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On asymptotic stability of discrete-time non-autonomous delayed Hopfield neural networks

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ABSTRACT

In this paper, we obtain some sufficient conditions for determining the asymptotic stability of discrete-time non-autonomous delayed Hopfield neural networks by utilizing the Lyapunov functional method. An example is given to show the validity of the results.

1. Introduction

The stability of continuous-time delayed Hopfield neural networks has received much attention due to its importance in many applications such as associative memories, pattern recognition, image processing, optimization problems. So far, many researchers have investigated the global asymptotic stability and/or global exponential stability of continuous-time delayed Hopfield neural networks and obtained various results, for example, see [1–7]. In conducting numerical simulation of continuous-time neural networks, it is necessary to formulate a discrete-time version which is an analogue of the continuous-time neural networks. Hence, stability for discrete-time delayed Hopfield neural networks has also received considerable attention from many researchers, see [8–13]. However, all the results are concerned with the autonomous neural network models and to the best of our knowledge, few results for the non-autonomous models have been obtained. In this paper, by making use of the construction of a suitable Lyapunov functional, several stability criteria are provided for asymptotic stability of discrete-time non-autonomous delayed Hopfield neural networks. An example is presented to illustrated the efficiency of the results.

2. Stability analysis

The dynamic behavior of discrete-time non-autonomous delayed Hopfield neural networks can be described as follows

$$y_i(n+1) = a_i(n)y_i(n) + \sum_{j=1}^{m} b_{ij}(n)g_j(y_j(n-\kappa)) \tag{2.1}$$

for $i \in \{1, 2, \ldots, m\}, n \in \{0, 1, 2, \ldots\}$, where m corresponds to the number of units in a neural network; $x(n) = [x_1(n), \ldots, x_m(n)]^{\mathrm{T}} \in R^m$ corresponds to the state vector; $f(x(n)) = [f_1(x_1(n)), \ldots, f_m(x_m(n))]^{\mathrm{T}} \in R^m$ denotes the activation function of the neurons; $f(x(n-\kappa)) = [f_1(x_1(n-\kappa)), \ldots, f_m(x_m(n-\kappa))]^{\mathrm{T}} \in R^m$; $A(n) = \mathrm{diag}(a_i(n))$ $(a_i(n) \in (0, 1))$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs. $B(n) = \{b_{ij}(n)\}$ represents the delayed feedback matrix, κ is a positive integer and denotes the transmission delay along the axon of the jth unit.

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The initial conditions associated with system (2.1) are of the form

$$y_i(l) = \varphi_i(l), \quad i \in \{1, 2, \dots, m\} \tag{2.2}$$

where l is an integer with $l \in [-\kappa, 0]$.

In this paper, we will assume that the activation functions g_i, $i = 1, 2, \dots, m$ satisfy the following condition

$$\begin{aligned} &|g_i(\xi_1) - g_i(\xi_2)| \le L_i|\xi_1 - \xi_2|, \quad \forall \xi_1, \xi_2 \in R. \\ &g_i(0) = 0. \end{aligned} \tag{2.3}$$

This type of activation function is clearly more general than both the usual sigmoid activation functions and the piecewise linear function (PWL): $g_i(x) = \frac{1}{2}(|x+1| - |x-1|)$ which is used in [14].

Throughout this paper, for symmetric matrices X and Y, the notation $X > Y$ ($X \ge Y$) means that matrix $X - Y$ is positive definite (positive semi-definite). Matrices, if not explicitly stated, are assumed to have compatible dimensions. For any real matrix A, A^{T} denotes the transpose of A. I denotes the identity matrix with appropriate dimension.

System (2.1) can be rewritten as

$$y(n+1) = A(n)y(n) + B(n)g(y(n-\kappa)) \tag{2.4}$$

where $y(n) = (y_1(n), y_2(n), \dots, y_m(n))^{\mathrm{T}}$, $A(n) = \mathrm{diag}(a_1(n), \dots, a_m(n))$, $B(n) = \left(b_{ij}(n)\right)_{m\times m}$, $g(y(n)) = (g_1(y_1(n)), g_2(y_2(n)), \dots, g_m(y_m(n)))^{\mathrm{T}}$.

Lemma 1. *Let $M(x) = M^{\mathrm{T}}(x)$, $P(x) = P^{\mathrm{T}}(x) > 0$ and $Q(x)$ depend affinely on x. then*

$$\begin{bmatrix} Q(x) & M(x) \\ M^{\mathrm{T}}(x) & -P(x) \end{bmatrix} < 0$$

is equivalent to

$$Q(x) + M(x)P^{-1}(x)M^{\mathrm{T}}(x) < 0.$$

Next, we will present a sufficient condition for ensuring asymptotic stability of Eq. (2.1).

Theorem 1. *Eq. (2.1) is asymptotically stable if there exist two positive definite matrices $P > 0$ and $Q > 0$ such that*

$$\begin{aligned} \Xi &= \begin{bmatrix} A(n)PA(n) - P + \lambda_{\max}(Q)L^2 & A(n)PB(n) \\ B^{\mathrm{T}}(n)PA(n) & B^{\mathrm{T}}(n)PB(n) - Q \end{bmatrix} \\ &< 0 \end{aligned} \tag{2.5}$$

where $L = \mathrm{diag}(L_i)$, $\lambda_{\max}(Q)$ denotes the largest eigenvalue of the positive definite matrix Q.

Proof. Consider the following function

$$V(n) = y^{\mathrm{T}}(n)Py(n) + \sum_{i=n-\kappa}^{n-1} g^{\mathrm{T}}(y(i))Qg(y(i)) \tag{2.6}$$

then, we get

$$\begin{aligned} V(n+1) &= y^{\mathrm{T}}(n+1)Py(n+1) + \sum_{i=n+1-\kappa}^{n} g^{\mathrm{T}}(y(i))Qg(y(i)) \\ &= [A(n)y(n) + B(n)g(y(n-\kappa))]^{\mathrm{T}} P\, [A(n)y(n) + B(n)g(y(n-\kappa))] \\ &\quad + \sum_{i=n+1-\kappa}^{n-1} g^{\mathrm{T}}(y(i))Qg(y(i)) + g^{\mathrm{T}}(y(n))Qg(y(n)) \\ &= \left[y^{\mathrm{T}}(n)A(n)P + g^{\mathrm{T}}(y(n-\kappa))B^{\mathrm{T}}(n)P\right] [A(n)y(n) + B(n)g(y(n-\kappa))] \\ &\quad + \sum_{i=n+1-\kappa}^{n-1} g^{\mathrm{T}}(y(i))Qg(y(i)) + g^{\mathrm{T}}(y(n))Qg(y(n)) \\ &= y^{\mathrm{T}}(n)A(n)PA(n)y(n) + 2y^{\mathrm{T}}(n)A(n)PB(n)g(y(n-\kappa)) + g^{\mathrm{T}}(y(n-\kappa))B^{\mathrm{T}}(n)PB(n)g(y(n-\kappa)) \\ &\quad + \sum_{i=n+1-\kappa}^{n-1} g^{\mathrm{T}}(y(i))Qg(y(i)) + g^{\mathrm{T}}(y(n))Qg(y(n)). \end{aligned}$$

Calculating the difference $\Delta V(n) = V(n+1) - V(n)$ along (2.4), we can obtain

$$\begin{aligned}\Delta V(n) &= V(n+1) - V(n)\\ &= y^{\mathrm{T}}(n)A(n)PA(n)y(n) + 2y^{\mathrm{T}}(n)A(n)PB(n)g(y(n-\kappa)) + g^{\mathrm{T}}(y(n-\kappa))B^{\mathrm{T}}(n)PB(n)g(y(n-\kappa))\\ &\quad + \sum_{i=n+1-\kappa}^{n-1} g^{\mathrm{T}}(y(i))Qg(y(i)) + g^{\mathrm{T}}(y(n))Qg(y(n)) - y^{\mathrm{T}}(n)Py(n) - \sum_{i=n-\kappa}^{n-1} g^{\mathrm{T}}(y(i))Qg(y(i))\\ &= y^{\mathrm{T}}(n)[A(n)PA(n) - P]y(n) + 2y^{\mathrm{T}}(n)A(n)PB(n)g(y(n-\kappa)) + g^{\mathrm{T}}(y(n-\kappa))B^{\mathrm{T}}(n)PB(n)g(y(n-\kappa))\\ &\quad + g^{\mathrm{T}}(y(n))Qg(y(n)) - g^{\mathrm{T}}(y(n-\kappa))Qg(y(n-\kappa))\\ &\le y^{\mathrm{T}}(n)[A(n)PA(n) - P + \lambda_{\max}(Q)L^2]y(n) + 2y^{\mathrm{T}}(n)A(n)PB(n)g(y(n-\kappa)) + g^{\mathrm{T}}(y(n-\kappa))\\ &\quad \times \left[B^{\mathrm{T}}(n)PB(n) - Q\right]g(y(n-\kappa))\\ &= \begin{bmatrix} y(n)\\ g(y(n-\kappa))\end{bmatrix}^{\mathrm{T}} \Xi \begin{bmatrix} y(n)\\ g(y(n-\kappa))\end{bmatrix}\\ &< 0.\end{aligned}$$

This completes the proof. □

If let $Q = I$ in Theorem 1, we can easily obtain Corollary 1.

Corollary 1. *Eq. (2.1) is asymptotically stable if there exists a positive definite matrix $P > 0$ such that*

$$\Xi_1 = \begin{bmatrix} A(n)PA(n) - P + L^2 & A(n)PB(n)\\ B^{\mathrm{T}}(n)PA(n) & B^{\mathrm{T}}(n)PB(n) - I\end{bmatrix} < 0. \tag{2.7}$$

Furthermore, if we let $P = 2I$ in Corollary 1, we can get the following Corollary 2.

Corollary 2. *Eq. (2.1) is asymptotically stable if*

$$\Xi_2 = \begin{bmatrix} 2A(n)A(n) - 2I + L^2 & 2A(n)B(n)\\ 2B^{\mathrm{T}}(n)A(n) & 2B^{\mathrm{T}}(n)B(n) - I\end{bmatrix} < 0. \tag{2.8}$$

If $L = I$ in (2.1), then Corollary 2 reduces to the following

Corollary 3. *Eq. (2.1) is asymptotically stable if*

$$\Xi_3 = \begin{bmatrix} 2A(n)A(n) - I & 2A(n)B(n)\\ 2B^{\mathrm{T}}(n)A(n) & 2B^{\mathrm{T}}(n)B(n) - I\end{bmatrix} < 0. \tag{2.9}$$

Remark 1. Based on Lemma 1, if $2B^{\mathrm{T}}(n)B(n) - I < 0$, then Corollary 3 can be rewritten as

$$\begin{aligned}\Xi_3' &= 2A(n)A(n) - I\\ &-4A(n)B(n)\left(2B^{\mathrm{T}}(n)B(n) - I\right)^{-1} B^{\mathrm{T}}(n)A(n) < 0.\end{aligned} \tag{2.10}$$

3. An example

In this section, we will give an example to show the validity of the results given in this paper.

Example 1. Consider the following discrete-time non-autonomous delayed Hopfield neural network with two neurons,

$$y(n+1) = A(n)y(n) + B(n)g(y(n-\kappa)) \tag{3.1}$$

with the matrices

$$A(n) = \begin{bmatrix} \sqrt{\dfrac{20}{91}} & 0\\ 0 & 0.05\mathrm{e}^{-n}\end{bmatrix}$$

$$B(n) = \begin{bmatrix} 0.2 & 0.4\\ 0.2 & -0.4\end{bmatrix} \tag{3.2}$$

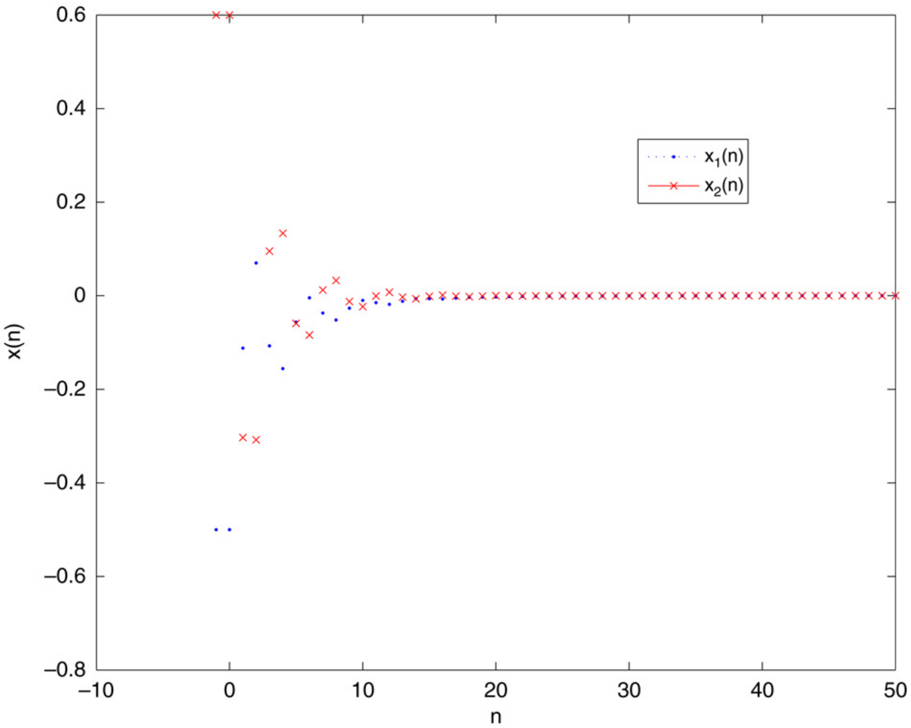

Fig. 1. The solution trajectories of Eq. (3.1).

and the nonlinear input–output function is chosen as $f(x) = \tanh(x)$. It can be verified that this function satisfies assumption (2.3) with $L_1 = L_2 = 1$. Furthermore, we have

$$2B^{\mathrm{T}}(n)B(n) - I = \begin{bmatrix} -0.6 & 0 \\ 0 & -0.36 \end{bmatrix} < 0$$

and one can easily check that

$$\Xi_3' = \begin{bmatrix} -\frac{1}{9} & -\frac{68}{45}\sqrt{\frac{20}{91}}0.05\mathrm{e}^{-n} \\ -\frac{68}{45}\sqrt{\frac{20}{91}}0.05\mathrm{e}^{-n} & \frac{91}{9000}\mathrm{e}^{-2n} - 1 \end{bmatrix}.$$

The determinants of the principal submatrices of Ξ_3' are $-\frac{1}{9} < 0$ and $\frac{1}{9} - 0.0024\mathrm{e}^{-2n} > 0$. Based on Hurwitz's theorem, we can conclude that $\Xi_3' < 0$. Therefore, from Remark 1, it follows that system (3.1) is asymptotically stable.

In the numerical simulation, let $\kappa = 1$ in (3.1), the initial state $(\varphi_1(l), \varphi_2(l))^{\mathrm{T}} = (-0.5, 0.6)^{\mathrm{T}}$. The numerical simulation is illustrated in Fig. 1.

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