JOURNAL OF COMBINATORIAL THEORY 6, 152–164 (1969)

A Theorem on Tait Colorings with an Application to the Generalized Petersen Graphs

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Communicated by W. T. Tutte

Received December 15, 1967

ABSTRACT

A family of graphs which includes the Petersen graph is postulated, and it is conjectured that the Petersen graph is the only member of this family not to have a Tait coloring. A general theorem about Tait colorings is proved and the conjecture is shown to be equivalent to a combinatorial assertion involving cyclically ordered arrays of n objects each belonging to one of 3 distinguishable classes. Finally, the combinatorial formulation is used to show that the conjecture is true wherever the parameters of the family satisfy any of a number of equalities or congruences.

1. A CONJECTURE

A *Tait coloring* of a trivalent graph G is an edge-coloring of G in 3 colors so that the 3 edges incident to any vertex are differently colored. An *isthmus* of a graph is an edge whose deletion disconnects the component in which it lies. It has been conjectured by Tutte [4] that "any trivalent graph with no isthmus and no Tait coloring can be reduced to a Petersen graph by deleting some edges and contracting other to single vertices."

The converse of this conjecture, however, is false. A counterexample appears in Figure 1, where the symbols α, β, and γ denote colors of a Tait coloring, although the two subgraphs generated by the vertices u_i together with the vertices v_i with even subscripts in one case and with odd subscripts in the other case are each contractible to the Petersen graph.

For integers k and n satisfying

$$1 \leqslant k \leqslant n-1, \qquad 2k \neq n, \tag{1-1}$$

one defines the *generalized Petersen graph* $G(n, k)$ with vertex set

$$V(G(n,k)) = \{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\},$$

and the set of edges $E(G(n, k))$ consisting of all those of the form

$$[u_i, u_{i+1}], \qquad [u_i, v_i], \qquad [v_i, v_{i+k}],$$

where i is an integer. *All subscripts in this note are to be read modulo n.* The particular value of n will be clear from the context, but n will always denote an integer $\geqslant 3$.

Thus $G(5, 2)$ is the Petersen graph, and $G(10, 4)$ is represented in Figure 1. The family of generalized Petersen graphs will be denoted by $\mathscr{P}$.

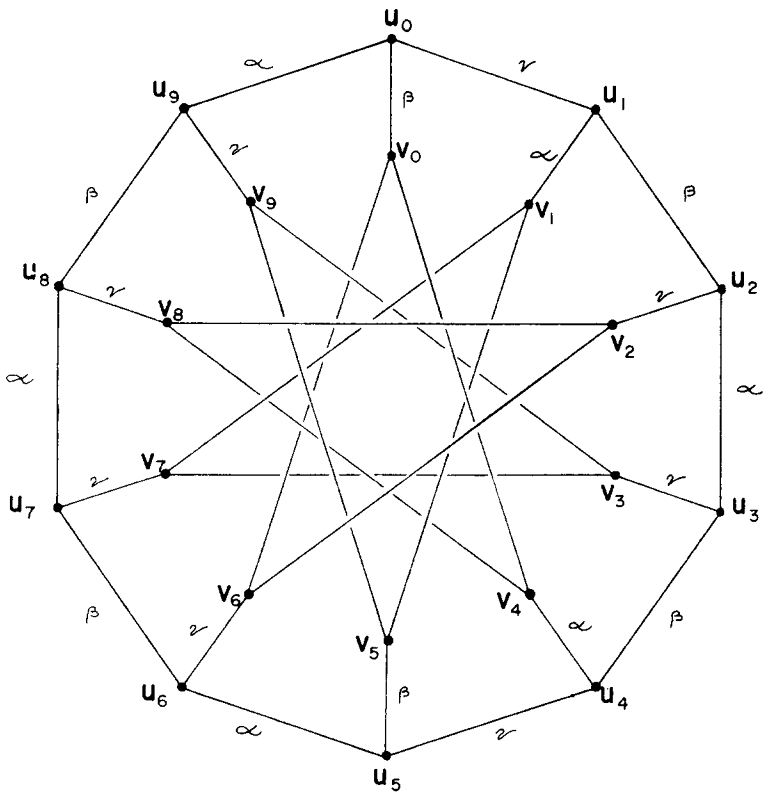

FIG. 1

CONJECTURE. The Petersen graph is the only member of $\mathscr{P}$ which does not have a Tait coloring.

2. Some Preliminary Terminology, Notation, and Remarks

It is clear that the vertex map

$$u_i \to u_{i+1} ,$$

$$v_i \to v_{i+1} ,$$

induces an automorphism of $G(n, k)$. Another automorphism is induced by the vertex map

$$u_i \to u_{n-i},$$
$$v_i \to v_{n-i}.$$

Thus the dihedral group D_n is a subgroup of the group of symmetries of $G(n, k)$. The existence of this latter automorphism implies

LEMMA 2.1. *$G(n, k)$ and $G(n, n-k)$ are isomorphic.*

If $2k = n$, then $G(n, k)$ is not trivalent anyway. Thus the condition (1-1) can be replaced by

$$1 \leqslant k < n/2. \tag{2-1}$$

The polygon generated by the vertices $u_0, u_1, \ldots, u_{n-1}$ will be called the *outer rim* of $G(n, k)$. Each connected component of the subgraph of $G(n, k)$ generated by the vertices $v_0, v_1, \ldots, v_{n-1}$ will be an *inner rim* of $G(n, k)$. If (n, k) denotes the greatest common divisor of the integers n and k, then it is seen that $G(n, k)$ will have precisely (n, k) inner rims, each of which is a polygon of length $n/(n, k)$. The edges $[u_i, v_i]$ will be called the *spokes* of $G(n, k)$.

LEMMA 2.2. *If $1 \leqslant k, m \leqslant n-1$ and if $km \equiv 1 \pmod n$, then $G(n, k)$ and $G(n, m)$ are isomorphic.*

PROOF: The hypothesis implies that $(n, k) = (n, m) = 1$ and so $G(n, k)$ and $G(n, m)$ have each a single inner rim. Let

$$V(G(n, m)) = \{u'_0, u'_1, \ldots, u'_{n-1}, v'_0, v'_1, \ldots, v'_{n-1}\},$$

where $E(G(n, m))$ consists of all edges of the form

$$[u'_i, u'_{i+1}], \qquad [u'_i, v'_i], \qquad [v'_i, v'_{i+m}].$$

The vertex map

$$u_i \to v'_{mi},$$
$$v_i \to u'_{mi},$$

is seen to induce a graph isomorphism of $G(n, k)$ onto $G(n, m)$.

Lower case Greek letters, particularly the symbols α, β, and γ, will be reserved for colors.

A *Hamilton circuit* in a graph is a circuit covering all the vertices of the graph. A *Tait cycle* in a graph is the union of two or more disjoint even circuits which cover all the vertices of the graph. It is well known and easily verified that a trivalent graph with a Hamilton circuit or a Tait cycle has Tait coloring.

If $G(n, k)$ has a Tait coloring and if the letters a, b, and c denote the number of spokes of $G(n, k)$ which have been colored α, β, and γ, respectively, then clearly

$$a + b + c = n. \tag{2-2}$$

By a *cyclic n-sequence* $\langle \xi_0, \xi_1, \ldots, \xi_{n-1} \rangle$, we mean the set of all ordered n-tuples

$$\{(\xi_m, \xi_{m+1}, \ldots, \xi_{m+n-1}) : m \text{ is an integer}\}. \tag{2-3}$$

(Recall the convention governing subscripts.)

The set $X(n; a, b, c)$ will consist of all cyclic n-sequences (2-3) where $\xi_i \in \{\alpha, \beta, \gamma\}$ for all integers i and α, β, and γ appear in any element of (2-3) with multiplicities a, b, and c, respectively. Thus (2-2) holds in this context, too.

If $(n, k) = 1$ and (1-1) holds, we define the function f_k from $X(n; a, b, c)$ into itself by the rule

$$f_k\langle \xi_0, \xi_1, \xi_2, \ldots, \xi_{n-1} \rangle = \langle \xi_0, \xi_k, \xi_{2k}, \ldots, \xi_{(n-1)k} \rangle$$

for each $\langle \xi_0, \ldots, \xi_{n-1} \rangle \in X(n; a, b, c)$. It is easily seen that the subscripts $0, k, 2k, \ldots, (n-1)k$ are a permutation of $0, 1, 2, \ldots, n-1$. Equivalently, f_k is well defined if and only if $(n, k) = 1$.

It is also easily verified that, if $(n, k) = (n, m) = 1$, then

$$f_k f_m = f_{km}.$$

Under the isomorphism $k \leftrightarrow f_k$ the semigroup $F_n = \{f_1, \ldots, f_{n-1}\}$ with the binary operation of composition is isomorphic to the multiplicative semigroup of residues modulo n. F_n is a group if and only if n is prime.

A *segment* of the cyclic n-sequence $\langle \xi_0, \ldots, \xi_{n-1} \rangle$ is an ordered j-tuple $(\rho_0, \ldots, \rho_{j-1})$ consisting of the first j terms in order from some ordered n-tuple belonging to $\langle \xi_0, \ldots, \xi_{n-1} \rangle$. If $\rho_1 = \cdots = \rho_{j-2} = \sigma$, the segment may be abbreviated by $(\rho_0, \sigma^{(j-2)}, \rho_{j-1})$, and $\langle \sigma_0^{(p_0)} \sigma_1^{(p_1)} \cdots \sigma_r^{(p_r)} \rangle$ denotes the cyclic n-sequence $\langle \xi_0, \ldots, \xi_{n-1} \rangle$ where

$$\begin{array}{l} \xi_0 = \cdots = \xi_{p_0 - 1} = \sigma_0 \\ \xi_{p_0} = \cdots = \xi_{p_1 - 1} = \sigma_1 \\ \cdots\cdots\cdots\cdots \\ \xi_{n-p_r} = \cdots = \xi_{n-1} = \sigma_r . \end{array}$$

We distinguish two types of segments which may occur in an element

$$\langle \xi_0, ..., \xi_{n-1} \rangle \in X(n; a, b, c).$$

Here (ζ, η, θ) is any permutation whatever of the symbols (α, β, γ):

Type I: $(\zeta, \eta^{(j)}, \zeta)$, where j is an odd positive integer.

Type II: $(\zeta, \eta^{(j)}, \theta)$, where j is an even positive integer.

We let $Y(n; a, b, c)$ denote the set of those cyclic n-sequences in $X(n, a, b, c)$ which have no segment of Type I or Type II.

3. A General Theorem on Tait Colorings and An Equivalent Combinatorial Problem

Let G be an arbitrary trivalent graph and let C be a circuit in G. Let the vertices of C be denoted in some cyclic order by

$$x_0, x_1, ..., x_{n-1}. \tag{3.1}$$

A positive sense in C is defined if the vertices are encountered in the order (3-1), extended cyclically. Incident to each vertex a_i there is an edge s_i not in C called the *spoke of C at x_i*.

Let the n spokes of C be assigned colors α, β, and γ arbitrarily. Pick a vertex x_i and proceed around C in the positive sense from x_i while noting the color $\xi_j \in \{\alpha, \beta, \gamma\}$ of the spoke s_j as each vertex x_j is encountered. This yields an ordered n-tuple

$$(\xi_i, ..., \xi_{n-1}, \xi_0, ..., \xi_{i-1}).$$

By repeating this process, varying only the starting point x_i, we obtain a set of n such ordered n-tuples. This set is a cyclic n-sequence

$$\langle \xi_0, ..., \xi_{n-1} \rangle \in X(n; a, b, c),$$

where a, b, and c have the same meaning as in the previous section, and (2-2) holds.

Thus to any coloring of the spokes of C in colors α, β, and γ, there corresponds a cyclic n-sequence in some set $X(n; a, b, c)$, and to any element of any set $X(n; a, b, c)$ there corresponds a coloring of the spokes of C in α, β, and γ.

Let $\langle \xi_0, ..., \xi_{n-1} \rangle \in X(n; a, b, c)$ and let m be any integer. Let the spoke s_{m+j} of C be colored ξ_j ($j = 0, 1, ..., n-1$). If it is possible to assign colors α, β and γ to the edges of C in such a way that each of the vertices $x_i \in V(C)$ is incident to precisely one edge of each color, then $\langle \xi_0, ..., \xi_{n-1} \rangle$ shall be said to be *extensible to C.*

LEMMA 3.1. *Let* $\Xi = \langle \xi_0, ..., \xi_{n-1} \rangle \in X(n; a, b, c)$ *and suppose* Ξ *has a segment of Type I or of Type II. Let C be any n-circuit in a trivalent graph G. Then* Ξ *is not extensible to C.*

PROOF: Let $V(C)$ be labeled as in (3-1). We may suppose that the spoke s_i of C at x_i has been colored ξ_i, $(i = 0, ..., n-1)$.

Suppose Ξ has a segment of Type I and, for definiteness, assume that $(\xi_0, ..., \xi_{j+1}) = (\alpha, \beta^{(j)}, \alpha)$ is such a segment, where j is an odd positive integer. The edges

$$[x_i, x_{i+1}], \qquad i = 0, 1, ..., j, \tag{3-2}$$

must receive alternately the colors γ and α, beginning with γ. There is an even number of edges in (3-2) since j is odd. Thus $[x_j, x_{j+1}]$ receives α. But s_{j+1} is also colored α; i.e., x_{j+1} is incident to two edges colored α. Hence Ξ is not extensible to C.

If Ξ has a segment of Type II, we assume for definiteness that

$$(\xi_0, ..., \xi_{j+1}) = (\alpha, \beta^{(j)}, \gamma),$$

where j is an even positive integer. Again the edges (3-2) must be colored alterately in γ and α, beginning with γ. But since there is now an odd number of these edges, $[x_j, x_{j+1}]$ receives γ, too, and x_{j+1} is incident to two edges colored γ.

LEMMA 3.2. *Let* $\Xi = \langle \xi_0, ..., \xi_{n-1} \rangle \in Y(n; a, b, c)$ *and let it not be true that*

$$\xi_0 = \cdots = \xi_{n-1},$$

Then Ξ *is extensible to any n-circuit in a trivalent graph.*

PROOF: Let C be a circuit in a trivalent graph where $V(C)$ is labeled as in (3-1). Assume that the spoke s_i of C at x_i is colored ξ_i $(i = 0, ..., n-1)$.

Since it is not true that $\xi_0 = \cdots = \xi_{n-1}$, we may assume without loss of generality that $\xi_0 = \alpha$ and $\xi_1 = \beta$. Let ξ_{j_2} be the next term after ξ_1 different from β. In general, let $\xi_{j_{i+1}}$ be the first term after ξ_{j_i} different from ξ_{j_i}.

Now color $[x_0, x_1]$ in γ and color the $j_2 - 1$ edges

$$[x_1, x_2], ..., [x_{j_2-1}, x_{j_2}]$$

alternately first in α then in γ. Thus $[x_{j_2-1}, x_{j_2}]$ will be colored α if and only if $j_2 - 1$ is odd. Now $j_2 - 1$ is odd if and only if $\xi_{j_2} = \gamma$, since $\langle \xi_0, ..., \xi_{n-1} \rangle$ has no segment of Type I or Type II. Now consider the edges

$$[x_{j_2}, x_{j_2+1}], ..., [x_{j_3-1}, x_{j_3}].$$

If $\xi_{j_2} = \gamma$ these edges are properly colorable alternately first in β then in α, whereas if $\xi_{j_2} = \alpha$, they are properly colorable first in β then in α. This process can be continued until all the edges of C are properly colored.

The above two lemmas imply

THEOREM 1. *Let $\Xi = \langle \xi_0, ..., \xi_{n-1} \rangle \in X(n; a, b, c)$, and let it not be true that $\xi_0 = \cdots = \xi_{n-1}$. Let C be any n-circuit in some trivalent graph G. Then Ξ is extensible to C if and only if $\Xi \in Y(n; a, b, c)$.*

Let $\Xi = \langle \xi_0, ..., \xi_{n-1} \rangle \in X(n; a, b, c)$. If Ξ is extensible to the outer rim of a generalized Petersen graph $G(n, k)$, then Ξ is said to be an *outer sequence* for $G(n, k)$.

COROLLARY 1A. *A cyclic n-sequence $\Xi \in X(n; a, b, c)$ is an outer sequence for $G(n, k)$ if and only if $\Xi \in Y(n; a, b, c)$.*

Let m be any integer and let the spoke s_{m+i} of $G(n, k)$ be colored $\xi_i \in \{\alpha, \beta, \gamma\}$ $(i = 0, 1, ..., n-1)$. If the edges of the inner rims of $G(n, k)$ can be assigned colors α, β, and γ in such a way that each vertex v_i on an inner rim is incident with precisely one edge of each color, then $\langle \xi_0, ..., \xi_{n-1} \rangle$ is an *inner sequence* for $G(n, k)$.

COROLLARY 1B. *Let n and k satisfy* (1-1) *and suppose* $(n, k) = 1$. *Let $\Xi = \langle \xi_0, ..., \xi_{n-1} \rangle \in X(n; a, b, c)$ and let it not hold that $\xi_0 = \cdots = \xi_{n-1}$. Then Ξ is an inner sequence for $G(n, k)$ if and only if $f_k\Xi \in Y(n; a, b, c)$.*

PROOF: Let m be any integer and let the spoke s_{m+j} of $G(n, k)$ be colored ξ_j $(j = 0, 1, ..., n-1)$. Let the positive sense on the inner rim C of $G(n, k)$ be such that its vertices are encountered in the cyclic order

$$v_0, v_k, v_{2k}, ..., v_{(n-1)k}.$$

Proceed in the positive sense around C from v_{ik} noting the color $\xi_{(j-m)k}$ of the spoke s_{jk} as each vertex v_{jk} is encountered $(j = 0, 1, ..., n-1)$. By repeating this process as $i = 0, 1, ..., n-1$, we obtain precisely the cyclic n-sequence

$$\langle \xi_0, \xi_k, ..., \xi_{(n-1)k} \rangle = f_k\Xi.$$

The definitions imply that Ξ is an inner sequence if and only if $f_k\Xi$ is extensible to C. But, by Theorem 1, $f_k\Xi$ is extensible to C if and only if it belongs to $Y(n; a, b, c)$.

THEOREM 2. *Let integers n and k satisfy* (1-1) *and suppose* $(n, k) = 1$. *The generalized Petersen graph* $G(n, k)$ *has a Tait coloring if and only if, for some positive integers* a, b, c *satisfying* (2-2), *there exists a cyclic n-sequence* $\Xi \in Y(n; a, b, c)$ *such that* $f_k\Xi \in Y(n; a, b, c)$.

PROOF: Let $\Xi = \langle \xi_0, \ldots, \xi_{n-1} \rangle \in X(n; a, b, c)$. Let m be any integer and let the spoke s_{j+m} of $G(n, k)$ be colored ξ_j.

Clearly $G(n, k)$ has a Tait coloring if and only if Ξ is both an outer sequence and inner sequence for $G(n, k)$. The theorem is now an immediate consequence of Corollaries 1A and 1B.

COROLLARY 2A. *Let n and k satisfy* (1-1) *and suppose* $(n, k) = 1$. *If for some cyclic n-sequence* Ξ, *both* Ξ *and* $f_k\Xi$ *are in* $Y(n; a, b, c)$, *then*

$$a \equiv b \equiv c \text{ (modulo 2).} \tag{3-3}$$

PROOF: The result follows from Theorem 2 and a result by Mlle. Blanche Descartes [1] that, if S is a separating set of edges of a trivalent graph G and if a, b, and c denote the numbers of edges in S colored each of the three colors in a Tait coloring of G, then (3-3) must hold.

REMARK. Unfortunately the subset $Y(n; a, b, c)$ does not appear to emerge "naturally" from the set $X(n; a, b, c)$ in any algebraic way. It is possibly of interest to consider the following related question:

Consider a set $X(n; a, b, c)$ and suppose n is prime so that $F_n = \{f_1, \ldots, f_{n-1}\}$ forms a group under composition, as mentioned above. Pick a generator f_k of F_n. What can we say then as to how $X(n; a, b, c)$ will be decomposed into orbits under f_k, and how might $Y(n; a, b, c)$ be distributed among these orbits?

This has been worked out below for the set $X(7; 1, 3, 3)$. f_3 is a generator of F_7. The columns of the table below are the orbits under f_3. The cyclic sequences followed by * belong to $Y(7; 1, 3, 3)$.

$\langle\alpha\gamma\beta\beta\gamma\gamma\beta\rangle$*	$\langle\alpha\beta\beta\gamma\gamma\beta\gamma\rangle$	$\langle\alpha\gamma\gamma\beta\beta\gamma\beta\rangle$	$\langle\alpha\gamma\gamma\beta\gamma\beta\beta\rangle$
$\langle\alpha\beta\beta\beta\gamma\gamma\gamma\rangle$*	$\langle\alpha\gamma\gamma\beta\beta\beta\gamma\rangle$	$\langle\alpha\beta\beta\gamma\gamma\gamma\beta\rangle$	$\langle\alpha\beta\beta\gamma\beta\gamma\gamma\rangle$
$\langle\alpha\beta\gamma\beta\gamma\beta\gamma\rangle$	$\langle\alpha\beta\gamma\gamma\beta\gamma\beta\rangle$	$\langle\alpha\gamma\beta\beta\gamma\beta\gamma\rangle$	
$\langle\alpha\beta\gamma\gamma\beta\beta\gamma\rangle$*	$\langle\alpha\gamma\beta\gamma\gamma\beta\beta\rangle$	$\langle\alpha\beta\gamma\beta\beta\gamma\gamma\rangle$	
$\langle\alpha\gamma\gamma\gamma\beta\beta\beta\rangle$*	$\langle\alpha\gamma\beta\beta\beta\gamma\gamma\rangle$	$\langle\alpha\beta\gamma\gamma\gamma\beta\beta\rangle$	
$\langle\alpha\gamma\beta\gamma\beta\gamma\beta\rangle$	$\langle\alpha\beta\gamma\beta\gamma\gamma\beta\rangle$	$\langle\alpha\gamma\beta\gamma\beta\beta\gamma\rangle$	

Note that $f_6 = f_3^3$ merely juxtaposes the β's and γ's in each cyclic sequence, not surprisingly, since f_6^2 is the group's identity f_1.

4. A Near Proof of the Conjecture

It is assumed in this section that n and k are integers satisfying (1-1).

THEOREM 3. *If n is even, then $G(n, k)$ has a Tait coloring.*

PROOF: There are two cases:

I. *$n/(n, k)$ is even.* The outer rim as well as each of the (n, k) inner rims is an even circuit, and together they cover all of the vertices of $G(n, k)$. So $G(n, k)$ has a Tait cycle and hence a Tait coloring.

II. *$n/(n, k)$ is odd.* In this case there is an even number of inner rims, each of odd length.

Consider the cyclic sequence

$$\langle \alpha\beta\gamma^{(n/(n,k)-2)} \rangle \in X(n/(n, k);\ 1, 1, n/(n, k) - 2). \tag{4-1}$$

Since $(n/(n, k), k/(n, k)) = 1$, $f_{k/(n,k)}$ is a unit in $F_{n/(n,k)}$. It has an inverse $f_{k/(n,k)}^{-1}$ which, when applied to the cyclic sequence (4-1), yields a cyclic sequence of the form

$$\langle \alpha\gamma^{(r)}\beta\gamma^{(n/(n,k)-(r+2))} \rangle, \tag{4-2}$$

where r is an integer uniquely determined by n and k and satisfies $1 \leqslant r \leqslant n/(n, k) - 3$.

Now let

$$\Xi = \langle \beta\alpha^{((n,k)-1)}\gamma^{((n,k)r)}\alpha\beta^{((n,k)-1)}\gamma^{(n-(n,k)(r+2))} \rangle. \tag{4-3}$$

Note that in (4-3) the maximal segments with a single iterated color have even length if the color is γ and odd length if the color is α or β. Thus Ξ has no segments of Type I or Type II. By Corollary 1A, Ξ is extensible to the outer rim of $G(n, k)$.

Now consider the cyclic $(n/(n, k))$ sequences formed by starting with an arbitrary term of Ξ and picking in cyclic order every (n, k)-th term. One so obtains one cyclic sequence

$$\langle \beta\gamma^{(r)}\alpha\gamma^{(n/(n,k)-(r+2))} \rangle \tag{4-4}$$

and $(n, k) - 1$ cyclic sequences all like (4-2). The application of $f_{k/(n,k)}$ to (4-4) gives

$$\langle \beta\alpha\gamma^{(n/(n,k)-2)} \rangle \tag{4-5}$$

and the application of $f_{k/(n,k)}$ to (4-2) yields (4-1). The cyclic $(n/(n, k))$ sequences (4-5) and (4-1) are extensible to the inner rims of $G(n, k)$ by Theorem 1, since they contain no segments of Type I or Type II. Thus $G(n, k)$ has a Tait coloring.

This theorem has been invoked in the coloring of $G(10, 4)$ in Figure 1. In this case $(n, k) = 2$ and $r = 1$.

THEOREM 4. *Let m be a positive integer. $G(5m, 2m)$ has a Tait coloring if and only if $m > 1$.*

PROOF: If m is even, then $G(5m, 2m)$ has a Tait coloring by Theorem 3. $G(5, 2)$ is the Petersen graph which is known [2] to have no Tait coloring. Hence suppose that m is an odd integer >1.

The cuclic n-sequence

$$\Xi = \langle \alpha^{(m)}\gamma\beta^{(m-1)}\gamma\alpha\beta^{(m-1)}\alpha\gamma^{(m)}\beta^{(m-2)} \rangle$$

has no segment of Type I or Type II and so is an outer sequence by Corollary 1A.

Now consider the cyclic 5-sequences formed by starting with an arbitrary term of Ξ and recording in cyclic order every m-th term. One obtains

1 cyclic 5-sequence	$\langle \alpha\gamma\gamma\beta\gamma \rangle$,
1 cyclic 5-sequence	$\langle \alpha\beta\alpha\alpha\gamma \rangle$,
$m - 2$ cyclic 5-sequences	$\langle \alpha\beta\beta\gamma\beta \rangle$.

The application of $f_3 = f_2^{-1}$ to these yields

1 cyclic 5-sequence	$\langle \alpha\beta\gamma\gamma\gamma \rangle$,
1 cyclic 5-sequence	$\langle \beta\gamma\alpha\alpha\alpha \rangle$,
$m - 2$ cyclic 5-sequences	$\langle \gamma\alpha\beta\beta\beta \rangle$,

extensible by Theorem 1 to the inner rims of $G(n, k)$.

THEOREM 5. *If n is odd and $G(n/(n, k), k/(n, k))$ has a Tait coloring, then so does $G(n, k)$.*

Proof: If $(n, k) = 1$, the theorem is trivial, so assume $(n, k) > 1$.

There exists an outer sequence $\Xi = \langle \xi_0, \ldots, \xi_{n/(n,k)-1} \rangle$ for $G(n/(n, k), k/(n, k))$. Let $\Sigma = \langle \sigma_0, \ldots, \sigma_{n-1} \rangle$ be the cyclic n-sequence

$$\Sigma = \langle \xi_0^{((n,k))} \xi_1^{((n,k))} \cdots \xi_{n-1}^{((n,k))} \rangle.$$

We show that Σ is an outer sequence for $G(n, k)$. Consider a maximal segment of Σ consisting of identical terms. There is no loss of generality in denoting the segment by

$$(\sigma_0, \sigma_1, \ldots, \sigma_{m(n,k)-1}), \tag{4-6}$$

where

$$1 \leqslant m < n/(n, k). \tag{4-7}$$

(Since n is odd, so is $n/(n, k)$, so the outer rim of $G(n/(n, k), k/(n, k))$ requires all three colors. Thus not all the ξ_i are the same, and strict inequality holds in the right-hand side of (4-7).) Let us suppose further that all the terms of (4-6) are β and that $\sigma_{n-1} = \alpha$. Thus $\xi_{n-1} = \alpha$ and $\xi_0 = \cdots = \xi_{m-1} = \beta$.

If m is odd, then, since Ξ contains no segment of Type I, $\xi_m = \gamma$. Thus $m(n, k)$ is odd and $\sigma_{m(n,k)} = \gamma$. The segment

$$(\sigma_{n-1}, \sigma_0, \ldots, \sigma_{m(n,k)-1}, \sigma_{m(n,k)}) = (\alpha, \beta^{(m(n,k))}, \gamma)$$

is not of Type II (and clearly not of Type I).

If m is even, then since Ξ contains no segment of Type II,

$$\xi_m = \alpha = \sigma_{m(n,k)}\,.$$

So $m(n, k)$ is even and

$$(\sigma_{n-1}, \sigma_0, \ldots, \sigma_{m(n,k)-1}, \sigma_{m(n,k)}) = (\alpha, \beta^{(m(n,k))}, \alpha)$$

is not of Type I (and clearly not of Type II). Hence Σ is extensible to the outer rim of $G(n, k)$.

The coloring induced by Σ can be extended to the (n, k) inner rims since Ξ is an inner sequence for $G(n/(n, k), k/(n, k))$, and Ξ is the cyclic $(n/(n, k))$-sequence obtained by starting with an arbitrary term of Σ and recording in order every (n, k)-th term.

Let us dispense with a trivial case:

Theorem 6. *$G(n, 1)$ has a Tait coloring.*

Proof: The outer rim with $[u_0, u_1]$ deleted plus the inner rim with

$[v_0, v_1]$ deleted, together with the spokes s_0 and s_1, form a Hamilton circuit of $G(n, 1)$. Hence $G(n, 1)$ has a Tait coloring.

In the light of the foregoing results, we may restrict our investigation of generalized Petersen graphs $G(n, k)$ to those for which:

(i) n is odd,

(ii) $(n, k) = 1$,

(iii) $n \geqslant 7$,

(iv) $2 \leqslant k < n/2$.

Unfortunately, we have found no general method for dispensing with all of the remaining cases. However, when n and k satisfy certain congruences, it is possible by the use of Theorem 2 to prove that $G(n, k)$ has a Tait coloring. We give three examples:

A. $n \equiv -1$ (modulo $2k$).

Let $\Xi = \langle \alpha\beta^{(2k-1)}\gamma^{(n-2k)} \rangle$. Since $2k - 1$ and $n - 2k$ are odd, $\Xi \in Y(n; 1, 2k - 1, n - 2k)$, and if $n = 2km - 1$, then

$$f_k\Xi = \langle \alpha\beta\gamma^{(2m-2)}\beta^{(2)}\gamma^{(2m-2)} \cdots \beta^{(2)}\gamma^{(2m-3)} \rangle,$$

which also belongs to $Y(n; 1, 2k - 1, n - 2k)$.

B. k is odd; $n \equiv 1$ (modulo $2k$).

Let $\Xi = \langle \alpha\beta\gamma^{(n-k-2)}\beta^{(2)}\gamma^{(k-2)} \rangle$. Then both Ξ and $f_k\Xi$ are in

$$Y(n; 1, 3, n - 4).$$

C. $n = 3m$ for some positive integer m.

Let $\Xi = \langle \alpha\beta\gamma\alpha\beta\gamma \cdots \alpha\beta\gamma \rangle$. Then $\Xi \in Y(3m; m, m, m)$. Since we assume $f_k\Xi = \Xi$ or

$$f_k\Xi = \langle \alpha\gamma\beta\alpha\gamma\beta \cdots \alpha\gamma\beta \rangle$$

according as $k \equiv 1$ (modulo 3) or $k \equiv 2$ (modulo 3), respectively.

Moreover, by Lemma 2.2, if $G(n, k)$ has a Tait coloring and $mk \equiv 1$ (modulo n) where $1 < m < n$, then $G(n, m)$ also has a Tait coloring.

Finally, if $G(n, k)$ has a Tait coloring then so does $G(n, n - k)$, by Lemma 2.1.

We conclude with an application of some of the results of this note to show that the conjecture holds, for example, when $n = 11$.

If $k = 3$, then since $11 \equiv -1$ (modulo 6), Case **A** above applies and $G(11, 3)$ has a Tait coloring. Since $3 \cdot 4 \equiv 1$ (modulo 11), G(11, 4) is also Tait colorable. By Lemma 2.1, so are $G(11, 8)$ and $G(11, 7)$.

If $k = 5$, then k is odd and $11 \equiv 1$ (modulo 10), so Case **B** above applies. Since $5 \cdot 9 \equiv 1$ (modulo 11), $G(11, 9)$ has a Tait coloring too. By Lemma 2.1, so do $G(11, 6)$ and $G(11, 2)$. Finally $G(11, 1)$ and $G(11, 10)$ are Tait colorable by Theorem 6.

These techniques are hardly exhaustive, however; for example, they offer no hint as to how to color $G(13, 5)$. This graph does incidently have a Tait coloring, and the outer sequence could be

$$\langle \alpha^{(3)} \gamma \beta^{(4)} \gamma^{(4)} \beta \rangle.$$

REMARK. The graphs $G(n, 2)$ where $n \geqslant 7$ is odd are a subclass of a class of graphs which Robertson [3] has shown to contain a Tait cycle if $n \equiv 5$ (modulo 6) and a Hamilton circuit otherwise. Thus $G(n, 2)$ has a Tait coloring for all $n \geqslant 7$.

Since $2(n + 1)/2 \equiv 1$ (modulo n), so do

$$G(n, (n+1)/2) \quad \text{and} \quad G(n, (n-1)/2).$$

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