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On the Cohomology of an Algebra Morphism

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INTRODUCTION

The major theorems of this paper constitute a special case of two theorems of [3]; however, the technique of proof is entirely new. We have isolated this special case for three reasons:

1. The techniques suggest a spectral sequence argument for a valuable generalization of the theorems (discussed below).

2. In [3] we introduced a cochain map of some importance but were unable to work with it directly. Here that is precisely what we do.

3. Of necessity [3] is rather densely packed. We hope that a detailed discussion of a special case—namely, algebra morphisms—will make it more accessible.

Here is a precis of the cohomological aspects of [3]: Define a *diagram* of k-algebras over a partially ordered set $\mathscr{I}$ to be a contravariant functor $\mathbb{A}$ from $\mathscr{I}$ to the category of k-algebras. (A presheaf of algebras.) An $\mathbb{A}$-bimodule is a presheaf of bimodules. The category of $\mathbb{A}$-bimodules is abelian and, so, has a Yoneda cohomology theory, $\mathrm{Ext}^*_{\mathbb{A}}(-,-)$. Since it has enough projectives and injectives, $\mathrm{Ext}^*_{\mathbb{A}}(-,-)$ is universal in each variable. We provide a natural generalization of Hochschild cochains and show $H^*(\mathbb{A},-)\cong \mathrm{Ext}^*_{\mathbb{A}}(\mathbb{A},-)$. Moreover, associated to each diagram $\mathbb{A}$ and $\mathbb{A}$-bimodule $\mathbb{M}$ there is an algebra $\mathbb{A}!$ and an $\mathbb{A}!$-bimodule $\mathbb{M}!$. We establish a natural transformation $\omega^*\colon \mathrm{Ext}^*_{\mathbb{A}}(\mathbb{N},\mathbb{M})\to \mathrm{Ext}^*_{\mathbb{A}!}(\mathbb{N}!,\mathbb{M}!)$. The *Cohomology Comparison Theorem* (CCT) asserts: when $\mathscr{I}$ is finite (and in

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certain infinite cases) ω^* is an isomorphism. The proof proceeds by the construction and comparison of particular projective resolutions of $\mathbb{N}$ and $\mathbb{N}!$. The existence of ω^* yields a morphism of Hochschild cohomologies $H^*(\mathbb{A}, \mathbb{M}) \to H^*(\mathbb{A}!, \mathbb{M}!)$. We provide a cochain map τ^* which effects this morphism but are unable to bypass ω^* in proving that $H^*(\tau)$ is an isomorphism when $\mathscr{I}$ is finite.

The work of [3] was initiated—as its title suggests—in order to study the deformation theory of diagrams. Indeed in a wide variety of cases—including that of an algebra morphism—the deformation theories of $\mathbb{A}$ and $\mathbb{A}!$ are the same. However, the importance of the cohomology transcends its applications to deformation theory. For example, in [4] we associate with any simplicial complex Σ a particular diagram $\mathbb{k}_\Sigma$. It is elementary that $H^*(\Sigma, k) = H^*(\mathbb{k}_\Sigma, \mathbb{k}_\Sigma)$. Hence, the results of [3] imply $H^*(\Sigma, k) \cong H^*(\mathbb{k}_\Sigma!, \mathbb{k}_\Sigma!)$; that is, every simplicial complex has a naturally associated k-algebra whose Hochschild cohomology is the simplicial cohomology.

We state—without details—our conjectured generalization of the CCT. Diagrams and bimodules can be defined over any category $\mathscr{C}$, as can their Hochschild and Yoneda cohomologies. We have proved that, as before, $H^*(\mathbb{A}, -) = \mathrm{Ext}^*_{\mathbb{A}}(\mathbb{A}, -)$. We associate to each small category $\mathscr{C}$ a "barycentric subdivision" $\mathrm{Sd}\,\mathscr{C}$ and a covariant functor $\mathrm{Sd}\,\mathscr{C} \to \mathscr{C}$. This induces (by composition) a functor subdividing diagrams, $\mathbb{A} \rightsquigarrow \mathrm{Sd}\,\mathbb{A}$, and a functor Sd: $\mathbb{A}$-bimodules $\to$ (Sd $\mathbb{A}$)-bimodules.

Conjecture 1. $\mathrm{Ext}^*_{\mathbb{A}}(\mathbb{N}, \mathbb{M}) \cong \mathrm{Ext}^*_{\mathrm{Sd}\,\mathbb{A}}(\mathrm{Sd}\,\mathbb{N}, \mathrm{Sd}\,\mathbb{M})$. Now subdivision has the properties: (i) Sd(Sd $\mathscr{C}$) is always a poset; (ii) if $\mathscr{C}$ is finite and has no loops then Sd $\mathscr{C}$ is a finite poset. Hence in case (ii) the CCT and the conjecture yield $\mathrm{Ext}^*_{\mathbb{A}}(\mathbb{N}, \mathbb{M}) \cong \mathrm{Ext}^*_{(\mathrm{Sd}\,\mathbb{A})!}((\mathrm{Sd}\,\mathbb{N})!, \mathrm{Sd}\,\mathbb{M})!)$.

Conjecture 2. When $\mathscr{C}$ = any poset (even infinite), $H^*(\mathbb{A}, -) \cong H^*(\mathbb{A}!, -!)$.

The two conjectures combine as: for any $\mathscr{C}$ and any $\mathbb{A}$ we have $H^*(\mathbb{A}, -) \cong H^*(\mathbb{A}!!, -!!)$, where $-!! = \mathrm{Sd}(\mathrm{Sd}(-))!$.

In this work we generalize the Hochschild cochain complex for a k-algebra to give one for a k-algebra morphism $\phi\colon B \to A$. We show that many standard results (discussed next) carry over (Section 1). In Section 2 we construct the ring $\phi!$ and the cochain map τ^*; then we show the latter to be a cohomology isomorphism. Finally, in Section 3 we use more sophisticated techniques—again differing from those in [3]—to show the full CCT (as described above) for the case of a morphism.

Let k be a commutative ring and Λ, a k-algebra. An epimorphism or monomorphism of Λ-bimodules is called *allowable* if it splits when considered merely as a k-module morphism. An arbitrary morphism is allowable if it has an epi-mono factorization by allowable morphisms [7, IX.4]. An exact sequence is allowable if every morphism appearing in it is allowable. Allowable exact sequence form the foundation of a *relative* Yoneda cohomology, $\mathrm{Ext}_{\Lambda}^{*}(-,-)$, [7, XII.4]. (Briefly: $\mathrm{Ext}_{\Lambda}^{0}(\mathcal{N},\mathcal{M})=\mathrm{Hom}_{\Lambda}(\mathcal{N},\mathcal{M})$; for $n>0$, $\mathrm{Ext}_{\Lambda}^{n}(\mathcal{N},\mathcal{M})=$ equivalence classes of allowable n-fold extensions $0\to\mathcal{M}\to\mathcal{E}_{n-1}\to\cdots\to\mathcal{E}_0\to\mathcal{N}\to 0$.) This is a relative δ-functor; that is, an allowable short exact sequence, $\mathcal{E}$, induces the usual long exact sequence of cohomology (by splicing with $\mathcal{E}$).

To be a *relative projective* a Λ-bimodule need only enjoy the usual lifting property with respect to allowable epimorphisms. *Relative injectives* are defined dually. The category of Λ-bimodules has enough of each [7, IX.6); that is, every Λ-bimodule has an allowable monomorphism into a relative injective, and, dually, an allowable epimorphism from a relative projective. It follows that $\mathrm{Ext}_{\Lambda}^{*}(-,-)$ is *universal* in each variable [7, XII.9]; that is, if F^* is a relative δ-functor then any natural transformation $\mathrm{Hom}_{\Lambda}(\mathcal{N},-)\to F^0$ extends uniquely to $\mathrm{Ext}_{\Lambda}^{*}(\mathcal{N},-)\to F^*$. (Similarly, $\mathrm{Hom}_{\Lambda}(-,\mathcal{M})\to F^0$ extends uniquely to $\mathrm{Ext}_{\Lambda}^{*}(-,\mathcal{M})\to F^*$). Of course, Λ also has a *Hochschild cochain complex* $C^*(\Lambda,-)$ [6; 7, X.3], whose cohomology is $H^*(\Lambda,-)$. Then $H^*(\Lambda,-)\cong\mathrm{Ext}_{\Lambda}^{*}(\Lambda,-)$ [7, X.1, 3]. There is a subcomplex of *normal* cochains, $C_n^*(\Lambda,-)$, whose cohomology is identical with $H^*(\Lambda,-)$, [7, X.2, 3]. A *singular* extension of Λ by a bimodule $\mathcal{M}$ is a k-split exact sequence $0\to\mathcal{M}\to\Lambda'\to^{\pi}\Lambda\to 0$ in which: Λ' is a k-algebra; π is an algebra morphism; $\mathcal{M}^2=0$ in Λ'; and $\lambda_1' m\lambda_2'=\pi(\lambda_1')\,m\pi(\lambda_2')$ [6; 7, X.3]. The Yoneda equivalence classes [7, VII.5, XII.4] of singular extensions form a k-module $\mathrm{exal}(\Lambda,\mathcal{M})$ under Baer sum [7, III.5]. Then $H^2(\Lambda,-)\cong\mathrm{exal}(\Lambda,-)$ [6; 7, X.3]. The complex $C^*(\Lambda,\Lambda)$ has several graded products: a pre-Lie product $\bar{\circ}$; a Lie bracket $[-,-]$; and a cup product $\smile$ [2]. The latter two induce products on $H^*(\Lambda,\Lambda)$; the former does not [2]. In Section 1 we shall replicate these results in the case of a morphism.

The following notational conventions will be in force throughout this paper: k will be a commutative ring with unit; all algebras and morphisms will be unital. If Λ and Λ' are algebras the category of left Λ-right Λ'-bimodules will be denoted $(\Lambda-\Lambda')$-MOD; when $\Lambda=\Lambda'$ we shorten this to Λ-MOD. We shall use $+$ to indicate direct sum for k-modules *only*; otherwise we use $\oplus$. Matrix notation will be used for morphisms between direct sums; $\binom{\alpha}{\beta}$; $X\to Y\oplus Z$ will usually be denoted $(\alpha\ \beta)'$. Finally, $\phi\colon B\to A$ will be a fixed k-algebra morphism. It is frequently convenient to write b^{ϕ} instead of $\phi(b)$. An A-module $\mathcal{M}$ can be viewed as a B-module (base change) *via* $b\cdot m=b^{\phi}m$ or $m\cdot b=mb^{\phi}$. Occasionally we shall denote M with this structure by ${}_{\phi}M$, M_{ϕ}, or ${}_{\phi}M_{\phi}$ (as appropriate); usually we forego the additional notation. Following tradition we refer to Lemma p of Section q as Lemma $q.\ p$; also, Theorem q will refer to the (unique) theorem in Section q. While the proofs in Sections 2, 3 are new we should comment that both the results and proofs in Section 1 first appeared in [9, Sect. 3].

1. ϕ-Bimodules and Hochschild Cohomology

A *ϕ-bimodule* is a triple $\langle N, M, T\rangle$ in which $N \in B$-MOD, $M \in A$-MOD, and $T: N \to {}_\phi M_\phi$ is a B-bimodule morphism. We habitually abbreviate these data to $T: N \to M$ or simply T. A morphism $T \to T'$ consists of a B-bimodule morphism $f: N \to N'$ and an A-bimodule morphism $g: M \to M'$ making the evident square commutative ($T'f = gT$). It is *allowable* if f and g are. Elementary axiom checking shows that the category of ϕ-bimodules, ϕ-MOD, is a bicomplete abelian category. (All constructions are performed "object-wise.") Though we shall never use this fact it is worth noting that ϕ-MOD is a comma category, [8, II.6]: $\phi\text{-MOD} = (id_{B\text{-Mod}} \downarrow {}_\phi - {}_\phi)$.

There are obvious exact restriction functors $\text{res}_B: \phi\text{-MOD} \to B\text{-MOD}$ and $\text{res}_A: \phi\text{-MOD} \to A\text{-MOD}$. Each of these has left and right adjoints (the *inflations*):

$$\text{linf}_B(N) = N \to A \otimes_B N \otimes_B A; \qquad \text{linf}_A(M) = 0 \to M. \tag{1.1}$$

$$\text{rinf}_B(N) = N \to 0; \qquad \text{rinf}_A(M) = {}_\phi M_\phi \to M. \tag{1.2}$$

(Note: only linf_B can fail to be exact). All six functors preserve allowability. The exactness of res_B and res_A implies that the left inflations preserve (relative) projectives and the right inflations preserve (relative) injectives. (Of course one can verify this directly from the definitions.) Consequently, ϕ-MOD has enough of each. For example, if T is a ϕ-bimodule pick allowable monomorphisms $N \to I \in B$-MOD and $M \to I' \in A$-MOD with I and I' relative injectives. Then $\text{rinf}_B(I) \oplus \text{rinf}_A(I')$ is a relative injective in ϕ-MOD and there is an allowable monomorphism $T \to \text{rinf}_B(I) \oplus \text{rinf}_A(I')$. (All relative injectives have this form [3, Sect. 1], a fact we shall not need until Section 3, where it appears as Lemma 3.2).

The *relative Yoneda cohomology* on ϕ-MOD, $\text{Ext}^*_\phi(-, -)$, is defined precisely as in the case of algebras. The presence of enough relative injectives implies that for each $T \in \phi$-MOD, $\text{Ext}^*_\phi(T, -)$ is a universal relative δ-functor. We wish, following Hochschild, to define a cochain complex whose cohomology is $\text{Ext}^*_\phi(\phi, -)$. The correct complex arises as a mapping cylinder.

If C^* and D^* are cochain complexes in an abelian category and $f: C^* \to D^*$ is a cochain map then the *algebraic mapping cylinder $Mc(f)$* is defined by $Mc(f)^* = C^* \oplus D^{*-1}$ and

$$\delta = \begin{pmatrix} \delta_C & 0 \\ f & -\delta_D \end{pmatrix}.$$

Note that the natural inclusion $D^* \to Mc(f)^{*+1}$ is *not* a cochain map. However, $D^n \xrightarrow{i} Mc(f)^{n+1}$, $d \mapsto (-1)^n d$, is. (The sign $(-1)^n$ reflects the dimension shift.) Then there is an exact sequence of cochain complexes $0 \to D^{*-1} \xrightarrow{i} Mc(f)^* \xrightarrow{\pi} C^* \to 0$, yielding the long exact cohomology sequence

$$\cdots \to H^{n-1}(D) \xrightarrow{H(i)} H^n(Mc(f)) \xrightarrow{H(\pi)} H^n(C) \to H^n(D) \to \cdots.$$

The connecting homomorphism $H^n(C) \to H^n(D)$ is usually defined *via* the snake lemma as: $c \in Z^n(C) \mapsto i^{-1}\delta\{c; 0\} = i^{-1}\{0; fc\} = (-1)^n fc$. To rid ourselves of the nettlesome $(-1)^n$ we take instead for the connecting homomorphism $c \mapsto (-1)^n i^{-1}\delta\{c; 0\}$; then it is just $H^n(f)$.

Given a ϕ-bimodule T we define the *Hochschild cochains* $C^*(\phi, T)$ to be the mapping cylinder of $\Phi: C^*(B, N) + C^*(A, M) \to C^*(B, M)$, $\Phi(\langle \Gamma^B, \Gamma^A \rangle) = T \circ \Gamma^B - \Gamma^A \circ \phi$. Thus, $C^n(\phi, T) = C^n(B, N) + C^n(A, M) + C^{n-1}(B, M)$ and $\delta\{\Gamma^B, \Gamma^A; \Gamma^{AB}\} = \{\delta\Gamma^B, \delta\Gamma^A; T\Gamma^B - \Gamma^A\phi - \delta\Gamma^{AB}\}$. We shall generally write Γ for $\{\Gamma^B, \Gamma^A; \Gamma^{AB}\}$. The sequence of complexes becomes

$$0 \to C^{*-1}(B, M) \xrightarrow{i} C^*(\phi, T) \to C^*(B, N) + C^*(A, M) \to 0. \quad (1.3)$$

Clearly, any morphism $T \to T'$ induces a cochain map $C^*(\phi, T) \to C^*(\phi, T')$. Moreover, for any short exact sequence $\mathscr{E} \in \phi$-MOD, $C^*(\phi, \mathscr{E}) = C^*(B, \mathrm{res}_B \mathscr{E}) + C^*(A, \mathrm{res}_A \mathscr{E}) + C^{*-1}(B, \mathrm{res}_A \mathscr{E})$. So $C^*(\phi, \mathscr{E})$ is exact and the snake lemma provides the connecting homomorphisms required to make $H^*(\phi, -)$ a relative δ-functor. From the definitions it is immediate that $C^*(\phi, T \oplus T') = C^*(\phi, T) + C^*(\phi, T')$ and, so, $H^*(\phi, T \oplus T') = H^*(\phi, T) + H^*(\phi, T')$. Observe that $H^0(\phi, T) = \mathrm{Hom}_\phi(\phi, T) = \mathrm{Ext}^0(\phi, T)$. Hence there is a unique morphism $\mathrm{Ext}^*_\phi(\phi, -) \to H^*(\phi, -)$ extending the identity. This is an isomorphism if $H^*(\phi, -)$ is universal.

THEOREM. $H^*(\phi, -) \cong \mathrm{Ext}^*_\phi(\phi, -)$.

Proof. We shall establish the required universality of $H^*(\phi, T)$ by showing that $H^{n+1}(\phi, -)$, $n \geq 0$, vanishes on enough relative injectives, namely products of inflations.

Let $I \in B$-MOD be a (relative) injective. Then (1.3) yields

$$\cdots \to H^n(B, 0) \to H^{n+1}(\phi, \mathrm{rinf}_B(I)) \to H^{n+1}(B, I) + H^{n+1}(A, 0) \to \cdots.$$

Since I is a B-relative-injective $H^{n+1}(B, I) = 0$ and, so $H^{n+1}(\phi, \mathrm{rinf}_B(I)) = 0$, $n \geq 0$.

Now let $I \in A$-MOD be a (relative) injective. This time (1.3) yields

$$\cdots \to H^n(B, I) + H^n(A, I) \xrightarrow{H(\Phi)} H^n(B, I) \xrightarrow{H(i)} H^{n+1}(\phi, \mathrm{rinf}_A(I)) \to \cdots.$$

For $n>0$, $H^n(A, I)=0$ and, so, $H^n(\Phi)=id$, an isomorphism. Also, $H^0(\Phi)$ is an epimorphism since $H^0(\Phi)(\langle\beta, 0\rangle)=\beta$. It follows that $H^{n+1}(\phi, \mathrm{rinf}_A(I))=0$, $n\geqslant 0$. ▮

Recall that a (standard) Hochschild cochain f is *normal* if $f(x_1,..., x_n)=0$ whenever any $x_i=1$. The normal cochains form a subcomplex and the inclusion of complexes induces a cohomology isomorphism. [8, X.2]. Let $C_n^*(B, -)$ and $C_n^*(A, -)$ be the normal cochain complexes. Clearly, Φ restricts to give $C_n^*(B, N)+C_n^*(A, M)\to C_n^*(B, M)$. The mapping cylinder $C_n^*(\phi, T)$ fits into a short exact sequence

$$0\to C_n^{*-1}(B, M)\to C_n^*(\phi, T)\to C_n^*(B, N)+C_n^*(A, M)\to 0. \quad (1.4)$$

The inclusion of complexes gives a map of sequences (1.4) → (1.3) having cohomology isomorphisms at each end. Hence the Five Lemma implies that the middle is a cohomology isomorphism. That is, $H^*(\phi, T)$ can be computed using *normal* cochains, (those Γ for which Γ^B, Γ^A, and Γ^{AB} are all normal).

One place the Hochschild theory for algebras and morphisms differs from that for arbitrary diagrams is the representation of singular extensions. A *singular extension* of ϕ by T is a short exact sequence ($\mathscr{E}$): $0\to T\to\phi'\to\phi\to 0$ in which ϕ' is an algebra morphism and $\mathrm{res}_B\mathscr{E}$, $\mathrm{res}_A\mathscr{E}$ are singular algebra extensions, (as defined in the Introduction). The Yoneda equivalence classes of these form a group $\mathrm{exal}(\phi, T)$ under Baer sum.

The obvious generalization of the next proposition is *not* true for diagrams over partially ordered sets. However, something close to it is [3, Sect. 8]. Consequencely, we merely sketch the details.

PROPOSITION. $H^2(\phi, -)\cong\mathrm{exal}(\phi, -)$.

Proof (Sketch). Given $\{\Gamma^B, \Gamma^A; \Gamma^{AB}\}\in Z_n^2(\phi, T)$ define $\phi'\colon B'\to A'$ *via*: $B'=B+N$ with $\langle b_1, n_1\rangle\cdot\langle b_2, n_2\rangle=\langle b_1b_2, b_1n_2+n_1b_2+\Gamma^B(b_1, b_2)\rangle$; $A'=A+M$ with $\langle a_1, m_1\rangle\cdot\langle a_2, m_2\rangle=\langle a_1a_2, a_1m_2+m_1a_2+\Gamma^A(a_1a_2)\rangle$; $\phi'(\langle b, n\rangle)=\langle\phi b, Tn+\Gamma^{AB}(b)\rangle$.

Given $0\to T\to\phi'\to\phi\to 0$ pick k-linear splittings $s_B\colon B\to B'$ and $s_A\colon A\to A'$. Define $\Gamma\in Z_n^2(\phi, T)$ *via*: $\Gamma^B(b_1, b_2)=s_B(b_1)\,s_B(b_2)-s_B(b_1b_2)$; $\Gamma^A(a_1, a_2)=s_A(a_1)\,s_A(a_2)-s_A(a_1a_2)$; $\Gamma^{AB}=\phi's_B-s_A\phi$. ▮

So far the Hochschild theories of algebras and morphisms appear identical. Thus encouraged one might anticipate that $C^*(\phi, \phi)$ carries graded Lie and graded cup products. We know of none. However, the parallel persists: $C^*(\phi, \phi)$ *does* carry graded pairings which induce such products on $H^*(\phi, \phi)$. For $\Gamma\in C^m(\phi, \phi)$, $\Delta\in C^n(\phi, \phi)$ these are

$$[\Gamma, \Delta]=\Gamma\cdot\Delta-(-1)^{(m-1)(n-1)}\Delta\cdot\Gamma \quad (1.5)$$

where

$$\Gamma \cdot \Delta = \{\Gamma^B \bar{\circ} \Delta^B, \Gamma^A \bar{\circ} \Delta^A; \Gamma^A \bar{\circ} \Delta^{AB} + (-1)^{n-1} \Gamma^{AB} \bar{\circ} \Delta^B + \Delta^{AB} \smile \Gamma^{AB}\} \tag{1.6}$$

$$\Gamma \smile \Delta = \{\Gamma^B \smile \Delta^B, \Gamma^A \smile \Delta^A; \Gamma^{AB} \smile \phi\Delta^B + (-1)^m \Gamma^A \phi \smile \Delta^{AB}\}. \tag{1.7}$$

Direct calculational proofs of the properties of [,] and $\smile$ are possible but unilluminating. [9, Sect. 4]. A less calculational proof using the CCT appears in [3, Sect. 18]. Both the generalization of (1.5)–(1.7) to diagrams over partially ordered sets and a still better proof of their properties appear in [4, Sects. 4, 5].

2. The Mapping Ring and the Hochschild Cohomology Isomorphism

The most economical description of the *mapping ring*, $\phi!$, is: $\phi! = \begin{pmatrix} B & 0 \\ A & A \end{pmatrix}$ with $(a \ \ 0)\begin{pmatrix} b \\ 0 \end{pmatrix} = (ab^\phi \ \ 0)$. For calculational purposes the following is a more convenient representation: as a k-module, $\phi! = B + A + A\phi$ (the suffix ϕ distinguishes the off-diagonal copy of A from the diagonal copy); the multiplication is determined by linearity, the products in B and A, and

$$\begin{aligned} &B \cdot A = B \cdot A\phi = A \cdot B = A\phi \cdot A = A\phi \cdot A\phi = 0 \\ &a\phi \cdot b = ab^\phi \phi \\ &a \cdot a'\phi = aa'\phi. \end{aligned} \tag{2.1}$$

Since $a \cdot \phi = a \cdot 1_A \phi$, we abbreviate $1_A \phi$ to ϕ and think of ϕ as an element of the ring. Observe that 1_B and 1_A are orthogonal idempotents and that $\phi!$ is a unital k-algebra with $1 = 1_B + 1_A$.

Since $1_B \cdot \phi! \subset \phi! \cdot 1_B$ and $\phi! \cdot 1_A \subset 1_A \cdot \phi!$ we see that $\phi! 1_B = B + A\phi$ and $1_A \phi! = A + A\phi$ are two sided ideals. Hence there are algebra epimorphisms $\phi! \to^{\pi_A} \phi!/\phi! 1_B = A$ and $\phi! \to^{\pi_B} \phi!/1_A \phi! = B$. These then induce change-of-base functors from, variously, A-MOD, $(A-B)$-MOD, B-MOD, and $(B-A)$-MOD to $\phi!$-MOD. All four base changes are exact and preserve allowability. We shall use them without further comment to view modules in any of the source categories as $\phi!$-bimodules.

The *mapping bimodule* of a ϕ-bimodule T is $T! = \begin{pmatrix} N & 0 \\ M & M \end{pmatrix} = N + M + M\phi$ with the nonobvious operation given by

$$a\phi \cdot n = an^T \phi; \ m\phi \cdot b = mb^\phi \phi. \tag{2.2}$$

It is immediate that $!: \phi\text{-MOD} \to \phi!\text{-MOD}$ is an exact embedding and preserves allowability. Hence there is a natural transformation

$$\omega^*: \mathrm{Ext}^*_\phi(-, -) \to \mathrm{Ext}^*_{\phi!}(-!, -!).$$

Of course, in dimension 0 this is just $\omega^0: \mathrm{Hom}_\phi(-, -) \to \mathrm{Hom}_{\phi!}(-!, -!)$.

PROPOSITION. *! is full; that is, ω^0 is an isomorphism.*

Proof. If $f \in \mathrm{Hom}_{\phi!}(T!, T'!)$ then $f(N) = f(1_B T!\, 1_B) \subset 1_B T'!\, 1_B = N'$; so $f|_N \in \mathrm{Hom}_B(N, N')$. Similarly, $f|_M \in \mathrm{Hom}_A(M, M')$. Then $\langle f|_N, f|_M \rangle$ is a ϕ-bimodule morphism $T \to T'$, as we now show: $f(\phi \cdot n) = \phi \cdot f(n) = \phi \cdot f|_N(n) = f|_N(n)^{T'}\phi = T' \circ f|_N(n)\phi$; also $f(\phi \cdot n) = f(n^T\phi) = f(n^T)\phi = f|_M(n^T)\phi = f|_M \circ T(n)\phi$. But right multiplication by ϕ is a k-isomorphism $M' \to M'\phi$. Hence $T' \circ f|_N = f|_M \circ T$, as required. It is trivial that $\langle f|_M, f|_M \rangle ! = f$. ∎

In fact ω^* is an isomorphism, a special case of the CCT of [3]. This, together with Theorem 1 and the comments in the Introduction, implies that there is an isomorphism $H^*(\phi, -) \to H^*(\phi!, -!)$. We shall soon define a cochain map $\tau^*: C^*(\phi, -) \to C^*(\phi!, -!)$ and prove—without invoking the CCT—that $H^*(\tau)$ is an isomorphism. But first we examine the $\phi!$-bimodule $T!$ more closely.

Observe that $1_A T!\, 1_B = M\phi$ is a submodule of $T!$. It is also an $(A-B)$-bimodule and its module structure over $\phi!$ is the same as that obtained through base change from its $(A-B)$-structure. The quotient module $T!/M\phi$ is isomorphic (over k) to $N + M$. This is immediately seen to be a $\phi!$-direct sum, where N and M are viewed as $\phi!$-bimodules through $\phi! \to^{\pi_B} B$ and $\phi! \to^{\pi_A} A$. Thus there is an allowable exact sequence in $\phi!$-MOD

$$0 \to M\phi \to T! \to N \oplus M \to 0. \tag{2.3}$$

Of course, (2.3) and the cochain isomorphism $C^*(\phi!, N \oplus M) = C^*(\phi!, N) + C^*(\phi!, M)$ induce

$$0 \to C^*(\phi!, M\phi) \to C^*(\phi!, T!) \to C^*(\phi!, N) + C^*(\phi!, M) \to 0. \tag{2.4}$$

We shall reserve the symbol x to represent *pure elements* of $\phi!$, i.e., those in B, A, and $A\phi$. A cochain is completely determined by its values on tuples of pure elements. Consequently, in (2.5)–(2.8) we shall define cochains by giving their values *only* on pure tuples.

We define $\tau\Gamma$ for $\Gamma = \{\Gamma^B, \Gamma^A; \Gamma^{AB}\}$ $C^n(\phi, T)$ by

$$\begin{aligned}
&\tau\Gamma|_B = \Gamma^B;\ \tau\Gamma|_A = \Gamma^A \\
&\tau\Gamma(a\phi, b_2,..., b_n) = \Gamma^A(a, b_2^\phi,..., b_n^\phi)\phi + a\Gamma^{AB}(b_2,..., b_n)\phi \\
&\tau\Gamma(a_1,..., a_{r-1}, a_r\phi, b_{r+1},..., b_n) = \Gamma^A(a_1,..., a_r, b_{r+1}^\phi,..., b_n^\phi)\phi \\
&\tau\Gamma(x_1,..., x_n) = 0 \qquad \text{otherwise.}
\end{aligned} \tag{2.5}$$

Routine but quite tedious calculations verify that τ is a cochain map. As a courtesy to the reader we omit them.

Now using i: $C^{*-1}(B, M) \to C^*(\phi, T)$ we may restrict τ. It is immediate from (2.5) that $\operatorname{im}(\tau i) \subset C^*(\phi!, M\phi)$. In fact τi is described by

$$\begin{aligned}
&\tau i\Gamma^{AB}(a\phi, b_2,..., b_n) = (-1)^n a\Gamma^{AB}(b_2,..., b_n)\phi \\
&\tau i\Gamma^{AB}(x_1,..., x_n) = 0 \qquad \text{otherwise.}
\end{aligned} \tag{2.6}$$

Putting (1.3) and (2.4) together we obtain

$$\begin{array}{ccccccc}
0 \to C^{*-1}(B, M) & \xrightarrow{i} & C^*(\phi, T) & \to & C^*(B, N) + C^*(A, M) & \to 0 \\
\downarrow{\scriptstyle \tau i} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \bar\tau} & \\
0 \to C^*(\phi!, M) & \to & C^*(\phi!, T!) & \to & C^*(\phi!, N) + C^*(\phi!, M) & \to 0
\end{array} \quad . \tag{2.7}$$

It is easy to check that

$$\bar\tau = \begin{pmatrix} \bar\tau_B & 0 \\ 0 & \bar\tau_A \end{pmatrix}$$

where $\bar\tau_B$ and $\bar\tau_A$ are defined by

$$\begin{aligned}
&\bar\tau_B\Gamma^B|_B = \Gamma^B;\ \bar\tau_B\Gamma^B(x_1,..., x_n) = 0 \qquad \text{otherwise.} \\
&\bar\tau_A\Gamma^A|_A = \Gamma^A;\ \bar\tau_A\Gamma^A(x_1,..., x_n) = 0 \qquad \text{otherwise.}
\end{aligned} \tag{2.8}$$

The cochain map τ appeared in [3] for arbitrary diagrams and may seem somewhat mysterious. However, the definitions of τi and $\bar\tau$ seem quite natural. This then removes some of the mystery concerning τ, for it is the "simplest" cochain map inducing τi and $\bar\tau$ as in (2.7); it respects the natural filtrations on $C^*(\phi, T)$ and $C^*(\phi!, T!)$.

We are now in a position to prove:

THEOREM. *$H^*(\tau)$ is an isomorphism.*

Proof. The Five Lemma implies that τ is a cohomology isomorphism if both τi and $\bar{\tau}$ are. Hence the theorem follows from Lemmas 1 and 2 below. ▮

LEMMA 1. *$H^*(\tau i)$ is an isomorphism.*

LEMMA 2. *$H^*(\bar{\tau})$ is an isomorphism.*

Note that Lemma 2 is equivalent to the conjunction of Lemmas 3 and 4:

LEMMA 3. *$H^*(\bar{\tau}_B)$ is an isomorphism.*

LEMMA 4. *$H^*(\bar{\tau}_A)$ is an isomorphism.*

To prove Lemma 1 we shall require two intermediate constructions, namely particular functors

$$\sim : (A - \phi!) - \text{MOD} \to \phi!\text{-MOD}, \mathcal{M} \rightsquigarrow \tilde{\mathcal{M}},$$

and

$$\frown : (\phi! - A) - \text{MOD} \to \phi!\text{-MOD}, \mathcal{N} \rightsquigarrow \hat{\mathcal{N}}.$$

As k-modules $\tilde{\mathcal{M}} = \mathcal{M} + {}_\phi\mathcal{M}$ and $\hat{\mathcal{N}} = \mathcal{N} + \mathcal{N}_\phi$; the operation of $\phi!$ on $\tilde{\mathcal{M}}$ and $\hat{\mathcal{N}}$ are given by

$$\begin{aligned} &\phi\langle m_1, m_2\rangle = \langle m_2, 0\rangle \\ &b\langle m_1, m_2\rangle = \langle 0, b^\phi m_2\rangle; \qquad a\langle m_1, m_2\rangle = \langle am_1, 0\rangle \\ &\langle m_1, m_2\rangle x = \langle m_1 x, m_2 x\rangle \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &\langle n_1, n_2\rangle \phi = \langle 0, n_1\rangle \\ &\langle n_1, n_2\rangle b = \langle 0, n_2 b^\phi\rangle; \qquad \langle n_1, n_2\rangle a = \langle n_1 a, 0\rangle \\ &x\langle n_1, n_2\rangle = \langle xn_1, xn_2\rangle. \end{aligned} \tag{2.10}$$

Of course, we may—as usual—consider $\mathcal{M}$ and ${}_\phi\mathcal{M}$ as $\phi!$-bimodules through $\phi! \to A$ and $\phi! \to B$. When so considered $\mathcal{M}$ is a submodule of $\tilde{\mathcal{M}}$—but ${}_\phi\mathcal{M}$ is not (since $\phi \cdot {}_\phi\mathcal{M} = 0$ while $\phi\langle 0, m\rangle = \langle m, 0\rangle$). However, there is an allowable exact sequence of $\phi!$-bimodules

$$0 \to \mathcal{M} \to \tilde{\mathcal{M}} \to {}_\phi\mathcal{M} \to 0. \tag{2.11}$$

Analogously, there is an allowable exact sequence $0 \to \mathcal{N}_\phi \to \hat{\mathcal{N}} \to \mathcal{N} \to 0$. Observe that if $\mathcal{N}$ is actually an A-bimodule M' then $\hat{\mathcal{N}} = (0 \to M')! = M' + M'\phi$.

In the long exact cohomology sequence induced by (2.11) the connecting homomorphism

$$\mathrm{Ext}^*_{\phi!}(-, {}_\phi\mathscr{M}) \to \mathrm{Ext}^{*+1}_{\phi!}(-, \mathscr{M}) \tag{2.12}$$

is given by splicing with (2.11). When $- = \phi!$ we also have:

LEMMA 5. *The connecting homomorphism* $H^*(\phi!, {}_\phi\mathscr{M}) \to H^{*+1}(\phi!, \mathscr{M})$ *is induced by the cochain map* $C^*(\phi!, {}_\phi\mathscr{M}) \to C^{*+1}(\phi!, \mathscr{M})$, $f \mapsto f'$, *where*

$$\begin{aligned} f'(z_0,..., z_n) &= (-1)^n af(z_1,..., z_n) \qquad &\text{if } z_0 = a\phi \\ &= 0 &\text{if } z_0 \in B + A. \end{aligned}$$

Proof. That $f \mapsto f'$ is a cochain map is merely a tedious computation. There is a short exact sequence of complexes induced by (2.11): $0 \to C^*(\phi!, \mathscr{M}) \to C^*(\phi!, \tilde{\mathscr{M}}) \to C^*(\phi!, {}_\phi\mathscr{M}) \to 0$. The connecting homomorphism is described by the snake lemma as follows: if $f \in Z^n(\phi!, {}_\phi\mathscr{M})$ then $(0\ f)' \in C^n(\phi!, \tilde{\mathscr{M}})$, $(-1)^n\delta(0\ f)' \in Z^{n+1}(\phi!, \mathscr{M})$, and $[f] \mapsto [(-1)^n\delta(0\ f)']$. Now $\delta(0\ f)'(z_0,..., z_n) = z_0(0\ f)'(z_1,..., z_n) + \sum(-1)^{i+1}(0\ f)'(..., z_iz_{i+1},...) + (-1)^{n+1}(0\ f)'(z_1,..., z_{n-1})\, z_n$.

If $z_0 \in B + A$ then $z_0(0\ f)' = (0\ z_0f)'$ and $\delta(0\ f)'(z_0,..., z_n) = \langle 0, \delta f(z_0,..., z_n)\rangle = 0$.

If $z_0 = a\phi$ then $a\phi(0\ f)' = (af\ 0)'$ while $(a\phi)f = 0$, (since $f \in C^n(\phi!, {}_\phi\mathscr{M})$ and $\phi \cdot {}_\phi\mathscr{M} = 0$). Hence $\delta(0\ f)'(a\phi, z_1,..., z_n) = \langle af(z_1,..., z_n), \delta f(a\phi, z_1,..., z_n)\rangle = \langle af(z_1,..., z_n), 0\rangle$. Thus $(-1)^n\delta(0\ f)' = f'$ and $f \mapsto f'$ induces the connecting homomorphism. ∎

We shall establish, as Lemma 7, that (2.12) and, so, $f \mapsto f'$ are isomorphisms when $\mathscr{M} \in (A - B) -$ MOD. First observe that $\mathfrak{A} \rightsquigarrow 1_A\mathfrak{A}$ and $\mathfrak{A} \rightsquigarrow \mathfrak{A}1_A$ define functors $\phi!$-MOD $\to (A - \phi!)$-MOD and $\phi!$-MOD $\to (\phi! - A)$-MOD, which are obviously exact and preserve allowability. It is also obvious that both $\sim$ and $\wedge$ are exact and preserve allowability. Moreover, we have

LEMMA 6. $\sim$ *is right adjoint to* $\mathfrak{A} \rightsquigarrow 1_A\mathfrak{A}$ *and* $\wedge$ *is left adjoint to* $\mathfrak{A} \rightsquigarrow \mathfrak{A}1_A$; *that is,* $\mathrm{Hom}_{\phi!}(\mathfrak{A}, \tilde{\mathscr{M}}) \to \mathrm{Hom}_{A-\phi!}(1_A\mathfrak{A}, \mathscr{M})$, $f \mapsto f|_{1_A\mathfrak{A}}$, and $\mathrm{Hom}_{\phi!}(\hat{\mathscr{N}}, \mathfrak{A}) \to \mathrm{Hom}_{\phi!-A}(\mathscr{N}, \mathfrak{A}1_A)$, $g \mapsto g|_{\mathscr{N}}$, *are natural isomorphisms.*

Proof. Clearly, $f \mapsto f|_{1_A\mathfrak{A}}$ and $g \mapsto g|_{\mathscr{N}}$ are natural transformations. We shall give their inverses, thereby establishing that they are isomorphisms.

Let $\mathfrak{A}$ be a $\phi!$-bimodule; as a k-module $\mathfrak{A} = 1_A\mathfrak{A} + 1_B\mathfrak{A}$. Given $h \in \mathrm{Hom}_{A-\phi!}(1_A\mathfrak{A}, \mathscr{M})$ define $h'\colon 1_B\mathfrak{A} \to {}_\phi\mathscr{M}$ by $h' = 1_B\mathfrak{A} \to^{\phi\cdot} 1_A\mathfrak{A} \to^{h} \mathscr{M} \to^{\cong} {}_\phi\mathscr{M}$ and let $\tilde{h} = \left(\begin{smallmatrix} h & 0 \\ 0 & h' \end{smallmatrix}\right)$. It is routine to check that $\tilde{h}$ is a $\phi!$-bimodule morphism and $\tilde{h}|_{1_A\mathfrak{A}}=h$ while $(\tilde{f}|_{1_A\mathfrak{A}})^{\sim}=f$. Hence $h\mapsto\tilde{h}$ is inverse to $f\mapsto f|_{1_A\mathfrak{A}}$.

Next, write $\mathfrak{A}=\mathfrak{A}1_A+\mathfrak{A}1_B$ and, for $h\in\mathrm{Hom}_{\phi!-A}(\mathcal{N},\mathfrak{A}1_A)$, set $h'=\mathcal{N}_\phi\to^{\cong}\mathcal{N}\to^{h}\mathfrak{A}1_A\to^{\cdot\phi}\mathfrak{A}1_B$ and define $\hat{h}\colon\hat{\mathcal{N}}\to\mathfrak{A}$ by $\hat{h}=\begin{pmatrix} h & 0\\ 0 & h'\end{pmatrix}$. As before, it is routine that $\hat{h}$ is a ϕ!-bimodule morphism and that $h\mapsto\hat{h}$ is inverse to $g\mapsto g|_{\mathcal{N}}$. ∎

We shall adhere to the following notational conventions: if $\mathscr{E}\colon 0\to K\to\mathscr{E}_{n-1}\to\cdots\to\mathscr{E}_1\to\mathscr{E}_0\to^{\partial_0}G\to 0$ is an n-fold extension and $\kappa\colon K\to K'$, $\gamma\colon G'\to G$ are morphisms then $\kappa\mathscr{E}$ and $\mathscr{E}\gamma$ are, respectively, the pushout and pullback extensions (e.g., $\mathscr{E}\gamma=0\to K\to\mathscr{E}_{n-1}\to\cdots\to\mathscr{E}_1\to\mathscr{E}_0\gamma\to G'\to 0$, where $\mathscr{E}_0\gamma=\{\langle e_0,g'\rangle\in\mathscr{E}_0\times G'\mid\partial_0(e_0)=\gamma(g')\}$.) We write $\mathscr{E}\equiv\mathscr{E}'$ to indicate a congruence of extensions. Such a congruence can always be represented by a pair of morphisms of extensions $\mathscr{E}\leftarrow\mathscr{F}\to\mathscr{E}'$, each having the identity at both ends. Note that if $\mathscr{E}\to\mathscr{E}'$ is a morphism of extensions having κ at the left end and γ at the right end then $\kappa\mathscr{E}\equiv\mathscr{E}'\gamma$.

LEMMA 7. *For $T'\in\phi$-MOD and $\mathcal{M}\in(A-B)$-MOD, $\mathrm{Ext}^*_{\phi!}(T'!,\tilde{\mathcal{M}})=0$. In this case (2.12) becomes an isomorphism $\mathrm{Ext}^*_{\phi!}(T'!,{}_\phi\mathcal{M})\cong\mathrm{Ext}^{*+1}_{\phi!}(T'!,\mathcal{M})$.*

Proof. The second statement follows trivially from the first, which we now prove.

We begin with the case of dimension zero. Since $\mathrm{Ext}^0_{\phi!}(T'!,\tilde{\mathcal{M}})=\mathrm{Hom}_{\phi!}(T'!,\tilde{\mathcal{M}})$ the first adjunction of Lemma 6 reduces this case to: $\mathrm{Hom}_{A-\phi!}(M'+M'\phi,\mathcal{M})=0$. So suppose $f\colon M'+M'\phi\to\mathcal{M}$ is an $(A-\phi!)$-bimodule morphism; then $f(M')\subset\mathcal{M}1_A=0$ and $f(M'\phi)=f(M')\phi=0$. That is, $f=0$ as required.

Next we consider the case of dimension $n>0$. The long exact cohomology sequence induced by $0\to M'+M'\phi\to T'!\to N'\to 0$ shows that the lemma will follow from: $\mathrm{Ext}^*_{\phi!}(M'+M'\phi,\tilde{\mathcal{M}})=0=\mathrm{Ext}^*_{\phi!}(N',\tilde{\mathcal{M}})$. (In fact, this is equivalent to the lemma since $M'+M'\phi=(0\to M')!$ and $N'=(N'\to 0)!$.)

Given $\mathscr{E}$: $\quad 0\to\tilde{\mathcal{M}}\to\mathscr{E}_{n-1}\to\cdots\to\mathscr{E}_0\to M'+M'\phi\to 0 \qquad (n>0),$

consider $\mathscr{E}1_A$. From Lemma 6 and the morphism *id*: $\mathscr{E}1_A\to\mathscr{E}1_A$ we obtain $(\mathscr{E}1_A)^\wedge\to\mathscr{E}$. But $(\tilde{\mathcal{M}}1_A)^\wedge=\hat{0}=0$ and $((M'+M'\phi)\,1_A)^\wedge=\hat{M}'=M'+M'\phi$. Hence the morphism $(\mathscr{E}1_A)^\wedge\to\mathscr{E}$ has zero at the left end and identity at the right. This yields $0=0(\mathscr{E}1_A)^\wedge\equiv\mathscr{E}\,\mathrm{id}=\mathscr{E}$ and, hence, $\mathrm{Ext}^*_{\phi!}(M'+M'\phi,\tilde{\mathcal{M}})=0$.

Now, suppose we are given $[\mathscr{E}]\in\mathrm{Ext}^n_{\phi!}(N',\tilde{\mathcal{M}})$ $(n>0)$. As above, from id: $1_A\mathscr{E}\to 1_A\mathscr{E}$ and Lemma 6 we obtain $\mathscr{E}\to(1_A\mathscr{E})^\sim$. But $(1_A\tilde{\mathcal{M}})^\sim=\tilde{\mathcal{M}}$ while $(1_AN')^\sim=\tilde{0}=0$. Hence, $\mathscr{E}\to(1_A\mathscr{E})^\sim$ has identity at the left end and zero at the right. This yields $\mathscr{E} = \mathrm{id}\,\mathscr{E} \equiv (1_A\mathscr{E})^{\sim}0 = 0$ and, hence, $\mathrm{Ext}^*_{\phi!}(N', \tilde{\mathscr{M}}) = 0$. ▮

The proof of Lemma 1 requires Lemma 3. Nonetheless, we present it now and then proceed to Lemmas 3 and 4.

Proof of Lemma 1. The B-bimodules M and $_\phi(M\phi)$ are identical. Also Lemma 3 applies to *any* B-bimodule N, in particular to $_\phi(M\phi)$. So Lemmas 3 and 7—with $T' = \phi$, $\mathscr{M} = M\phi$—combine to give an isomorphism $H^*(B, M) \to H^*(\phi!, {}_\phi(M\phi)) \to H^{*+1}(\phi!, M\phi)$. Invoking (2.8) and Lemma 5 we see that the isomorphism is induced by τi. ▮

We isolate two more lemmas to aid in the proofs of Lemmas 3 and 4.

LEMMA 8. *Let T' be a ϕ-bimodule. Then* $\mathrm{Ext}^*_B(N', N) \to \mathrm{Ext}^*_{\phi!}(N', N)$, $[\mathscr{E}] \mapsto [\mathscr{E}]$, *and* $\mathrm{Ext}^*_A(M', M) \to \mathrm{Ext}^*_{\phi!}(M', M)$, $[\mathscr{E}] \mapsto [\mathscr{E}]$, *are isomorphisms.*

Proof. If $[\mathscr{E}] \in \mathrm{Ext}^*_B(N', N)$ then each bimodule in $\mathscr{E}$ becomes a $\phi!$-bimodule *via* $\phi! \to B$. Plainly, if $\mathscr{E} \leftarrow \mathscr{F} \to \mathscr{E}'$ is a congruence in B-MOD then it is one in $\phi!$-MOD as well. Thus, $[\mathscr{E}] \mapsto [\mathscr{E}]$ is indeed a morphism.

Suppose that $[\mathscr{E}] \in \mathrm{Ext}^*_{\phi!}(N', N)$. For any $\phi!$-bimodule $\mathfrak{A}$ both $\mathfrak{A}1_B$ and $1_A\mathfrak{A}1_B$ are submodules. Hence there are monomorphisms of $\phi!$-bimodule extensions: $\mathscr{E}1_B \subsetneq \mathscr{E}$ and $1_A\mathscr{E}1_B \subsetneq \mathscr{E}1_B$. The first of these is a congruence $\mathscr{E}1_B \equiv \mathscr{E}$, as it has equality at each end ($N1_B = N$; $N'1_B = N'$). The second has quotient $\mathscr{E}1_B \to 1_B\mathscr{E}1_B$, which again has equality at both ends and, so, is a congruence $\mathscr{E}1_B \equiv 1_B\mathscr{E}1_B$. Therefore, $\mathscr{E} \equiv 1_B\mathscr{E}1_B$. To establish that this gives a morphism $\mathrm{Ext}^*_{\phi!}(N', N) \to \mathrm{Ext}^*_B(N', N)$ we must show: $\mathscr{E} \equiv \mathscr{E}'$ in $\phi!$-MOD implies $1_B\mathscr{E}1_B \equiv 1_B\mathscr{E}'1_B$ in B-MOD. So suppose that $\mathscr{E} \leftarrow \mathscr{F} \to \mathscr{E}'$ is a congruence in $\phi!$-MOD. Then we have congruences $1_A\mathscr{E}1_B \leftarrow 1_A\mathscr{F}1_B \to 1_A\mathscr{E}'1_B$ and $\mathscr{E}1_B \leftarrow \mathscr{F}1_B \to \mathscr{E}'1_B$. Taking quotients we find $1_B\mathscr{E}1_B \leftarrow 1_B\mathscr{F}1_B \to 1_B\mathscr{E}'1_B$ is a congruence in B-MOD. Clearly, $[\mathscr{E}] \mapsto [1_B\mathscr{E}1_B]$ is an inverse to $\mathrm{Ext}^*_B(N', N) \to \mathrm{Ext}^*_{\phi!}(N', N)$.

The second isomorphism is established similarly. The inverse is effected by the congruence $\mathscr{E} \leftarrow 1_A\mathscr{E} \to 1_A\mathscr{E}1_A$. ▮

LEMMA 9. *Let T' be a ϕ-bimodule. Then*

$$\mathrm{Ext}^*_{\phi!}(N', N) \to \mathrm{Ext}^*_{\phi!}(T'!, N),\ [\mathscr{E}] \mapsto [\mathscr{E}\pi_{N'}],$$

and

$$\mathrm{Ext}^*_{\phi!}(M', M) \to \mathrm{Ext}^*_{\phi!}(T'!, M),\ [\mathscr{E}] \mapsto [\mathscr{E}\pi_{M'}]$$

are isomorphisms.

Proof. The morphism $[\mathscr{E}] \mapsto [E\pi_{N'}]$ is induced by the allowable short exact sequence $0 \to M' + M'\phi \to T'! \to N' \to 0$. Hence it will be an isomorphism if and only if $\mathrm{Ext}^*_{\phi!}(M' + M'\phi, N) = 0$. Note that there is also an allowable exact sequence $0 \to M'\phi \to M' + M'\phi \to M' \to 0$. So the triviality of $\mathrm{Ext}^*_{\phi!}(M' + M'\phi, N)$ will follow from $\mathrm{Ext}^*_{\phi!}(M', N) = 0 = \mathrm{Ext}^*_{\phi!}(M'\phi, N)$. Let $\mathscr{E}$ represent a class in either of these groups. If $\mathfrak{A}$ is any $\phi!$-bimodule then $1_A \mathfrak{A}$ is a submodule. Hence there is a morphism of extensions, $1_A \mathscr{E} \subsetneq \mathscr{E}$, having equality at the right end. ($1_A M' = M'$; $1_A M'\phi = M'\phi$). But the left end is $0 \to N$ ($1_A N = 0$). Thus, $0 = 0(1_A \mathscr{E}) \equiv \mathscr{E}\,\mathrm{id} = \mathscr{E}$; that is, $[\mathscr{E}] = 0$.

The other isomorphism is established similarly. It arises from $0 \to N' + M'\phi \to T'! \to M' \to 0$. The triviality of $\mathrm{Ext}^*_{\phi!}(N' + M'\phi, M)$ is revealed by the exact sequence $0 \to M'\phi \to N' + M'\phi \to N' \to 0$ and the morphism of extensions $\mathscr{E} 1_B \subsetneq \mathscr{E}$. ∎

At last everything is in place to give:

Proof of Lemmas 3 *and* 4. For any k-algebra Λ and Λ-bimodule $\mathscr{M}$ the isomorphism $H^*(\Lambda, \mathscr{M}) \to \mathrm{Ext}^*_\Lambda(\Lambda, \mathscr{M})$ is achieved as follows. Let $\mathscr{P}\colon 0 \to \partial\mathscr{P}_n \to \Lambda^{\otimes n+1} \to \cdots \to \Lambda^{\otimes 2} \to \Lambda \to 0$ be the usual nth-stage truncation of the Hochschild resolution [7, X.2]. Then every class in $\mathrm{Ext}^n_\Lambda(\Lambda, \mathscr{M})$ is represented by an extension of the form $[\lambda\mathscr{P}]$ with $\lambda \in Z^n(\Lambda, \mathscr{M})$ and $[\lambda] \mapsto [\lambda\mathscr{P}]$ is the isomorphism [7, III.6].

Lemmas 8 and 9 combine to give an isomorphism

$$H^*(B, N) \to \mathrm{Ext}^*_B(B, N) \to \mathrm{Ext}^*_{\phi!}(\phi!, N) \to H^*(\phi!, N) \tag{2.13}$$

which we claim is $H^*(\bar{\tau}_B)$. Let $\mathscr{P}$ and $\mathscr{P}'$ be, respectively, the nth-stage Hochschild resolutions of $\phi!$ and B. Then $\pi_B\colon \phi! \to B$ induces an obvious morphism of extensions $\mathscr{P} \to \mathscr{P}'$. If $\Gamma^B \in Z^n(B, N)$ then

$$\bar{\tau}_B \Gamma^B = \partial\mathscr{P}_n \xrightarrow{\pi_B^{\otimes n+2}} \partial\mathscr{P}'_n \xrightarrow{\Gamma^B} N$$

and, so, the composite morphism of extensions $\mathscr{P} \to \mathscr{P}' \to \Gamma^B\mathscr{P}'$ has $\bar{\tau}_B\Gamma^B\colon \partial\mathscr{P}_n \to N$ at the left end and $\pi_B\colon \phi! \to B$ at the right end. But this means that $\mathscr{P} \to \Gamma^B\mathscr{P}'$ gives a congruence $(\bar{\tau}_B\Gamma^B)\mathscr{P} \equiv (\Gamma^B\mathscr{P}')\,\pi_B$. Hence (2.13) is $[\Gamma^B] \mapsto [\Gamma^B\mathscr{P}'] \mapsto [(\Gamma^B\mathscr{P}')\,\pi_B] = [(\bar{\tau}_B\Gamma^B)] \mapsto [\bar{\tau}_B\Gamma^B]$; that is, it is $H^*(\bar{\tau}_B)$.

That $H^*(\bar{\tau}_A)$ is an isomorphism follows by the systematic substitution of A for B and M for N throughout the last paragraph. ∎

3. The Yoneda Cohomology Isomorphism

In this section we prove:

Theorem (CCT). *$\omega^*: \mathrm{Ext}^*_\phi(T', T) \to \mathrm{Ext}^*_{\phi!}(T'!, T!)$, $[\mathscr{E}] \mapsto [\mathscr{E}!]$, is an isomorphism for all T', $T \in \phi$-MOD.*

Note that Theorems 1 and 2 (together with an obvious universality argument) imply the CCT in the case $T' = \phi$. Conversely, the CCT in conjunction with either of the earlier theorems will give the others. (Again, universality arguments are needed.)

The CCT would be trivial if ! preserved either enough relative projectives or enough relative injectives; unfortunately, it does neither. [3, Sect. 11]. (See the comments following Lemma 4 below.) The proof of the CCT in [3] used projective resolutions of T' and $T'!$ while that in Section 2 applies only to the case $T' = \phi$. The critical lemma for the one we give here is:

Lemma 1. *If $T'' \in \phi$-MOD is a relative injective then $T''!$ is a relative $\mathrm{Hom}_{\phi!}(T'!, -)$-acyclic bimodule; that is,*

$$\mathrm{Ext}^p_{\phi!}(T'!, T''!) = (R^p \,\mathrm{Hom}_{\phi!}(T'!, -))(T''!) = 0 \qquad \text{for} \quad p > 0.$$

Note that since the right derived functors are computed using only allowable resolutions we could not assert more than that $T''!$ be a *relative* acyclic bimodule.

Of course, Lemma 1 is an immediate consequence of the theorem. In a moment we shall show that it also implies the theorem, and, so, they are equivalent. But first we cite—without proof—a general, though quite standard result. Suppose: $\mathscr{C}$ and $\mathscr{D}$ are abelian categories, $\mathscr{C}$ has enough (relative) injectives, $F: \mathscr{C} \to \mathscr{D}$ is a covariant left exact functor, and $0 \to C \to I_\bullet \in \mathscr{C}$ is an (allowable) resolution of C by (relative) F-acyclic objects. Then $(R^pF)(C) = H^p(F(I_\bullet))$; that is, (relative) cohomology can be computed using (relative) acyclic resolutions [1, XVII.3; 5, Theorem 2.4.1, Remark 3].

Proof (CCT). Let $0 \to T \to T''_0 \to T''_1 \to \cdots$ be an allowable relative injective resolution of T in ϕ-MOD. Then $0 \to T! \to T''_\bullet!$ is an allowable resolution of $T!$ in $\phi!$-MOD. We have: $\mathrm{Ext}^*_\phi(T', T) = H^* \,\mathrm{Hom}_\phi(T', T''_\bullet) = H^* \,\mathrm{Hom}_{\phi!}(T'!, T''_\bullet) = (R^*(\mathrm{Hom}_{\phi!}(T'!, -))(T!) = \mathrm{Ext}^*_{\phi!}(T'!, T!)$. The second equality holds because ! is full (Proposition 2); the third follows from Lemma 1 and the comments above; the other two are simply the assertions that $\mathrm{Ext}^*_\phi(T', -)$ and $\mathrm{Ext}^*_{\phi!}(T'!, -)$ are given by relative right derived functions. ∎

The first reduction of Lemma 1 is a classification of the injectives in ϕ-MOD. We use the right inflation functors of (1.2): $\mathrm{rinf}_B(N) = N \to 0$; $\mathrm{rinf}_A(M) = {}_\phi M_\phi \to M$. These preserve (relative) injectives.

LEMMA 2. *If $T'' \in \phi$-MOD is a relative injective then $T'' = \mathrm{rinf}_B(\ker T'') \oplus \mathrm{rinf}_A(M'')$ and $\ker T'' \in B$-MOD, $M'' \in A$-MOD are relative injectives.*

Proof. First observe that $\mathrm{Hom}_\phi(N \to 0, T'') = \mathrm{Hom}_B(N, \ker T'')$. Hence $\mathrm{Hom}_\phi(-, T'')$ is exact on allowable exact sequences of the form $\mathrm{rinf}_B\, \mathscr{E}$ if and only if $\ker T''$ is a relative injective in B-MOD. That is, the relative injectivity of T'' implies that of $\ker T''$ which, in turn, implies that if $\mathrm{rinf}_B(\ker T'')$. Thus the allowable exact sequence

$$0 \to \mathrm{rinf}_B(\ker T'') \to T'' \to \bar{T}'' \to 0 \tag{3.1}$$

splits and $\bar{T}''$ is also a relative injective. Note that $\bar{T}'' = N''/\ker T'' \to M''$ and $\ker \bar{T}'' = 0$. So there is an inclusion $\langle \bar{T}'', \mathrm{id} \rangle$: $\bar{T}'' \subsetneq \mathrm{rinf}_A(M'')$ which must then be split; the cokernel has the form $T''': N''' \to 0$ and is a summand of $\mathrm{rinf}_A(M'') = {}_\phi M''_\phi \to^{\mathrm{id}} M''$. But then $0 = \ker(\mathrm{id}) = \ker \bar{T}'' \oplus \ker T''' = N'''$ and we see that $\bar{T}'' = \mathrm{rinf}_A(M'')$. Finally, referring back to the splitting of (3.1) we have the lemma. ▌

Lemma 2 shows that Lemma 1 is equivalent to:

LEMMA 3. *If $I \in B$-MOD and $I' \in A$-MOD are relative injectives then $(\mathrm{rinf}_B I)!$ and $(\mathrm{rinf}_A I')!$ are relative $\mathrm{Hom}_{\phi!}(T'!, -)$-acyclic bimodules.*

Half of Lemma 3—and also Lemma 2.3— follow instantly from the stronger result:

LEMMA 4. *The functor B-MOD $\to \phi!$-MOD induced by $\phi! \to B$ factors as B-MOD $\to^{\mathrm{rinf}_B} \phi$-MOD $\to^{!}$ $\phi!$-MOD. It preserves relative injectives.*

Proof. The factorization is easy: $(\mathrm{rinf}_B N)! = (N \to 0)! = N + 0 + 0\phi = N$. For the rest: suppose that $0 \to \mathfrak{A} \to \mathfrak{B} \in \phi!$-MOD is allowable, $I \in B$-MOD is a relative injective, and $f \in \mathrm{Hom}_{\phi!}(\mathfrak{A}, I)$. Then $f(1_A \mathfrak{A} + 1_B \mathfrak{A} 1_A) \subset 1_A I + 1_B I 1_A = 0$ while $0 \to 1_B \mathfrak{A} 1_B \to 1_B \mathfrak{B} 1_B$ is allowable in B-MOD. Let $f' \in \mathrm{Hom}_B(1_B \mathfrak{B} 1_B, I)$ be an extension of f—at least one is guaranteed by the relative injectivity of I—and set $f'(1_A \mathfrak{B} + 1_B \mathfrak{B} 1_A) = 0$. One easily checks that f' is a well-defined $\phi!$-bimodule morphism extending f. ▌

Naturally, Lemma 4 raises the question: if $I \in A$-Mod is a relative injective will $(\mathrm{rinf}_A I)!$ also be a relative injective? The (negative) answer is a special case of the:

PROPOSITION. *$(\mathrm{rinf}_A M)!$ is a (relative) injective if and only if $M = 0$.*

Before proving this we note that together with Lemmas 2 and 4 it implies: the only injectives preserved by ! are those of the form $\operatorname{rinf}_B(I)$. Since $0 \to M$ cannot be injected into one of these, ! cannot preserve enough injectives. In [3] we failed to make this simple observation—indeed, we posed it as an open problem.

Proof. (of the Proposition). Consider the submodules $\mathfrak{A} \subset \mathfrak{B}$ of $\phi! \otimes_k \phi!$ given by: $\mathfrak{B} = B \otimes A + A\phi \otimes A + B \oplus A\phi + A\phi \otimes A\phi$ and $\mathfrak{A} = A\phi \otimes A + A\phi \otimes A\phi$. Note that $\mathfrak{A} \subsetneq \mathfrak{B}$ is an (allowable) monomorphism. For each $M \in A$-Mod let F_M be the functor $\operatorname{Hom}_{\phi!}(-, (\operatorname{rinf}_A M)!)$. We shall show that $F_M(\mathfrak{B}) \to F_M(\mathfrak{A})$ is an epimorphism if and only if $M = 0$; this yields the proposition.

Suppose $f \in F_M(\mathfrak{A})$. Then $f(a\phi \otimes a') = af(\phi \otimes 1_A)\, a'$ and $f(a\phi \otimes a'\phi) = af(\phi \otimes 1_A)\, a'\phi$. That is, f is completely determined by $f(\phi \otimes 1_A) = 1_A f(\phi \otimes 1_A)\, 1_A \in M$ and, so, $F_M(\mathfrak{A}) = \operatorname{Hom}_A(A\phi \otimes A, M) = M$. Meanwhile, any $g \in F_M(\mathfrak{B})$ must have $g(1_B \otimes 1_A) \in 1_B(\operatorname{rinf}_A M)! 1_A = 0$. This is quickly seen to imply $g = 0$ and, so, $F_M(\mathfrak{B}) = 0$. But then $F_M(\mathfrak{B}) \to F_M(\mathfrak{A})$ is $0 \to M$. ∎

For each $M \in A$-MOD there is an allowable exact sequence

$$\mathscr{E}: 0 \to M\phi \to (\operatorname{rinf}_A M)! \to {}_\phi M_\phi \oplus M \to 0. \tag{3.2}$$

Of course, (3.2) induces a long exact sequence in which the connecting homomorphism is "splice with $\mathscr{E}$," which we denote by $\mathscr{E}\smile$:

$$\operatorname{Ext}^*_{\phi!}(T'!, {}_\phi M_\phi \oplus M) \to \operatorname{Ext}^{*+1}_{\phi!}(T'!, M\phi), \qquad [\mathscr{F}] \mapsto [\mathscr{E} \smile \mathscr{F}]. \tag{3.3}$$

We shall compute $\operatorname{Ext}^*_{\phi!}(T'!, (\operatorname{rinf}_A M)!)$ by examining (3.3).

As always with a direct sum, there are natural inclusions and projections: ${}_\phi M_\phi \to^{i_1} {}_\phi M_\phi \oplus M \leftarrow^{i_2} M$, ${}_\phi M_\phi \leftarrow^{p_1} {}_\phi M_\phi \oplus M \to^{p_2} M$. These induce a natural isomorphism

$$\operatorname{Ext}^*_{\phi!}(T'!, {}_\phi M_\phi) + \operatorname{Ext}^*_{\phi!}(T'!, M) \xrightarrow{(i_1\ i_2)} \operatorname{Ext}^*_{\phi!}(T'!, {}_\phi M_\phi \oplus M), \tag{3.4}$$

namely: $\langle [\mathscr{F}], [\mathscr{F}'] \rangle \mapsto i_1[\mathscr{F}] + i_2[\mathscr{F}']$. Of course, the inverse to $(i_1\ i_2)$ is $(p_1\ p_2)^t$.

Composing (3.4) and (3.3) gives a morphism

$$\operatorname{Ext}^*_{\phi!}(T'!, {}_\phi M_\phi) + \operatorname{Ext}^*_{\phi!}(T'!, M) \to \operatorname{Ext}^{*+1}_{\phi!}(T'!, M\phi), \tag{3.5}$$

specifically:

$$\langle [\mathscr{F}], [\mathscr{F}'] \rangle \mapsto [\mathscr{E}] \smile (i_1[\mathscr{F}] + i_2[\mathscr{F}']) = [\mathscr{E} i_1 \smile \mathscr{F}] + [\mathscr{E} i_2 \smile \mathscr{F}'].$$

LEMMA 5. $\operatorname{Ext}^*_{\phi!}(T'!, (\operatorname{rinf}_A M)!) = \ker \mathscr{E}\smile$.

Proof. First we examine the submodule $M\phi \subset (\text{rinf}_A\, M)!$. It is naturally an $(A-B)$-bimodule and as such $M\phi \to M_\phi$, $m\phi \mapsto m$, is an isomorphism. If we now view M_ϕ as a $\phi!$-bimodule through $\phi! \to B$, $\phi! \to A$ then $M\phi \to M_\phi$ becomes a $\phi!$-isomorphism. Thus (2.11) yields an allowable exact sequence

$$\mathscr{E}' : 0 \to M\phi \to (M\phi)^{\sim} \to {}_\phi M_\phi \to 0. \tag{3.6}$$

Observe that $\mathscr{E} i_1 = \mathscr{E}'$. Also Lemma 2.7 and (2.12) imply $\text{Ext}^*_{\phi!}(T'!, {}_\phi M_\phi) \to \text{Ext}^{*+1}_{\phi!}(T'!, M\phi)$, $[\mathscr{F}] \mapsto [\mathscr{E}' \smile \mathscr{F}]$, is an isomorphism.

Now (3.5) is just $(\mathscr{E}' \smile\ \mathscr{E} i_2 \smile)$, i.e., $\langle [\mathscr{F}], [\mathscr{F}'] \rangle \mapsto [\mathscr{E}' \smile \mathscr{F}] + [\mathscr{E} i_2 \smile \mathscr{F}']$. Since the first component is an isomorphism, it follows that $(\mathscr{E}' \smile\ \mathscr{E} i_2 \smile)$ is an epimorphism. But (3.5) differs from (3.3) by an isomorphism; hence $\mathscr{E} \smile$ is also an epimorphism. Also $\text{Hom}_{\phi!}(T'!, M\phi) = 0$. The last two facts and the long exact sequence induced by $\mathscr{E}$ easily imply $\text{Ext}^*_{\phi!}(T'!, (\text{rinf}_A\, M)!) \cong \ker \mathscr{E} \smile$, as required. ∎

We now have all the ingredients for the:

Proof of Lemma 3. First note that since (3.3) differs from (3.5) by an isomorphism we have $\ker \mathscr{E} \smile = \ker(\mathscr{E}' \smile\ \mathscr{E} i_2 \smile)$. Now let I be a relative injective A-bimodule. Lemmas 2.8 and 2.9 imply $\text{Ext}^p_{\phi!}(T'!, I) \cong \text{Ext}^p_A(M', I) = 0$ for $p > 0$. Thus (3.5), for $* > 0$, reduces to $\mathscr{E}' \smile$, an isomorphism. But then Lemma 5 asserts: for $p > 0$, $\text{Ext}^p_{\phi!}(T'!, (\text{rinf}_A\, I)!) = \ker \mathscr{E} \smile \cong \ker \mathscr{E}' \smile = 0$; that is, $(\text{rinf}_A\, I)!$ is a relative $\text{Hom}_{\phi!}(T'!, -)$-acyclic bimodule. ∎

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