$ denote the group of automorphisms of $\mathscr{S}$ that are ρ-invariant and stabilize $\{\mathscr{U}_i\}$. We show that the kernel of the morphism

$$\mathrm{Aut}_{\rho,\{\mathscr{U}_i\}}\,\mathscr{S} \to \mathrm{Aut}\,\mathbf{gr}_{\{\mathfrak{u}_i\}}\mathfrak{u}, \tag{15}$$

like $\ker\,\Phi$ above (13), is prounipotent. Fix (by Proposition 4.3) a diagonalizable subgroup $T_0 \subseteq \mathscr{S}$ that maps isomorphically onto T. Let $\ker\,\Phi$ act on $\mathscr{S} = T_0 \ltimes \mathscr{U}$ leaving T_0 fixed, and let $\mathscr{U}$ act on $\mathscr{S}$ by conjugation. These actions induce a morphism,

$$\begin{aligned} &\ker\,\Phi \ltimes \mathscr{U} \to \mathrm{Aut}_{\rho,\{\mathscr{U}_i\}}\mathscr{S}, \\ &\quad (\psi, u) \mapsto \psi(u(\cdot)u^{-1}). \end{aligned}$$

We claim that the image of this morphism is exactly the kernel of (15). Suppose that $\psi \in \mathrm{Aut}_{\rho,\{\mathscr{U}_i\}}\,\mathscr{S}$ is an automorphism that fixes $\mathbf{gr}_{\{\mathfrak{u}_i\}}\mathfrak{u}$. Since $\psi(T_0) \subseteq \mathscr{S}$ is another closed subgroup that maps isomorphically onto T, we may find $u \in \mathscr{U}$ such that $u\psi(T_0)u^{-1} = T_0$. Since $u\psi(\cdot)u^{-1}$ is ρ-invariant, it therefore fixes T_0;

[4] To be precise, $\mathrm{Aut}_{\{\mathscr{U}_i\}}\mathscr{U}$ is the functor that takes a $\mathbb{C}$-algebra A to the group of automorphisms of $\mathscr{U} \times_{\mathbb{C}} \mathrm{Spec}\,A$ that preserve $\{\mathscr{U}_i \times_{\mathbb{C}} \mathrm{Spec}\,A\}$.

and since it also fixes $\mathbf{gr}_{\{\sqcap_i\}}\sqcap$, we may find it in the image of ker Φ. This proves the claim. And indeed ker $\Phi \ltimes \mathscr{U}$ is prounipotent since both ker Φ and $\mathscr{U}$ are.

We summarize this discussion.

Lemma 7.1. *Suppose that*

$$1 \longrightarrow \mathscr{U} \longrightarrow \mathscr{S} \xrightarrow{\rho} T \longrightarrow 1,$$

is an exact sequence of groups in which T is diagonalizable and $\mathscr{U}$ is prounipotent. Suppose that $\mathscr{U} = \mathscr{U}_0 \supseteq \mathscr{U}_1 \supseteq \mathscr{U}_2 \supseteq \cdots$ is a filtration by closed normal subgroups such that $\bigcap_{i=0}^{\infty} \mathscr{U}_i = \{1\}$. Then the kernel of the morphism

$$\mathrm{Aut}_{\rho,\{\mathscr{U}_i\}}\, \mathscr{S} \rightarrow \mathrm{Aut}\,\mathbf{gr}_{\{\sqcap_i\}}\sqcap$$

is prounipotent.

We are now ready to state the main theorem. Suppose that $1 \to K \to G \to A \to 1$ is an exact sequence of groups with A abelian and finitely generated. Note that on the vector space $H_1(K;\mathbb{C})$ (or equivalently, $\mathscr{U}(K)/[\mathscr{U}(K),\mathscr{U}(K)]$) there is an induced conjugation action $G \to \mathrm{Aut}\, H_1(K;\mathbb{C})$, which is abelian. We are interested in the case when this action is algebraic, and thus may be written as

$$G \to T \times (\mathbb{G}_a)^m \to \mathrm{Aut}\, H_1(K;\mathbb{C}),$$

where T is a diagonalizable group.

Theorem 7.2. *Suppose* $1 \to K \to G \to A \to 1$ *is an exact sequence of groups with A abelian and finitely generated, and suppose that the conjugation action* $G \to \mathrm{Aut}\, H_1(K;\mathbb{C})$ *factors as*

$$G \to T \times (\mathbb{G}_a)^m \to \mathrm{Aut}\, H_1(K;\mathbb{C}),$$

where $\rho: G \to T$ is a diagonalizable representation with Zariski dense image. Then

$$1 \to \mathscr{U}(K) \to \mathscr{S}_\rho(G) \to \mathscr{S}_\rho(\mathrm{im}\,\rho) \to 1 \tag{16}$$

is an exact sequence.

Proof. Let

$$A = \mathbb{Z}/(n_1) \oplus \mathbb{Z}/(n_2) \oplus \cdots \oplus \mathbb{Z}/(n_r),$$

with $n_i \in \mathbb{Z}$. We induct on r. Let $G' \subseteq G$ be the inverse image of $\mathbb{Z}/(n_1) \oplus \cdots \oplus \mathbb{Z}/(n_{r-1})$, and assume that $\mathscr{U}(K) \hookrightarrow \mathscr{S}_\rho(G')$.

Let $\overline{G}$ denote the pushout of the diagram

$$\begin{array}{ccc} G' & \longrightarrow & G\,. \\ \big\downarrow & & \\ \mathscr{S}_\rho(G') & & \end{array}$$

Then $\overline{G}$ fits into an exact sequence

$$1 \to \mathscr{S}_\rho(G') \to \overline{G} \to \mathbb{Z}/(n_r) \to 0. \tag{17}$$

For convenience, we consider $\mathscr{U}(K)$ and $\mathscr{S}_\rho(G')$ as subgroups of $\overline{G}$.

We desire a splitting of the sequence (17). If $n_r = 0$ this is straightforward. Otherwise, choose an element of G that maps to $(0, 0, \ldots, 0, 1) \in A$, and let g denote its image in $\overline{G}$. Then g^{n_r} is contained in the prounipotent group $\mathscr{U}(K)$, so it has a unique n_rth root u in $\mathscr{U}(K)$, which commutes with g. Replacing g with $u^{-1}g$ we have $g^{n_r} = 1$, hence

$$\overline{G} = \langle g \rangle \ltimes \mathscr{S}_\rho(G').$$

Lemma 7.1will show that the conjugation action of g on $\mathscr{S}_\rho(G')$ is algebraic. Filter the prounipotent kernel of $\mathscr{S}_\rho(G') \to T$ via the central series of $\mathscr{U}(K)$:

$$\begin{aligned}
&\mathscr{U}_0 = \ker[\mathscr{S}_\rho(G') \to T]\\
&\mathscr{U}_1 = \mathscr{U}(K)\\
&\mathscr{U}_2 = [\mathscr{U}_1, \mathscr{U}_1]\\
&\mathscr{U}_3 = [\mathscr{U}_1, \mathscr{U}_2]\\
&\mathscr{U}_4 = [\mathscr{U}_1, \mathscr{U}_3]\\
&\quad \ldots
\end{aligned}$$

This filtration is evidently stabilized by g. The action of g on $\sqcap_0/\sqcap_1$ is trivial, and the action of g on the graded Lie algebra $\mathbf{gr}_{\{\sqcap_1,\sqcap_2,\ldots\}}\sqcap$ is isomorphic to the action of g on $\sqcap_1/\sqcap_2 \cong H_1(K; \mathbb{C})$ (recall the discussion prior to (14)), hence the commutative diagram

$$\begin{array}{ccccc}
 & & & & \operatorname{Aut}_{\rho,\{\mathscr{U}_i\}} \mathscr{S}_\rho(G') \\
 & \nearrow & & & \downarrow \\
\langle g \rangle & \longrightarrow & T \times (\mathbb{G}_a)^m & \longrightarrow & \operatorname{Aut} \mathbf{gr}_{\{\sqcap_i\}}\sqcap.
\end{array}$$

The horizontal map $\langle g \rangle \to \operatorname{Aut} \mathbf{gr}_{\{\sqcap_i\}}\sqcap$ factors through $\mathscr{S}_\rho(\langle g \rangle)$, and the downward map has prounipotent kernel (Lemma 7.1), therefore the diagonal map also factors through $\mathscr{S}_\rho(\langle g \rangle)$ by the universal mapping property. Thus we can "thicken" $\overline{G}$ further:

$$\overline{G} = \langle g \rangle \ltimes \mathscr{S}_\rho(G') \hookrightarrow \mathscr{S}_\rho(\langle g \rangle) \ltimes \mathscr{S}_\rho(G').$$

This completes the proof, as $\mathscr{S}_\rho(\langle g \rangle) \ltimes \mathscr{S}_\rho(G')$ contains $\mathscr{U}(K)$ and is isomorphic to $\mathscr{S}_\rho(G)$, as can be seen from the universal mapping property. □

8. Exponential iterated integrals on complex curves and fibered knot spaces

In this section we consider two examples that illustrate the π_1 de Rham theorems and Theorem 7.2.

We are interested in manifolds for which the ring of closed exponential iterated integrals can be calculated explicitly. We have earlier referred to the problem that the same map $P_{x,x}M \to \mathbb{C}$ may be computed by two different integral expressions. This problem can sometimes be solved by putting a complex structure on M and restricting attention to integrals composed from holomorphic 1-forms.

Consider first the case when M is a smooth affine complex curve. Since $\pi_1(M, x)$ is free, the unipotent completion $\pi_1(M, x) \to \mathscr{U}(\pi_1(M, x))$ is a faithful representation,[5] hence ordinary iterated integrals are adequate to distinguish any two elements of $\pi_1(M, x)$. The next proposition provides a description for $H^0(B(M, x))$.

Proposition 8.1. *Suppose* (M, x) *is a smooth affine complex curve. Closed iterated integrals separate the elements of* $\pi_1(M, x)$*. Let* $\{\omega_1, \ldots, \omega_n\}$ *be a basis for the space of closed holomorphic* 1*-forms on M. Then*

$$\mathscr{B} = \{1\} \cup \left\{ \int \omega_{j_1} \ldots \omega_{j_k} \mid k > 0, (j_i) \in \{1, \ldots, n\}^k \right\} \tag{18}$$

is a basis for $H^0(B(M, x))$.

Proof. The key observations are that each class in $H^1(M; \mathbb{C})$ has a unique holomorphic representative, and that the wedge product of any two closed holomorphic 1-forms is zero. The fact that $\mathscr{B}$ is a basis for $H^0(B(M, x))$ actually follows from a general theorem of Chen's [2, Theorem 4.1.1] which deals with the complex of higher iterated integrals. We provide here an elementary proof for the sake of clarity.

Lemma 8.2. *Any connection on M of the form* $d - \theta$*, where* θ *is a matrix of closed holomorphic* 1*-forms, is flat. Any flat connection* $d - \omega$*, with* ω *a nilpotent upper triangular matrix of* 1*-forms, is conjugate via a matrix* $G: M \to \mathbf{U}_n(\mathbb{C})$ *to such a connection* $d - \theta$.

Proof. The first assertion is immediate since the curvature of $d - \theta$ is $\mathrm{d}\theta + \theta \wedge \theta = 0$. For the second, take any $k \in \{1, \ldots, n-1\}$ and suppose

$$d - \begin{bmatrix} 0 & \omega_{12} & \omega_{13} & \omega_{14} & \cdots & \omega_{1n} \\ 0 & 0 & \omega_{23} & \omega_{24} & \cdots & \omega_{2n} \\ 0 & 0 & 0 & \omega_{34} & \cdots & \omega_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{19}$$

is a flat connection such that each entry ω_{ij} with $j - i < k$ is closed and holomorphic. The equation $\mathrm{d}\omega + \omega \wedge \omega = 0$ implies that the 1-forms ω_{ij} for which $j - i = k$ must be closed. Choose for each such entry an exact 1-form df_{ij} such that $\omega_{ij} - \mathrm{d}f_{ij}$ is holomorphic. Let F be the $n \times n$ matrix with F_{ij} equal to f_{ij} if $j - i = k$ and zero otherwise. The matrix ω' satisfying

$$d - \omega' = (I + F)^{-1}(d - \omega)(I + F) \tag{20}$$

has $\omega'_{ij} = \omega_{ij} - \mathrm{d}F_{ij}$ for $j - i \leqslant k$. Continuing by induction on k we obtain the desired matrix θ. □

Now since any closed iterated integral I on M arises from the monodromy of such a connection $d - \omega$ (recall the proof of Theorem 6.1), it may be expressed in terms of closed holomorphic 1-forms.

[5] For a proof see Appendix A3 in [7], Proposition 3.6(a) in particular.

And linearity

$$\int \gamma_1 \ldots (z\gamma_i + z'\gamma_i') \ldots \gamma_n = z \int \gamma_1 \ldots \gamma_i \ldots \gamma_n + z' \int \gamma_1 \ldots \gamma_i' \ldots \gamma_n,$$

allows us then to express I as a sum of elements from $\mathscr{B}$.

It remains only to show that $\mathscr{B}$ is a linearly independent set; this follows from a straightforward manipulation. Let $\lambda_1, \ldots, \lambda_n$ be free generators for $\pi_1(M, x)$; it suffices to prove the proposition for any chosen basis $\{\omega_j\}_j$, so let us assume $\{\omega_j\}_j$ is such that $\{[\omega_j]\}_j \subseteq H^1(M; \mathbb{C})$ is the dual basis for $\{[\lambda_j]\}_j \subseteq H_1(M; \mathbb{C})$. Suppose

$$\sum_{I \in \mathscr{B}' \subseteq \mathscr{B}} z_I I = 0 \tag{21}$$

is a nontrivial finite sum with each z_I nonzero. Evidently such an expression must include a term of length at least 2. Suppose $z_{I_0} I_0 = z_{I_0} \int \omega_{i_1} \ldots \omega_{i_k}$ is a term of maximal length. The transformation

$$I \mapsto I - \langle \Delta I, (\cdot, \lambda_{i_k}) \rangle$$

(applying the comultiplication formula (10) formally) turns (21) into a nontrivial sum with terms of length $\leqslant k - 1$. Continuing in this manner yields a contradiction. This completes the proof. □

Now we consider the example of a fibered knot complement. Suppose $K \subseteq S^3$ is a tame knot and that $S^3 \backslash K$ has an infinite-cyclic covering map

$$\phi: (\mathbb{R} \times F, (0, \overline{x})) \to (S^3 \backslash K, x),$$

whose deck transformations $\Psi_n, n \in \mathbb{Z}$ are given by $\Psi_1(t, f) = (t + 1, \psi(f))$ where $\psi: F \to F$ is a homeomorphism. F is a noncompact 2-manifold which we may take to be an affine complex curve, making the previous discussion useful. Note that pulling back an exponential iterated integral on $S^3 \backslash K$ with closed exponents,

$$\phi^* \int e^{\delta_0} \omega_{01} \ldots \omega_{(n-1)n} e^{\delta_n} = \int e^{\phi^* \delta_0} \phi^* \omega_{01} \ldots \phi^* \omega_{(n-1)n} e^{\phi^* \delta_n},$$

produces an exponential iterated integral with exact exponents, which may be rewritten via Proposition 3.12 as an ordinary iterated integral. Let L be the $\mathbb{Z}$-module of closed 1-forms that defines the diagonal part of the conjugation representation

$$\pi_1(S^3 \backslash K, x) \to \mathrm{Aut}\, H_1(\mathbb{R} \times F; \mathbb{C}).$$

The content of Theorem 7.2 is that the map

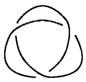
$$\phi^*: H^0(EB(S^3 \backslash K, x)^L) \to H^0(B(\mathbb{R} \times F, (0, \overline{x}))) = H^0(B(F, \overline{x})) \tag{22}$$

is surjective. Combining this fact with Proposition 8.1 shows that $H^0(EB(S^3 \backslash K, x)^L)$ separates the elements of $\pi_1(S^3 \backslash K, x)$. It also allows one to describe $H^0(EB(S^3 \backslash K, x)^L)$ by looking at the pre-image of the basis $\mathscr{B}$. We demonstrate this approach with the particular example of the trefoil knot.

It is convenient (for coordinates) to consider a complex manifold that is of the same homotopy type as the complement of the trefoil. Assume

$$S^3 = \{(x, y) \mid |x|^2 + |y|^2 = 1\} \subseteq \mathbb{C}^2.$$

The knot,



can be specified as the intersection of S^3 with the singular curve

$$C = \{(x, y) \mid x^3 + y^2 = 0\},$$

and $\mathbb{C}^2 \backslash C$ deformations retracts onto $S^3 \backslash C \cap S^3$. The infinite cyclic cover of $\mathbb{C}^2 \backslash C$ may written as $\mathbb{C} \times F$, where $F = \{(x, y) \mid x^3 + y^2 = 1\} \subseteq \mathbb{C}^2$:

$$\begin{aligned} \phi\colon \mathbb{C} \times F &\to \mathbb{C}^2 \backslash C \\ (t, x, y) &\mapsto (\mathrm{e}^{(2\pi \mathrm{i}/3)t} x, \mathrm{e}^{\pi \mathrm{i} t} y). \end{aligned}$$

The homeomorphism $\psi\colon F \to F$ that generates the group of deck transformations is given by

$$\psi(x, y) = (\zeta_6^{-2} x, \zeta_6^{-3} y).$$

The closed holomorphic 1-forms

$$\omega_{-1} = \frac{\mathrm{d}x}{y}, \quad \omega_1 = \frac{x\, \mathrm{d}x}{y}$$

on F diagonalize the action of ψ^* on $H^1(F; \mathbb{C})$, for

$$\psi^* \omega_{-1} = \zeta_6 \omega_{-1}, \quad \psi^* \omega_1 = \zeta_6^{-1} \omega_1.$$

Extend the 1-forms ω_{-1} and ω_1 to $\mathbb{C} \times F$ by pulling back along the projection map to F. The 1-forms

$$\mathrm{e}^{-(\pi \mathrm{i}/3)t} \omega_{-1}, \quad \mathrm{e}^{-(\pi \mathrm{i}/3)t} \omega_1, \quad \frac{\pi \mathrm{i}}{3} \mathrm{d}t \in E^1(\mathbb{C} \times F; \mathbb{C})$$

are each invariant under the action of the group of deck transformations. Let ω_1°, ω_{-1}°, and δ denote the 1-forms that they induce on $\mathbb{C}^2 \backslash C$, respectively.

Consider the integral

$$I = \int \omega_{\varepsilon_1}^\circ \mathrm{e}^{\varepsilon_1 \delta} \omega_{\varepsilon_2}^\circ \mathrm{e}^{(\varepsilon_1 + \varepsilon_2)\delta} \ldots \omega_{\varepsilon_n}^\circ \mathrm{e}^{(\varepsilon_1 + \cdots + \varepsilon_n)\delta}, \tag{23}$$

where $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$. Applying Proposition 3.12 we find

$$\phi^* I = \int \omega_{\varepsilon_1} \omega_{\varepsilon_2} \ldots \omega_{\varepsilon_n}.$$

Integrals of this form provide a basis for $H^0(B(\mathbb{C} \times F, (0, \overline{x})))$, as we know. The exact sequence (16) from Theorem 7.2 implies that $H^0(EB(\mathbb{C}^2 \backslash C, x)^{\langle \delta \rangle})$ is isomorphic as a $\mathbb{C}$-algebra to $H^0(B(\mathbb{C} \times F, (0, \overline{x}))) \otimes \mathscr{O}(\mathscr{S}_\rho(\mathrm{im}\, \rho))$, hence the following.

Proposition 8.3. *The set of integrals*

$$\int \delta^m \int e^{k\delta} \int \omega_{\varepsilon_1}^{\circ} e^{\varepsilon_1\delta} \omega_{\varepsilon_2}^{\circ} e^{(\varepsilon_1+\varepsilon_2)\delta} \dots \omega_n^{\circ} e^{(\varepsilon_1+\cdots+\varepsilon_n)\delta} \tag{24}$$

with $k \in \mathbb{Z}$, $m, n \geqslant 0$, *and* $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$, *is a basis for* $H^0(EB(\mathbb{C}^2 \backslash C, x)^{\langle\delta\rangle})$.

The comultiplication formula (10) is easily applied to these integrals, giving us an effective description for the Hopf algebra $H^0(EB(\mathbb{C}^2 \backslash C, x)^{\langle\delta\rangle})$.

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