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CHARACTERIZATIONS OF OUTERPLANAR GRAPHS

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The paper presents several characterizations of outerplanar graphs, some of them are counterparts of the well-known characterizations of planar graphs and the other provide very efficient tools for outerplanarity testing, coding (i.e. isomorphism testing), and counting such graphs. Finally, we attempt to generalize these results for k-outerplanar graphs.

1. Introduction

The main purpose of this paper is to provide some further characterizations of outerplanar graphs.

A *graph* $G=(V,E)$ consists of a set V of n vertices and a set E of m edges. Only *simple graphs*, i.e. without loops and multiple edges are considered in this paper.

A *simple path* $p: v \to w$ in G is a sequence of distinct vertices and edges leading from v to w. A closed simple path is a *cycle*. If all interior vertices of a simple path p are of degree 2 in G then p is a *series of edges*, and it is a *maximal series of edges* if each of its ends is of degree not equal to 2 in G.

A planar graph is *outerplanar* if it can be embedded in the plane so that all its vertices lie on the same face, hereafter we assume this face to be the exterior. Without loss of generality we may consider only biconnected graphs, since a graph G is outerplanar if and only if all its biconnected components are outerplanar.

With every graph G we can associate the *cycle space* of G consisting of all cycles and unions of edge-disjoint cycles of G. The cycle space is a vector space over the field of integers modulo 2 with addition of elements defined as the ring sum of sets. A *cycle basis* of G is defined as a basis for the cycle space of G which consists entirely of cycles. There are special cycle bases of a graph which can be derived from spanning trees of a graph. Let T be a spanning tree of G. Then, the set of cycles obtained by inserting each of the remaining edges of G into T is a *fundamental cycle set* of G with respect to T.

Consider a biconnected planar graph G. The set of the interior faces of a plane graph G we shall call a *planar cycle basis* of G. Planar cycle bases of G are well defined, since every face of a planar embedding of a biconnected planar graph is a cycle. The set of the interior faces of a biconnected outerplane graph G we shall call an *outerplanar cycle basis* of G.

The *length* of a cycle basis $\mathscr{C}=\{C_i\}$ of a graph G is defined as follows $|\mathscr{C}|=\sum_{C_i\in\mathscr{C}}|C_i|$, where $|C_i|$ is the number of edges in C_i.

Outerplanar graphs were characterized firstly by Chartrand and Harary, see [1] and [4, pp. 102–116].

Theorem 1. *A graph G is outerplanar if and only if it contains no subgraph homeomorphic to K_4 or $K_{2,3}$ except K_4-x, where x denotes an edge of K_4.*

For a given graph, finding all cycles (see [8, 10, 17]) or finding the longest cycle (see [17]) using the vector space approach, and finding a minimum cycle basis (see [5, 7]) are some of the problems related to cycle bases of a graph which still have some open questions and no efficient algorithms.

Let $\mathscr{C}$ be a cycle basis of G. Then the intersection graph of $\mathscr{C}$ over the set of edges E is called a *cycle graph of G with respect to $\mathscr{C}$*. A graph F is said to be a *cycle graph* if there exist a graph G and its cycle basis $\mathscr{C}$ such that F is isomorphic to the cycle graph of G with respect to $\mathscr{C}$. The cycle graph of G with respect to a fundamental cycle set of G is called a *fundamental cycle graph*. The notion of a cycle graph was introduced by Mateti and Deo [8] to present a new approach to and a new algorithm for the first of the above mentioned problems.

Outerplanar graphs without triangles are of a special interest in the characterization of fundamental cycle graphs, see Theorem 9 and [16].

We can simplify Theorem 1 for such graphs as follows.

Corollary (to Theorem 1). *A graph G without triangles is outerplanar if and only if it contains no subgraph homeomorphic to $K_{2,3}$.*

Proof. The proof is similar to that of Theorem 1 presented in [1]. On the other hand, it suffices to note that if G without triangles has a subgraph homeomorphic to K_4 then there should exist a path (longer than 1) in the subgraph connecting a pair of vertices of K_4. Thus, the subgraph has a subgraph homeomorphic to $K_{2,3}$.

All definitions not given here may be found in [4].

2. Characterizations

The following characterization of outerplanar graphs has been communicated to the author by Nebeský [9].

Theorem 2. *A graph G is outerplanar if and only if it can be embedded in the plane so that for arbitrary four vertices v_1, v_2, v_3 and v_4 in G there exists a face F such that v_i $(1\leq i\leq 4)$ lie on the face F.*

Proof. The necessity is evident. Conversely, if G is not outerplanar then on the base of Theorem 1 there exist a subgraph H of G and four distinct vertices v_1, v_2, v_3 and v_4 in H such that either H is homeomorphic to K_4 and v_i $(1 \leqslant i \leqslant 4)$ are of degree 3 in H, or H is homeomorphic to $K_{2,3}$, v_1 is of degree 3 in H, and v_2, v_3, v_4 are adjacent to v_1 in H. It is clear that for no embedding of G in the plane, the vertices v_1, v_2, v_3 and v_4 can lie on the same face. This completes the proof.

The following characterization, but in terms of weak duals has been independently proved in [3].

Theorem 3 [12]. *A graph G is outerplanar if and only if it has a dual G^* having a vertex v such that $G^* - v$ contains no cycle.*

This characterization has been applied to count all outerplanar graphs with a given number of vertices, see [14].

We can formulate the last theorem in terms of cycle graphs as follows.

Theorem 4. *A biconnected graph G is outerplanar if and only if it has a cycle basis with respect to which the cycle graph of G contains no cycle and every pair of basic cycles has at most one edge in common.*

Proof. If G is outerplanar, then the cycle graph of G with respect to the outerplanar cycle basis of G is isomorphic to $G^* - v$ which, by Theorem 3 has no cycle, where G^* is the dual graph of G and v corresponds to the exterior face of the outerplane graph G.

To prove the converse, note that if a graph G has a cycle basis with respect to which the cycle graph contains no cycle then every edge of G belongs to at most two cycles of the cycle basis which according to MacLane's characterization of planar graphs is a planar cycle basis of G. Since every pair of basic cycles has at most one edge in common, the cycle graph of G with respect to this basis is isomorphic to $G^* - v$. Hence G is outerplanar.

There are several necessary conditions for a graph G to be outerplanar, for instance,

(a) the numbers of edges m and vertices n in G satisfy $m \leqslant 2n - 3$, and

(b) G has at least two vertices of degree not exceeding 2 and at least three vertices of degree not exceeding 3.

As a consequence of property (b) we can see that a biconnected outerplanar graph G is either isomorphic to the n-vertex cycle or there exist at least two series of edges from v_i to v_j and from v_k to v_l such that (v_i, v_j) and (v_k, v_l) are edges in G. If G is outerplanar and there exists in G a series of edges p from v to w then it is evident that G remains outerplanar if we replace p by edge (v, w). It leads to the following *inductive and algorithmic characterization* of outerplanar graphs proved in [11].

Theorem 5. *A n-vertex biconnected graph G $(n > 4)$ is outerplanar if and only if it is isomorphic to the cycle with n vertices or if a cycle can be obtained from G by a sequence of the following reductions (initially, all edges of G are unmarked):*

Let $p: v \to w$ be a maximal series of edges (marked and/or unmarked) of length greater than or equal to 2 then perform either (i) *or* (ii):

(i) *if (v, w) is an unmarked edge of G then mark (v, w) and remove p from G;*

(ii) *if $(v, w) \notin E$ then add this edge to G as a marked edge and remove p from G.*

The algorithmic characterization has been successfully used to produce linear time algorithms for testing and coding outerplanar graphs, for details see [11] and [13], respectively.

It is evident that the connectivity of an outerplanar graph can not exceed 2, so that every biconnected outerplanar graph has at least one *disconnecting pair* of vertices. Moreover, outerplanar graphs can be characterized in terms of disconnecting pairs of vertices.

Theorem 6. *A biconnected graph G is outerplanar if and only if for every vertex v of G the set of vertices adjacent to v contains exactly two vertices of G which do not form the disconnecting pairs with v.*

Proof. If a biconnected graph G is outerplanar then the boundary of the exterior face of its outerplanar embedding is a Hamiltonian cycle, say $(v_1, v_2, v_3, \ldots, v_n, v_1)$. Thus, for every vertex v_i, only two of the vertices adjacent to v_i namely v_{i-1} and v_{i+1} do not form the disconnecting pairs with v_i.

We prove the converse by induction on the number of vertices of a graph G. Suppose, that all k-vertex graphs $(k < n)$ which satisfy the assumption of the theorem are outerplanar. If all vertices of G are of degree 2 then G is isomorphic to the n-vertex cycle, so that it is outerplanar. Otherwise, there exists a vertex v of degree greater than 2 and at least one pair of adjacent vertices v and w which disconnects graph G. This pair of vertices cuts G into two biconnected proper subgraphs of G which satisfy the assumption of the theorem, therefore they are outerplanar on the base of the inductive hypothesis. It follows that the entire graph G is outerplanar, since those subgraphs have one common edge (v, w) which belongs to the exterior faces of the outerplane embeddings of the both subgraphs.

Corollary (to Theorem 6). *A biconnected outerplanar graph has the unique Hamiltonian cycle.*

It is clear that a biconnected outerplanar graph has the unique outerplane embedding in the plane, and in this sense, in the case of outerplanar graphs, the biconnectivity plays a similar role as the triconnectivity does in the case of planar graphs. This property of biconnected outerplanar graphs has been successfully used to simplify methods for coding and counting such graphs.

If G is a biconnected graph then every edge of G belongs to at least one cycle of G. If G is a biconnected planar graph, then the set $\mathscr{C}=\{C_i\}$ of the interior faces of any of its embeddings in the plane forms a cycle basis which on the base of MacLane's characterization of planar graphs satisfies

$$|\mathscr{C}|=\sum_{C_i\in\mathscr{C}}|C_i|+|C_{ex}|-|C_{ex}|=2m-|C_{ex}|,$$

where C_{ex} denotes the exterior face (cycle) of the embedding. Thus, every planar cycle basis $\mathscr{C}$ of an n-vertex biconnected graph G satisfies the inequality

$$|\mathscr{C}|\geqslant 2m-n,$$

and the following theorem is evident

Theorem 7. *A biconnected planar graph G is outerplanar if and only if its minimum planar cycle basis is of length $2m-n$.*

3. Applications and extensions

Given that a biconnected graph G is a planar, that is G has a dual. Let G^* denote the geometric dual graph of G obtained for a given embedding of G in the plane. It is readily seen that the cycle graph of G with respect to the planar cycle basis of G is isomorphic to the simple graph of G^*-v, where v denotes the vertex in G^* corresponding to the exterior face of the embedding and a *simple graph of a multigraph H* is defined as a maximal spanning subgraph of H. Thus, one can consider a cycle graph as a generalization of the notion of the dual graph. On the other hand, it is easy to show that every biconnected planar graph is a cycle graph, every complete graph K_n is a cycle graph, in particular K_5 is a cycle graph, but another minimal non-planar graph $K_{3,3}$ is not a cycle graph.

The following partial characterizations of cycle graphs and fundamental cycle graphs have been proved in [15], see also [16].

Theorem 8. *If a graph F is a cycle graph then every triangle-free induced subgraph of F is planar.*

Theorem 9. *If a graph F is a fundamental cycle graph then every triangle-free induced subgraph of F is outerplanar.*

The complete characterizations of cycle graphs and fundamental cycle graphs are still open problems.

Kane and Basu generalized in [6] the notion of the outerplanar graph. Graph G with n vertices is called *k-outerplanar* $(3\leqslant k\leqslant n)$ if it can be embedded in the plane so that k vertices lie on the boundary of the exterior face and it cannot be embedded in the plane so that more than k vertices lie on the boundary of the same face.

One can easily notice that the k-outerplanarity of a graph G is not a property of G which may be considered separately for biconnected components of G as it is in the case of planarity, outerplanarity and minimal outerplanarity of graphs.

For the sake of simplicity, in the sequel, we shall consider only biconnected graphs.

No characterization of k-outerplanar graphs has appeared so far, and here we are able to generalize only Theorems 2 and 7 of the previous section.

Lemma 1. *Let a plane graph G have a subset W of vertices such that for every four vertices $v_1, v_2, v_3, v_4 \in W$ there exists a face of G which contains each v_i $(1 \leq i \leq 4)$ on its boundary. Then G has a face H such that all vertices of W lie on the boundary of H.*

Proof. Suppose that for a plane graph G which satisfies the assumption of the theorem there exist a vertex $w \in W$, a subset $U \subseteq W$, and a face H of G such that $|U| \geq 4$, $w \notin U$, all vertices of U lie on the boundary of H and $U \cup \{w\}$ do not lie on the boundary of any face of G. Thus there exist $v_1, v_2, v_3, v_4 \in U$ and a face H_0 such that v_1, v_2, v_3 and w lie on the boundary of H_0, $v_4 \neq v_i$ $(i = 1, 2, 3)$, and v_4 does not lie on the boundary of H_0. Therefore the vertices v_1, v_2 and v_3 lie on the path which is the common part of boundaries of H_0 and H, and let us suppose that v_2 is an internal vertex of this path with respect to v_1 and v_3. Hence, there is no face of G on whose boundary lie vertices v_1, v_2, v_4 and w, which is a contradiction.

Theorem 10. *A graph $G = (V, E)$ is k-outerplanar $(k \geq 4)$ if and only if G can be embedded in the plane so that there exists a subset of k vertices $W \subseteq V$ such that for arbitrary four vertices $v_1, v_2, v_3, v_4 \in W$ there exists a face of G on which boundary lie v_i $(1 \leq i \leq 4)$, and W is a maximum set possessing such a property.*

Another characterization of biconnected k-outerplanar graphs can be easily obtained by generalization of Theorem 7.

Theorem 11. *A biconnected graph is k-outerplanar $(3 \leq k \leq n)$ if and only if it has a minimum planar cycle basis of length $2m - k$.*

This theorem gives the method for k-outerplanarity testing provided there exists a method for finding a minimum planar cycle basis of a planar graph.

Deo [2] has suggested that the algorithms and methods which have been developed for outerplanar graphs can be extended to those for k-outerplanar graphs.

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