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Persistence of embedded eigenvalues

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Abstract

We consider conditions under which an embedded eigenvalue of a self-adjoint operator remains embedded under small perturbations. In the case of a simple eigenvalue embedded in continuous spectrum of multiplicity $m < \infty$ we show that in favorable situations, the set of small perturbations of a suitable Banach space which do not remove the eigenvalue form a smooth submanifold of codimension m. We also have results regarding the cases when the eigenvalue is degenerate or when the multiplicity of the continuous spectrum is infinite.

Keywords: Embedded eigenvalues; Perturbation

1. Introduction

An eigenvalue in the continuous spectrum of an operator typically disappears under small perturbations. Or if there is enough analyticity for some sort of analytic continuation of the resolvent, it typically becomes a resonance, that is a pole in the analytic continuation of certain matrix elements of the resolvent.

The simplest mechanism which has been used to prove that embedded eigenvalues disappear under perturbation is Fermi's Golden Rule, which in physics gives the lifetime of the decaying

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unperturbed state. See for example [24,23,25,3]. See also the earlier work [9,10], which relies on analytic continuation. For the basic mathematical ideas behind Fermi's Golden Rule, see [28].

On the contrary, in this article we are interested in the structure of the set of perturbations which do not remove an embedded eigenvalue. No analyticity assumptions are made to allow the analytic continuation of the resolvent. A simple example of the kind of theorem we are after is given in [11]: Suppose V is real and in $L^1(\mathbb{R})$. Consider the self-adjoint operator in $L^2(\mathbb{R})$ given by

$$H = -\frac{d^2}{dx^2} + \lambda \frac{\sin(kx)}{x} + V(x),$$

where $\lambda > k > 0$. Then the set of such V for which H has an embedded eigenvalue is a smooth codimension 1 submanifold of real $L^1(\mathbb{R})$. Such a global result is difficult to obtain in more general cases. We restrict ourselves to small bounded perturbations of a given operator, but the methods can be extended to small H-bounded perturbations.

In Sections 2 to 5, we consider a simple eigenvalue embedded in continuous spectrum of multiplicity $m < \infty$. Under favorable assumptions including the smoothness of the boundary values of the resolvent (after the pole term corresponding to the embedded eigenvalue has been removed) we show that small perturbations which do not remove the eigenvalue form a smooth submanifold (of appropriate Banach spaces) of codimension m. See Theorem 1. In Section 6, we give two applications of this theorem.

In Section 7, a smooth manifold of perturbations of codimension $m + n - 1$ is shown not to remove a degenerate eigenvalue of multiplicity n embedded in continuous spectrum of multiplicity m. The set of small perturbations which do not remove the degenerate eigenvalue is a much larger set, but its structure is not known.

In Section 8 we give a weak theorem, but one which covers a very general class of operators of the form $-\Delta + V$, where V is a real function on $\mathbb{R}^n$. This theorem shows that the set of small local perturbations which do not remove a (simple or degenerate) eigenvalue is quite large. Of course if $n \geqslant 2$, the continuous spectrum will in general have infinite multiplicity.

See [12] for another approach to the problem where the continuous spectrum of the operators involved has infinite multiplicity. In [12], the structure of the set of local perturbations which do not remove an embedded simple eigenvalue is determined for a specific example.

2. Assumptions and result in the case of a simple eigenvalue and finite multiplicity of the continuous spectrum

Let $\mathcal{H}$ be a Hilbert space, and let $C : \mathcal{H} \to \mathcal{H}$ be an antiunitary involution, i.e. a conjugate-linear mapping satisfying $C^2 = I$ and $\langle Cf, Cg\rangle = \overline{\langle f, g\rangle}$. An element $f \in \mathcal{H}$ is said to be *real* if $Cf = f$, and we say that an operator H on $\mathcal{H}$ is *real* if $HC = CH$. We assume that:

(H1) H is a real, self-adjoint operator acting in $\mathcal{H}$.

We introduce a scale of Hilbert spaces $\mathcal{H}_s$ for $s \in \mathbb{R}$ such that $\mathcal{H}_0 = \mathcal{H}$, the dual space of $\mathcal{H}_s$ is $\mathcal{H}_{-s}$ (using the inner product of $\mathcal{H}$) and $\mathcal{H}_s$ is continuously embedded in $\mathcal{H}_t$ for $s \geqslant t$. We also assume that $\mathcal{H}_s$ is dense in $\mathcal{H}_t$ if $s > t$. If $s \geqslant 0$, then $\mathcal{H}_s \subset \mathcal{H} \subset \mathcal{H}_{-s}$. We denote the inner product of $\mathcal{H}$ and also the duality pairing of $\mathcal{H}_s$ with $\mathcal{H}_{-s}$ by $\langle\cdot,\cdot\rangle$. For notational simplicity we assume that for $s > 0$, $\|f\|_{\mathcal{H}} \leqslant \|f\|_{\mathcal{H}_s}$. We assume that $C : \mathcal{H}_s \to \mathcal{H}_s$ is bounded for every $s \in \mathbb{R}$. It then follows that

$$\langle Cf, Cg\rangle = \overline{\langle f, g\rangle}$$

for $f \in \mathcal{H}_{-s}$ and $g \in \mathcal{H}_s$.

Let $\sigma_{pp}(H)$ be the pure point spectrum of H, i.e. the set of eigenvalues of H.

(H2) H has an eigenvalue at $\lambda = \lambda_0$ of finite multiplicity which is embedded in the continuous spectrum and isolated in $\sigma_{pp}(H)$, and the corresponding eigenspace is a subspace of $\bigcap_{s\geqslant 0} \mathcal{H}_s$.

The condition that the eigenspace is a subspace of $\bigcap_{s\geqslant 0} \mathcal{H}_s$ can be relaxed, and it is enough that it is a subspace of $\mathcal{H}_{s_*}$ for a certain $s_* > 0$. In examples, the condition can be checked by using methods from [13,14,7,21].

We denote by P_0 the orthogonal projection in $\mathcal{H}$ onto the eigenspace of H corresponding to the eigenvalue λ_0. Let $\overline{H} := H + P_0$. If H satisfies conditions (H1) and (H2) then the continuous spectra of H and $\overline{H}$ coincide, and $\overline{H}$ does not have any eigenvalues in a neighborhood of λ_0 (see Proposition 2).

$(\mathrm{H3})_k$ There exist $k \geqslant 0$ and $s_1 \geqslant 0$ and such that for any $s > s_1$ there is a $\delta_1 > 0$ such that the norm limits $\lim_{\epsilon\downarrow 0}(\overline{H} - \lambda \pm i\epsilon)^{-1} = (\overline{H} - \lambda \pm i0)^{-1}$ exist in $\mathcal{L}(\mathcal{H}_s, \mathcal{H}_{-s})$ and are C^k in λ in the norm topology for $\lambda \in (\lambda_0 - \delta_1, \lambda_0 + \delta_1)$.

In examples, $(\mathrm{H3})_k$ can be verified using methods from [22,18], as is done in Examples 1 and 2 of this paper. $(\mathrm{H3})_0$ is called the limiting absorption principle for $\overline{H}$. Suppose that $(\mathrm{H3})_k$ holds. We will consider perturbations W in the space X_s, where X_s is a real Banach space whose elements are bounded self-adjoint operators on $\mathcal{H}$ and such that

$$X_s \subset \{W \in \mathcal{L}(\mathcal{H}_{-s}, \mathcal{H}_s);\ W \text{ is real and self-adjoint on } \mathcal{H}\},$$

where $s > s_1$ and the inclusion is continuous.

If H satisfies (H1), (H2), and $(\mathrm{H3})_0$, we introduce the notation

$$\delta(\overline{H} - \lambda) := \frac{1}{2\pi i}\big((\overline{H} - \lambda - i0)^{-1} - (\overline{H} - \lambda + i0)^{-1}\big).$$

Note that if $\lambda \neq \lambda_0$ but $|\lambda - \lambda_0|$ is small, then $\delta(\overline{H} - \lambda) = \delta(H - \lambda)$. The multiplicity of the continuous spectrum of H at λ is by definition the dimension of $\operatorname{Ran}\delta(\overline{H} - \lambda) \subset \mathcal{H}_{-s}$. Using the density of $\mathcal{H}_s$ in $\mathcal{H}_t$ for $s > t$ it is easy to show that the multiplicity is independent of s for $s > s_1$.

(H4) The multiplicity of the continuous spectrum of H in $(\lambda_0 - \delta_1, \lambda_0 + \delta_1)$ is $m < \infty$.

Our last assumption is a condition that the set of perturbations is not too small. We will eventually need one version of this condition (H5) when λ_0 is a simple eigenvalue of H, and a stronger condition (H5′) when λ_0 is a degenerate eigenvalue.

(H5) λ_0 is a simple eigenvalue and φ_0 is a corresponding real normalized eigenvector of H. The complex linear span of $\{\delta(\overline{H}-\lambda_0)W\varphi_0;\ W \in X_s\}$ is $\operatorname{Ran}\delta(\overline{H}-\lambda_0)$.

(H5′) There exists a real vector $\psi_1 \in \operatorname{Ran} P_0$ such that the complex linear span of $\{W\psi_1;\ W \in X_s\}$ is dense in $\mathcal{H}_s$.

In Section 6 we give examples of operators for which the assumptions (H1)–(H5) are satisfied. For $\delta > 0$, let $\mathcal{M}_{\delta,s}$ be the set

$$\mathcal{M}_{\delta,s} := \bigl\{W \in X_s;\ \text{there exists a } \lambda \in (\lambda_0-\delta, \lambda_0+\delta) \text{ such that } \lambda \text{ is an eigenvalue of } H+W\bigr\}.$$

Theorem 1. *Suppose that H satisfies conditions* (H1), (H2), $(\mathrm{H3})_k$ *with $k \geqslant 1$ and* (H4) *and* (H5). *Let s_1 and m be as in assumptions* $(\mathrm{H3})_k$ *and* (H4), *and let $s > s_1$ and* $\dim X_s \geqslant m$. *Then there exist a number $\delta > 0$ and a neighborhood $\mathcal{O}$ of 0 in X_s such that $\mathcal{M}_{\delta,s} \cap \mathcal{O}$ is a C^k manifold in X_s of codimension m. Moreover, if $W \in \mathcal{M}_{\delta,s} \cap \mathcal{O}$, then $H+W$ has exactly 1 eigenvalue in the interval $(\lambda_0-\delta, \lambda_0+\delta)$, and it is simple.*

3. Some preliminary lemmas

Some of the propositions of this section are similar to results found elsewhere in the literature, see e.g. [3,16,17,19]. Furthermore some of the propositions and lemmas needed for proving Theorem 1 are valid without all of the assumptions (H1)–(H5).

We first remark that we have not assumed a condition of uniformity in $(\mathrm{H3})_k$. That this assumption is unnecessary follows from the following lemma:

Lemma 1. *Let $k \geqslant 0$ and suppose* $(\mathrm{H3})_k$ *holds. Then in any compact subinterval $J_1 \subset (\lambda_0-\delta_1, \lambda_0+\delta_1)$ and any $j \leqslant k$ the convergence of $\frac{d^j(\overline{H}-\lambda\pm i\epsilon)^{-1}}{d\lambda^j}$ to its boundary value $\frac{d^j(\overline{H}-\lambda\pm i0)^{-1}}{d\lambda^j}$ is uniform for $\lambda \in J_1$.*

Proof. If J is any compact subinterval of $(\lambda_0-\delta_1, \lambda_0+\delta_1)$, let $Q_J = \{z \in \mathbb{C};\ \operatorname{Im} z > 0, \operatorname{Re} z \in J\}$. For z with $\operatorname{Im} z > 0$, define $F(z) = (\overline{H}-z)^{-1}$. We shall consider F as an operator valued function with values in $\mathcal{L}(\mathcal{H}_s, \mathcal{H}_{-s})$, $s > s_1$. Note that F is analytic in the half-plane $\operatorname{Im} z > 0$ and satisfies $\|F(z)\|_{\mathcal{L}(\mathcal{H}_s,\mathcal{H}_{-s})} \leqslant C(\operatorname{Im} z)^{-1}$. We shall show that F is bounded in Q_J and that F admits a continuous extension to $\overline{Q_J}$. For this purpose, if $f \in \mathcal{H}_s$ let

$$h(z) = \bigl(\langle F(z)f, f\rangle + i\bigr)^{-1}$$

and note that $|h(z)| \leqslant 1$. A well-known theorem asserts that

$$h(x) := \lim_{y\downarrow 0} h(x+iy)$$

exists for a.e. $x \in \mathbb{R}$ and that

$$h(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} P(x-t, y)h(t)\,dt \tag{1}$$

where P is the Poisson kernel

$$P(x,y)=\frac{y}{x^2+y^2}$$

(see for example [27], Theorems 11.24 and 11.30). Note that it follows from assumption $(\mathrm{H3})_k$ (and the definition of $h(z)$) that the limit defining $h(x)$ exists for all $x\in J$, that $h(x)$ is continuous for $x\in J$, and that $h(x)\neq 0$ for $x\in J$. Hence, it follows from the representation (1) that $h(z)$ admits a continuous extension to $\overline{Q_J}$ and that for some $\delta>0$,

$$\left|h(z)\right|\geqslant\delta,\quad z\in Q_J\cap\{z\colon \operatorname{Im}z\leqslant 1\}$$

and thus by polarization $\langle F(z)f,g\rangle$ is continuous on $\overline{Q_J}$ for all $f,g\in\mathcal{H}_s$. In particular the uniform boundedness principle implies that $\|F(z)\|_{\mathcal{L}(\mathcal{H}_s,\mathcal{H}_{-s})}\leqslant C_J$ for $z\in\overline{Q_J}$, where J is any compact subinterval of $(\lambda_0-\delta_1,\lambda_0+\delta_1)$.

With J and Q_J as above, let $\zeta(t)$, $t\in\mathbb{R}$, be a C^1 curve in $\overline{Q_J}\cup\{\operatorname{Im}z>0\}$ satisfying $\zeta(t)=t$ for $t\in J$ and $\zeta(t)=t+i$ for $t\notin(\lambda_0-\delta_1,\lambda_0+\delta_1)$ and $\operatorname{Im}\zeta(t)>0$ if $t\notin J$. Let $z=x+iy$. Integrating the function $F(\zeta)(2\pi i(\zeta-z)(\zeta-\bar{z}))^{-1}$ along the curve $\zeta(\cdot)$ we obtain (by the residue theorem) the representation

$$F(x+iy)=\frac{1}{\pi}\int\limits_{\mathcal{C}}\frac{yF(\zeta)}{(x-\zeta)^2+y^2}\,d\zeta, \tag{2}$$

where $\mathcal{C}$ is the curve $\zeta(\cdot)$. It is clear from (2) and the continuity of F on $(\lambda_0-\delta_1,\lambda_0+\delta_1)$ that F is continuous in the topology of $\mathcal{L}(\mathcal{H}_s,\mathcal{H}_{-s})$ for $x+iy$ on or above $\mathcal{C}$.

Differentiating (2) we obtain

$$\frac{\partial F(x+iy)}{\partial x}=\frac{1}{\pi}\int\limits_{\mathcal{C}}\frac{\partial((x-\zeta)^2+y^2)^{-1}}{\partial x}yF(\zeta)\,d\zeta.$$

If $J=[a,b]$ we integrate by parts on $[a,b]$ to obtain

$$\begin{aligned}\frac{\partial F(x+iy)}{\partial x}&=\frac{1}{\pi}\int\limits_{\mathcal{C}\setminus[a,b]}\frac{\partial((x-\zeta)^2+y^2)^{-1}}{\partial x}yF(\zeta)\,d\zeta+\frac{1}{\pi}\int\limits_{[a,b]}\frac{yF'(\zeta)}{(x-\zeta)^2+y^2}\,d\zeta\\&\quad+\frac{F(b)y}{\pi((x-b)^2+y^2)}-\frac{F(a)y}{\pi((x-a)^2+y^2)}.\end{aligned} \tag{3}$$

If $[a',b']\subset(a,b)$ it follows that uniformly for $x\in[a',b']$, $\lim_{y\downarrow 0}\frac{\partial F(x+iy)}{\partial x}=F'(x)$. Note also the equality for the one-sided derivative

$$-i\frac{\partial F(x+iy)}{\partial y}\bigg|_{y=0}=F'(x)$$

which we will use in Proposition 4. This follows from taking $y' \downarrow 0$ in

$$F(x+iy) - F\big(x+iy'\big) = \int_{y'}^{y} \frac{\partial F(x+it)}{\partial t}\,dt = i\int_{y'}^{y} \frac{\partial F(x+it)}{\partial x}\,dt.$$

We have thus proved the uniform convergence for $j = 0, 1$. In the same way, differentiating (3) as many times as necessary and integrating by parts, the lemma follows for limits from the upper half plane. A similar proof works for the lower half plane. □

The following proposition is basically a corollary of Lemma 1.

Proposition 1. *Suppose that* (H3)$_k$ *holds for some* $k \geqslant 0$ *and* $s > s_1$. *Then given a compact subinterval* $J_1 \subset (\lambda_0 - \delta_1, \lambda_0 + \delta_1)$ *there exists* $\gamma = \gamma_{J_1} > 0$ *so that if* $\|W\|_{\mathcal{L}(\mathcal{H}_{-s},\mathcal{H}_s)} < \gamma$ *and* $j \leqslant k$ *the limits*

$$\lim_{\epsilon \downarrow 0} \frac{d^j(\overline{H} + W - \lambda \pm i\epsilon)^{-1}}{d\lambda^j} = \frac{d^j(\overline{H} + W - \lambda \pm i0)^{-1}}{d\lambda^j} \tag{4}$$

are uniform in λ *for* $\lambda \in J_1$.

Proof. Let $z = \lambda + i\epsilon$. Define γ so that

$$\gamma \sup\big\{ \big\|(\overline{H} - z)^{-1}\big\|_{\mathcal{L}(\mathcal{H}_s,\mathcal{H}_{-s})};\ \operatorname{Re} z \in J_1,\ \operatorname{Im} z \geqslant 0\big\} < 1.$$

Then

$$(\overline{H} + W - z)^{-1} = (\overline{H} - z)^{-1}\big(I + W(\overline{H} - z)^{-1}\big)^{-1}. \tag{5}$$

Then noting that $A \mapsto A^{-1}$ is C^∞ we can differentiate (5) k times using the chain rule and take limits as $\epsilon \downarrow 0$ to obtain the result. □

We note that by Theorem XIII.20 of [26], if condition (H3)$_0$ holds and $s > s_1$, then for $\|W\|_{\mathcal{L}(\mathcal{H}_{-s},\mathcal{H}_s)} < \gamma_{J_1}$ where J_1 is a compact subinterval of $(\lambda_0 - \delta_1, \lambda_0 + \delta_1)$, $\overline{H}$ and $\overline{H} + W$ have purely absolutely continuous spectrum in J_1.

Recall that P_0 is the orthogonal projection in $\mathcal{H}$ onto the eigenspace corresponding to λ_0 and that $\overline{H} = H + P_0$. We have the following proposition:

Proposition 2. *Let* H *satisfy* (H1), (H2), *and* (H3)$_0$ *and assume* $s > s_1$. *Then*

(i) $\sigma_c(\overline{H}) = \sigma_c(H)$,
(ii) $\overline{H}$ *has no eigenvalues in* $(\lambda_0 - \delta_1, \lambda_0 + \delta_1)$.

Proof. To prove (i), let $\psi \in \operatorname{Dom} H \cap \operatorname{Ran}(I - P_0) = \operatorname{Dom} H \cap \ker P_0$. Then

$$\overline{H}\psi = (H + P_0)\psi = H\psi.$$

In other words, on $\mathrm{Dom}\, H \cap \ker P_0$, the operators H and $\overline{H}$ coincide. In particular, their continuous spectra are the same.

According to the remark after Proposition 1, $\overline{H}$ has purely absolutely continuous spectrum in $(\lambda_0 - \delta_1, \lambda_0 + \delta_1)$. Thus (ii) follows. □

Our proof of Theorem 1 is based on the study of the operator $Q(z, W)$ defined by

$$Q(z, W) = P_0(\overline{H} + W - z)^{-1} P_0 \tag{6}$$

for $\mathrm{Im}\, z \neq 0$ and $W \in X_s$ ($s \geqslant 0$). Note that $Q(z, W)$ is a finite dimensional operator on $\mathrm{Ran}\, P_0$, since λ_0 has finite multiplicity.

The resolvent equation shows that for $\mathrm{Im}\, z \neq 0$,

$$(H + W - z)^{-1} = (\overline{H} + W - z)^{-1} + (H + W - z)^{-1} P_0 (\overline{H} + W - z)^{-1}, \tag{7}$$

which implies that

$$(H + W - z)^{-1} P_0 = (\overline{H} + W - z)^{-1} P_0 + (H + W - z)^{-1} Q(z, W),$$

or equivalently

$$(H + W - z)^{-1} P_0\big(I - Q(z, W)\big) = (\overline{H} + W - z)^{-1} P_0. \tag{8}$$

This formula gives a one-to-one correspondence between the two operators $(H + W - z)^{-1} P_0$ and $(\overline{H} + W - z)^{-1} P_0$. In fact, we have:

Proposition 3. *Suppose that* (H1) *and* (H2) *are satisfied, and for* $\mathrm{Im}\, z \neq 0$ *and* $W \in X_s$ ($s \geqslant 0$) *let* $Q(z, W) : \mathcal{H} \to \mathcal{H}$ *be given by* (6). *Then* $I - Q(z, W)$ *is invertible.*

Proof. If $I - Q(z, W)$ is not invertible, then by the Fredholm alternative, there exists an $f \neq 0$ such that

$$\big(I - Q(z, W)\big) f = 0.$$

But then by (8),

$$(\overline{H} + W - z)^{-1} P_0 f = 0,$$

which implies that $P_0 f = 0$. This means that $f = (I - P_0) f$, and so

$$0 = \big(I - Q(z, W)\big)(I - P_0) f = (I - P_0) f, \tag{9}$$

and we see that $f = 0$. □

Proposition 3 together with Eq. (8) show that

$$(H + W - z)^{-1} P_0 = (\overline{H} + W - z)^{-1} P_0 \big(I - Q(z, W)\big)^{-1}.$$

Then by (7) we have

$$(H+W-z)^{-1} = (\overline{H}+W-z)^{-1} + (\overline{H}+W-z)^{-1} P_0\bigl(I-Q(z,W)\bigr)^{-1} P_0(\overline{H}+W-z)^{-1}. \tag{10}$$

Assuming $(\mathrm{H3})_0$ and $s > s_1$, it follows that

$$Q(\lambda+i0,W) = A(\lambda,W) + i\pi P_0\delta(\overline{H}+W-\lambda)P_0, \tag{11}$$

where

$$A(\lambda,W) := \frac{1}{2}\bigl(Q(\lambda+i0,W)+Q(\lambda-i0,W)\bigr) \tag{12}$$

and $\delta(\overline{H}+W-\lambda)$ is given by

$$\begin{aligned}\delta(\overline{H}+W-\lambda) &:= \lim_{\substack{z=\lambda+i\epsilon\\ \epsilon\downarrow 0}} \frac{1}{2\pi i}\bigl((\overline{H}+W-z)^{-1}-(\overline{H}+W-\bar{z})^{-1}\bigr)\\ &= \lim_{\substack{z=\lambda+i\epsilon\\ \epsilon\downarrow 0}} (\overline{H}+W-z)^{-1}\frac{\epsilon}{\pi}(\overline{H}+W-\bar{z})^{-1}.\end{aligned} \tag{13}$$

Proposition 4. *Let H satisfy* (H1)*,* (H2)*, and* $(\mathrm{H3})_1$*. Let $s > s_1$ be fixed, and let J_1 and γ be as in Proposition* 1 *with $\|W\|_{\mathcal{L}(\mathcal{H}_{-s},\mathcal{H}_s)} < \gamma$. If $\lambda \in J_1$, then* 1 *is an eigenvalue of $Q(\lambda+i0,W)$ if and only if λ is an eigenvalue of $H+W$.*

Proof. The limit $Q(\lambda+i0,W)$ is a compact operator, so if $Q(\lambda+i0,W)$ does not have eigenvalue 1, then $(I-Q(\lambda+i0,W))^{-1}$ exists, and we can take the limit in (10). Hence also $(H+W-\lambda-i0)^{-1}$ exists. It was shown above that this implies that λ is not an eigenvalue of $H+W$. To prove the converse, we first use Proposition 1 for $j=1$ to see that $Q(z,W)$ has a C^1 extension to the real axis for $\lambda \in J_1$. In particular we have the Taylor expansion

$$Q(z,W) = Q(\lambda+i0,W) + Q'_z(\lambda+i0,W)(z-\lambda) + o\bigl(|z-\lambda|\bigr). \tag{14}$$

Suppose that $Q(\lambda+i0,W)f = f$ for some f. Then by the definition of $Q(z,W)$, $(I-P_0)f = 0$. If λ is not an eigenvalue of $H+W$, then by (8) and (14)

$$\begin{aligned}P_0 f &= \lim_{\substack{z=\lambda+i\epsilon\\ \epsilon\downarrow 0}} \bigl(I+P_0(H+W-z)^{-1}\bigr)P_0\bigl(I-Q(z,W)\bigr)f\\ &= -\lim_{\substack{z=\lambda+i\epsilon\\ \epsilon\downarrow 0}} \bigl(I+P_0(H+W-z)^{-1}\bigr)P_0\bigl(Q'_z(\lambda+i0,W)(z-\lambda)+o(|z-\lambda|)\bigr)f\\ &= -\lim_{\substack{z=\lambda+i\epsilon\\ \epsilon\downarrow 0}} P_0(H+W-z)^{-1}(z-\lambda)P_0 Q'_z(\lambda+i0,W)f\\ &= P_0 E_{\{\lambda\}}(H+W)P_0 Q'_z(\lambda+i0,W)f = 0\end{aligned}$$

since $E_{\{\lambda\}}(H+W) = 0$. This shows that $f = 0$. □

Corollary 1. *Let H satisfy* (H1), (H2), *and* $(\mathrm{H3})_1$. *Let $s > s_1$ be fixed, and let J_1 and γ be as in Proposition* 1 *with $\|W\|_{\mathcal{L}(\mathcal{H}_{-s},\mathcal{H}_s)} < \gamma$. It follows that if λ is not an eigenvalue of $H + W$, then the limiting absorption principle holds for $H + W$ at λ.*

Proof. The statement follows from Proposition 4 and (10). □

In the case when the multiplicity of the eigenvalue λ_0 is 1, we get a simple expression for the eigenvector of $H + W$ corresponding to λ.

Proposition 5. *Suppose that all conditions of Proposition* 4 *are satisfied and that λ_0 is a simple eigenvalue of H. Suppose that λ is an eigenvalue of $H + W$ with $\lambda \in J_1$. Then λ is a simple eigenvalue of $H + W$ and a corresponding eigenvector is given by*

$$\psi = (\overline{H} + W - \lambda - i0)^{-1}\varphi_0. \tag{15}$$

Proof. We first prove that ψ as defined by (15) is nonzero. This follows from Proposition 4 since

$$P_0\psi = Q(\lambda + i0, W)\varphi_0 = \varphi_0 \neq 0.$$

Next we use (14), and let the number $d = d(\lambda, W)$ be defined by $Q_z'(\lambda + i0, W) = dP_0$. Hence, by (10) and Proposition 4, we have for $f \in \mathcal{H}_s$,

$$\begin{aligned}-i\epsilon(H + W - \lambda - i\epsilon)^{-1} f = {} & -i\epsilon(\overline{H} + W - \lambda - i\epsilon)^{-1} f \\ & + \big(d + o(1)\big)^{-1}(\overline{H} + W - \lambda - i\epsilon)^{-1} \\ & \times P_0(\overline{H} + W - \lambda - i\epsilon)^{-1} f\end{aligned} \tag{16}$$

as $\epsilon \downarrow 0$. Choosing $f = \varphi_0$ and passing to the limit in $\mathcal{H}_{-s}$ as $\epsilon \downarrow 0$, it follows that $d \neq 0$, since otherwise the right-hand side of (16) would blow up (using that $\psi \neq 0$ and $P_0\psi = \varphi_0$), while the left-hand side tends to $E_{\{\lambda\}}(H + W)\varphi_0$. Indeed, we have

$$E_{\{\lambda\}}(H + W)\varphi_0 = \frac{1}{d}(\overline{H} + W - \lambda - i0)^{-1}\varphi_0, \tag{17}$$

which also shows that the right-hand side of (17) belongs to $\mathcal{H}$. Multiplying by d yields the expression (15).

Finally, by (16) for a general $f \in \mathcal{H}_s$, and passing to the limit as $\epsilon \downarrow 0$, the right-hand side is always a multiple of ψ, and so λ is a simple eigenvalue of $H + W$. □

Proposition 6. *Suppose that* (H1), (H2), *and* $(\mathrm{H3})_k$ *are satisfied, where $k \geqslant 0$, and let $s > s_1$, where s_1 is defined in* $(\mathrm{H3})_k$. *Then for some neighborhood J of λ_0 and some neighborhood $\mathcal{O}$ of $0 \in X_s$, $(\overline{H} + W - \lambda - i0)^{-1} : J \times \mathcal{O} \to \mathcal{L}(\mathcal{H}_s, \mathcal{H}_{-s})$ is C^k as a function of λ and W.*

Proof. We first compute the partial Fréchet derivative of $R(\lambda, W) := (\overline{H} + W - \lambda - i0)^{-1}$ with respect to W. We will see that it is given by

$$R_W'(\lambda, W)\widetilde{W} = -R(\lambda, W)\widetilde{W}R(\lambda, W). \tag{18}$$

Indeed, by the resolvent equation we have

$$\begin{aligned}
&\big\|R(\lambda, W+\widetilde{W}) - R(\lambda, W) + R(\lambda, W)\widetilde{W}R(\lambda, W)\big\|_{\mathcal{L}(\mathcal{H}_s,\mathcal{H}_{-s})} \\
&\quad = \big\|-R(\lambda, W+\widetilde{W})\widetilde{W}R(\lambda, W) + R(\lambda, W)\widetilde{W}R(\lambda, W)\big\|_{\mathcal{L}(\mathcal{H}_s,\mathcal{H}_{-s})} \\
&\quad = \big\|R(\lambda, W+\widetilde{W})\widetilde{W}R(\lambda, W)\widetilde{W}R(\lambda, W)\big\|_{\mathcal{L}(\mathcal{H}_s,\mathcal{H}_{-s})} \\
&\quad \leqslant \big\|R(\lambda, W+\widetilde{W})\big\|_{\mathcal{L}(\mathcal{H}_s,\mathcal{H}_{-s})}\big\|R(\lambda, W)\big\|^2_{\mathcal{L}(\mathcal{H}_s,\mathcal{H}_{-s})}\|\widetilde{W}\|^2_{\mathcal{L}(\mathcal{H}_{-s},\mathcal{H}_s)}.
\end{aligned} \tag{19}$$

Since

$$R(\lambda, W) = R(\lambda, W+\widetilde{W})\big(I + \widetilde{W}R(\lambda, W)\big),$$

and since $R(\lambda, W)$ is bounded from $\mathcal{H}_s$ to $\mathcal{H}_{-s}$, we have for $\|\widetilde{W}\|_{\mathcal{L}(\mathcal{H}_{-s},\mathcal{H}_s)}$ small

$$R(\lambda, W+\widetilde{W}) = R(\lambda, W)\big(I + \widetilde{W}R(\lambda, W)\big)^{-1},$$

so that

$$\begin{aligned}
&\big\|R(\lambda, W+\widetilde{W})\big\|_{\mathcal{L}(\mathcal{H}_s,\mathcal{H}_{-s})} \\
&\quad \leqslant \big\|R(\lambda, W)\big\|_{\mathcal{L}(\mathcal{H}_s,\mathcal{H}_{-s})}\big(1 - \|\widetilde{W}\|_{\mathcal{L}(\mathcal{H}_{-s},\mathcal{H}_s)}\big\|R(\lambda, W)\big\|_{\mathcal{L}(\mathcal{H}_s,\mathcal{H}_{-s})}\big)^{-1},
\end{aligned}$$

and so $\|R(\lambda, W+\widetilde{W})\|_{\mathcal{L}(\mathcal{H}_s,\mathcal{H}_{-s})}$ is uniformly bounded with respect to $\widetilde{W}$, for $\widetilde{W}$ small. This proves (18). By induction in (19) it follows that R is C^∞ in the W-variable, and that $R_W^{(j)}(\lambda, W)$ is a multilinear map such that $R_W^{(j)}(\lambda, W)(\widetilde{W}_1, \dots, \widetilde{W}_j)$ is of the form $M(\lambda, W; \widetilde{W}_1, \dots, \widetilde{W}_j)$, where $M(\lambda, W; \widetilde{W}_1, \dots, \widetilde{W}_j)$ is a sum of products with $2j+1$ factors where every second factor is $(\overline{H} + W - \lambda - i0)^{-1}$ and every second factor is $\widetilde{W}_l$ for some $l \in \{1, \dots, j\}$.

By Proposition 1 the derivatives $\frac{\partial^j}{\partial\lambda^j}R(\lambda, W)$ exist for $j \leqslant k$, and since the terms of $R_W^{(j)}(\lambda, W; \widetilde{W}_1, \dots, \widetilde{W}_j)$ are compositions of resolvents and $\widetilde{W}_l$, it follows from the product rule that the mixed derivatives exist and are continuous up to order k in λ when we apply the W-derivatives first and then the λ-derivatives.

To prove that the partial derivatives taken in another order exist and are continuous, we will use a corresponding result from calculus. Let f be a function of $x_1, \dots, x_n$ such that all the mixed partial derivatives up to order m exist and are continuous when the partial derivatives are taken in the order of increasing index of the variables. Hence, we assume that $\partial^r f/\partial x_{l_1} \dots \partial x_{l_r}$ exist and are continuous for every $r \leqslant m$ and every $l_1, \dots, l_r$ such that $l_1 \leqslant l_2 \leqslant \dots \leqslant l_r$. Then $f \in C^m(\mathbb{R}^n)$. The proof when $m = 2$ follows from Theorem 1 of [6, p. 163], and the general case follows by induction on the order of the derivative.

Let $1 \leqslant m \leqslant k$. The mth Gateaux derivative of R (if it exists) is given by

$$\frac{\partial}{\partial t_1}\cdots\frac{\partial}{\partial t_m}R\left((\lambda, W) + \sum_{l=1}^{m} t_l(\widetilde{\lambda}_l, \widetilde{W}_l)\right)\Bigg|_{\substack{t_l=0\\ l=1,\dots,m}}. \tag{20}$$

Let $0 \leqslant r \leqslant m$ be arbitrary. By choosing $\widetilde{\lambda}_l = 0$ for $l = 0, \ldots, r$ and $\widetilde{W}_l = 0$ for $l = r + 1, \ldots, m$, we get from (20) the mixed partial derivative where we first differentiate $l - r$ times with respect to W and then r times with respect to λ. This is the derivative which we know exists and is continuous for $|\lambda - \lambda_0|$, $\|W\|_X$, s_l, t_l small. By the calculus result quoted above, we may change the order of differentiation in (20), and we see that all the mixed partial derivatives of total order m of R exist and are continuous. To conclude that R is k times Gateaux differentiable, we let g be the function

$$g(s_1, \ldots, s_m, t_1, \ldots, t_m) := R\left((\lambda, W) + \sum_{l=1}^{m} \bigl(s_l(\widetilde{\lambda}_l, 0) + t_l(0, \widetilde{W}_l)\bigr)\right),$$

for an arbitrary choice of $\widetilde{\lambda}_l$ and $\widetilde{W}_l$, $l = 1, \ldots, m$. Note that g has continuous partial derivatives of order m. Let $h(t_1, \ldots, t_m) := g(t_1, \ldots, t_m, t_1, \ldots, t_m)$ and note that by (20) that the mth order Gateaux derivative of R is just a mixed partial derivative of h. We obtain from the chain rule that h is m times continuously differentiable, and so (20) holds for any choice of $\widetilde{\lambda}_l$, $\widetilde{W}_l$, $l = 1, \ldots, m$, and since $m \in \{1, \ldots, k\}$ was arbitrary we see that all the Gateaux derivatives up to order k are continuous with respect to λ and W in a neighborhood of $(\lambda_0, 0)$ and multilinear. By [5, p. 73], R is also k times continuously Fréchet differentiable in this neighborhood. □

4. The equation $Q(\lambda + i0, W)f = f$

Proposition 4 leads us to the study of the equation

$$Q(\lambda + i0, W)f = f. \tag{21}$$

If (21) holds, then

$$\langle f, f\rangle = \bigl\langle Q(\lambda + i0, W)f, f\bigr\rangle = \bigl\langle f, Q(\lambda - i0, W)f\bigr\rangle = \bigl\langle Q(\lambda - i0, W)f, f\bigr\rangle.$$

By (13) it follows that $\langle \delta(\overline{H} + W - \lambda)f, f\rangle = 0$.

Note that $\langle \delta(\overline{H} + W - \lambda)f, g\rangle$ defines a sesquilinear form on $\mathcal{H}_s$, for which the Schwarz inequality holds. Hence, for every $g \in \mathcal{H}_s$

$$\bigl|\bigl\langle \delta(\overline{H} + W - \lambda)f, g\bigr\rangle\bigr|^2 \leqslant \bigl\langle \delta(\overline{H} + W - \lambda)f, f\bigr\rangle\bigl\langle \delta(\overline{H} + W - \lambda)g, g\bigr\rangle = 0.$$

It follows that (21) is equivalent to

$$\delta(\overline{H} + W - \lambda)f = 0, \qquad A(\lambda, W)f = f, \tag{22}$$

where $A(\lambda, W)$ is given by (12).

We first study the second equation of (22) for $f \in \operatorname{Ran} P_0$. We focus on the non-degenerate case, i.e. we assume that λ_0 has multiplicity 1.

Proposition 7. *Suppose that* (H1), (H2)*, and* (H3)$_k$ *with* $k \geqslant 1$ *are satisfied, and that the eigenvalue* λ_0 *has multiplicity* 1. *Suppose* $s > s_1$. *Then the second equation of* (22) *defines* $\lambda = \lambda(W)$ *in a neighborhood of* $(\lambda, W) = (\lambda_0, 0) \in \mathbb{R} \times X_s$. *Moreover,* $\lambda(\cdot)$ *is a* C^k *function and* $\lambda'(0)\widetilde{W} = \langle \varphi_0, \widetilde{W}\varphi_0\rangle$.

Proof. It is natural to identify the operator $A(\lambda, W)$ with the function $\langle \varphi_0, A(\lambda, W)\varphi_0 \rangle$, where $P_0\varphi_0 = \varphi_0$ and $\|\varphi_0\|_{\mathcal{H}} = 1$. We then have

$$A(\lambda, W) = \frac{1}{2}\big\langle \varphi_0, \big((\overline{H} + W - \lambda - i0)^{-1} + (\overline{H} + W - \lambda + i0)^{-1}\big)\varphi_0 \big\rangle. \tag{23}$$

Since by Proposition 6, $Q \in C^k(J \times \mathcal{O}; \mathbb{C})$ where $J \times \mathcal{O}$ is a neighborhood of $(\lambda_0, 0)$ in $\mathbb{R} \times X_s$ it follows that also $A \in C^k(J \times \mathcal{O}; \mathbb{C})$. By self-adjointness of H and W, we have for every $\epsilon > 0$ that

$$\big((\overline{H} + W - \lambda - i\epsilon)^{-1}\big)^* = (\overline{H} + W - \lambda + i\epsilon)^{-1}.$$

It follows that $A(\lambda, W) = \overline{A(\lambda, W)}$, and so $A \in C^k(J \times \mathcal{O}; \mathbb{R})$.

By (23) and since φ_0 is an eigenvector of H with eigenvalue λ_0,

$$A(\lambda, 0) = \frac{1}{\lambda_0 + 1 - \lambda} =: c(\lambda).$$

Observing that $c'(\lambda) = 1/(\lambda_0 + 1 - \lambda)^2$, we see that

$$A'_\lambda(\lambda_0, 0) = 1,$$

and $A(\lambda_0, 0) = 1$. By the implicit function theorem $A(\lambda, W) = 1$ defines λ as a C^k function of W in a neighborhood of $\lambda = \lambda_0$ and for $W \in X_s$ small.

By (18), and since φ_0 is an eigenvector of H with eigenvalue λ_0, we obtain

$$A'_W(\lambda_0, 0)W = -\langle \varphi_0, W\varphi_0 \rangle.$$

Since $A'_\lambda(\lambda_0, 0) = 1$ it follows that $\lambda'(0)W = \langle \varphi_0, W\varphi_0 \rangle$. □

Proposition 8. *Suppose that* (H1), (H2) *and* $(\mathrm{H3})_0$ *are satisfied and* J_1, γ, s_1, *and* δ_1 *are as in Proposition* 1. *Then if* $s > s_1$, $\|W\|_{\mathcal{L}(\mathcal{H}_{-s}, \mathcal{H}_s)} < \gamma$, *and* $\lambda \in J_1$, *we then have the following perturbation formula for* $\delta(\overline{H} + W - \lambda)$:

$$\delta(\overline{H} + W - \lambda) = \big(I - (\overline{H} + W - \lambda - i0)^{-1}W\big)\delta(\overline{H} - \lambda)\big(I - W(\overline{H} + W - \lambda + i0)^{-1}\big).$$

Proof. We have

$$(\overline{H} + W - \lambda - i\epsilon)^{-1} = \big(I - (\overline{H} + W - \lambda - i\epsilon)^{-1}W\big)(\overline{H} - \lambda - i\epsilon)^{-1},$$
$$(\overline{H} + W - \lambda + i\epsilon)^{-1} = (\overline{H} - \lambda + i\epsilon)^{-1}\big(I - W(\overline{H} + W - \lambda + i\epsilon)^{-1}\big),$$

which imply

$$
\begin{aligned}
&(\overline{H}+W-\lambda-i\epsilon)^{-1}\frac{\epsilon}{\pi}(\overline{H}+W-\lambda+i\epsilon)^{-1}\\
&\quad=\big(I-(\overline{H}+W-\lambda-i\epsilon)^{-1}W\big)(\overline{H}-\lambda-i\epsilon)^{-1}\\
&\qquad\times\frac{\epsilon}{\pi}(\overline{H}-\lambda+i\epsilon)^{-1}\big(I-W(\overline{H}+W-\lambda+i\epsilon)^{-1}\big).
\end{aligned}
$$

By (13), this proves the proposition. □

5. Finite multiplicity of the continuous spectrum

Proof of Theorem 1. By $(\mathrm{H3})_k$, $\operatorname{Ran}\delta(\overline{H}-\lambda)\subset\mathcal{H}_{-s}$ for $s>s_1$. By (H4), $\operatorname{Ran}\delta(\overline{H}-\lambda_0)$ is m-dimensional. Let $\varphi_1,\dots,\varphi_m\in\mathcal{H}_s$ and $f_1,\dots,f_m\in\mathcal{H}_{-s}$ be linearly independent and satisfy

$$\delta(\overline{H}-\lambda_0)\varphi_j=f_j. \tag{24}$$

We may without loss of generality assume that f_j for $j=1,\dots,m$ are real. Indeed, suppose that there are only $j\leqslant m-1$ real linearly independent vectors $f_1,\dots,f_j\in\operatorname{Ran}\delta(\overline{H}-\lambda_0)$. Since $\operatorname{Ran}\delta(\overline{H}-\lambda_0)$ is m-dimensional, we can choose $f\in\operatorname{Ran}\delta(\overline{H}-\lambda_0)$ such that $f_1,\dots,f_j,f$ are linearly independent. Let $\operatorname{Re}f=(f+Cf)/2$ and $\operatorname{Im}f=(f-Cf)/2i$ so that $f=\operatorname{Re}f+i\operatorname{Im}f$. It is not possible that both $\operatorname{Im}f$ and $\operatorname{Re}f$ are linear combinations of $f_1,\dots,f_j$, since if they are then so is f. Hence one of $\operatorname{Im}f$ and $\operatorname{Re}f$ is not a linear combination of $f_1,\dots,f_j$, say $\operatorname{Re}f$. But then there are $j+1$ real linearly independent vectors that span $\operatorname{Ran}\delta(\overline{H}-\lambda_0)$, contradicting our assumption that only j such vectors exist.

For W in a sufficiently small neighborhood of $0\in X_s$, let

$$f_j(W):=\delta\big(\overline{H}+W-\lambda(W)\big)\varphi_j,$$

where $\lambda(W)$ is defined as in Proposition 7. Note that $f_j(0)=f_j$. By Proposition 6, $f_j(\cdot)\in C^k(X_s;\mathcal{H}_{-s})$. Note that $(I-W(\overline{H}+W-\lambda+i0)^{-1}):\mathcal{H}_s\to\mathcal{H}_s$ and $(I-(\overline{H}+W-\lambda-i0)^{-1}W):\mathcal{H}_{-s}\to\mathcal{H}_{-s}$ are invertible. Indeed, the inverses are given by $(I+W(\overline{H}-\lambda+i0)^{-1})$ and $(I+(\overline{H}-\lambda-i0)^{-1})$, respectively. Then by Proposition 8 and (H4), $\{f_j(W);\ j=1,\dots,m\}$ span the m-dimensional subspace $\operatorname{Ran}\delta(\overline{H}+W-\lambda(W))\subset\mathcal{H}_{-s}$ if $\|W\|_{\mathcal{L}(\mathcal{H}_{-s},\mathcal{H}_s)}$ is sufficiently small. Let $g_l\in\mathcal{H}_s$, $l=1,\dots,m$, be such that

$$\big\langle f_j(0),g_l\big\rangle=\delta_{jl} \tag{25}$$

for $j,l\in\{1,\dots,m\}$. Note that we may assume that also the g_l are real, since

$$\big\langle f_j(0),Cg_l\big\rangle=\langle Cf_j,Cg_l\rangle=\overline{\langle f_j,g_l\rangle}=\overline{\delta_{jl}}=\delta_{jl}.$$

Hence we may replace g_l by $(g_l+Cg_l)/2$.

We claim that for $W\in X_s$ small, the equation $\delta(\overline{H}+W-\lambda(W))\varphi_0=0$ is equivalent to

$$F_j(W):=\big\langle g_j,\delta\big(\overline{H}+W-\lambda(W)\big)\varphi_0\big\rangle=0,\quad j=1,\dots,m. \tag{26}$$

To verify this, it suffices to show that (26) implies that $\delta(\overline{H}+W-\lambda(W))\varphi_0=0$ for $\|W\|_{\mathcal{L}(\mathcal{H}_{-s},\mathcal{H}_s)}$ small, since the other implication is trivial. Write $\delta(\overline{H}+W-\lambda(W))\varphi_0=\sum_{l=1}^m \alpha_l(W)f_l(W)$, and suppose that (26) holds. Then for every $j\in\{1,\dots,m\}$,

$$\sum_{l=1}^m \alpha_l(W)\langle g_j, f_l(W)\rangle=0. \tag{27}$$

Note that the $m\times m$-matrix with entries $\langle g_j, f_l(W)\rangle$ is continuous and equal to the identity matrix when $W=0$. Hence it is invertible for $\|W\|_{\mathcal{L}(\mathcal{H}_{-s},\mathcal{H}_s)}$ small, and from (27) we obtain $\alpha_j(W)=0$ for $\|W\|_{\mathcal{L}(\mathcal{H}_{-s},\mathcal{H}_s)}$ small.

By Proposition 6 we have for some neighborhood $\mathcal{O}$ of $0\in X_s$, $F\in C^k(\mathcal{O};\mathbb{C}^m)$, where F_j are the components of F. Note that φ_0 can be chosen real since otherwise we may replace φ_0 by its real or imaginary parts (i.e. $(\varphi_0+C\varphi_0)/2$ or $(\varphi_0-C\varphi_0)/2$). From our choice of g_j it now follows that $F\in C^k(\mathcal{O};\mathbb{R}^m)$.

We need to calculate $F_j'(0)W$. Note that $\delta(\overline{H}-\lambda)\varphi_0=0$ for every real λ in a neighborhood of λ_0, and that $(\overline{H}-\lambda_0+i0)^{-1}\varphi_0=\varphi_0$, and so it follows from the chain rule, Proposition 6 and Proposition 8 that for $\lambda=\lambda_0$,

$$\begin{aligned} F_j'(0)W &= -\langle g_j, \delta(\overline{H}-\lambda)W(\overline{H}-\lambda+i0)^{-1}\varphi_0\rangle - \langle g_j, (\overline{H}-\lambda-i0)^{-1}W\delta(\overline{H}-\lambda)\varphi_0\rangle \\ &\quad + \langle\varphi_0, W\varphi_0\rangle\left\langle g_j, \frac{d}{d\lambda}\delta(\overline{H}-\lambda)\varphi_0\right\rangle \\ &= -\langle g_j, \delta(\overline{H}-\lambda_0)W\varphi_0\rangle. \end{aligned}$$

We need to show that $F_1'(0),\dots,F_m'(0)$ are linearly independent. To see this, let

$$\sum_{j=1}^m \alpha_j F_j'(0)=0,$$

and let $g:=\sum_{j=1}^m \alpha_j g_j$. Then for every $W\in X_s$,

$$\langle g, \delta(\overline{H}-\lambda_0)W\varphi_0\rangle=0.$$

By (H5) and by the definition of g it follows that $g=0$, and so by the linear independence of $g_1,\dots,g_m$, we obtain that $\alpha_j=0$ for $j=1,\dots,m$, and we conclude that $F_1'(0),\dots,F_m'(0)$ are linearly independent.

We are now able to make the decomposition $X_s=\ker F'(0)\oplus\mathcal{M}$, where $\mathcal{M}$ has dimension m. Moreover, the map $F'(0):\mathcal{M}\to\mathbb{R}^m$ is a linear homeomorphism and $F(0)=0$. For $W\in X_s$, we write $W=\xi+\eta$ where $\xi\in\ker F'(0)$ and $\eta\in\mathcal{M}$. We also use the notation $F(\xi,\eta)=F(\xi+\eta)$. By the implicit function theorem the equation $F(W)=0$ can be solved for η in terms of ξ, i.e. $\eta=\eta(\xi)$ in a neighborhood of 0, and for some neighborhood U of $0\in\ker F'(0)$, $\eta\in C^k(U;\mathcal{M})$. This defines a C^k manifold of codimension m in a neighborhood of 0. □

6. Applications to elliptic differential operators

Here we present some examples for which the assumptions (H1)–(H5) can be verified.

Example 1. Let $\mathcal{H} := L^2(\mathbb{R})$, and let $\mathcal{H}_s := L^2_s(\mathbb{R})$, where $L^2_s(\mathbb{R})$ is the Hilbert space of functions ψ such that $(1+x^2)^{s/2}\psi(x)$ is square integrable. Let $H := d^4/dx^4 + V(x)$, where $V \in C^{k+2}(\mathbb{R})$ is a real potential satisfying

(i) $\sup_{x\in\mathbb{R}} |V^{(j)}(x)|(1+|x|^2)^{j/2} < \infty$ for $j = 0, \dots, k+2$,
(ii) $\sup_{x\in\mathbb{R}} |V(x)|(1+|x|^2)^q < \infty$, where $q > 1/2$.

We consider H as an unbounded operator in $L^2(\mathbb{R})$ with its domain being the Sobolev space $W^{4,2}(\mathbb{R})$ (also denoted by $H^4(\mathbb{R})$), where $Hu = \frac{d^4u}{dx^4} + Vu$ for $u \in \mathrm{Dom}(H)$. Denote by H_0 the same operator with $V \equiv 0$. It is readily seen (by Fourier transform) that H_0 is a self-adjoint operator with $\sigma(H_0) = \sigma_c(H_0) = \mathbb{R}_+$. Since V is a bounded function on $\mathbb{R}$ which tends to 0 as $|x| \to \infty$, it follows by well-known results that H is a self-adjoint operator and that $\sigma_c(H) = \sigma_c(H_0) = \mathbb{R}_+$. Thus in particular H satisfies assumption (H1). We shall assume now that

(iii) H has an embedded eigenvalue $\lambda_0 > 0$ with multiplicity 1.[1]

We take $s > k + 1/2$ and $k \geqslant 1$. As the space of perturbations X_s, we choose the set of real multiplication operators in $\mathcal{L}(\mathcal{H}_{-s}, \mathcal{H}_s)$, i.e. multiplication by real functions W on $\mathbb{R}$ which satisfy

$$\sup_{x\in\mathbb{R}} \left(1+|x|^2\right)^s \left|W(x)\right| < \infty.$$

As the antiunitary involution we choose complex conjugation. Below we show that the conditions (H2)–(H5) are satisfied for this operator with $s_1 = k + 1/2$ in condition $(\mathrm{H3})_k$ and $m = 2$ in condition (H4). Theorem 1 then implies that the set of small perturbations which do not remove the embedded eigenvalue is a C^k manifold in X_s of codimension 2.

To verify (H2), we use Theorem 4.1 and inequality (4.1) of [4] or Theorem 30.2.10 of [15], which shows that the eigenvalues of H can only accumulate at 0 and that the eigenfunctions belong to $\mathcal{H}_s$ for every s. Theorem 30.2.10 of [15] shows that $(\mathrm{H3})_0$ is satisfied for $s > 1/2$.[2]

To prove $(\mathrm{H3})_k$, we need to verify the assumptions of Theorem 2.2 of [18] with $A = \frac{1}{8}(xD + Dx)$, where $D := -i\,d/dx$. The calculations are similar to those in Section I of [22] and Section 5 of [18], but we include them here for the convenience of the reader.

Note that $\mathscr{S}(\mathbb{R})$ is a common core for $\overline{H}$ and A, and so we may compute the commutators on $\mathscr{S}(\mathbb{R})$. Hence (a) of Definition 2.1 in [18] is satisfied, i.e. $\mathrm{Dom}(A) \cap \mathrm{Dom}(\overline{H})$ is a core for $\overline{H}$.

---

[1] That this can be achieved can be seen for the example when V is given by $V(x) = 20/\cosh^2 x - 24/\cosh^4 x$. A short calculation shows that $\lambda_0 = 1$ is an embedded eigenvalue and that the corresponding eigenfunction is $\varphi_0(x) = 1/\cosh x$. It follows from ODE theory [8] that λ_0 is a simple eigenvalue, since the equation $d^4\psi/dx^4 + V(x)\psi = \lambda_0\psi$ has exactly one linearly independent solution which decays as $x \to +\infty$ (or $x \to -\infty$).

[2] (H2) and $(\mathrm{H3})_0$ could also be verified by the methods of [2].

Condition (b) of [18] is that $e^{i\theta A}$ maps $\mathrm{Dom}(\overline{H})$ into $\mathrm{Dom}(\overline{H})$ and for each $\psi \in \mathrm{Dom}(\overline{H})$,

$$\sup_{|\theta|\leqslant 1} \left\| \overline{H} e^{i\theta A}\psi \right\| < \infty. \tag{28}$$

To prove this, we use the formula

$$e^{i\theta A} f(x) = e^{\theta/8} f\left(e^{\theta/4}x\right), \tag{29}$$

which holds since the left- and right-hand sides of (29) define C_0 semigroups with the same infinitesimal generator iA. By the Hille–Yosida Theorem, the semigroups must be equal. By using (29), it is easy to see that $e^{i\theta A}$ maps $\mathrm{Dom}(\overline{H})$ into $\mathrm{Dom}(\overline{H})$ and that (28) holds.

Let $B_0 = \overline{H}$. The condition (c_{k+1}) of [18] requires that the forms $i^j B_j$ defined on $\mathrm{Dom}(\overline{H}) \cap \mathrm{Dom}(A)$ are all bounded from below and closable, and that $\mathrm{Dom}(B_j) \supset \mathrm{Dom}\,\overline{H}$, where B_j is the closure of $[B_{j-1}, A]$ for $j = 1, \dots, k+1$, and the commutator $[B_{j-1}, A]$ is interpreted as a quadratic form, i.e.

$$\left\langle \varphi, [B_{j-1}, A]\psi \right\rangle := \langle B_{j-1}\varphi, A\psi\rangle - \langle A\varphi, B_{j-1}\psi\rangle,$$

for $\varphi, \psi \in \mathrm{Dom}(A) \cap \mathrm{Dom}(\overline{H})$. To verify this, we first use Theorem 8.1 of [8] to show that φ_0 and its derivatives are exponentially decaying as $|x| \to \infty$. Indeed, after rewriting the eigenvalue equation as a system of four linear ODE's in the standard way, this theorem implies that φ_0, φ_0', φ_0'' and $\varphi_0^{(3)}$ are all exponentially decaying. From the eigenvalue equation $\varphi_0^{(4)} = \lambda_0\varphi_0 - V\varphi_0$ and by (i), it follows that $\varphi_0^{(4)}$ is exponentially decaying. We now differentiate this equation and proceed by induction. We see that $\varphi_0^{(j)}$ is exponentially decaying for $j = 1, \dots, k+6$. It also follows that $A^j\varphi_0$ is exponentially decaying for each $j \leqslant k+6$. In particular, $\varphi_0 \in \mathrm{Dom}(A^j)$ for $j = 1, \dots, k+6$ and $A^j P_0$ is defined for those j. A calculation shows that

$$i^j B_j = \frac{d^4}{dx^4} + \frac{(-1)^j}{4^j}\left(x\frac{d}{dx}\right)^j V(x) + i^j \sum_{l=0}^{j} (-1)^l \binom{j}{l} A^l P_0 A^{j-l} \tag{30}$$

is bounded from below and closable when $j \leqslant k+1$, and its closure (also denoted by B_j) has the domain $\mathrm{Dom}\,(B_j) = \mathrm{Dom}(H) = \mathrm{Dom}\,(\overline{H})$. Hence (c_{k+1}) of [18] is satisfied.

Condition (d_{k+1}) of [18] states that the form $[B_{k+1}, A]$ defined on $\mathrm{Dom}(\overline{H}) \cap \mathrm{Dom}(A)$ extends to a bounded operator from $\mathrm{Dom}(\overline{H})$ equipped with the graph norm to its dual obtained by the inner product on $\mathcal{H}$. Using (30), this is straightforward to check, since B_{k+2} is a bounded operator from $\mathrm{Dom}(\overline{H})$ to $L^2(\mathbb{R})$.

Finally, we verify the Mourre estimate (e) of [18], i.e. we need to verify that there exist $\alpha > 0$, $\delta > 0$, and a compact operator K on $\mathcal{H}$ such that

$$E_J(\overline{H}) i B_1 E_J(\overline{H}) \geqslant \alpha E_J(\overline{H}) + E_J(\overline{H}) K E_J(\overline{H}),$$

where $J := (\lambda_0 - \delta, \lambda_0 + \delta)$. Let $0 < \delta < \lambda_0$, and let $K := (-V + i[V, A] - P_0 + i[P_0, A])E_J(\overline{H})$. The assumption (ii) on V ensures that $(-V + i[V, A] - P_0 + i[P_0, A])$ is $\overline{H}$-compact, and hence K is compact. By (30) for $j = 1$,

$$\begin{aligned} E_J(\overline{H})iB_1E_J(\overline{H}) &= E_J(\overline{H})\big(\overline{H} - V + i[V,A] - P_0 + i[P_0,A]\big)E_{\overline{H}}(J) \\ &= E_J(\overline{H})\overline{H}E_J(\overline{H}) + E_J(\overline{H})KE_J(\overline{H}) \\ &\geqslant (\lambda_0 - \delta)E_J(\overline{H}) + E_J(\overline{H})KE_J(\overline{H}). \end{aligned}$$

According to [18] this shows that the limits

$$\lim_{\epsilon\downarrow 0}\big(1+A^2\big)^{-s/2}(\overline{H} - \lambda \pm i\epsilon)^{-1}\big(1+A^2\big)^{-s/2} = \big(1+A^2\big)^{-s/2}(\overline{H} - \lambda \pm i0)^{-1}\big(1+A^2\big)^{-s/2}$$

exist in $\mathcal{L}(\mathcal{H})$ and are C^k in λ in the norm topology of $\mathcal{L}(\mathcal{H})$ for λ in some interval around λ_0. We must now prove that A can be replaced by x in the latter statement. It suffices to take $s = k+1$. Using the resolvent equation repeatedly we see that it is enough to show that $A^s(\overline{H} + N)^{-n}(1+x^2)^{-s/2}$ is bounded for some large N and n. By interpolation it suffices to take $s = k+1$. Since A^sP_0 is bounded it is enough to show boundedness of $D^lx^l(H+N)^{-n}(1+x^2)^{-s/2}$, where $D = -id/dx$ and $l \leqslant s$. Let $R = (H+N)^{-1}$. We will control the terms generated by taking commutators with x^l by the following lemma, which is easily proved by induction.

Lemma 2. *Suppose that $l \geqslant 0$, $k \in \{0,1,2,3,4\}$, and $V \in C^{(l+k-4)_+}(\mathbb{R})$ with bounded derivatives. Then*

$$R : \mathcal{H}^l(\mathbb{R}) \to \mathcal{H}^{l+k}(\mathbb{R})$$

is bounded.

Commuting x^l through R^n produces terms with factors of n resolvents interspersed with ($\leqslant s$) factors of the form D^jR where $j = 0, 1, 2$ or 3. The string of factors always begins with R on the left. Let us use the new factors D^jR to map $H^l \to H^l$ and the old factors R to increase the Sobolev index to s which we write as $s = 4k_0 + m$ where $m = 0, 1, 2$, or 3. We will then know that D^s times the operator string is bounded. The only question is how many bounded derivatives of V this requires. Suppose after applying a string including r of the original R's and any number of the new D^jR's to $L^2(\mathbb{R})$ we find ourselves in H^{4r} needing at most $4r - 1$ bounded derivatives of V. According to Lemma 2, applying r' additional R's brings us to $H^{4r+4r'}$ needing $4r + 4r' - 4$ bounded derivatives of V. Applying any number of D^jR's requires at most $4r + 4r' - 1$ bounded derivatives to stay in $H^{4r+4r'}$. Thus inductively we can reach H^{4k_0} needing at most $4k_0 - 1$ bounded derivatives. We use the last R to reach H^{4k_0+m} with no further derivatives needed. Thus the requirement that $A^s(\overline{H} + N)^{-n}(1+x^2)^{-s/2}$ is bounded requires at most $s - 1 = k$ bounded derivatives which we have by assumption (i).

This concludes the proof of $(\mathrm{H3})_k$.

We proceed by verifying (H4). More precisely, we will check that $\dim \operatorname{Ran} \delta(\overline{H} - \lambda) = 2$ for λ in a neighborhood of λ_0. Let $H_0 := d^4/dx^4$. It is clear that $\dim \operatorname{Ran}(H_0 - \lambda) = 2$ if $\lambda > 0$. Indeed, the range is the span of the functions $e^{i\lambda^{1/4}\cdot}$ and $e^{-i\lambda^{1/4}\cdot}$. We now apply Proposition 8 with H_0 taking the place of H and $P_0 + V$ taking the place of W. We then get

$$\delta(\overline{H} - \lambda) = \big(I - (\overline{H} - \lambda - i0)^{-1}(P_0 + V)\big)\delta(H_0 - \lambda)\big(I - (P_0 + V)(\overline{H} - \lambda + i0)^{-1}\big).$$

Since $(I - (P_0 + V)(\overline{H} - \lambda + i0)^{-1})$ and $(I - (\overline{H} - \lambda - i0)^{-1}(P_0 + V))$ are invertible (with inverses $(I + (P_0 + V)(H_0 - \lambda + i0)^{-1})$ and $(I + (H_0 - \lambda - i0)^{-1}(P_0 + V))$, respectively), this gives a one-to-one correspondence between the range of $\delta(H_0 + V - \lambda)$ and the range of $\delta(\overline{H} + V - \lambda)$. We conclude that the dimensions of the ranges must be equal.

(H5) is satisfied since $\{W\varphi_0;\ W \in X_s\}$ is dense in $\mathcal{H}_s$, which follows since the zero set of φ_0 is at most countable with no accumulation points.

Example 2. Let M be the infinite cylinder $\mathbb{R} \times S^1$ with generic point $x = (z, \theta)$. M is considered as a Riemannian manifold with the metric $dx^2 = dz^2 + d\theta^2$. Let H be a Schrödinger operator on M of the form

$$H := -\Delta + V = -\left(\frac{d^2}{dz^2} + \frac{d^2}{d\theta^2}\right) + V(z),$$

where V is a real function in $C_0^\infty(\mathbb{R})$, and consider H as an unbounded operator in $L^2(M)$ with the domain being the Sobolev space $W^{2,2}(\mathbb{R} \times S^1)$. Viewing H as a perturbation of the operator $H_0 := -\Delta$, it is easy to see (as in Example 1) that H is a self-adjoint operator and that $\sigma_c(H) = \sigma_c(H_0) = \mathbb{R}_+$. Now, we shall choose the potential $V \in C_0^\infty(\mathbb{R})$ so that $h := -d^2/dz^2 + V$ on $L^2(\mathbb{R})$ has exactly one eigenvalue $e < 0$ of multiplicity 1. It is possible to choose V such that $e > -1$. We denote the corresponding eigenfunction by f. Let $n \geqslant 1$ and let $F(z, \theta) = \cos(n\theta) f(z)$. Then $HF = \lambda_0 F$ with $\lambda_0 = n^2 + e > 0$, and so λ_0 is an embedded eigenvalue. Note that the multiplicity of λ_0 is 2 since $\sin(n\theta) f(z)$ is also an eigenfunction. This degeneracy can be removed by letting $\mathcal{H}$ be the subspace of $L^2(M)$ consisting of functions which are even in the θ-variable. Let

$$\mathcal{H}_s = \left\{f \in \mathcal{H};\ \left(1 + z^2\right)^{s/2} f \in L^2(M)\right\},$$

and let X_s be the space of real multiplication operators W which are even in θ and satisfy

$$\sup_{\substack{z\in\mathbb{R},\\ \theta\in S^1}} \left(1 + |z|^2\right)^s \left|W(z, \theta)\right| < \infty.$$

The antiunitary involution is complex conjugation. Below we show that conditions (H1)–(H4) are satisfied for this operator for any fixed $k \geqslant 1$ with $s_1 = k + 1/2$ in condition $(\mathrm{H3})_k$ and $m = 2n$ in condition (H4). Then Theorem 1 tells us that the set of small perturbations for which the embedded eigenvalue persists is a C^k manifold in X_s of codimension $2n$.

Assumptions (H2) and $(\mathrm{H3})_0$ are verified (via separation of variables) using, as in Example 1, well-known limiting absorption results for the Schrödinger operator on $\mathbb{R}$ (e.g. [2,15]). We omit the details.

To verify $(\mathrm{H3})_k$, we apply the result from [18] with $A = (zD + Dz)/4$, where $D = -i\partial/\partial z$. Note that $i[H_0, A] = D^2$, and let $J = (\lambda_0 - \delta, \lambda_0 + \delta)$, where $\delta < \min(1, \min_{j\in\mathbb{Z}} |\lambda_0 - j^2|) =: \alpha$. Let $E_J(\overline{H})$ be the spectral projection of $\overline{H}$ in $L^2(\mathbb{R} \times S^1)$ corresponding to the interval J, and let $1_J(h)$ be the spectral projection of h in $L^2(\mathbb{R})$ corresponding to J. Let $Q_n = 1_{\{n^2\}}(-\partial_\theta^2)$ in $L^2(S^1)$. Note that

$$E_J(\overline{H}) = E_J(\overline{H})P_0 + E_J(\overline{H})(I - P_0) = E_J(H)(I - P_0),$$

since $J \cap \{\lambda_0 + 1\} = \emptyset$. By the choice of J,

$$J \cap \sigma\big(h + j^2\big) = J \cap \big(\big[j^2, \infty\big) \cup \big\{e + j^2\big\}\big) = \begin{cases} \{\lambda_0\} & \text{if } j^2 = n^2, \\ \emptyset & \text{if } j^2 > \lambda_0 \text{ and } j^2 \neq n^2, \\ J & \text{if } j^2 < \lambda_0, \end{cases}$$

where $\sigma(h + j^2)$ is the spectrum of $h + j^2$ in $L^2(\mathbb{R})$. Note that

$$P_0 = 1_{\{\lambda_0\}}\big(h + n^2\big) \otimes Q_n,$$

and that for $j^2 < \lambda_0$, we actually have $j^2 \leqslant \lambda_0 - \alpha$. A calculation using that $P_0(I - P_0) = 0$ shows that

$$\begin{aligned} E_J(\overline{H}) &= \bigg(\sum_{j^2 \leqslant \lambda_0 - \alpha} 1_J\big(h + j^2\big) \otimes Q_j + 1_{\{\lambda_0\}}\big(h + n^2\big) \otimes Q_n\bigg)(I - P_0) \\ &= \bigg(\sum_{j^2 \leqslant \lambda_0 - \alpha} 1_J\big(h + j^2\big) \otimes Q_j\bigg)(I - P_0). \end{aligned}$$

Moreover,

$$\begin{aligned} E_J(\overline{H})\big(-\partial_z^2 + V\big)E_J(\overline{H}) &= \bigg(\sum_{j^2 \leqslant \lambda_0 - \alpha} \big(h 1_J\big(h + j^2\big)\big) \otimes Q_j\bigg)(I - P_0) \\ &\geqslant \bigg(\sum_{j^2 \leqslant \lambda_0 - \alpha} \big(\lambda_0 - j^2 - \delta\big) 1_J\big(h + j^2\big) \otimes Q_j\bigg)(I - P_0) \\ &\geqslant (\alpha - \delta)\bigg(\sum_{j^2 \leqslant \lambda_0 - \alpha} 1_J\big(h + j^2\big) \otimes Q_j\bigg)(I - P_0) \\ &= (\alpha - \delta) E_J(\overline{H}). \end{aligned}$$

Note that

$$\begin{aligned} i B_1 E_{\overline{H}}(J) &= \big(\big(-\partial_z^2 + V\big) + i[V, A] - V + i[P_0, A] - P_0\big) E_{\overline{H}}(J) \\ &= \big(\big(-\partial_z^2 + V\big) + K\big) E_{\overline{H}}(J), \end{aligned}$$

where K is compact. Hence

$$\begin{aligned} E_J(\overline{H}) i B_1 E_J(\overline{H}) &= E_J(\overline{H})\big(-\partial_z^2 + V\big)E_J(\overline{H}) + E_J(\overline{H}) K E_J(\overline{H}) \\ &\geqslant (\alpha - \delta) E_J(\overline{H}) + E_J(\overline{H}) K E_J(\overline{H}), \end{aligned}$$

and $(\mathrm{H3})_k$ follows from [18] and an argument similar to the one in Example 1 which converts $(1 + A^2)^{s/2}$ to $(1 + z^2)^{s/2}$.

Condition (H4) is verified in the same way as in Example 1. Since $0 < \lambda_0^{1/2}$ and $\lambda_0^{1/2}$ is not an integer, $\operatorname{Ran}\delta(H_0 - \lambda)$ is the span of the functions $e^{ikz}\cos(j\theta)$, where $j^2 + k^2 = \lambda_0$, $j = 0, \ldots, n-1$, $k \in \mathbb{R}$. The number of such functions is $2n$, so $\dim \operatorname{Ran}\delta(H_0 - \lambda_0) =: m = 2n$. Now we proceed as in Example 1 and conclude that the multiplicity of the continuous spectrum is constant when $\lambda_0 \neq j^2$ for $j \in \mathbb{Z}$.

To prove (H5), we note that the set Z_f of zeroes of f is at most countable and has no accumulation points, and so the zero set Z_F of F is

$$\left(Z_f \times S^1\right) \cup \left(\mathbb{R} \times \left\{(1/2 + j)\pi/n;\ j = 0, \ldots, 2n-1\right\}\right),$$

which is a union of straight lines and circles which do not accumulate. Let

$$Z_F^\epsilon := \left\{(z,\theta) \in \mathbb{R} \times S^1;\ \left|F(z,\theta)\right| < \epsilon\right\},$$

and let χ_ϵ be a smooth cutoff function such that $0 \leqslant \chi_\epsilon \leqslant 1$ and

$$\chi_\epsilon(z,\theta) = \begin{cases} 0 & \text{for } (z,\theta) \in Z_F^\epsilon, \\ 1 & \text{for } (z,\theta) \notin Z_F^{2\epsilon}. \end{cases}$$

We also require that χ_ϵ is even in the θ-variable. Let $\varphi \in \mathcal{H}_s$, and let

$$W_\epsilon(z,\theta) := \begin{cases} \frac{\chi_\epsilon(z,\theta)\varphi(z,\theta)}{F(z,\theta)} & \text{if } F(z,\theta) \neq 0, \\ 0 & \text{if } F(z,\theta) = 0. \end{cases}$$

Then $W_\epsilon \in X_s$ and $\|W_\epsilon F - \varphi\|_{\mathcal{H}_s} \to 0$ as $\epsilon \to 0$. It follows that $\{WF;\ W \in X_s\}$ is dense in $\mathcal{H}_s$, and so (H5) holds.

7. On perturbations of degenerate embedded eigenvalues

In this section we assume that (H1), (H2), and $(\mathrm{H3})_k$, are satisfied. In the last theorem of this section we also assume (H4) and (H5′). Let n denote the multiplicity of the eigenvalue λ_0. We assume that $n \geqslant 2$.

We start by fixing an orthonormal basis $\psi_1, \ldots, \psi_n$ in the eigenspace of H at λ_0. We assume that all ψ_i are real. We first show that this is possible: Suppose that $\psi_1, \ldots, \psi_n$ is an ON-basis for $\operatorname{Ran} P_0$. If ψ_1 is not real, then we replace ψ_1 by $(\psi_1 + C\psi_1)/2$ or $(\psi_1 - C\psi_1)/(2i)$, and choose a new ON-basis $\psi_1, \ldots, \psi_n$ for $\operatorname{Ran} P_0$, where $C\psi_1 = \psi_1$. Now

$$C : \operatorname{span}\{\psi_2, \ldots, \psi_n\} \to \operatorname{span}\{\psi_2, \ldots, \psi_n\},$$

since if $C\psi_2 = \alpha\psi_1 + \psi_\perp$ where $\langle\psi_\perp, \psi_1\rangle = 0$, then

$$\langle C\psi_1, C\psi_2\rangle = \langle\psi_2, \psi_1\rangle = 0 = \langle\psi_1, C\psi_2\rangle = \overline{\alpha},$$

and so $\alpha = 0$. Now we replace ψ_2 by $(\psi_2 + C\psi_2)/2$ or $(\psi_2 - C\psi_2)/(2i)$ then renormalize the result and repeat. This shows that $\psi_1, \ldots, \psi_n$ can be assumed to be real.

Denote by P_i the orthogonal projection in $\mathcal{H}$ onto $\operatorname{span}\{\psi_i\}$. Recall from Section 2 that P_0 is the orthogonal projection onto the full eigenspace of H at λ_0 and $\overline{H} = H + P_0$.

Let $H_1 := H + P_0 - P_1 = \overline{H} - P_1$, and note that λ_0 is an embedded eigenvalue of multiplicity 1 of H_1, and that ψ_1 is a corresponding normalized eigenvector.

We give sufficient conditions for $H + W$ to have at least one embedded eigenvalue close to λ_0. The eigenvalue and eigenvector of $H + W$ that we will construct coincide with the eigenvalue and eigenvector of the operator $H_1 + W$.

To this end we first notice that by the proof of Proposition 7 there exists a C^k real valued function $\lambda_1(W)$, defined in a neighborhood of 0 in X_s, such that $\langle\psi_1, A(\lambda_1(W), W)\psi_1\rangle = 1$, $\lambda_1(0) = \lambda_0$ and $\lambda'(0)\widetilde{W} = \langle\psi_1, \widetilde{W}\psi_1\rangle$, where as usual $A(\lambda, W)$ denotes the operator (12) associated with H and the eigenvalue λ_0.

Remark 1. Note that the operator $Q_1(\lambda + i0, W)$ which is the operator $Q(\lambda + i0, W)$ corresponding to H_1 is given by $Q_1(\lambda + i0, W) = P_1(H_1 + P_1 + W - \lambda - i0)^{-1}P_1 = P_1(\overline{H} + W - \lambda - i0)^{-1}P_1$. It follows that $\langle\psi_1, A(\lambda, W)\psi_1\rangle = \langle\psi_1, A_1(\lambda, W)\psi_1\rangle$, where $A_1(\lambda, W)$ denotes the operator (12) associated with the operator H_1 and the eigenvalue λ_0.

Proposition 9. *Let $n \geqslant 2$, and assume that* (H1)*,* (H2)*, and* (H3)$_k$ *are satisfied for some $k \geqslant 1$. Suppose $s > s_1$. If $W \in X_s$ is sufficiently small, then $H_1 + W$ has an embedded eigenvalue in a neighborhood of λ_0 if and only if*

$$\delta\big(\overline{H} + W - \lambda_1(W)\big)\psi_1 = 0,$$

where λ_1 is the function defined above. Moreover the corresponding eigenvector is given by

$$\psi_1^W = \big(\overline{H} + W - \lambda_1(W) - i0\big)^{-1}\psi_1.$$

Proof. The result follows directly from Eq. (22), Propositions 7 and 5 and Remark 1. □

Theorem 2. *Suppose that* (H1)*,* (H2)*, and* (H3)$_k$ *are satisfied for some $k \geqslant 1$. Let $n \geqslant 2$ and $s > s_1$, and let λ_1 be the function defined above. Then there exists a neighborhood $\mathcal{O}$ of 0 in X_s such that if $W \in \mathcal{O}$, then a sufficient condition that $\lambda_1(W)$ is an eigenvalue of $H + W$ is that*

$$\big\langle\psi_i, A\big(\lambda_1(W), W\big)\psi_1\big\rangle = 0, \quad i = 2, \dots, n, \tag{31}$$

and

$$\delta\big(\overline{H} + W - \lambda_1(W)\big)\psi_1 = 0. \tag{32}$$

Proof. By (32) and Proposition 9, $\lambda_1(W)$ is an eigenvalue of $H_1 + W$ with corresponding eigenfunction ψ_1^W.

The conditions (31) and (32) together with Proposition 9 imply that

$$\big\langle\psi_i, \psi_1^W\big\rangle = \big\langle\psi_i, \big(\overline{H} + W - \lambda_1(W) - i0\big)^{-1}\psi_1\big\rangle = \big\langle\psi_i, A\big(\lambda_1(W), W\big)\psi_1\big\rangle = 0 \tag{33}$$

for $i = 2, \dots, n$ and W sufficiently small. In particular the eigenvector ψ_1^W of $H_1 + W$ is orthogonal to ψ_i, $i \neq 1$. Finally we note that $\lambda_1(W)$ is also an eigenvalue of $H + W$, and that ψ_1^W is a corresponding eigenvector. Indeed,

$$\big(H + W - \lambda_1(W)\big)\psi_1^W = \big(H_1 + W - \lambda_1(W)\big)\psi_1^W - \sum_{i=2}^{n} P_i\psi_1^W = 0. \qquad \square$$

For our final theorem in this section, we need the additional conditions (H4) and (H5′):

Theorem 3. *Suppose that* (H1), (H2), $(\mathrm{H3})_k$, (H4) *and* (H5′) *are satisfied for some* $k \geqslant 1$*, and suppose that* $n \geqslant 2$ *and* $s > s_1$*. Then there exists a* C^k *manifold* $\mathcal{M} \subset X_s$ *of codimension* $\nu := m + n - 1$*, a neighborhood* $\mathcal{O}$ *of* $0 \in X_s$ *and a* $\delta > 0$ *such that if* $W \in \mathcal{M} \cap \mathcal{O}$ *then* $H + W$ *has an embedded eigenvalue* $\lambda_1(W) \in (\lambda_0 - \delta, \lambda_0 + \delta)$.

Proof. The manifold $\mathcal{M}$ will be the set of $W \in X_s$ such that (31) and (32) are satisfied. Let φ_i, f_i and g_i, $i = 1, \dots, m$, be the functions defined in the proof of Theorem 1: The vectors $\varphi_i \in \mathcal{H}_s$, $i = 1, \dots, m$, are chosen so that $f_i = \delta(\overline{H} - \lambda_0)\varphi_i$ are real and span $\mathrm{Ran}(\delta(\overline{H} - \lambda_0))$. Then the vectors g_i, $i = 1, \dots, m$, are chosen to be real and so that they satisfy $\langle f_j, g_l\rangle = 0$ for $j, l = 1, \dots, m$. We also have to make sure that the first eigenvector $\psi_1 \in \mathrm{Ran}\, P_0$ is chosen so that (H5′) is satisfied for this ψ_1. This means that the other basis vectors which were chosen in the beginning of this section may have to be modified so that $\psi_1, \dots, \psi_n$ form an ON-basis.

As in the proof of Theorem 1, (32) is equivalent to $\langle g_i, \delta(\overline{H} + W - \lambda_1(W))\psi_1\rangle = 0$, $i = 1, \dots, m$, and hence we need to study the equations

$$\begin{aligned} &\big\langle \psi_i, A\big(\lambda_1(W), W\big)\psi_1\big\rangle = 0, \quad i = 2, \dots, n, \\ &\big\langle g_i, \delta\big(\overline{H} + W - \lambda_1(W)\big)\psi_1\big\rangle = 0, \quad i = 1, \dots, m. \end{aligned} \tag{34}$$

Note that there are ν conditions to be satisfied. We write (34) as

$$F(W) = 0,$$

where F maps a neighborhood of 0 in X_s to $\mathbb{R}^\nu$, and the components of F are the left-hand side of Eqs. (34) in some order.

By Proposition 6, F is a C^k function of W, and a calculation shows that the components of $F'(0)$ are given by the functionals

$$\begin{aligned} &W \mapsto -\big\langle g_i, \delta(\overline{H} - \lambda_0)W\psi_1\big\rangle, \quad i = 1, \dots, m, \\ &W \mapsto -\langle \psi_i, W\psi_1\rangle, \quad i = 2, \dots, n. \end{aligned} \tag{35}$$

We must show that these functionals are linearly independent, and so we let $g := \sum_{i=1}^{m} \alpha_i g_i$ and $\psi_\perp := \sum_{i=2}^{n} \beta_i \psi_i$. Then

$$\big\langle g, \delta(\overline{H} - \lambda_0)W\psi_1\big\rangle + \langle \psi_\perp, W\psi_1\rangle = 0$$

for every $W \in X_s$ holds if and only if

$$\big\langle \delta(\overline{H} - \lambda_0)g + \psi_\perp, W\psi_1\big\rangle = 0$$

for every $W \in X_s$. If this is true, then by (H5′),

$$\psi_\perp + \delta(\overline{H} - \lambda_0)g = 0.$$

But

$$\langle \psi_\perp, \delta(\overline{H} - \lambda_0)g \rangle = \langle \delta(\overline{H} - \lambda_0)\psi_\perp, g \rangle = 0$$

since $H\psi_\perp = \lambda_0 \psi_\perp$. Thus $\psi_\perp = 0$ (and hence $\beta_i = 0$) and $\delta(\overline{H} - \lambda_0)g = 0$. But

$$\langle g_j, \delta(\overline{H} - \lambda_0)\varphi_i \rangle = \delta_{ij}.$$

Thus

$$\langle g, \delta(\overline{H} - \lambda_0)\varphi_i \rangle = \alpha_i = \langle \delta(\overline{H} - \lambda_0)g, \varphi_i \rangle = 0,$$

which shows that the functionals in (35) are linearly independent.

Finally, we make the decomposition $X_s = (\ker F'(0)) \oplus \mathcal{V}$, where $\mathcal{V}$ has dimension ν and the map $F'(0) : \mathcal{V} \to \mathbb{R}^\nu$ is a linear homeomorphism. For $W \in X_s$, we write $W = \xi + \eta$ where $\xi \in \ker F'(0)$ and $\nu \in \mathcal{V}$. By the implicit function theorem, there is a neighborhood U of $0 \in \ker F'(0)$ and a C^k function $\eta : U \to \mathcal{V}$, $\xi \mapsto \eta(\xi)$ such that $\eta(0) = 0$ and $F(\xi + \eta(\xi)) = 0$ for $\|\xi\|_{X_s}$ small. □

Example 3. We revisit Example 2 when $H := -\Delta + V$ on $L^2(\mathbb{R} \times S^1)$, but this time we do not restrict to the subspace of functions which are even in the θ-variable. The multiplicity of the eigenvalue λ_0 is 2. Let $s = k + 1/2$, where k is a positive integer. The conditions (H1)–(H4) are verified as in Example 2, except that the multiplicity of the continuous spectrum is now $m := 4n - 2$ since $(n-1)^2 < \lambda_0 < n^2$. Let f be as in Example 2. Following the notation of this section, we choose

$$\psi_1(z, \theta) = \frac{1}{\sqrt{\pi}\|f\|_{L^2(\mathbb{R})}} f(z)\cos(n\theta),$$

$$\psi_2(z, \theta) = \frac{1}{\sqrt{\pi}\|f\|_{L^2(\mathbb{R})}} f(z)\sin(n\theta).$$

Then ψ_1 and ψ_2 are normalized eigenfunctions with eigenvalue $\lambda_0 = n^2 + e > 0$.

Theorem 3 guarantees the existence of a C^k manifold $\mathcal{M}$ of codimension $\nu := m + 1 = 4n - 1$ such that if W belongs to this manifold and is sufficiently small, then $H + W$ has an embedded eigenvalue close to λ_0.

Note that with a different choice of ψ_1, we would in general get a different manifold $\mathcal{M}$, and that there is a 1 parameter family of such normalized ψ_1. The set of small perturbations making the embedded eigenvalue persist is included in the union of these manifolds. The structure of the set of small perturbations making the embedded eigenvalue persist is not yet fully understood in this case.

8. Extensions to the case of infinite multiplicity of the continuous spectrum

In this section we study the example of the self-adjoint Schrödinger operator $H := -\Delta + V$ in the space $\mathcal{H} := L^2(\mathbb{R}^n)$, $n \geqslant 2$, in which case the continuous spectrum may have infinite multiplicity. We impose some conditions on V. First we assume that V is a real measurable locally bounded function on $\mathbb{R}^n$. We denote by $\dot{H}$ the symmetric operator in $L^2(\mathbb{R}^n)$ with $\mathrm{Dom}(\dot{H}) = C_0^\infty(\mathbb{R}^n)$ such that $\dot{H}u = -\Delta u + Vu$ for $u \in C_0^\infty(\mathbb{R}^n)$.

We assume that $\dot{H}$ is essentially self-adjoint. We note that a simple sufficient condition for $\dot{H}$ to be essentially self-adjoint is that $V_-(x) := \min\{V(x), 0\}$ is a bounded function. (For general conditions ensuring essential self-adjointness, see [20].)

We denote by H the self-adjoint operator which is the closure of $\dot{H}$ in $L^2(\mathbb{R}^n)$. We observe that if φ_0 is an eigenfunction of H, associated with the eigenvalue λ_0, then φ_0 is a continuous function. Indeed, since

$$\big\langle(-\Delta + V - \lambda_0)\psi, \varphi_0\big\rangle = \big\langle(H - \lambda_0)\psi, \varphi_0\big\rangle = \big\langle\psi, (H - \lambda_0)\varphi_0\big\rangle = 0$$

for all $\psi \in C_0^\infty(\mathbb{R}^n) \subset \mathrm{Dom}(H)$ and since $V \in L^\infty_{loc}(\mathbb{R}^n)$, it follows, using the L^p regularity theory of weak solutions of elliptic equations, that the eigenfunction φ_0 belongs to the Sobolev space $W^{2,p}_{loc}(\mathbb{R}^n)$ for any p, $1 < p < \infty$ (see e.g. Theorem 6.1 in [1]). It then follows from the Sobolev embedding theorem that the eigenfunction φ_0 is continuous. (More precisely, it follows that $\varphi_0 \in C^1(\mathbb{R}^n)$ and that the first order derivatives of φ_0 satisfy a local Hölder condition of any order < 1.)

We consider the Schrödinger operator H in the setup of Section 2. We choose for the $\mathcal{H}_s$ spaces the weighted L^2 spaces on $\mathbb{R}^n$ with weight $(1 + |x|^2)^s$ and we let the antiunitary involution C be complex conjugation. We shall show in the following that under assumptions (H2)–(H3)$_k$, embedded eigenvalues of H persists for a large class of perturbations W.

In the following, we denote by λ_0 some fixed embedded eigenvalue of H verifying assumption (H2). As usual, P_0 denotes the orthogonal projection on the eigenspace at λ_0. We denote by $\varphi_0(x)$ some fixed eigenfunction corresponding to λ_0. We assume that φ_0 is real and normalized. We also fix some ball $B_r(x_0) := \{x \in \mathbb{R}^n;\ |x - x_0| < r\}$ such that $\varphi_0(x) \neq 0$ on $\overline{B_r(x_0)}$. We introduce the following function spaces:

$$K_0 := \big\{f \in L^\infty\big(\mathbb{R}^n\big);\ f \text{ is real and } f = 0 \text{ a.e. in the complement of } B_r(x_0)\big\},$$
$$K_1 := \big\{f \in C^2\big(\mathbb{R}^n\big);\ f \text{ is real, } \operatorname{supp} f \subset \overline{B_r(x_0)} \text{ and } \langle f, \varphi\rangle = 0 \text{ for every } \varphi \in \operatorname{Ran} P_0\big\}.$$

K_0 and K_1 are considered as real Banach spaces with norms

$$\|f\|_{K_0} := \|f\|_{L^\infty(\mathbb{R}^n)}$$

and

$$\|f\|_{K_1} := \sup_{x \in \mathbb{R}^n} \sum_{|\alpha| \leqslant 2} \left|\left(\frac{\partial}{\partial x}\right)^\alpha f(x)\right|.$$

Theorem 4. *Let $H := -\Delta + V$ and let λ_0 and φ_0 be as above. Assume that* (H2)–(H3)$_k$ *are satisfied. Then there exist positive numbers δ, η, a C^k injective map*

$$g : \{u \in K_1;\ \|u\|_{K_1} < \delta\} \to K_{0,\eta} := \{W \in K_0;\ \|W\|_{K_0} < \eta\},$$

and a C^k map

$$\lambda : K_{0,\eta} \to \mathbb{R} \tag{36}$$

such that $g(0) = 0$, $\lambda(0) = \lambda_0$ and if $W = g(u)$ and $\|u\|_{K_1} < \delta$ then $\lambda(W)$ is an eigenvalue of $H + W$.

Proof. We note that the analysis and results of Sections 3, 4 and 7 (except for Theorem 3) are valid also when the continuous spectrum of H has infinite multiplicity in a neighborhood of λ_0. This leads us to define the function $\lambda = \lambda(W)$ (the map (36)) to be the solution $\lambda = \lambda(W)$ of the equation

$$\langle \varphi_0, A(\lambda, W)\varphi_0 \rangle = 1, \qquad \lambda(0) = \lambda_0 \tag{37}$$

for $W \in K_{0,\eta}$ for some $\eta > 0$.

Remark 2. Here and in the following, η denotes a generic small positive number which may change throughout the proof. All statements involving η hold under the assumption that η is chosen sufficiently small.

Now, if λ is a simple eigenvalue, then it follows from Proposition 7 that there exists a unique solution $\lambda = \lambda(W)$ of (37) for all $W \in K_{0,\eta}$, where η is sufficiently small, such that $\lambda(W)$ is of class C^k in $K_{0,\eta}$. Moreover, by the proof of Proposition 7, the same result holds if λ_0 is a degenerate eigenvalue.

Next, let $F : K_1 \times K_{0,\eta} \to K_0$ be defined by

$$F(u, W) := W\varphi_0 - \big(H + W - \lambda(W)\big)u.$$

Note that F is well defined since $C_0^2(\mathbb{R}^n) \subset \mathrm{Dom}(H)$ and φ_0 is continuous. Using that $\lambda \in C^k$, it follows that $F \in C^k(K_1 \times K_{0,\eta})$. A short calculation shows that

$$F'_W(0, 0)\widetilde{W} = \widetilde{W}\varphi_0.$$

From the definition of K_0, it follows that the map $F'_W(0, 0) : K_0 \to K_0$ is invertible. Hence by the implicit function theorem, there exist $\delta > 0$ and a C^k map $g : \{u \in K_1;\ \|u\|_{K_1} < \delta\} \to K_{0,\eta}$ such that $F(u, W) = 0$ for $\|u\|_{K_1} < \delta$ and $W \in K_{0,\eta}$ if and only if $W = g(u)$.

We claim that $\lambda(W)$ with $W = g(u)$, $\|u\|_{K_1} < \delta$ is an eigenvalue of $H + W$. For the claim to hold, we need to show in the case that λ_0 is a simple eigenvalue, that the two Eqs. (22) with $f = \varphi_0$ and $W = g(u)$ hold. Now the second equation (22) in our case coincides with equation of (37). Thus, to prove the result for a simple eigenvalue λ_0, we only need to verify that the first equation of (22) holds; i.e. we need to show that

$$\delta\big(\overline{H} + W - \lambda(W)\big)\varphi_0 = 0 \tag{38}$$

for $W = g(u)$, $\|u\|_{K_1} < \delta$.

To prove the claim when λ_0 is a degenerate eigenvalue, we observe that it follows from Proposition 9 and Theorem 2 (with ψ_1 replaced by φ_0 and $\lambda_1(W)$ replaced by $\lambda(W)$) that $\lambda(W)$ with $W = g(u)$, $\|u\|_{K_1} < \delta$, is an eigenvalue of $H + W$ if (38) holds and in addition:

$$\big\langle \varphi, A\big(\lambda(W), W\big)\varphi_0\big\rangle = 0 \tag{39}$$

for all $\varphi \in \operatorname{Ran} P_0$ such that $\langle \varphi, \varphi_0\rangle = 0$.

We proceed to show that (38) and (39) hold, thus proving our claim. To this end, we consider the functions $(\overline{H} + W - \lambda(W) \pm i0)^{-1}\varphi_0$ for $W = g(u)$, $\|u\|_{K_1} < \delta$ (δ small as above). Using the second resolvent equation, we find that

$$\begin{aligned}
&\big(\overline{H} + W - \lambda(W) \pm i0\big)^{-1}\varphi_0 \\
&\quad = \big(\overline{H} - \lambda(W) \pm i0\big)^{-1}\varphi_0 - \big(\overline{H} + W - \lambda(W) \pm i0\big)^{-1} W \big(\overline{H} - \lambda(W) \pm i0\big)^{-1}\varphi_0 \\
&\quad = \big(1 + \lambda_0 - \lambda(W)\big)^{-1}\varphi_0 - \big(1 + \lambda_0 - \lambda(W)\big)^{-1}\big(\overline{H} + W - \lambda(W) \pm i0\big)^{-1} W\varphi_0 \\
&\quad = \big(1 + \lambda_0 - \lambda(W)\big)^{-1}\varphi_0 - \big(1 + \lambda_0 - \lambda(W)\big)^{-1}\big(\overline{H} + W - \lambda(W) \pm i0\big)^{-1}\big(\overline{H} + W - \lambda(W)\big)u \\
&\quad = \big(1 + \lambda_0 - \lambda(W)\big)^{-1}(\varphi_0 - u),
\end{aligned} \tag{40}$$

where the third equality of (40) follows since $W\varphi_0 = (\overline{H} + W - \lambda(W))u$ for $W = g(u)$, which follows from the definition of g since $P_0 u = 0$. The last equality in (40) follows since u has compact support, and thus $u \in \mathcal{H}_s$ for all s.

Now it follows from (40) that $(\overline{H} + W - \lambda(W) + i0)^{-1}\varphi_0 = (\overline{H} + W - \lambda(W) - i0)^{-1}\varphi_0$ for $W = g(u)$, where $\|u\|_{K_1} < \delta$ for δ sufficiently small, which implies (by the definition of $\delta(\cdot)$) that (38) holds. Also, let $\varphi \in \operatorname{Ran} P_0$ and assume that $\langle \varphi, \varphi_0\rangle = 0$. Using the definition of $A(\cdot,\cdot)$ and (40) we find that

$$\begin{aligned}
\big\langle \varphi, A\big(\lambda(W), W\big)\varphi_0\big\rangle &= \frac{1}{2}\big\langle \varphi, \big(\big(\overline{H} + W - \lambda(W) - i0\big)^{-1} + \big(\overline{H} + W - \lambda(W) + i0\big)^{-1}\big)\varphi_0\big\rangle \\
&= \big(1 + \lambda_0 - \lambda(W)\big)^{-1}\langle \varphi, \varphi_0 - u\rangle = 0,
\end{aligned}$$

since $\langle \varphi, \varphi_0\rangle = 0$ by assumption and $\langle \varphi, u\rangle = 0$ by the definition of K_1. This completes the proof that $\lambda(W)$ is an eigenvalue of $H + W$ for $W = g(u)$, $\|u\|_{K_1} < \delta$, δ sufficiently small.

It remains to check that the map $g : \{u \in K_1;\ \|u\|_{K_1} < \delta\} \to K_0$ is injective. If $g(u_1) = g(u_2) = W$ then

$$0 = F(u_2, W) - F(u_1, W) = \big(H + W - \lambda(W)\big)(u_1 - u_2),$$

i.e. $u_1 - u_2$ is an eigenfunction of $H + W$ with eigenvalue $\lambda(W)$. But since $P_0 u_1 = P_0 u_2 = 0$, it follows that $(H + W - \lambda(W))(u_1 - u_2) = (\overline{H} + W - \lambda(W))(u_1 - u_2)$, and so $u_1 - u_2$ is also an eigenfunction of $\overline{H} + W$. Since $\overline{H} + W$ has no eigenvalues in a neighborhood of λ_0, the only possibility is that $u_1 - u_2 = 0$. Hence g is injective. □

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