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The motivic zeta function and its smallest poles ☆

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Abstract

Let f be a regular function on a nonsingular complex algebraic variety of dimension d. We prove a formula for the motivic zeta function of f in terms of an embedded resolution. This formula is over the Grothendieck ring itself, and specializes to the formula of Denef and Loeser over a certain localization. We also show that the space of n-jets satisfying $f=0$ can be partitioned into locally closed subsets which are isomorphic to a cartesian product of some variety with an affine space of dimension $\ulcorner dn/2 \urcorner$. Finally, we look at the consequences for the poles of the motivic zeta function.

Keywords: Motivic zeta function; Jet spaces

1. Introduction

(1.1) All schemes that are considered in this paper have base field $\mathbb{C}$.

The topological Euler–Poincaré characteristic χ has the following properties on complex algebraic varieties: $\chi(V)=\chi(V')$ if V is isomorphic to V', $\chi(V)=\chi(V\setminus W)+\chi(W)$ if W is a closed subset of V, and $\chi(V\times W)=\chi(V)\chi(W)$. The Hodge–Deligne polynomial of com-

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plex algebraic varieties (see (1.5)) is a finer invariant which has also these properties. The finest invariant with these properties is the class of a variety in the Grothendieck ring.

We recall this notion. The Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ of complex algebraic varieties is the abelian group generated by the symbols $[V]$, where V is a complex algebraic variety, subject to the relations $[V]=[V']$, if V is isomorphic to V', and $[V]=[V\setminus W]+[W]$, if W is closed in V. One can extend the Grothendieck bracket in the obvious way to constructible sets. The ring structure of $K_0(\mathrm{Var}_{\mathbb{C}})$ is given by $[V]\cdot[W]:=[V\times W]$. We denote by $\mathbb{L}$ the class of the affine line, and by $\mathcal{M}_{\mathbb{C}}$ the localization of $K_0(\mathrm{Var}_{\mathbb{C}})$ with respect to $\mathbb{L}$.

(1.2) Let Sch be the category of separated schemes of finite type over $\mathbb{C}$ and let $n\in\mathbb{Z}_{\geqslant 0}$. The functor $\cdot\times_{\mathrm{Spec}\,\mathbb{C}}\mathrm{Spec}\,\mathbb{C}[t]/(t^{n+1}):\mathrm{Sch}\to\mathrm{Sch}$ has a right adjoint, which we denote by $\mathcal{L}_n$. We call $\mathcal{L}_n(V)$ the scheme of n-jets of V and we define an n-jet on V as a closed point on $\mathcal{L}_n(V)$. If $f:W\to V$ is a morphism of schemes, then we get an induced morphism $f_n:=\mathcal{L}_n(f):\mathcal{L}_n(W)\to\mathcal{L}_n(V)$. For $m,n\in\mathbb{Z}_{\geqslant 0}$ satisfying $m\geqslant n$, the canonical embeddings $V\times_{\mathrm{Spec}\,\mathbb{C}}\mathrm{Spec}\,\mathbb{C}[t]/(t^{n+1})\hookrightarrow V\times_{\mathrm{Spec}\,\mathbb{C}}\mathrm{Spec}\,\mathbb{C}[t]/(t^{m+1})$ induce canonical (projection) morphisms $\pi^m_n:\mathcal{L}_m(V)\to\mathcal{L}_n(V)$. Because we want $\mathcal{L}_n(V)$ to be a variety (e.g. to take its class in the Grothendieck ring), we always endow $\mathcal{L}_n(V)$ with its reduced structure, and we interpret the morphisms above in this context. For more information about these constructions, see [DL1] or [Mu].

If V is a closed subscheme of $\mathbb{A}^m$, the closed points of $\mathcal{L}_n(V)$ are the $(a_{ij})_{1\leqslant i\leqslant m,\,0\leqslant j\leqslant n}\in\mathbb{A}^{m(n+1)}$ for which $(a_{1,0}+a_{1,1}t+\cdots+a_{1,n}t^n,\dots,a_{m,0}+a_{m,1}t+\cdots+a_{m,n}t^n)\in(\mathbb{C}[t]/(t^{n+1}))^m$ satisfies the equations of V. Actually, we get a set of equations of (the original possibly nonreduced version of) $\mathcal{L}_n(V)$ by substituting an arbitrary point of $(\mathbb{C}[t]/(t^{n+1}))^m$ in the equations of V. If V is an arbitrary separated scheme of finite type over $\mathbb{C}$, we apply the construction above to the elements of an affine cover, and we glue them together. Note that an n-jet can be seen as a parameterized curve modulo t^{n+1}.

(1.3) Let X be a nonsingular irreducible algebraic variety of dimension d, and let $f:X\to\mathbb{A}^1$ be a nonconstant regular function. Put $V=\mathrm{div}(f)$. For each $n\in\mathbb{Z}_{\geqslant 0}$, we consider the induced morphism $f_n:\mathcal{L}_n(X)\to\mathcal{L}_n(\mathbb{A}^1)$. For $n\in\mathbb{Z}_{\geqslant 0}$, the set

$$\mathcal{X}_n:=\{\gamma\in\mathcal{L}_n(X)\mid\gamma\cdot V=n\}$$

is a locally closed subvariety of $\mathcal{L}_n(X)$. Note that $\gamma\cdot V=\mathrm{ord}_t(f_n(\gamma))$. The motivic zeta function $Z(t)$ of $f:X\to\mathbb{A}^1$ is by definition

$$Z(t):=\sum_{n\geqslant 0}[\mathcal{X}_n]t^n\in\mathcal{M}_{\mathbb{C}}[\![t]\!].$$

In a lot of papers, there is a normalization factor $\mathbb{L}^{-dn}$ in the $(n+1)$th term in the definition of the motivic zeta function. Note that $Z(\mathbb{L}^{-d}t)-[X\setminus V]$ is the naive motivic zeta function from [DL2].

(1.4) We now describe a formula for $Z(t)$ in terms of an embedded resolution. Denef and Loeser deduced it by using motivic integration. Let $h:Y\to X$ be an embedded resolution of f, i.e. h is a proper birational morphism from a nonsingular variety Y such that h is an isomorphism on $Y \setminus h^{-1}(f^{-1}\{0\})$ and $h^{-1}(f^{-1}\{0\})$ is a normal crossings divisor. Let E_i, $i \in S$, be the irreducible components of $h^{-1}(f^{-1}\{0\})$. Let $K_{Y/X}$ be the relative canonical divisor supported in the exceptional locus of h. We define the numerical data N_i and ν_i by the equalities $\operatorname{div}(f \circ h) = \sum_{i \in S} N_i E_i$ and $K_{Y/X} = \sum_{i \in S}(\nu_i - 1)E_i$. For $I \subset S$, denote $E_I^\circ := (\bigcap_{i \in I} E_i) \setminus (\bigcup_{i \notin I} E_i)$. Then, the announced formula for $Z(t)$ is

$$Z(t) = \sum_{I \subset S} [E_I^\circ] \prod_{i \in I} \frac{(\mathbb{L}-1)\mathbb{L}^{dN_i - \nu_i} t^{N_i}}{1 - \mathbb{L}^{dN_i - \nu_i} t^{N_i}}.$$

In particular, $Z(t)$ is rational and belongs to the subring of $\mathcal{M}_{\mathbb{C}}[\![t]\!]$ generated by $\mathcal{M}_{\mathbb{C}}$ and the elements $t^N/(1 - \mathbb{L}^{dN-\nu}t^N)$, with $\nu, N \in \mathbb{Z}_{>0}$. Note that the equation implies, in particular, that the right-hand side does not depend on the embedded resolution.

(1.5) We now introduce the Hodge zeta function. Recall that the Hodge–Deligne polynomial of a complex algebraic variety W is

$$H(W) := \sum_{p,q} \left(\sum_{i \geqslant 0} (-1)^i h^{p,q}\big(H_c^i(W, \mathbb{C})\big) \right) u^p v^q \in \mathbb{Z}[u, v],$$

where $h^{p,q}(H_c^i(W, \mathbb{C}))$ is the dimension of the (p, q)-Hodge component of the ith cohomology group with compact support of W. The Hodge zeta function of f is

$$Z_{\mathrm{Hod}}(t) := \sum_{I \subset S} H\big(E_I^\circ\big) \prod_{i \in I} \frac{(uv-1)(uv)^{dN_i - \nu_i} t^{N_i}}{1 - (uv)^{dN_i - \nu_i} t^{N_i}}.$$

The right-hand side does not depend on the embedded resolution because it is obtained from the right-hand side of the formula in (1.4) by specializing to Hodge–Deligne polynomials. Note that the Hodge zeta function is in a lot of papers a normalization of this one.

(1.6) Let f and V be as in (1.3). We consider the power series

$$J(t) := \sum_{n \geqslant 0} \big[\mathcal{L}_n(V)\big] t^n \in \mathcal{M}_{\mathbb{C}}[\![t]\!].$$

Because $[\mathcal{X}_n] = \mathbb{L}^d[\mathcal{L}_{n-1}(V)] - [\mathcal{L}_n(V)]$ for $n \geqslant 1$ and $[\mathcal{X}_0] = [X] - [V]$, we have the relation

$$J(t) = \frac{Z(t) - [X]}{\mathbb{L}^d t - 1}.$$

Consequently, the series $J(t)$ and $Z(t)$ determine each other.

(1.7) In Section 2, we prove the formula of (1.4) without using motivic integration. We will actually prove a stronger result which is over $K_0(\mathrm{Var}_{\mathbb{C}})$ instead of $\mathcal{M}_{\mathbb{C}}$: for an integer c satisfying $(\nu_i - 1)/N_i \leqslant c$ for all $i \in S$, we have

$$\sum_{n \geqslant 0} [\mathcal{X}_n] \big(\mathbb{L}^{2cd+c-d} t\big)^n = \sum_{I \subset S} [E_I^\circ] \prod_{i \in I} \frac{(\mathbb{L}-1)\mathbb{L}^{(2cd+c)N_i - \nu_i} t^{N_i}}{1 - \mathbb{L}^{(2cd+c)N_i - \nu_i} t^{N_i}}$$

in $K_0(\mathrm{Var}_{\mathbb{C}})[\![t]\!]$. After localizing with respect to $\mathbb{L}$, we can indeed deduce the formula of (1.4) because $\mathbb{L}$ is not a zero-divisor in $\mathcal{M}_{\mathbb{C}}$. However, it is unknown whether $\mathbb{L}$ is a zero-divisor in $K_0(\mathrm{Var}_{\mathbb{C}})$. This implies that we cannot deduce our formula straightforward from the one in (1.4) and that we do not know whether the formula of (1.4) holds in $K_0(\mathrm{Var}_{\mathbb{C}})[\![t]\!]$ whenever $dN_i - \nu_i \geqslant 0$ for all $i \in S$. (There always exists an embedded resolution for which this condition is satisfied.)

In Section 3, we will prove that $[\mathcal{L}_n(V)]$ is a multiple of $\mathbb{L}^{\ulcorner dn/2 \urcorner}$ in $K_0(\mathrm{Var}_{\mathbb{C}})$ for all $n \in \mathbb{Z}_{\geqslant 0}$. (We use the notation $\ulcorner x \urcorner$ for the smallest integer larger than or equal to $x \in \mathbb{R}$.) We will actually construct a partition of $\mathcal{L}_n(V)$ into locally closed subsets which are isomorphic to $W \times \mathbb{A}^{\ulcorner dn/2 \urcorner}$ for some variety W depending on the locally closed subset. The first author proved already an analogous result for the number of solutions of polynomial congruences in [Se2]. The difficulty here is that we do not have to count solutions, but that we have to construct isomorphisms. We also note that our setting of (1.3) is more general than polynomials, i.e. regular functions on affine space.

In Section 4, we will consider $Z(t)$ as a power series over a ring R which is a quotient of the image of the localization map in $\mathcal{M}_{\mathbb{C}}$. Using the previous result, we prove that $Z(t)$ belongs to the subring of $R[\![t]\!]$ generated by $R[t]$ and the elements $1/(1 - \mathbb{L}^{dN-\nu} t^N)$, with $\nu, N \in \mathbb{Z}_{>0}$ and $\nu/N \leqslant d/2$. An analogous result was already proved in [Se2] for the topological zeta function (and for Igusa's p-adic zeta function), where it says that there are no poles (with real part) less than $-d/2$. See [RV] for a possible definition of the notion of pole for the motivic zeta function. Because the ring R specializes to Hodge–Deligne polynomials, this result is also true for the Hodge zeta function.

In Section 5, we adapt the previous results in a relative setting.

2. The motivic zeta function over the Grothendieck ring

(2.1) If $g : Y \to Z$ is étale, one can sometimes reduce a problem about $\mathcal{L}_m(Y)$ to an analogous problem for $\mathcal{L}_m(Z)$ because of the following proposition. For a proof, see for example [Bl, Proposition 2.2].

Proposition. *Let $g : Y \to Z$ be étale and $m \in \mathbb{Z}_{>0}$. Then the natural map $\mathcal{L}_m(Y) \to Y \times_Z \mathcal{L}_m(Z)$ is an isomorphism.*

If $g : Y \to \mathbb{A}^d$ is étale, we obtain

$$\mathcal{L}_m(Y) \cong Y \times_{\mathbb{A}^d} \mathcal{L}_m\big(\mathbb{A}^d\big) = Y \times_{\mathbb{A}^d} \big(\mathbb{A}^d \times \mathbb{A}^{dm}\big) \cong Y \times \mathbb{A}^{dm}.$$

If Y is an arbitrary nonsingular irreducible algebraic variety of dimension d, we can cover Y with open subsets U for which $\mathcal{L}_m(U) \cong U \times \mathbb{A}^{dm}$. Consequently, $[\mathcal{L}_m(Y)] = [Y]\mathbb{L}^{dm}$.

Note that also the equality $[\mathcal{X}_n] = \mathbb{L}^d[\mathcal{L}_{n-1}(V)] - [\mathcal{L}_n(V)]$ for $n \geqslant 1$ of (1.6) can be proved by using this proposition.

(2.2) We need a theorem of Denef and Loeser [DL1, Lemma 3.4], see also [ELM], to obtain our formula.

Theorem. *Let X and Y be nonsingular irreducible algebraic varieties of dimension d, and let $h : Y \to X$ be a proper birational morphism. For $e, m \in \mathbb{Z}_{\geqslant 0}$ satisfying $m \geqslant e$, the set*

$$\Delta_{e,m} := \big\{\gamma \in \mathcal{L}_m(Y) \bigm| \gamma \cdot K_{Y/X} = e\big\}$$

is a locally closed subset of $\mathcal{L}_m(Y)$. If $m \geqslant 2e$, then $\Delta_{e,m}$ is the union of fibers of h_m and the restriction $\Delta_{e,m} \to h_m(\Delta_{e,m})$ of h_m is a piecewise trivial fibration with fiber $\mathbb{A}^e$. Moreover, two elements of the same fiber have the same image in $\mathcal{L}_{m-e}(Y)$.

(2.3) Let X be a nonsingular irreducible algebraic variety of dimension d, and let $f: X \to \mathbb{A}^1$ be a nonconstant regular function. Let $h: Y \to X$ be an embedded resolution of f. Let E_i, $i \in S$, be the irreducible components of $h^{-1}(f^{-1}\{0\})$, and let N_i and ν_i, with $i \in S$, be the numerical data.

Theorem. *If c is an integer satisfying $(\nu_i - 1)/N_i \leqslant c$ for all $i \in S$, then*

$$\sum_{n \geqslant 0} [\mathcal{X}_n] \big(\mathbb{L}^{2cd+c-d} t\big)^n = \sum_{I \subset S} \big[E_I^\circ\big] \prod_{i \in I} \frac{(\mathbb{L}-1)\mathbb{L}^{(2cd+c)N_i - \nu_i} t^{N_i}}{1 - \mathbb{L}^{(2cd+c)N_i - \nu_i} t^{N_i}} \tag{1}$$

in $K_0(\mathrm{Var}_{\mathbb{C}})[\![t]\!]$.

Proof. For $n, m \in \mathbb{Z}_{\geqslant 0}$ satisfying $m \geqslant n$, the set

$$\mathcal{X}_{n,m} := \big\{ \gamma \in \mathcal{L}_m(X) \,\big|\, \gamma \cdot \mathrm{div}(f) = n \big\}$$

is a locally closed subvariety of $\mathcal{L}_m(X)$. Note that $\mathcal{X}_n = \mathcal{X}_{n,n}$. Because $\mathcal{X}_{n,m} \cong \mathcal{X}_n \times \mathbb{A}^{d(m-n)}$ if X admits an étale map $X \to \mathbb{A}^d$, we have $[\mathcal{X}_{n,m}] = [\mathcal{X}_n]\mathbb{L}^{d(m-n)}$ for general X.

Let $\gamma \in h_m^{-1}(\mathcal{X}_{n,m})$. We have that

$$\sum_{i \in S} N_i (\gamma \cdot E_i) = \gamma \cdot \left(\sum_{i \in S} N_i E_i \right) = h_m(\gamma) \cdot \mathrm{div}(f) = n,$$

and that

$$\gamma \cdot K_{Y/X} = \gamma \cdot \left(\sum_{i \in S} (\nu_i - 1) E_i \right) = \sum_{i \in S} (\nu_i - 1)(\gamma \cdot E_i).$$

Let c be an integer satisfying $(\nu_i - 1)/N_i \leqslant c$ for all $i \in S$. Such an integer exists because S is finite. Note that there exists an embedded resolution h (which is a composition of blowing-ups with well chosen center) for which $\nu_i/N_i \leqslant d - 1$ for all $i \in S$ [Se1, Proof of Theorem 2.4.0]. We obtain

$$\gamma \cdot K_{Y/X} = \sum_{i \in S} (\nu_i - 1)(\gamma \cdot E_i) \leqslant c \sum_{i \in S} N_i (\gamma \cdot E_i) = cn.$$

In particular, there are only a finite number of possibilities for $\gamma \cdot K_{Y/X}$. In view of (2.2), we will try to find a formula for

$$\sum_{n \geqslant 0} \mathbb{L}^{cn} [\mathcal{X}_{n,2cn}] t^n = \sum_{n \geqslant 0} [\mathcal{X}_n] \big(\mathbb{L}^{2cd+c-d} t\big)^n \in K_0(\mathrm{Var}_{\mathbb{C}})[\![t]\!]$$

in terms of h.

Fix $n \in \mathbb{Z}_{>0}$. For an $|S|$-tuple of positive integers $a = (a_i)_{i \in S}$, we define $S_a := \{i \in S \mid a_i > 0\}$, $E_a^\circ := E_{S_a}^\circ$ and $|a| := \operatorname{Card}(S_a)$. We define a set A by

$$A := \left\{ (a_i)_{i \in S} \;\middle|\; \forall i \in S\colon\ a_i \in \mathbb{Z}_{\geqslant 0},\ \sum_{i \in S} a_i N_i = n \text{ and } E_a^\circ \neq \emptyset \right\},$$

and obtain a disjoint union

$$h_{2cn}^{-1}(\mathcal{X}_{n,2cn}) = \bigsqcup_{a \in A} \bigl\{ \gamma \in \mathcal{L}_{2cn}(Y) \bigm| \forall i \in S\colon\ \gamma \cdot E_i = a_i \bigr\}.$$

Fix $a \in A$. Put $e = \sum_{i \in S} a_i(\nu_i - 1)$. Denote the origin of γ by γ_0. If U_t, $t \in T$, is a partition of E_a°, then

$$\bigl\{ \gamma \in \mathcal{L}_{2cn}(Y) \bigm| \forall i \in S\colon\ \gamma \cdot E_i = a_i \bigr\} = \bigsqcup_{t \in T} \bigl\{ \gamma \in \mathcal{L}_{2cn}(Y) \bigm| \forall i \in S\colon\ \gamma \cdot E_i = a_i \text{ and } \gamma_0 \in U_t \bigr\}.$$

Using (2.1), we can take [Cr, Proof of Proposition 2.5] a partition U_t, $t \in T$, of E_a° into locally closed subset such that for each $t \in T$, the set

$$F_{a,t} := \bigl\{ \gamma \in \mathcal{L}_{2cn}(Y) \bigm| \forall i \in S\colon\ \gamma \cdot E_i = a_i \text{ and } \gamma_0 \in U_t \bigr\}$$

is isomorphic to $U_t \times \mathbb{A}^{2cdn - \sum_{i \in S_a} a_i} \times (\mathbb{A}^1 \setminus \{0\})^{|a|}$, and hence

$$[F_{a,t}] = [U_t] \mathbb{L}^{2cdn - \sum_{i \in S_a} a_i} (\mathbb{L} - 1)^{|a|}.$$

Because $F_{a,t} \subset \Delta_{e,2cn}$ is a union of fibers of h_{2cn} (one can use the last statement in Theorem 2.2 to prove this), we obtain from (2.2) that the restriction $F_{a,t} \to h_{2cn}(F_{a,t})$ of h_{2cn} is a piecewise trivial fibration with fiber $\mathbb{A}^e$. Hence,

$$[F_{a,t}] = \mathbb{L}^e \bigl[h_{2cn}(F_{a,t}) \bigr].$$

By summing over all $t \in T$, we obtain

$$\begin{aligned}
&\mathbb{L}^e \bigl[h_{2cn}\bigl(\bigl\{ \gamma \in \mathcal{L}_{2cn}(Y) \bigm| \forall i \in S\colon\ \gamma \cdot E_i = a_i \bigr\} \bigr) \bigr] \\
&\quad = \mathbb{L}^e \left[h_{2cn}\left(\bigsqcup_{t \in T} F_{a,t} \right) \right] = \sum_{t \in T} \mathbb{L}^e \bigl[h_{2cn}(F_{a,t}) \bigr] \\
&\quad = \sum_{t \in T} [F_{a,t}] = \bigl[E_a^\circ \bigr] \mathbb{L}^{2cdn - \sum_{i \in S_a} a_i} (\mathbb{L} - 1)^{|a|}.
\end{aligned}$$

Now we multiply both sides of the obtained equality with $\mathbb{L}^{cn-e}$ and sum over all $a \in A$. Note that e depends on a. We obtain

$$\begin{aligned}\mathbb{L}^{cn}[\mathcal{X}_{n,2cn}] &= \sum_{a\in A}\big[E_a^\circ\big]\mathbb{L}^{2cdn-\sum_{i\in S_a}a_i}\mathbb{L}^{cn-e}(\mathbb{L}-1)^{|a|}\\ &= \sum_{a\in A}\big[E_a^\circ\big](\mathbb{L}-1)^{|a|}\mathbb{L}^{(2cd+c)n-e-\sum_{i\in S_a}a_i}\\ &= \sum_{a\in A}\big[E_a^\circ\big](\mathbb{L}-1)^{|a|}\mathbb{L}^{\sum_{i\in S_a}((2cd+c)N_i-\nu_i)a_i}\\ &= \sum_{a\in A}\big[E_a^\circ\big]\prod_{i\in S_a}(\mathbb{L}-1)\mathbb{L}^{((2cd+c)N_i-\nu_i)a_i}.\end{aligned}$$

The last expression is the coefficient of t^n in the formal power series

$$\sum_{I\subset S}\big[E_I^\circ\big]\prod_{i\in I}\frac{(\mathbb{L}-1)\mathbb{L}^{(2cd+c)N_i-\nu_i}t^{N_i}}{1-\mathbb{L}^{(2cd+c)N_i-\nu_i}t^{N_i}},$$

and consequently, we have finished our proof. □

Remark. If we consider (1) as an equality in $\mathcal{M}_{\mathbb{C}}[\![t]\!]$ and if we replace t by $\mathbb{L}^{-(2cd+c-d)}t$, we obtain

$$Z(t)=\sum_{I\subset S}\big[E_I^\circ\big]\prod_{i\in I}\frac{(\mathbb{L}-1)\mathbb{L}^{dN_i-\nu_i}t^{N_i}}{1-\mathbb{L}^{dN_i-\nu_i}t^{N_i}}$$

in $\mathcal{M}_{\mathbb{C}}[\![t]\!]$. This is the formula of (1.4) which was first proved by Denef and Loeser using motivic integration.

3. Divisibility of jet spaces in $K_0(\mathrm{Var}_{\mathbb{C}})$

(3.1) Let X be a d-dimensional complex analytic manifold with analytic coordinates $(u_1,\dots,u_d)$. These coordinates induce tangent vector fields $\partial/\partial u_1,\dots,\ \partial/\partial u_d$ along X. Let f be a complex analytic function on X and let $b\in X$. By Taylor's theorem, we have for points x in a small enough neighborhood of b that

$$\begin{aligned}f(x) &= \lim_{m\to\infty}\sum_{\alpha\in\mathbb{Z}^d_{\geqslant 0}:|\alpha|\leqslant m}\frac{(\partial^{|\alpha|}f/\partial u^\alpha)(b)}{\alpha!}(x-b)^\alpha\\ &= f(b)+\sum_{j=1}^d(\partial f/\partial u_j)(b)(x_j-b_j)+\cdots.\end{aligned}$$

We explain the notation. The coordinates of b are $(b_1,\dots,b_d)$ and those of x are $(x_1,\dots,x_d)$. For $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{Z}^d_{\geqslant 0}$, we put $\alpha!=\alpha_1!\cdots\alpha_d!$, $(x-b)^\alpha=(x_1-b_1)^{\alpha_1}\cdots(x_d-b_d)^{\alpha_d}$, $|\alpha|=\alpha_1+\cdots+\alpha_d$ and $\partial^{|\alpha|}f/\partial u^\alpha=\partial^{\alpha_1+\cdots+\alpha_d}f/\partial u_1^{\alpha_1}\cdots\partial u_d^{\alpha_d}$.

Let γ be a convergent arc in X, i.e. a d-tuple of convergent power series in t with origin in X. Let $l\in\mathbb{Z}_{>0}$. Write $\gamma=b+t^l z$, where b is a d-tuple of polynomials in t of degree less than l and where z is a d-tuple of convergent power series. For every $t \in \mathbb{C}$ in the convergence domain of γ (and for which $\gamma(t) \in X$), we have

$$f\bigl(\gamma(t)\bigr) = \lim_{m\to\infty} \sum_{\alpha\in\mathbb{Z}^d_{\geqslant 0}:|\alpha|\leqslant m} \frac{(\partial^{|\alpha|} f/\partial u^\alpha)(b(t))}{\alpha!} t^{|\alpha|l}\bigl(z(t)\bigr)^\alpha,$$

and consequently, we obtain the equality

$$f(\gamma) = \lim_{m\to\infty} \sum_{\alpha\in\mathbb{Z}^d_{\geqslant 0}:\ |\alpha|\leqslant m} \frac{(\partial^{|\alpha|} f/\partial u^\alpha)(b)}{\alpha!} t^{|\alpha|l} z^\alpha$$

as formal power series in t.

Since every n-jet is liftable to a convergent arc, we get for an n-jet of the form $\gamma = b + t^l z$ that

$$f(\gamma) = \sum_{|\alpha|=0}^{\lceil n/l\rceil} \frac{(\partial^{|\alpha|} f/\partial u^\alpha)(b)}{\alpha!} t^{|\alpha|l} z^\alpha \mod t^{n+1}.$$

(3.2) Let X be a nonsingular irreducible algebraic variety of dimension d and let $g : X \to \mathbb{A}^d$ be an étale map. We identify $\mathcal{L}_n(X)$ and $X \times_{\mathbb{A}^d} \mathcal{L}_n(\mathbb{A}^d)$ by using the canonical isomorphism of (2.1).

The coordinates $(u_1, \dots, u_d)$ on $\mathbb{A}^d$ induce analytic coordinates on a complex neighborhood of every point of X. The tangent vector fields $\partial/\partial u_1, \dots, \partial/\partial u_d$, which we define along the whole of X, are actually algebraic. This implies that all first and higher order partial derivatives of a regular function on X with respect to $u_1, \dots, u_d$ are regular functions on X.

Let f be a regular function on X. Let $l \in \{1, \dots, n\}$. Let $(x, b) \in X \times_{\mathbb{A}^d} \mathcal{L}_n(\mathbb{A}^d) = \mathcal{L}_n(X)$ and let $z = (z_1, \dots, z_d) \in \mathcal{L}_{n-l}(\mathbb{A}^d)$. Then

$$f\bigl(x, b + t^l z\bigr) = \sum_{|\alpha|=0}^{\lceil n/l\rceil} \frac{(\partial^{|\alpha|} f/\partial u^\alpha)(x, b)}{\alpha!} t^{|\alpha|l} z^\alpha \mod t^{n+1}.$$

(3.3) Theorem. *Let X be a nonsingular irreducible algebraic variety of dimension $d \in \mathbb{Z}_{>1}$ and let $f : X \to \mathbb{A}^1$ be a regular function. Put $V := \operatorname{div}(f)$. Then $[\mathcal{L}_n(V)]$ is a multiple of $\mathbb{L}^{\lceil dn/2\rceil}$ in $K_0(\mathrm{Var}_{\mathbb{C}})$ for all $n \in \mathbb{Z}_{\geqslant 0}$.*

Remark. It follows that $[\mathcal{X}_n]$ is also a multiple of $\mathbb{L}^{\lceil dn/2\rceil}$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

Proof. We are going to partition $\mathcal{L}_n(V)$ into a finite number of locally closed subsets which are isomorphic to $W \times \mathbb{A}^{\lceil dn/2\rceil}$ for some variety W depending on the locally closed subset.

If the theorem holds for the members of an open cover of X and all their intersections, then it holds for X, by additivity of the Grothendieck bracket. Hence, we may assume that there exists an étale map $g : X \to \mathbb{A}^d$.

Let r be $n/2$ if n is even and $(n + 1)/2$ if n is odd. We define

$$\begin{aligned}\mathcal{L}_{n,r}(V) = \big\{(x,b) \in \mathcal{L}_n(X) \bigm| \forall j \in \{1,\dots,d\}\colon\ (\partial f/\partial u_j)(x,b) \equiv 0 \bmod t^r \text{ and}\\ f(x,b) \equiv 0 \bmod t^{n+1}\big\}\end{aligned}$$

and for $k \in \{0,1,\dots,r-1\}$, we define

$$\begin{aligned}\mathcal{L}_{n,k}(V) = \big\{(x,b) \in \mathcal{L}_n(X) \bigm| \forall j \in \{1,\dots,d\}\colon\ (\partial f/\partial u_j)(x,b) \equiv 0 \bmod t^k,\\ \exists j \in \{1,\dots,d\}\colon\ (\partial f/\partial u_j)(x,b) \not\equiv 0 \bmod t^{k+1} \text{ and } f(x,b) \equiv 0 \bmod t^{n+1}\big\}.\end{aligned}$$

Then, the sets $\mathcal{L}_{n,k}(V)$, $k \in \{0,1,\dots,r\}$, are locally closed subsets of $\mathcal{L}_n(V)$ which partition $\mathcal{L}_n(V)$.

We prove that $\mathcal{L}_{n,r}(V)$ is isomorphic to $W \times \mathbb{A}^{\ulcorner dn/2 \urcorner}$ for some variety W. Let $(x,b) \in \mathcal{L}_{n,r}(V)$. For $z = (z_1,\dots,z_d) \in \mathcal{L}_{r-1}(\mathbb{A}^d)$, we have that

$$\begin{aligned}f\big(x, b + t^{n-r+1}z\big) &= f(x,b) + \sum_{j=1}^{d} (\partial f/\partial u_j)(x,b) t^{n-r+1} z_j + t^{2(n-r+1)}(\dots)\\ &\equiv 0 \mod t^{n+1},\end{aligned}$$

such that $(x, b + t^{n-r+1}\mathcal{L}_{r-1}(\mathbb{A}^d)) \subset \mathcal{L}_n(V)$. Because $n - r + 1 \geqslant r$, we obtain that $(x, b + t^{n-r+1}\mathcal{L}_{r-1}(\mathbb{A}^d)) \subset \mathcal{L}_{n,r}(V)$. Consequently, $\mathcal{L}_{n,r}(V) \cong \pi^n_{n-r}(\mathcal{L}_{n,r}(V)) \times \mathbb{A}^{dr}$. This proves our assertion for $\mathcal{L}_{n,r}(V)$ because $dr \geqslant \ulcorner dn/2 \urcorner$.

Let $k \in \{0,1,\dots,r-1\}$. We study $\mathcal{L}_{n,k}(V)$. Let $p \in \{1,\dots,d\}$ and let $l \in \{k,\dots,n-k\}$. We define

$$\begin{aligned}\mathcal{L}_{n,k,p}(V) = \big\{(x,b) \in \mathcal{L}_n(X) \bigm| \forall j \in \{1,\dots,d\}\colon\ (\partial f/\partial u_j)(x,b) \equiv 0 \bmod t^k\\ \text{and } (\partial f/\partial u_p)(x,b) \not\equiv 0 \bmod t^{k+1} \text{ and } f(x,b) \equiv 0 \bmod t^{n+1}\big\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{O}_{l,k,p}(V) = \big\{(x,b) \in \mathcal{L}_l(X) \bigm| \forall j \in \{1,\dots,d\}\colon\ (\partial f/\partial u_j)(x,b) \equiv 0 \bmod t^k\\ \text{and } (\partial f/\partial u_p)(x,b) \not\equiv 0 \bmod t^{k+1} \text{ and } f(x,b) \equiv 0 \bmod t^{k+l+1}\big\}.\end{aligned}$$

Note that $\mathcal{L}_{n,0,p}(V) = \mathcal{O}_{n,0,p}(V)$. We check that $f(x,b)$ is well defined modulo t^{k+l+1} in the definition of $\mathcal{O}_{l,k,p}(V)$. Suppose that $(x,b) \in \mathcal{L}_l(X)$ satisfies $(\partial f/\partial u_j)(x,b) \equiv 0 \bmod t^k$ for all $j \in \{1,\dots,d\}$. Then

$$\begin{aligned}f\big(x, b + t^{l+1}z\big) &= f(x,b) + \sum_{j=1}^{d} (\partial f/\partial u_j)(x,b) t^{l+1} z_j + t^{2(l+1)}(\dots)\\ &\equiv f(x,b) \mod t^{k+l+1},\end{aligned}$$

and consequently, $f(x,b)$ is well defined modulo t^{k+l+1}.

The following isomorphisms can be checked easily:

$$\mathcal{L}_{n,k,p}(V) \cong \mathcal{O}_{n-k,k,p}(V) \times \mathbb{A}^{dk},$$
$$\mathcal{O}_{l+1,k,p}(V) \cong \mathcal{O}_{l,k,p}(V) \times \mathbb{A}^{d-1}, \quad \text{for } l \in \{k, \ldots, n-k-1\},$$

where the projections on the first factor are respectively π^n_{n-k} and π^{l+1}_l. The isomorphism $\mathcal{O}_{l,k,p}(V) \times \mathbb{A}^{d-1} \to \mathcal{O}_{l+1,k,p}(V)$ maps $((x; b_{10} + b_{11}t + \cdots + b_{1l}t^l, \ldots, b_{d0} + b_{d1}t + \cdots + b_{dl}t^l), (a_1, \ldots, a_{d-1}))$ to the unique element of $\mathcal{O}_{l+1,k,p}(V)$ of the form $(x; b_{10} + b_{11}t + \cdots + b_{1l}t^l + a_1 t^{l+1}, \ldots, b_{p0} + b_{p1}t + \cdots + b_{pl}t^l + ct^{l+1}, \ldots, b_{d0} + b_{d1}t + \cdots + b_{dl}t^l + a_{d-1}t^{l+1})$. The unique value $c \in \mathbb{C}$ is obtained by solving a linear equation.

Consequently,

$$\mathcal{L}_{n,k,p}(V) \cong \mathcal{O}_{k,k,p}(V) \times \mathbb{A}^{(d-1)(n-2k)+dk},$$

where the projection on the first factor is π^n_k. This implies

$$\mathcal{L}_{n,k,1}(V) \cong \mathcal{O}_{k,k,1}(V) \times \mathbb{A}^{(d-1)(n-2k)+dk},$$
$$\mathcal{L}_{n,k,2}(V) \setminus \mathcal{L}_{n,k,1}(V) \cong \big(\mathcal{O}_{k,k,2}(V) \setminus \mathcal{O}_{k,k,1}(V)\big) \times \mathbb{A}^{(d-1)(n-2k)+dk},$$
$$\vdots$$
$$\mathcal{L}_{n,k,d}(V) \Big\backslash \Big(\bigcup_{1 \leqslant p \leqslant d-1} \mathcal{L}_{n,k,p}(V) \Big) \cong \Big(\mathcal{O}_{k,k,d}(V) \Big\backslash \Big(\bigcup_{1 \leqslant p \leqslant d-1} \mathcal{O}_{k,k,p}(V) \Big) \Big) \times \mathbb{A}^{(d-1)(n-2k)+dk}.$$

This finishes our proof because the left-hand sides form a partition of $\mathcal{L}_{n,k}(V)$ into locally closed subsets and because $(d-1)(n-2k) + dk \geqslant \ulcorner dn/2 \urcorner$. □

4. The smallest poles of motivic zeta functions

(4.1) Let X be a nonsingular irreducible algebraic variety of dimension $d \in \mathbb{Z}_{>1}$ and let $f : X \to \mathbb{A}^1$ be a regular function. Put $V := \mathrm{div}(f)$. In this section, we fix an embedded resolution for which $dN_i - \nu_i \geqslant 0$ for every $i \in S$. Note that we mentioned already in (2.3) that there always exists an embedded resolution which satisfies the stronger condition $\nu_i / N_i \leqslant d-1$ for every $i \in S$.

Consider the ideal

$$I' = \big\{ \alpha \in K_0(\mathrm{Var}_{\mathbb{C}}) \bigm| \exists k \in \mathbb{Z}_{\geqslant 0} \colon \mathbb{L}^k \alpha = 0 \big\}$$

of $K_0(\mathrm{Var}_{\mathbb{C}})$ and put $R' = K_0(\mathrm{Var}_{\mathbb{C}})/I'$. The image of $\mathbb{L}$ in R', which is also denoted by $\mathbb{L}$, is not a zero-divisor in R'. Note that I' is the kernel of the localization map $K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathcal{M}_{\mathbb{C}}$, such that R' can be considered as the image of this map in $\mathcal{M}_{\mathbb{C}}$. Consequently, the formula of (1.4) still holds if we consider $Z(t)$ as a power series over R'. Although Poonen [Po] proved that $K_0(\mathrm{Var}_{\mathbb{C}})$ is not a domain, it is still not known whether the localization map is injective.

We want to use for instance that $\bigcap_{k \in \mathbb{Z}_{\geqslant 0}} (\mathbb{L}^k) = \{0\}$ and that a number $k \in \mathbb{Z} \setminus \{0\}$ is not a zero-divisor. Because we do not know whether these are true in R', we will work in an appropriate quotient of R'. Consider the ideal

$$I = \bigcap_{k\in\mathbb{Z}_{\geqslant 0}} \{\alpha \in R' \mid \exists n \in \mathbb{Z}\setminus\{0\}\colon\ n\alpha \in (\mathbb{L}^k)\}$$
$$= \bigcap_{k\in\mathbb{Z}_{\geqslant 0}} \{\alpha \in R' \mid \exists n \in \mathbb{Z}\setminus\{0\}\colon\ n\alpha \text{ is divisible by } \mathbb{L}^k \text{ in } R'\}$$

of R' and put $R = R'/I$. Note that R specializes to Hodge–Deligne polynomials and that we do not know whether $I \neq \{0\}$. One verifies easily that an element of $\mathbb{Z}\setminus\{0\}$ is not a zero-divisor in R. One also checks that the image in R of $\{\alpha \in R' \mid \exists n \in \mathbb{Z}\setminus\{0\}\colon\ n\alpha \in (\mathbb{L}^k)\}$ contains $(\mathbb{L}^k)$, and consequently $\bigcap_{k\in\mathbb{Z}_{\geqslant 0}}(\mathbb{L}^k) = \{0\}$ in R. Thus, if α is a nonzero element of R, there exists a $k \in \mathbb{Z}_{\geqslant 0}$ such that α is divisible by $\mathbb{L}^k$ but not by $\mathbb{L}^{k+1}$ in R. Moreover, if α is a nonzero element of R, there exists a positive integer k which has for every $n \in \mathbb{Z}\setminus\{0\}$ the property that $n\alpha$ is not divisible by $\mathbb{L}^k$. We also have that $1-\mathbb{L}^k$, with $k \in \mathbb{Z}_{>0}$, is not a zero-divisor in R. Indeed, if $\alpha \in R$ satisfies $(1-\mathbb{L}^k)\alpha = 0$, then $\alpha = \mathbb{L}^k\alpha = \mathbb{L}^{2k}\alpha = \mathbb{L}^{3k}\alpha = \cdots$, and thus $\alpha \in \bigcap_{k\in\mathbb{Z}_{\geqslant 0}}(\mathbb{L}^k) = \{0\}$.

From now on, we will consider the motivic zeta function $Z(t)$ as a power series over R. The formula of $Z(t)$ in terms of an embedded resolution also holds over R. We write the motivic zeta function in the form

$$Z(t) = \frac{B(t)}{\prod_{i\in I}(1-\mathbb{L}^{dN_i-\nu_i}t^{N_i})},$$

where $I \subset S$ and where $B(t)$ is not divisible by any of the $1-\mathbb{L}^{dN_i-\nu_i}t^{N_i}$, with $i \in I$. Put $l := \min\{-\nu_i/N_i \mid i \in I\}$.

In the next paragraphs, we will work in a more general context. By abuse of notation, we will use the symbols of this particular situation.

(4.2) Let $Z(t)$ be an *arbitrary* element of $R[\![t]\!]$ of the form

$$Z(t) = \frac{B(t)}{\prod_{i\in I}(1-\mathbb{L}^{dN_i-\nu_i}t^{N_i})},$$

where every $(\nu_i, N_i) \in \mathbb{Z}_{>0}\times\mathbb{Z}_{>0}$ satisfies $dN_i - \nu_i \geqslant 0$ and where $B(t) \in R[t]$ is not divisible by any of the $1-\mathbb{L}^{dN_i-\nu_i}t^{N_i}$, with $i \in I$. Put $l := \min\{-\nu_i/N_i \mid i \in I\}$. Define the elements $\gamma_n \in R$ by the equality

$$Z(t) = \sum_{n\geqslant 0}\gamma_n t^n.$$

(4.3) Proposition. *There exists an integer a which is independent of n such that γ_n is a multiple of $\mathbb{L}^{\ulcorner (d+l)n-a\urcorner}$ in R for all integers n satisfying $(d+l)n-a \geqslant 0$.*

Remark. (i) The statement in the proposition is obviously equivalent to the following. If $l' \leqslant l$, then there exists an integer a which is independent of n such that γ_n is a multiple of $\mathbb{L}^{\ulcorner (d+l')n-a\urcorner}$ for all integers n satisfying $(d+l')n-a \geqslant 0$.

(ii) Suppose that we are in the situation of (4.1). It follows from (4.6) that $d+l > 0$, so that $(d+l)n-a$ rises linearly as a function of n with a slope depending on l. The condition $(d+l)n-a \geqslant 0$ is thus satisfied for n large enough.

Proof. We will say that a formal power series in t has the divisibility property if the coefficient of t^n is a multiple of $\mathbb{L}^{\ulcorner (d+l)n \urcorner}$ for every n.

For $i \in I$, the series

$$\frac{1}{1-\mathbb{L}^{dN_i-\nu_i}t^{N_i}} = \sum_{n\geqslant 0} \mathbb{L}^{n(dN_i-\nu_i)}t^{nN_i}$$

has the divisibility property because $dN_i - \nu_i$ is an integer larger than or equal to $N_i(d+l)$.

One can easily check that the product of a finite number of power series with the divisibility property also has the divisibility property. Let g be the degree of $B(t)$. For $n \geqslant g$, we will have that γ_n is a multiple of $\mathbb{L}^{\ulcorner (d+l)(n-g)\urcorner}$. This implies our statement. □

(4.4) We will decompose $Z(t)$ into partial fractions in (4.6). To this end, we need to apply the following lemma several times.

Lemma.

(a) *Let $i, j \in I$ such that $\nu_i/N_i \neq \nu_j/N_j$. Then, there exist polynomials $g(x,t), h(x,t) \in \mathbb{Z}[x,t]$ and an integer $k \in \mathbb{Z}_{>0}$ such that*

$$g(x,t)\big(1-x^{dN_i-\nu_i}t^{N_i}\big) + h(x,t)\big(1-x^{dN_j-\nu_j}t^{N_j}\big) = 1-x^k$$

holds in $\mathbb{Z}[x,t]$, and consequently such that

$$g(\mathbb{L},t)\big(1-\mathbb{L}^{dN_i-\nu_i}t^{N_i}\big) + h(\mathbb{L},t)\big(1-\mathbb{L}^{dN_j-\nu_j}t^{N_j}\big) = 1-\mathbb{L}^k$$

holds in $R[t]$.

(b) *Let $D(t) \in R[t]$. There exist polynomials $g(t), h(t) \in R[t]$ with $\deg(h) < N_i$ and a $k \in \mathbb{Z}_{\geqslant 0}$ such that*

$$\mathbb{L}^k D(t) = \big(1-\mathbb{L}^{dN_i-\nu_i}t^{N_i}\big)g(t) + h(t).$$

Proof. (a) Although $\mathbb{Z}[x,t]$ is not a PID, we can obtain the first relation by applying the algorithm of Bezout–Bachet in number theory to the polynomials $1-x^{dN_i-\nu_i}t^{N_i}$ and $1-x^{dN_j-\nu_j}t^{N_j}$ in the variable t. The number k is different from 0 because otherwise the polynomials $1-x^{dN_i-\nu_i}t^{N_i}$ and $1-x^{dN_j-\nu_j}t^{N_j}$ would have a nontrivial common divisor, and this is not the case because $\nu_i/N_i \neq \nu_j/N_j$. (b) This is straightforward by applying the division algorithm. □

(4.5) For $r \in \mathbb{Z}_{>0}$, we define a function $f_r : \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ by the relation

$$\frac{1}{(1-x)^r} = \sum_{n=0}^{\infty} f_r(n)x^n.$$

One proves by induction on r that

$$f_r(n) = \frac{(n+r-2)!}{(r-1)!(n-1)!} = \frac{n(n+1)\dots(n+r-2)}{(r-1)!}.$$

Lemma. *Let $m \in \mathbb{Z}_{>1}$. Let $n_1, \ldots, n_m$ be m different natural numbers. Then, the determinant of the matrix with the elements*

$$\begin{aligned} v_1 &= \big(f_1(n_1), f_2(n_1), \ldots, f_m(n_1)\big), \\ v_2 &= \big(f_1(n_2), f_2(n_2), \ldots, f_m(n_2)\big), \\ &\vdots \\ v_m &= \big(f_1(n_m), f_2(n_m), \ldots, f_m(n_m)\big) \end{aligned}$$

of $\mathbb{Z}^m$ in the rows is equal to

$$\frac{\prod_{j>i}(n_j - n_i)}{\prod_{i=1}^{m-1} i!}.$$

Remark.

(i) This determinant is different from zero, so the set $\{v_1, v_2, \ldots, v_m\}$ is linearly independent. Consequently, every element e_i of the standard basis of the $\mathbb{Z}$-module $\mathbb{Z}^m$ has a multiple which is generated by it.
(ii) This lemma is probably known. We include its proof by lack of reference.

Proof. The proof is by induction on m. The statement is trivial for $m = 2$. Let now $m > 2$. We expand the determinant along the last column, apply the induction hypothesis to the cofactors, use Vandermonde determinants, and obtain that it is equal to

$$\frac{1}{\prod_{i=1}^{m-2} i!} \begin{vmatrix} 1 & n_1 & n_1^2 & \cdots & n_1^{m-2} & f_m(n_1) \\ 1 & n_2 & n_2^2 & \cdots & n_2^{m-2} & f_m(n_2) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & n_m & n_m^2 & \cdots & n_m^{m-2} & f_m(n_m) \end{vmatrix}.$$

By using properties of determinants and the Vandermonde determinant, we see that this is equal to

$$\frac{1}{\prod_{i=1}^{m-1} i!} \begin{vmatrix} 1 & n_1 & n_1^2 & \cdots & n_1^{m-2} & n_1^{m-1} \\ 1 & n_2 & n_2^2 & \cdots & n_2^{m-2} & n_2^{m-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & n_m & n_m^2 & \cdots & n_m^{m-2} & n_m^{m-1} \end{vmatrix} = \frac{\prod_{j>i}(n_j - n_i)}{\prod_{i=1}^{m-1} i!}. \qquad \square$$

(4.6) Proposition. *There exist an integer a which is independent of n and positive integers N and b such that γ_{nN+b} is not a multiple of $\mathbb{L}^{\ulcorner (d+l)(nN+b)+a \urcorner}$ in R for n large enough.*

Proof. Put $I_1 = \{j \in I \mid -\nu_i/N_i = l\}$ and $I_2 = I \setminus I_1$. Let N be the lowest common multiple of the N_i, $i \in I_1$, and let ν be the lowest common multiple of the ν_i, $i \in I_1$. Remark that $\nu/N =$

ν_i/N_i for all $i \in I_1$. Let m be the cardinality of I_1. Because $1 - \mathbb{L}^{dN-\nu} t^N$ is a multiple of $1 - \mathbb{L}^{dN_i - \nu_i} t^{N_i}$ for all $i \in I_1$, we can write

$$Z(t) = \frac{D(t)}{(1 - \mathbb{L}^{dN-\nu} t^N)^m \prod_{i \in I_2} (1 - \mathbb{L}^{dN_i - \nu_i} t^{N_i})},$$

where $D(t) \in R[t]$. Applying decomposition into partial fractions (see Lemma 4.4), we can write

$$\begin{aligned} wZ(t) &= \frac{\mu_{m,0} + \mu_{m,1} t + \cdots + \mu_{m,N-1} t^{N-1}}{(1 - \mathbb{L}^{dN-\nu} t^N)^m} + \frac{\mu_{m-1,0} + \mu_{m-1,1} t + \cdots + \mu_{m-1,N-1} t^{N-1}}{(1 - \mathbb{L}^{dN-\nu} t^N)^{m-1}} \\ &\quad + \cdots + \frac{\mu_{1,0} + \mu_{1,1} t + \cdots + \mu_{1,N-1} t^{N-1}}{1 - \mathbb{L}^{dN-\nu} t^N} + \frac{E(t)}{\prod_{i \in I_2} (1 - \mathbb{L}^{dN_i - \nu_i} t^{N_i})} \\ &= \sum_{b=0}^{N-1} \sum_{n=0}^{\infty} \bigl(f_m(n)\mu_{m,b} + \cdots + f_1(n)\mu_{1,b}\bigr) \mathbb{L}^{ndN - n\nu} t^{nN+b} \qquad (2) \\ &\quad + \frac{E(t)}{\prod_{i \in I_2} (1 - \mathbb{L}^{dN_i - \nu_i} t^{N_i})}, \qquad (3) \end{aligned}$$

where $\mu_{i,j} \in R$, where $E(t) \in R[t]$ and where w is a product of elements of the form $1 - \mathbb{L}^k$ and $\mathbb{L}^k$, with $k > 0$. Note that $wD(t)$ is not divisible by $1 - \mathbb{L}^{dN-\nu} t^N$ because w is not a zero divisor in R, the constant term of $1 - \mathbb{L}^{dN-\nu} t^N$ is a unit in R and $D(t)$ is not divisible by $1 - \mathbb{L}^{dN-\nu} t^N$.

We now consider the first part (2) of $wZ(t)$. Because $wD(t)$ is not divisible by $(1 - \mathbb{L}^{dN-\nu} t^N)^m$, there exists a $b \in \{0, \ldots, N-1\}$ for which the coefficient of t^{nN+b} is different from 0 for infinitely many n. Fix from now on such a b and a $j \in \{1, \ldots, m\}$ for which $\mu_{j,b} \neq 0$. Take a positive integer c such that we have for every $n \in \mathbb{Z} \setminus \{0\}$ that $n\mu_{j,b}$ is not divisible by $\mathbb{L}^c$. There do not exist m positive integers $n_1, \ldots, n_m$ for which $f_m(n_1)\mu_{m,b} + \cdots + f_1(n_1)\mu_{1,b}, \ldots, f_m(n_m)\mu_{m,b} + \cdots + f_1(n_m)\mu_{1,b}$ are multiples of $\mathbb{L}^c$, because otherwise, we can use Lemma 4.5 to obtain that $\mu_{j,b}$ has an integer multiple which is a multiple of $\mathbb{L}^c$. Consequently, for n large enough, $f_m(n)\mu_{m,b} + \cdots + f_1(n)\mu_{1,b}$ is not a multiple of $\mathbb{L}^c$. The coefficient of t^{nN+b} in the power series expansion of (2) is equal to $(f_m(n)\mu_{m,b} + \cdots + f_1(n)\mu_{1,b})\mathbb{L}^{(d+l)nN}$, which is not a multiple of $\mathbb{L}^{(d+l)nN+c} = \mathbb{L}^{(d+l)(nN+b)-(d+l)b+c}$ for n large enough. So let a be the largest integer smaller than or equal to $c - (d+l)b$.

Now we consider the remaining part (3) of $wZ(t)$. We obtain from Proposition 4.3 that there exists an $l' > l$ and an integer a' such that the coefficient of t^n in the power series expansion of (3) is a multiple of $\mathbb{L}^{\ulcorner (d+l')n - a' \urcorner}$ for n large enough. Consequently, this coefficient is a multiple of $\mathbb{L}^{\ulcorner (d+l)n + a \urcorner}$ for n large enough.

Because $w\gamma_{nN+b}$ is the sum of two elements of which exactly one is a multiple of $\mathbb{L}^{\ulcorner (d+l)(nN+b)+a \urcorner}$ for n large enough, we obtain that $w\gamma_{nN+b}$, and thus also γ_{nN+b}, is not a multiple of $\mathbb{L}^{\ulcorner (d+l)(nN+b)+a \urcorner}$ for n large enough. $\square$

Corollaries.

(i) *If there exists an integer a such that γ_n is a multiple of $\mathbb{L}^{\ulcorner (d+l')n - a \urcorner}$ for all n satisfying $(d+l')n - a \geqslant 0$, then $l' \leqslant l$. This is the converse of Proposition* 4.3.

(ii) *Because we saw in the previous section that $[\mathcal{X}_n]$ is a multiple of $\mathbb{L}^{\ulcorner dn/2 \urcorner}$ if we are in the situation of* (4.1)*, we obtain that $l \geqslant -d/2$.*

Because of the second corollary, we have proved the following theorem.

Theorem. *The motivic zeta function $Z(t) \in R[\![t]\!]$ belongs to the subring of $R[\![t]\!]$ generated by $R[t]$ and the elements $1/(1-\mathbb{L}^{dN-\nu}t^N)$, with $\nu, N \in \mathbb{Z}_{>0}$ and $\nu/N \leqslant d/2$.*

(4.7) In (4.1), we denoted the image of the localization map $K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathcal{M}_{\mathbb{C}}$ by R'. We introduced an ideal I of R' and put $R = R'/I$. The previous theorem is a priori weaker than the analogous statement over R' (or $\mathcal{M}_{\mathbb{C}}$), but it is not if $I = \{0\}$. We do not know whether $I \neq \{0\}$, but at any rate the theorem specializes to Hodge–Deligne polynomials. This gives us the following.

Theorem. *The Hodge zeta function $Z_{\mathrm{Hod}}(t)$ belongs to the subring of $\mathbb{Q}(u,v)(t)$ generated by $\mathbb{Q}(u,v)[t]$ and the elements $1/(1-(uv)^{dN-\nu}t^N)$, with $\nu, N \in \mathbb{Z}_{>0}$ and $\nu/N \leqslant d/2$.*

5. The relative setting

The generalization to the relative setting was suggested by the referee. Let X be a nonsingular irreducible algebraic variety of dimension d, and let $f : X \to \mathbb{A}^1$ be a nonconstant regular function. Let X_0 be the reduced scheme determined by $f = 0$. Note that $X_0 = \mathcal{L}_0(V)$, where $V = \mathrm{div}(f)$ as before. For $n \geqslant 1$, we have that $\mathcal{X}_n$ is an X_0 variety because of the canonical morphism $\pi_0^n : \mathcal{X}_n \to X_0$. Therefore, we can consider the class $[\mathcal{X}_n/X_0]$ of $\mathcal{X}_n$ in the relative Grothendieck ring $K_0(\mathrm{Var}_{X_0})$ of X_0-varieties. The definition is the straightforward generalization of the usual one, see for example [DL2].

One obtains analogously as in Section 2 that

$$\sum_{n\geqslant 1}[\mathcal{X}_n/X_0]\big(\mathbb{L}^{2cd+c-d}t\big)^n = \sum_{\emptyset\neq I\subset S}\big[E_I^\circ/X_0\big]\prod_{i\in I}\frac{(\mathbb{L}-1)\mathbb{L}^{(2cd+c)N_i-\nu_i}t^{N_i}}{1-\mathbb{L}^{(2cd+c)N_i-\nu_i}t^{N_i}}$$

in $K_0(\mathrm{Var}_{X_0})[\![t]\!]$. Here, c is an arbitrary integer satisfying $(\nu_i - 1)/N_i \leqslant c$ for all $i \in S$ and $\mathbb{L}$ is the class of $\mathbb{A}^1 \times X_0$ in $K_0(\mathrm{Var}_{X_0})$.

The main result of Section 3 can also be adapted to this context. Suppose that the dimension d of X is in $\mathbb{Z}_{>1}$. Then $[\mathcal{L}_n(V)/X_0]$ is a multiple of $\mathbb{L}^{\lceil dn/2\rceil}$ in $K_0(\mathrm{Var}_{X_0})$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

Also Section 4 can be generalized. One analogously constructs a ring R from $K_0(\mathrm{Var}_{X_0})$ such that $Z(t)$, considered as an element of $R[\![t]\!]$, belongs to the subring of $R[\![t]\!]$ generated by $R[t]$ and the elements $1/(1-\mathbb{L}^{dN-\nu}t^N)$, with $\nu, N \in \mathbb{Z}_{>0}$ and $\nu/N \leqslant d/2$.

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