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NOTE

On Generalized Variational Inequalities Involving Relaxed Lipschitz and Relaxed Monotone Operators

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We consider the solvability, based on iterative algorithms, of the generalized variational inequality (GVI) problems involving the relaxed Lipschitz and relaxed monotone operators.

1. INTRODUCTION

The theory of variational inequalities [1, 2] turned out to be significant in the sense that it provides us with a unified framework for dealing with a wide class of problems in physics, mathematical economics, and engineering sciences. Variational inequality techniques also deal with free-boundary value problems that arise in the study of flow through porous media, hydrodynamic lubrication, heat conduction and diffusion theory, optimization theory, and elastoplastic analysis. While studying nonlinear variational inequalities, the theory of associated nonlinear equations is equally important in the same way that the class of variational inequalities is equivalent to associated equations involving strongly monotone operators. For selected details on nonlinear equations, we refer to [3–7, 9].

Here we consider the solvability of a generalized variational inequality (GVI) problem involving single-valued strongly monotone and multivalued

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relaxed Lipschitz and monotone operators in a Hilbert space setting. The obtained GVI problem is of interest in a way that it not only generalizes the GVI problems recently studied by the author [7] and Yao [8] involving relaxed operators, but sharpens them as well.

2. PRELIMINARIES

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let K be a nonempty closed convex subset of H and P_K be the projection of H onto K.

We consider, for a given single-valued operator $f\colon H \to H$ and multivalued operators $S, T\colon H \to 2^H$, the generalized variational inequality (GVI) problem: Find x in H, w in $S(x)$, and z in $T(x)$ such that $f(x)$ is in K and

$$\langle w - z, v - f(x) \rangle \geq 0 \qquad \text{for all } v \text{ in } K. \tag{1}$$

Next, we consider the corresponding GVI problem: Find x in H, w in $S(x)$, and z in $T(x)$ such that $f(x)$ is in K and

$$\langle f(x) - (w - z), v - f(x) \rangle \geq 0 \qquad \text{for all } v \text{ in } K. \tag{2}$$

DEFINITION 2.1. An operator $S\colon H \to 2^H$ is said to be relaxed Lipschitz if for given $k \geq 0$,

$$\langle w_1 - w_2, u - v \rangle \leq -k\|u - v\|^2 \tag{3}$$

for all w_1 in $S(u)$ and w_2 in $S(v)$.

An operator $S\colon H \to 2^H$ is Lipschitz continuous if there exists a constant $m > 0$ such that for all w_1 in $S(u)$ and w_2 in $S(v)$,

$$\|w_1 - w_2\| \leq m\|u - v\|.$$

DEFINITION 2.2. An operator $f\colon H \to H$ is called Lipschitz continuous if for $s > 0$,

$$\|fu - fv\| \leq s\|u - v\| \qquad \text{for all } u, v \text{ in } H. \tag{4}$$

The operator f is strongly monotone if for all u, v in H there exists $r > 0$ such that

$$\langle fu - fv, u - v \rangle \geq r\|u - v\|^2. \tag{5}$$

DEFINITION 2.3. An operator $T\colon H \to 2^H$ is called relaxed monotone if for all u, v in H there exists $c > 0$ such that

$$\langle z_1 - z_2, u - v\rangle \geq -c\|u - v\|^2, \tag{6}$$

where z_1 is in $T(u)$ and z_2 is in $T(v)$.

3. GENERALIZED VARIATIONAL INEQUALITIES

In this section we first recall some auxiliary results on the equivalence of variational inequalities to some nonlinear equations. Then we consider the main GVI problem (2) on the solvability of the generalized variational inequalities.

LEMMA 3.1 [2]. *For a given z in H, x in K satisfies*

$$\langle x - z, v - x\rangle \geq 0 \qquad \textit{for all } v \textit{ in } K, \tag{7}$$

iff $x = P_K z$.

LEMMA 3.2. *Elements x in H, w in $S(x)$, and z in $T(x)$ such that $f(x)$ is in K, are a solution set of the GVI problem (2) iff x in H, w in $S(x)$, and z in $T(x)$ with $f(x)$ in K satisfy the equation for $t > 0$,*

$$f(x) = P_K[(1 - t)f(x) + t(w - z)]. \tag{8}$$

Proof. The proof follows from an application of Lemma 3.1.

Based on (8) we can generate an iterative algorithm:

ALGORITHM 3.1. For $n = 0, 1, 2, \ldots,$

$$f(x_{n+1}) = P_K[(1 - t)f(x_n) + t(w_n - z_n)]. \tag{9}$$

THEOREM 3.1. *Let K be a nonempty closed convex subset of a real Hilbert space H and let $f\colon H \to H$ be strongly monotone and Lipschitz continuous with corresponding constants $r > 0$ and $s > 0$. Let $S\colon H \to 2^H$ be relaxed Lipschitz and Lipschitz continuous with corresponding constants $k \geq 0$ and $m \geq 1$. Let $T\colon H \to 2^H$ be relaxed monotone and Lipschitz continuous with* *corresponding constants $c > 0$ and $d > 0$. Then the sequences $\{x_n\}$, $\{w_n\}$, $\{z_n\}$, and $\{f(x_n)\}$, as generated by Algorithm 3.1 with x_0 in H, w_0 in $S(x_0)$, z_0 in $T(x_0)$, $f(x_0)$ in K, and*

$$\left| t - \frac{1 + k - c - p(p - r)}{1 + 2(k - c) + (m + d)^2 - p^2} \right|$$
$$< \Big\{ [1 + (k - c) - p(p - r)]^2$$
$$- \left[1 + 2(k - c) + (m + d)^2 - p^2\right]\left[1 - (r - p)^2\right] \Big\}^{1/2}$$
$$\times \left[1 + 2(k - c) + (m + d)^2 - p^2\right]^{-1},$$

where $1 + (k - c) > p(p - r) + \{[1 + 2(k - c) + (m + d)^2 - p^2][1 - (r - p)^2]\}^{1/2}$ for $c < k$ and $p = (1 - 2r + s^2)^{1/2}$, converge to x, w, z, and $f(x)$, respectively, the solution of Eq. (8).

COROLLARY 3.1. *Let S: $H \to 2^H$ be relaxed Lipschitz and Lipschitz continuous with constants $k \geq 0$ and $m \geq 1$, and f: $H \to H$ be expanding and nonexpanding. Let T: $H \to 2^H$ be relaxed monotone and Lipschitz continuous with constants $c > 0$ and $d > 0$. Then sequences $\{x_n\}$, $\{w_n\}$, $\{z_n\}$, and $\{f(x_n)\}$, as generated by Algorithm* 3.1 *for x_0 in H, w_0 in $S(x_0)$, z_0 in $T(x_0)$, and $f(x_0)$ in K, and $0 < t < 2[1 + (k - c)]/[1 + 2(k - c) + (m + d)^2]$, converge, respectively, to x, w, z, and $f(x)$, the solution of* (8).

COROLLARY 3.2 [8, Theorem 3.6]. *Let f be the identity and S: $K \to H$ be relaxed Lipschitz and Lipschitz continuous. Then the sequence $\{x_n\}$ generated by*

$$x_{n+1} = P_K\left[(1 - t)x_n + tSx_n\right] \qquad \text{for } n = 0, 1, 2, \dots, \tag{10}$$

where x_0 is in K and $0 < t < 2(1 + k)/(1 + 2k + m^2)$, converges to the unique fixed point of S.

COROLLARY 3.3. *Let f be the identity and S: $K \to H$ be Lipschitz continuous with Lipschitz constant $m \geq 1$. Then the sequence $\{x_n\}$ generated by* (10) *for x_0 in K and $0 < t < 2/(1 + m)$, converges to the unique fixed point of S.*

Proof of Theorem 3.1. Since P_K is nonexpanding, we have

$$\begin{aligned}
&\|f(x_{n+1}) - f(x_n)\| \\
&\quad \le \|(1-t)f(x_n) + t(w_n - z_n) - (1-t)f(x_{n-1}) - t(w_{n-1} - z_{n-1})\| \\
&\quad = \|(1-t)\big(f(x_n) - f(x_{n-1})\big) + t(w_n - w_{n-1}) - t(z_n - z_{n-1})\| \\
&\quad \le \|(1-t)\big[x_n - x_{n-1} - \big(f(x_n) - f(x_{n-1})\big)\big] + (1-t)(x_n - x_{n-1}) \\
&\qquad + t(w_n - w_{n-1}) - t(z_n - z_{n-1})\|.
\end{aligned}$$

Since

$$\begin{aligned}
&(1-t)^2 \|x_n - x_{n-1} - \big(f(x_n) - f(x_{n-1})\big)\|^2 \\
&\qquad = (1-t)^2 \big[\|x_n - x_{n-1}\|^2 - 2\langle f(x_n) - f(x_{n-1}), x_n - x_{n-1}\rangle \\
&\qquad\qquad + \|f(x_n) - f(x_{n-1})\|^2\big] \\
&\qquad \le (1-t)^2 (1 - 2r + s^2) \|x_n - x_{n-1}\|^2,
\end{aligned}$$

and

$$\begin{aligned}
&\|(1-t)(x_n - x_{n-1}) + t(w_n - w_{n-1}) - t(z_n - z_{n-1})\|^2 \\
&\qquad = (1-t)^2 \|x_n - x_{n-1}\|^2 + 2t(1-t)\langle w_n - w_{n-1}, x_n - x_{n-1}\rangle \\
&\qquad\quad - 2t(1-t)\langle z_n - z_{n-1}, x_n - x_{n-1}\rangle \\
&\qquad\quad + t^2 \|w_n - w_{n-1} - (z_n - z_{n-1})\|^2 \\
&\qquad \le \big[(1-t)^2 - 2t(1-t)(k-c) + t^2(m+d)^2\big] \|x_n - x_{n-1}\|^2,
\end{aligned}$$

this implies that

$$\begin{aligned}
&\|f(x_{n+1}) - f(x_n)\| \\
&\qquad \le \Big[(1-t)(1 - 2r + s^2)^{1/2} \\
&\qquad\quad + \big\{(1-t)^2 - 2t(1-t)(k-c) + t^2(m+d)^2\big\}^{1/2}\Big] \|x_n - x_{n-1}\|.
\end{aligned}$$

Since f is strongly monotone and hence r-expanding, it follows that

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \le (1/r)\Big[(1-t)p + \big\{(1-t)^2 - 2t(1-t)(k-c) + t^2(m+d)^2\big\}^{1/2}\Big] \\
&\qquad \times \|x_n - x_{n-1}\|,
\end{aligned}$$

where $p = (1 - 2r + s^2)^{1/2}$. Therefore,

$$\|x_{n+1} - x_n\| \leq \theta \|x_n - x_{n-1}\|, \tag{11}$$

where $\theta = (1/r)[(1-t)p + \{(1-t)^2 - 2t(1-t)(k-c) + t^2(m+d)^2\}^{1/2}]$. Now under the assumptions, it follows that $0 < \theta < 1$, and consequently, for all q in N,

$$\|x_{n+q} - x_n\| \leq \theta^n/(1-\theta)\|x_1 - x_0\|. \tag{12}$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since H is complete, there exists an x in H such that $x_n \to x$. Now the Lipschitz continuity of the operators f, S, and T implies that $w_n \to w$, $z_n \to z$, and $f(x_n) \to f(x)$. Consequently, this leads to the solvability of the GVI problem (2). This completes the proof.

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