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Local and global well-posedness of SPDE with generalized coercivity conditions ☆

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ABSTRACT

In this paper we establish the local and global existence and uniqueness of solutions for general nonlinear evolution equations with coefficients satisfying some local monotonicity and generalized coercivity conditions. An analogous result is obtained for stochastic evolution equations in Hilbert space with additive noise. As applications, the main results are applied to obtain simpler proofs in known cases as the stochastic 3D Navier–Stokes equation, the tamed 3D Navier–Stokes equation and the Cahn–Hilliard equation, but also to get new results for stochastic surface growth PDE and stochastic power law fluids.

1. Main results

Let $(H, \langle\cdot,\cdot\rangle_H)$ be a real separable Hilbert space and identified with its dual space H^* by the Riesz isomorphism. Let V be a real reflexive Banach space such that it is continuously and densely

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embedded into H. Then we have the following Gelfand triple

$$V \subseteq H \equiv H^* \subseteq V^*.$$

If $\langle \cdot,\cdot \rangle_V$ denotes the dualization between V and its dual space V^*, then it is easy to show that

$$\langle u, v\rangle_V = \langle u, v\rangle_H, \quad u \in H, \ v \in V.$$

Now we consider the general nonlinear evolution equation

$$u'(t) = A\big(t, u(t)\big), \quad 0 < t < T, \qquad u(0) = u_0 \in H, \tag{1.1}$$

where $T > 0$, u' is the generalized derivative of u on $(0, T)$ and $A : [0, T] \times V \to V^*$ is restrictedly measurable, i.e. for each dt-version of $u \in L^1([0, T]; V)$, $t \mapsto A(t, u(t))$ is V^*-measurable on $[0, T]$.

A classical result says that (1.1) has a unique solution if A satisfies the monotonicity and coercivity conditions (see e.g. [1,8,30,49,53] for more detailed exposition and references). The proof is mainly based on the Galerkin approximation and the Minty (monotonicity) trick. In [32], the existence and uniqueness result was established by replacing the monotonicity condition with a local version (see (H2) below). The result was applied to many new fundamental equations within this variational framework such as Burgers type equations, 2D Navier–Stokes equation and the 3D Leray-α model. One of the main steps in the proof in [32] was to show that any operator satisfying local monotonicity is pseudo-monotone. One should remark that the notion of a pseudo-monotone operator is one of the most important extensions of the notion of a monotone operator and it was first introduced by Brézis in [7]. The prototype of a pseudo-monotone operator is the sum of a monotone operator and a strongly continuous operator (i.e. an operator maps a weakly convergent sequence into a strongly convergent sequence). Hence the theory of pseudo-monotone operators unifies both monotonicity arguments and compactness arguments (cf. [49,53]).

Also for stochastic partial differential equations (SPDE), the above approach, also called the variational approach, has been used extensively by many authors. The existence and uniqueness of solutions for SPDE was first investigated by Pardoux [40], Krylov and Rozovskii [28]. We refer to e.g. [25,44,54] for some further generalizations. In particular, the local monotonicity condition has been used to establish well-posedness for SPDE in [33,12]. For further references on various types of properties established for SPDE within the variational framework, we refer to [15,24,33,54].

In this work we establish existence, uniqueness and continuous dependence on initial conditions of solutions to (1.1) by using the local monotonicity condition (see (H2) below) and the generalized coercivity condition (H3) defined below. An analogous result for stochastic PDE with general additive noise is also obtained. The standard growth condition on A (cf. [1,28,30,53]) is also replaced by a much weaker condition such that the main result can be applied to a larger class of examples. This result seems new even in the finite dimensional case. The main result can be applied to establish the local/global existence and uniqueness of solutions for a large class of classical (stochastic) nonlinear evolution equations such as the stochastic 2D and 3D Navier–Stokes equations, the tamed 3D Navier–Stokes equation and the Cahn–Hilliard equation. Through our generalized framework we give new and significantly simpler proofs for all these well known results. Moreover, the main result is also applied to stochastic surface growth PDE and stochastic power law fluids to obtain some new existence and uniqueness results for these models (see Section 3 for more details). We emphasize that by applying the main result we obtain both the known local existence and uniqueness of strong solutions to the stochastic 3D Navier–Stokes equation and new local existence and uniqueness results for stochastic surface growth PDE. Here the meaning of strong solution is in the sense of both PDE and stochastic analysis.

In particular, the (stochastic) 2D and 3D Navier–Stokes equations are now included in this extended variational framework using the local monotonicity and generalized coercivity condition. The study of stochastic Navier–Stokes equations dates back to the work of Bensoussan and Temam [2]. Although we have quite satisfactory results for 2D stochastic Navier–Stokes equations such as well-posedness, small noise asymptotics and ergodicity (cf. [33,15,26] and the references therein), the results for the three dimensional case are still quite incomplete due to the lack of uniqueness (cf. [13,14,17,18,21,22,35,36]). Concerning the existence of solutions, in [21] Flandoli and Gatarek proved the existence of martingale solutions and stationary solutions for any dimensional stochastic Navier–Stokes equations in a bounded domain. Subsequently, Mikulevicius and Rozovskii in [36] showed the existence of martingale solutions to stochastic Navier–Stokes equations in $\mathbb{R}^d$ ($d \geqslant 2$) under weaker assumptions on the coefficients.

Replacing the standard coercivity assumption (i.e. $g(x) = Cx$ in (H3) below) by a more general version is motivated by many reasons. One motivation is trying to investigate the 3D Navier–Stokes equation by applying our new result since we know that the local monotonicity hold for both the 2D and 3D Navier–Stokes equation. However, as pointed out in [32,33], the growth condition (see (H4) below and Remark 3.3) fails to hold for the 3D Navier–Stokes equation. On the other hand, inspired by a series of works on the stochastic tamed 3D Navier–Stokes equation [46–48], we realized that, instead of working on the usual Gelfand triple $H^1 \subseteq H^0 \subseteq H^{-1}$ (see Section 3 for details), one may use the following Gelfand triple

$$H^2 \subseteq H^1 \subseteq H^0.$$

On this triple one can verify the growth condition and also the local monotonicity for 3D Navier–Stokes equation, but the usual coercivity condition does not hold anymore. Therefore, we introduce the generalized coercivity condition (H3) in order to overcome this difficulty. However, under this general form of coercivity one is only able to get the local existence and uniqueness of solutions. We should remark that our main result can also be applied to the tamed 3D Navier–Stokes equation to get the global existence and uniqueness of solutions (see Section 3 for more examples).

Another reason of using this generalized coercivity condition is coming from the proof of existence and uniqueness results for stochastic evolution equations with general additive type noise. It is well known that stochastic equations (see (1.4) below) can be reduced to deterministic evolution equations with a random parameter by a standard transformation (substitution). Then one can apply the result that we have already established for deterministic equations (cf. [32]). However, (H3) with the form of $g(x) = Cx$ fails to hold in some examples due to the more general growth condition (H4) (see the proof of Theorem 1.3). But in such cases one will see that (H3) still holds with a certain non-decreasing continuous function g (e.g. $g(x) = Cx^\gamma$ for some $C, \gamma > 0$). We refer to Section 3 for many examples only satisfying this generalized coercivity condition.

Now let us formulate the precise conditions on the coefficients in (1.1).

Suppose for fixed $\alpha > 1$, $\beta \geqslant 0$ there exist constants $\delta > 0$, C and a positive function $f \in L^1([0,T];\mathbb{R})$ such that the following conditions hold for all $t \in [0,T]$ and $v, v_1, v_2 \in V$.

(H1) (Hemicontinuity) The map $s \mapsto \langle A(t, v_1 + s v_2), v\rangle_V$ is continuous on $\mathbb{R}$.

(H2) (Local monotonicity)

$$\big\langle A(t,v_1) - A(t,v_2), v_1 - v_2\big\rangle_V \leqslant \big(f(t) + \rho(v_1) + \eta(v_2)\big)\|v_1 - v_2\|_H^2,$$

where $\rho, \eta : V \to [0, +\infty)$ are measurable and locally bounded functions on V.

(H3) (Generalized coercivity)

$$2\big\langle A(t,v), v\big\rangle_V \leqslant -\delta\|v\|_V^\alpha + g\big(\|v\|_H^2\big) + f(t),$$

where $g : [0,\infty) \to [0,\infty)$ is a non-decreasing continuous function.

(H4) (Growth)

$$\big\|A(t,v)\big\|_{V^*} \leqslant \big(f(t)^{\frac{\alpha-1}{\alpha}} + C\|v\|_V^{\alpha-1}\big)\big(1 + \|v\|_H^\beta\big).$$

Remark 1.1. (1) If $\rho = \eta \equiv 0$, $g(x) = Cx$ and $\beta = 0$, then (H1)–(H4) are the classical monotonicity and coercivity conditions in [53, Theorem 30.A] (see also [1,28,30,41]). It can be applied to many quasilinear PDE such as porous medium equations and the p-Laplace equation (cf. [53,41]).

(2) If $f(t) \equiv C$ in (H2) and $g(x) = Cx$ in (H3), existence and uniqueness is obtained in [32] and the result is applied to many examples such as Burgers type equations, the 2D Navier–Stokes equation, the 3D Leray-α model and the p-Laplace equation with non-monotone perturbations. For readers interested in stochastic partial differential equations we refer to [33,12] where the existence and uniqueness of strong solutions is established under another form of local monotonicity condition (namely $\rho \equiv 0$).

(3) We remark that (H2) also covers other non-Lipschitz conditions used in the literature (cf. e.g. [20]). Moreover, with small modifications to the proof, (H3) can be replaced by the following slightly modified condition:

$$2\langle A(t, v), v\rangle_V \leqslant -\delta\|v\|_V^\alpha + h(t)g\big(\|v\|_H^2\big) + f(t),$$

where $h : [0, T] \to [0, \infty)$ is an integrable function.

Now we can state the main result, which gives a more general framework to analyze various classes of nonlinear evolution equations.

Theorem 1.1. *Suppose that $V \subseteq H$ is compact and* (H1)–(H4) *hold.*

(i) *For any $u_0 \in H$, there exists a constant $T_0 \in (0, T]$ such that* (1.1) *has a solution on* $[0, T_0]$, *i.e.*

$$u \in L^\alpha\big([0, T_0]; V\big) \cap C\big([0, T_0]; H\big), \qquad u' \in L^{\frac{\alpha}{\alpha-1}}\big([0, T_0]; V^*\big)$$

and

$$\big\langle u(t), v\big\rangle_H = \langle u_0, v\rangle_H + \int_0^t \big\langle A\big(s, u(s)\big), v\big\rangle_V \, ds, \quad t \in [0, T_0], \ v \in V.$$

Moreover, if there exist nonnegative constants C and γ such that

$$\rho(v) + \eta(v) \leqslant C\big(1 + \|v\|_V^\alpha\big)\big(1 + \|v\|_H^\gamma\big), \quad v \in V, \tag{1.2}$$

then the solution of (1.1) *is unique on* $[0, T_0]$.

(ii) *If* (H3) *holds with $g(x) = Cx$ for some constant C, then all assertions in* (i) *hold on* $[0, T]$ (*i.e.* $T_0 = T$).

Remark 1.2. (1) In the proof one can see that T_0 is a constant depending on u_0, f and g. More precisely, one can take any constant T_0 which satisfies the following property:

$$0 < T_0 \leqslant T \quad \text{and} \quad T_0 < \sup_{x\in(0,\infty)} G(x) - G\left(\|u_0\|_H^2 + \int_0^{T_0} f(s)\, ds\right),$$

where $G(x) := \int_{x_0}^x \frac{1}{g(s)}\, ds$ for some $x_0 > 0$.

In particular, if $g(x) = c_0 x^\gamma$ ($\gamma \geqslant 1$), then one can take any $T_0 \in (0, T]$ satisfying

$$T_0 < \frac{c_0}{(\gamma - 1)(\|u_0\|_H^2 + \int_0^{T_0} f(s)\, ds)^{\gamma-1}}.$$

(2) If $\rho \equiv 0$ or $\eta \equiv 0$ in (H2), then the compactness assumption of $V \subseteq H$ can be removed by using a different proof (cf. [33]). Therefore, the result can also be applied to many nonlinear evolution equations with unbounded underlying domains.

The next result shows the continuous dependence of solution of (1.1) on the initial condition u_0.

Theorem 1.2. *Suppose that $V \subseteq H$ is compact and* (H1)–(H4) *hold, and u_i are solutions of* (1.1) *on* $[0, T_0]$ *for initial conditions $u_{i,0} \in H$, $i = 1, 2$ respectively and satisfying*

$$\int_0^{T_0} \big(\rho\big(u_1(s)\big) + \eta\big(u_2(s)\big)\big)\,ds < \infty.$$

Then there exists a constant C such that

$$\big\|u_1(t) - u_2(t)\big\|_H^2 \leqslant \|u_{1,0} - u_{2,0}\|_H^2 \exp\left[\int_0^t \big(f(s) + \rho\big(u_1(s)\big) + \eta\big(u_2(s)\big)\big)\,ds\right], \quad t \in [0, T_0]. \tag{1.3}$$

Now we formulate the analogous result for SPDE in Hilbert space with additive type noise. Suppose that U is a Hilbert space and $W(t)$ is a U-valued cylindrical Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. We consider the following type of stochastic evolution equations on H,

$$dX(t) = \big[A_1\big(t, X(t)\big) + A_2\big(t, X(t)\big)\big]\,dt + B(t)\,dW(t), \quad 0 < t < T, \qquad X(0) = X_0, \tag{1.4}$$

where $A_1, A_2 : [0, T] \times V \to V^*$ and $B : [0, T] \to L_2(U; H)$ (here $(L_2(U; H), \|\cdot\|_2)$ denotes the space of all Hilbert–Schmidt operators from U to H) are measurable.

Now we give the definition of a local solution to (1.4). We use τ to denote a stopping time in the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

Definition 1.1. (i) An H-valued $\mathcal{F}_t$-adapted process $\{X(t)\}_{t\in[0,\tau]}$ is called a local solution of (1.4) if $X(\cdot, \omega) \in L^1([0, \tau(\omega)]; V) \cap L^2([0, \tau(\omega)]; H)$ and $\mathbb{P}$-*a.s.* $\omega \in \Omega$,

$$X(t) = X_0 + \int_0^t \big[A_1\big(s, X(s)\big) + A_2\big(s, X(s)\big)\big]\,ds + \int_0^t B(s)\,dW(s), \quad 0 < t < \tau(\omega),$$

where τ is a stopping time satisfying $\tau(\omega) > 0$, $\mathbb{P}$-*a.e.* $\omega \in \Omega$ and $X_0 \in L^2(\Omega \to H; \mathcal{F}_0; \mathbb{P})$.

(ii) Local solution is called unique if for any two local solutions $\{X_1(t)\}_{t\in[0,\tau_1]}$ and $\{X_2(t)\}_{t\in[0,\tau_2]}$ we have

$$\mathbb{P}\big\{\omega\colon\ X_1(t) = X_2(t),\ t \in [0, \tau_1 \wedge \tau_2]\big\} = 1.$$

Theorem 1.3. *Suppose that $V \subseteq H$ is compact, A_1 satisfies* (H1)–(H4) *with $\rho \equiv 0$, $\beta = 0$ and $g(x) = Cx$, A_2 satisfies* (H1)–(H4)*, $B \in L^2([0, T]; L_2(U; H))$, and there exist nonnegative constants C and γ such that*

$$\rho(u + v) \leqslant C\big(\rho(u) + \rho(v)\big), \quad u, v \in V;$$

$$\eta(u + v) \leqslant C\big(\eta(u) + \eta(v)\big), \quad u, v \in V;$$

$$\rho(v) + \eta(v) \leqslant C\big(1 + \|v\|_V^\alpha\big)\big(1 + \|v\|_H^\gamma\big), \quad v \in V.$$

Then for any $X_0 \in L^2(\Omega \to H; \mathcal{F}_0; \mathbb{P})$, there exists a unique local solution $\{X(t)\}_{t\in[0,\tau]}$ to (1.4) satisfying

$$X(\cdot) \in L^\alpha\big([0,\tau]; V\big) \cap C\big([0,\tau]; H\big), \quad \mathbb{P}\text{-a.s.}$$

Moreover, if $g(x) = Cx$ in (H3) *and $\alpha\beta \leqslant 2$, then all assertions above hold for $\tau \equiv T$.*

Remark 1.3. (1) The main idea of the proof is to use a transformation to reduce SPDE (1.4) to a deterministic evolution equation (with some random parameter) which Theorem 1.1 can be applied to. More precisely, we consider the process Y which solves the following SPDE:

$$dY(t) = A_1\big(t, Y(t)\big)\,dt + B(t)\,dW(t), \quad 0 < t < T, \qquad Y(0) = 0. \tag{1.5}$$

Since A_1 satisfies (H1)–(H4) with $\rho \equiv 0$ and $g(x) = Cx$, then the existence and uniqueness of $Y(t)$ follows from Theorem 1.1 in [33]. Let $u(t) = X(t) - Y(t)$, then it is easy to show that $u(t)$ satisfies a deterministic evolution equation of type (1.1) for each fixed $\omega \in \Omega$.

(2) Unlike in [24], here we do not need to assume the noise to take values in V (i.e. $B \in L_2(U; V)$). The reason is that here we use the auxiliary process Y instead of subtracting the noise part directly as in [24] and that $A_1 \neq 0$ because it satisfies (H3).

(3) One can replace the Wiener process $W(t)$ in (1.4) by a Lévy type noise $L(t)$. Then the existence and uniqueness of solutions to (1.5) can be obtained from the main result in [12], and the rest of the proof can be carried out similarly.

More generally, one might replace $W(t)$ in (1.4) by a U-valued adapted stochastic process $N(t)$ with càdlàg paths. $N(t)$ can be various types of noises here. For instance, one can take $N(t)$ as cylindrical Wiener process, fractional Brownian motion or Lévy process (cf. [24]). This subject and some further applications will be investigated in future work.

(4) Comparing with the result obtained in [33,12], the theorem above can be applied to SPDE with more general drifts (see Section 3 for many examples) provided the noise is of additive type. On the other hand, the result in [33,12] is applicable to SPDE with general multiplicative Wiener noise or Lévy noise if $\rho \equiv 0$ in (H2) and $g(x) = Cx$ in (H3).

The rest of the paper is organized as follows. The proofs of the main results are given in the next section. In Section 3 we apply the main results to several concrete (stochastic) semilinear and quasilinear evolution equations in Banach space. Throughout the paper, we use C to denote some generic constant which might change from line to line.

2. Proofs of the main theorems

2.1. Proof of Theorem 1.1

We will first consider the Galerkin approximation to (1.1). However, even in the finite dimensional case, the existence and uniqueness of solutions to (1.1) seems not obvious because of the local monotonicity (H2) and the generalized coercivity condition (H3). Here we prove it by using a classical existence theorem of Carathéodory for ordinary differential equations. Another difference is that we cannot apply Gronwall's lemma directly for this general form of coercivity condition (H3). Instead, we will use Bihari's inequality, which is a generalized version of Gronwall's lemma (cf. [3,43]).

Lemma 2.1 *(Bihari's inequality). Let $g : (0,\infty) \to (0,\infty)$ be a non-decreasing continuous function. If p, q are two positive functions on $\mathbb{R}^+$ and $K \geqslant 0$ is a constant such that*

$$p(t) \leqslant K + \int_0^t q(s) g\big(p(s)\big)\,ds, \quad t \geqslant 0,$$

(i) *Then we have*

$$p(t) \leqslant G^{-1}\left(G(K) + \int_0^t q(s)\,ds\right), \quad 0 \leqslant t \leqslant T_0, \tag{2.1}$$

where $G(x) := \int_{x_0}^x \frac{1}{g(s)}\,ds$ is well defined for some $x_0 > 0$, G^{-1} is the inverse function and $T_0 \in (0,\infty)$ is a constant such that $G(K) + \int_0^{T_0} q(s)\,ds$ belongs to the domain of G^{-1}.

(ii) *If $K = 0$ and there exists some $\varepsilon > 0$ such that*

$$\int_0^\varepsilon \frac{1}{g(s)}\,ds = +\infty,$$

then $p(t) \equiv 0$.

Remark 2.1. It is obvious that the interval $[G(K), \sup_{x\in(0,\infty)} G(x))$ is contained in the domain of G^{-1}, hence (2.1) holds for $t \in [0, T_0]$, where T_0 satisfies

$$\int_0^{T_0} q(s)\,ds < \sup_{x\in(0,\infty)} G(x) - G(K).$$

In particular, if $q \equiv 1$ and $g(x) = C_0 x^\gamma$ for some constants $C_0 > 0$ and $\gamma \geqslant 1$, then

$$G(x) = \frac{C_0}{\gamma - 1}\left(x_0^{1-\gamma} - x^{1-\gamma}\right); \qquad G^{-1}(x) = \left(x_0^{1-\gamma} - \frac{\gamma - 1}{C_0} x\right)^{\frac{1}{1-\gamma}}.$$

Hence (2.1) holds on $[0, T_0]$ for any $T_0 \in [0, \frac{C_0}{\gamma-1} K^{1-\gamma})$ (in particular, for any $T_0 \in [0,\infty)$ if $\gamma = 1$).

Another difficulty is due to the local monotonicity. It is well known that the hemicontinuity and (global) monotonicity implies demicontinuity (cf. [41,53]), which implies continuity in the finite dimensional case. This is crucially used in the proof of existence of solutions for the finite dimensional equations of the Galerkin approximation. In order to show the demicontinuity of locally monotone operators, we need to use the techniques of pseudo-monotone operators. We first recall the definition of a pseudo-monotone operator, which is a very useful generalization of a monotone operator and was first introduced by Brézis in [7]. We use the notation "$\rightharpoonup$" for weak convergence in Banach spaces.

Definition 2.1. The operator $A : V \to V^*$ is called pseudo-monotone if $v_n \rightharpoonup v$ in V as $n \to \infty$ and

$$\liminf_{n\to\infty} \langle A(v_n), v_n - v\rangle_V \geqslant 0$$

implies for all $u \in V$

$$\langle A(v), v - u\rangle_V \geqslant \limsup_{n\to\infty} \langle A(v_n), v_n - u\rangle_V.$$

Remark 2.2. Browder introduced a slightly different definition of a pseudo-monotone operator in [11]: An operator $A : V \to V^*$ is called pseudo-monotone if $v_n \rightharpoonup v$ in V as $n \to \infty$ and

$$\liminf_{n\to\infty} \langle A(v_n), v_n - v\rangle_V \geqslant 0$$

implies

$$A(v_n) \rightharpoonup A(v) \quad \text{and} \quad \lim_{n\to\infty} \langle A(v_n), v_n\rangle_V = \langle A(v), v\rangle_V.$$

In particular, under assumption (H4), these two definitions are equivalent (cf. [32]).

Lemma 2.2. *If the embedding $V \subseteq H$ is compact, then* (H1) *and* (H2) *imply that $A(t,\cdot)$ is pseudo-monotone for any $t \in [0,T]$.*

Proof. For the proof we refer to [32, Lemma 2.5]. □

The proof of Theorem 1.1 is split into a few lemmas. We first consider the Galerkin approximation to (1.1).

Let $\{e_1, e_2, \ldots\} \subset V$ be an orthonormal basis in H and let $H_n := span\{e_1, \ldots, e_n\}$ such that $span\{e_1, e_2, \ldots\}$ is dense in V. Let $P_n : V^* \to H_n$ be defined by

$$P_n y := \sum_{i=1}^{n} \langle y, e_i\rangle_V e_i, \quad y \in V^*.$$

Obviously, $P_n|_H$ is just the orthogonal projection onto H_n in H and we have

$$\langle P_n A(t,u), v\rangle_V = \langle P_n A(t,u), v\rangle_H = \langle A(t,u), v\rangle_V, \quad u \in V,\ v \in H_n.$$

For each finite $n \in \mathbb{N}$ we consider the following evolution equation on H_n:

$$u_n'(t) = P_n A\big(t, u_n(t)\big), \quad 0 < t < T, \qquad u_n(0) = P_n u_0 \in H_n. \tag{2.2}$$

From now on, we fix T_0 as a positive constant satisfying

$$0 < T_0 \leqslant T \quad \text{and} \quad T_0 < \sup_{x\in(0,\infty)} G(x) - G\left(\|u_0\|_H^2 + \int_0^{T_0} f(s)\,ds\right),$$

where the functions f and G are as in (H3) and Lemma 2.1 respectively.

In particular, if $g(x) = C_0 x^\gamma$ $(\gamma \geqslant 1)$, then one can take any $T_0 \in (0,T]$ satisfying

$$T_0 < \frac{C_0}{(\gamma - 1)(\|u_0\|_H^2 + \int_0^{T_0} f(s)\,ds)^{\gamma-1}}.$$

Lemma 2.3. *Suppose that $V \subseteq H$ is compact and* (H1)–(H4) *hold, then* (2.2) *has a solution on* $[0,T_0]$. *Moreover, the solution is unique on* $[0,T_0]$ *if additionally* (1.2) *holds.*

Proof. For any $t \in [0,T]$, it is easy to show that $A(t,\cdot)$ is demicontinuous by (H1) and (H2) (cf. [41, Remark 4.1.1] or [53, Proposition 26.4]), i.e.

$$u_n \to u \text{ (strongly) in } V \text{ as } n \to \infty$$

implies that

$$A(t,u_n) \rightharpoonup A(t,u) \text{ in } V^* \text{ as } n \to \infty.$$

In fact, one can first show that A is locally bounded by using similar arguments as in [41]. This implies that $\{A(t,u_n)\}$ is bounded in V^*. Hence there exist a subsequence $(n_k)_{k\in\mathbb{N}}$ and $w \in V^*$ such that $A(t,u_{n_k}) \rightharpoonup w$ in V^* as $k \to \infty$.

Since $u_{n_k} \to u$ strongly in V as $k \to \infty$, we have

$$\lim_{k\to\infty} \langle A(t,u_{n_k}), u_{n_k}\rangle_V = \langle w, u\rangle_V.$$

By Lemma 2.2 we know that $A(t,\cdot)$ is a pseudo-monotone operator. Then by Remark 2.2 we can conclude that $A(u) = w$. Since for all such subsequences their weak limit is $A(u)$, we have

$$A(t,u_n) \rightharpoonup A(t,u) \text{ in } V^* \text{ as } n \to \infty.$$

In particular, the demicontinuity implies that $P_n A(t,\cdot): H_n \to H_n$ is continuous and hence the functions

$$(t,u) \to \langle P_n A(t,u), e_j\rangle_V, \quad j = 1,2,\ldots,n,$$

satisfy the Carathéodory condition on $[0,T]\times H_n$, i.e. for all $j = 1,2,\ldots,n$

$$t \to \langle P_n A(t,u), e_j\rangle_V \text{ is measurable on } [0,T] \text{ for all } u \in H_n;$$
$$u \to \langle P_n A(t,u), e_j\rangle_V \text{ is continuous on } H_n \text{ for almost all } t \in [0,T].$$

By (H3) and Lemma 2.1 we get the following a priori estimate for (2.2) (see Lemma 2.4):

There exist positive constants T_0 and c such that if $u: I_0 \to H_n$ is a solution of (2.2) on an arbitrary subinterval I_0 of $[0,T_0]$, then

$$\|u(t)\|_H \leqslant c \quad \text{for all } t \in I_0.$$

Therefore, according to the classical existence theorem of Carathéodory for ordinary differential equations in $\mathbb{R}^n$ (cf. [53, pp. 799–800]), there exists a unique solution u_n to (2.2) on $[0,T_0]$ such that

$$u_n \in L^\alpha([0,T_0];H_n) \cap C([0,T];H_n), \qquad u_n' \in L^{\frac{\alpha}{\alpha-1}}([0,T_0];H_n). \qquad \square$$

Remark 2.3. From the proof it is clear that the constant T_0 comes from the application of Bihari's inequality. It only depends on u_0, g, f and is independent of n.

For the constant $T_0 \in (0, T]$, let $X := L^\alpha([0, T_0]; V)$, then $X^* = L^{\frac{\alpha}{\alpha-1}}([0, T_0]; V^*)$. We denote by $W^1_\alpha(0, T_0; V, H)$ the Banach space

$$W^1_\alpha(0, T_0; V, H) = \{u \in X\colon u' \in X^*\},$$

where u' is the weak derivative of

$$t \mapsto u(t) \in V \subseteq H \subseteq V^*$$

and on $W^1_\alpha(0, T_0; V, H)$ the norm is defined by

$$\|u\|_W := \|u\|_X + \|u'\|_{X^*} = \left(\int_0^{T_0} \|u(t)\|_V^\alpha \, dt\right)^{\frac{1}{\alpha}} + \left(\int_0^{T_0} \|u'(t)\|_{V^*}^{\frac{\alpha}{\alpha-1}} \, dt\right)^{\frac{\alpha-1}{\alpha}}.$$

It's well known that $W^1_\alpha(0, T_0; V, H)$ is a reflexive Banach space and it is continuously embedded into $C([0, T_0]; H)$ (cf. [53]). Moreover, we also have the following integration by parts formula

$$\langle u(t), v(t)\rangle_H - \langle u(0), v(0)\rangle_H = \int_0^t \langle u'(s), v(s)\rangle_V \, ds + \int_0^t \langle v'(s), u(s)\rangle_V \, ds,$$
$$t \in [0, T_0],\ u, v \in W^1_\alpha(0, T_0; V, H).$$

Lemma 2.4. *Suppose that $V \subseteq H$ is compact and* (H1)–(H4) *hold, we have for any solution u_n to* (2.2)

$$\|u_n(t)\|_H^2 + \delta \int_0^t \|u_n(s)\|_V^\alpha \, ds \leqslant G^{-1}\left(G\left(\|u_0\|_H^2 + \int_0^{T_0} f(s)\, ds\right) + t\right), \quad t \in [0, T_0], \tag{2.3}$$

where $G(x) := \int_{x_0}^x \frac{1}{g(r)}\, dr$ is well defined for some $x_0 > 0$.

In particular, there exists a constant $K > 0$ such that

$$\|u_n\|_X + \sup_{t\in[0,T_0]} \|u_n(t)\|_H + \|A(\cdot, u_n)\|_{X^*} \leqslant K, \quad n \geqslant 1. \tag{2.4}$$

Proof. By the integration by parts formula and (H3) we have

$$\begin{aligned}
\|u_n(t)\|_H^2 - \|u_n(0)\|_H^2 &= 2\int_0^t \langle u_n'(s), u_n(s)\rangle_V \, ds \\
&= 2\int_0^t \langle P_n A(s, u_n(s)), u_n(s)\rangle_V \, ds \\
&= 2\int_0^t \langle A(s, u_n(s)), u_n(s)\rangle_V \, ds
\end{aligned}$$

$$\leqslant \int_0^t \left(-\delta \|u_n(s)\|_V^\alpha + g\left(\|u_n(s)\|_H^2\right) + f(s)\right) ds. \tag{2.5}$$

Hence we have for $t \in [0, T_0]$,

$$\|u_n(t)\|_H^2 + \delta \int_0^t \|u_n(s)\|_V^\alpha \, ds \leqslant \|u_0\|_H^2 + \int_0^{T_0} f(s)\, ds + \int_0^t g\left(\|u_n(s)\|_H^2\right) ds.$$

Then by Lemma 2.1 and Remark 2.1 we know that (2.3) holds.

Therefore, there exists a constant C_2 such that

$$\|u_n\|_X + \sup_{t\in[0,T_0]} \|u_n(t)\|_H \leqslant C_2, \quad n \geqslant 1.$$

Then by (H4) there exists a constant C_3 such that

$$\|A(\cdot, u_n)\|_{X^*} \leqslant C_3, \quad n \geqslant 1.$$

Hence the proof is complete. $\square$

Note that X, X^* and H are reflexive spaces. Then by Lemma 2.4 there exists a subsequence, again denoted by u_n, such that as $n \to \infty$

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } X \text{ and } W_\alpha^1(0, T_0; V, H);\\ &A(\cdot, u_n) \rightharpoonup w \quad \text{in } X^*;\\ &u_n(T_0) \rightharpoonup z \quad \text{in } H. \end{aligned}$$

Recall that $u_n(0) = P_n u_0 \to u_0$ in H as $n \to \infty$.

Lemma 2.5. *Suppose that $V \subseteq H$ is compact and* (H1)–(H4) *hold, then the limit elements u, w and z satisfy $u \in W_\alpha^1(0, T_0; V, H)$ and*

$$u'(t) = w(t), \quad 0 < t < T_0, \qquad u(0) = u_0, \qquad u(T_0) = z.$$

Proof. See [32, Lemma 2.3]. $\square$

The next crucial step in the proof of Theorem 1.1 is to verify $w = A(u)$. In the case of monotone operators, this is the well known Minty's lemma (or monotonicity trick) (cf. [37,38,9,10]). In the case of locally monotone operators, we use the following integrated version of Minty's lemma which holds due to pseudo-monotonicity. The following lemma has first been proved in [32, Lemma 2.6]. We include the proof here for the reader's convenience.

Lemma 2.6. *Suppose that $V \subseteq H$ is compact and* (H1)–(H4) *hold, and assuming that*

$$\liminf_{n\to\infty} \int_0^{T_0} \left\langle A\left(t, u_n(t)\right), u_n(t)\right\rangle_V dt \geqslant \int_0^{T_0} \left\langle w(t), u(t)\right\rangle_V dt, \tag{2.6}$$

we have for any $v \in X$

$$\int_0^{T_0} \langle A(t, u(t)), u(t) - v(t) \rangle_V \, dt \geqslant \limsup_{n \to \infty} \int_0^{T_0} \langle A(t, u_n(t)), u_n(t) - v(t) \rangle_V \, dt. \tag{2.7}$$

In particular, we have $A(t, u(t)) = w(t)$, *a.e.* $t \in [0, T_0]$.

Proof. Since $W^1_\alpha(0, T_0; V, H) \subset C([0, T_0]; H)$ is a continuous embedding, we have that $u_n(t)$ converges to $u(t)$ weakly in H for all $t \in [0, T_0]$.

Claim 1. For all $t \in [0, T_0]$ we have

$$\limsup_{n \to \infty} \langle A(t, u_n(t)), u_n(t) - u(t) \rangle_V \leqslant 0. \tag{2.8}$$

Suppose there exists a $t_0 \in [0, T_0]$ such that

$$\limsup_{n \to \infty} \langle A(t_0, u_n(t_0)), u_n(t_0) - u(t_0) \rangle_V > 0.$$

Then we can take a subsequence such that

$$\lim_{i \to \infty} \langle A(t_0, u_{n_i}(t_0)), u_{n_i}(t_0) - u(t_0) \rangle_V > 0.$$

By (H3) and (H4) there exists a constant K such that

$$\begin{aligned} 2\langle A(t_0, u_{n_i}(t_0)), u_{n_i}(t_0) - u(t_0) \rangle_V \leqslant{} & -\frac{\delta}{2} \|u_{n_i}(t_0)\|_V^\alpha + K\big(f(t) + g(\|u_{n_i}(t_0)\|_H^2)\big) \\ & + K\big(1 + \|u_{n_i}(t_0)\|_H^{\alpha\beta}\big) \|u(t_0)\|_V^\alpha. \end{aligned}$$

Hence we know that $\{u_{n_i}(t_0)\}$ is bounded in V (w.r.t. $\|\cdot\|_V$), so there exists a subsequence of $\{u_{n_i}(t_0)\}$ that converges to some limit weakly in V.

Note that $u_{n_i}(t_0)$ converges to $u(t_0)$ weakly in H, it is easy to show that $u_{n_i}(t_0)$ converges to $u(t_0)$ weakly in V.

Since $A(t_0, \cdot)$ is pseudo-monotone, we have

$$\langle A(t_0, u(t_0)), u(t_0) - v \rangle_V \geqslant \limsup_{i \to \infty} \langle A(t_0, u_{n_i}(t_0)), u_{n_i}(t_0) - v \rangle_V, \quad v \in V.$$

In particular, we have

$$\limsup_{i \to \infty} \langle A(t_0, u_{n_i}(t_0)), u_{n_i}(t_0) - u(t_0) \rangle_V \leqslant 0,$$

which is a contradiction to the definition of the subsequence $\{u_{n_i}(t_0)\}$.

Hence (2.8) holds.

Similarly, by (H3) and (H4) there exists a constant K such that

$$\begin{aligned} 2\langle A(t, u_n(t)), u_n(t) - v(t) \rangle_V \leqslant{} & -\frac{\delta}{2} \|u_n(t)\|_V^\alpha + K\big(f(t) + g(\|u_n(t)\|_H^2)\big) \\ & + K\big(1 + \|u_n(t)\|_H^{\alpha\beta}\big) \|v(t)\|_V^\alpha, \quad v \in X. \end{aligned}$$

Then by Lemma 2.4, Fatou's lemma, (2.6) and (2.8) we have

$$\begin{aligned} 0 &\leqslant \liminf_{n\to\infty} \int_0^{T_0} \big\langle A\big(t, u_n(t)\big), u_n(t) - u(t)\big\rangle_V \, dt \\ &\leqslant \limsup_{n\to\infty} \int_0^{T_0} \big\langle A\big(t, u_n(t)\big), u_n(t) - u(t)\big\rangle_V \, dt \\ &\leqslant \int_0^{T_0} \limsup_{n\to\infty} \big\langle A\big(t, u_n(t)\big), u_n(t) - u(t)\big\rangle_V \, dt \leqslant 0. \end{aligned} \tag{2.9}$$

Hence

$$\lim_{n\to\infty} \int_0^{T_0} \big\langle A\big(t, u_n(t)\big), u_n(t) - u(t)\big\rangle_V \, dt = 0.$$

Claim 2. There exists a subsequence $\{u_{n_i}\}$ such that

$$\lim_{i\to\infty} \big\langle A\big(t, u_{n_i}(t)\big), u_{n_i}(t) - u(t)\big\rangle_V = 0 \quad \text{for } a.e.\ t \in [0, T_0]. \tag{2.10}$$

Define $g_n(t) := \langle A(t, u_n(t)), u_n(t) - u(t)\rangle_V$, $t \in [0, T]$. Then

$$\lim_{n\to\infty} \int_0^{T_0} g_n(t)\, dt = 0, \qquad \limsup_{n\to\infty} g_n(t) \leqslant 0, \quad t \in [0, T_0].$$

Then by Lebesgue's dominated convergence theorem we have

$$\lim_{n\to\infty} \int_0^{T_0} g_n^+(t)\, dt = 0,$$

where $g_n^+(t) := \max\{g_n(t), 0\}$.

Note that $|g_n(t)| = 2g_n^+(t) - g_n(t)$, hence we have

$$\lim_{n\to\infty} \int_0^{T_0} \big|g_n(t)\big|\, dt = 0.$$

Therefore, we can take a subsequence $\{g_{n_i}(t)\}$ such that

$$\lim_{i\to\infty} g_{n_i}(t) = 0 \quad \text{for } a.e.\ t \in [0, T_0],$$

i.e. (2.10) holds.

Therefore, for any $v \in X$, we can choose a subsequence $\{u_{n_i}\}$ such that

$$\lim_{i\to\infty}\int_0^{T_0}\big\langle A\big(t,u_{n_i}(t)\big),u_{n_i}(t)-v(t)\big\rangle_V\,dt=\limsup_{n\to\infty}\int_0^{T_0}\big\langle A\big(t,u_n(t)\big),u_n(t)-v(t)\big\rangle_V\,dt;$$

$$\lim_{i\to\infty}\big\langle A\big(t,u_{n_i}(t)\big),u_{n_i}(t)-u(t)\big\rangle_V=0\quad\text{for } a.e.\ t\in[0,T_0].$$

Since A is pseudo-monotone, we have

$$\big\langle A\big(t,u(t)\big),u(t)-v(t)\big\rangle_V\geqslant\limsup_{i\to\infty}\big\langle A\big(t,u_{n_i}(t)\big),u_{n_i}(t)-v(t)\big\rangle_V,\quad t\in[0,T_0].$$

By Fatou's lemma we obtain

$$\begin{aligned}\int_0^{T_0}\big\langle A\big(t,u(t)\big),u(t)-v(t)\big\rangle_V\,dt&\geqslant\int_0^{T_0}\limsup_{i\to\infty}\big\langle A\big(t,u_{n_i}(t)\big),u_{n_i}(t)-v(t)\big\rangle_V\,dt\\&\geqslant\limsup_{i\to\infty}\int_0^{T_0}\big\langle A\big(t,u_{n_i}(t)\big),u_{n_i}(t)-v(t)\big\rangle_V\,dt\\&=\limsup_{n\to\infty}\int_0^{T_0}\big\langle A\big(t,u_n(t)\big),u_n(t)-v(t)\big\rangle_V\,dt.\end{aligned}\tag{2.11}$$

In particular, we have for any $v \in X$,

$$\begin{aligned}\int_0^{T_0}\big\langle A\big(t,u(t)\big),u(t)-v(t)\big\rangle_V\,dt&\geqslant\limsup_{n\to\infty}\int_0^{T_0}\big\langle A\big(t,u_n(t)\big),u_n(t)-v(t)\big\rangle_V\,dt\\&\geqslant\liminf_{n\to\infty}\int_0^{T_0}\big\langle A\big(t,u_n(t)\big),u_n(t)-v(t)\big\rangle_V\,dt\\&\geqslant\int_0^{T_0}\big\langle w(t),u(t)\big\rangle_V\,dt-\int_0^{T_0}\big\langle w(t),v(t)\big\rangle_V\,dt\\&=\int_0^{T_0}\big\langle w(t),u(t)-v(t)\big\rangle_V\,dt.\end{aligned}$$

Since $v \in X$ is arbitrary, we have $A(\cdot,u)=w$ as elements in X^*.
Hence the proof is complete. □

Now we can give the complete proof of Theorem 1.1.

Proof of Theorem 1.1. (i) Existence: The integration by parts formula implies that

$$\|u_n(T_0)\|_H^2 - \|u_n(0)\|_H^2 = 2\int_0^{T_0} \langle A(t, u_n(t)), u_n(t)\rangle_V \, dt;$$

$$\|u(T_0)\|_H^2 - \|u(0)\|_H^2 = 2\int_0^{T_0} \langle w(t), u(t)\rangle_V \, dt.$$

Since $u_n(T_0) \rightharpoonup z$ in H, by the lower semicontinuity of $\|\cdot\|_H$ we have

$$\liminf_{n\to\infty} \|u_n(T_0)\|_H^2 \geqslant \|z\|_H^2 = \|u(T_0)\|_H^2.$$

Hence we have

$$\begin{aligned}\liminf_{n\to\infty} \int_0^{T_0} \langle A(t, u_n(t)), u_n(t)\rangle_V \, dt &\geqslant \frac{1}{2}\left(\|u(T_0)\|_H^2 - \|u(0)\|_H^2\right) \\ &= \int_0^{T_0} \langle w(t), u(t)\rangle_V \, dt.\end{aligned}$$

By Lemma 2.6 we know that u is a solution to (1.1).

(ii) Uniqueness: Suppose $u(\cdot, u_0), v(\cdot, v_0)$ are the solutions to (1.1) with starting points u_0, v_0 respectively, then by the integration by parts formula we have for $t \in [0, T_0]$,

$$\begin{aligned}\|u(t) - v(t)\|_H^2 &= \|u_0 - v_0\|_H^2 + 2\int_0^t \langle A(s, u(s)) - A(s, v(s)), u(s) - v(s)\rangle_V \, ds \\ &\leqslant \|u_0 - v_0\|_H^2 + 2\int_0^t \left(f(s) + \rho(u(s)) + \eta(v(s))\right)\|u(s) - v(s)\|_H^2 \, ds.\end{aligned}$$

By (1.2) we know that

$$\int_0^{T_0} \left(f(s) + \rho(u(s)) + \eta(v(s))\right) ds < \infty.$$

Then by Gronwall's lemma we obtain

$$\|u(t) - v(t)\|_H^2 \leqslant \|u_0 - v_0\|_H^2 \exp\left[2\int_0^t \left(f(s) + \rho(u(s)) + \eta(v(s))\right) ds\right], \quad t \in [0, T_0]. \tag{2.12}$$

In particular, if $u_0 = v_0$, this implies the uniqueness of the solution to (1.1). □

2.2. Proof of Theorem 1.2

The proof is similar to the arguments in the proof of Theorem 1.1(ii). □

2.3. Proof of Theorem 1.3

We first consider the process Y which solves the following SPDE:

$$dY(t) = A_1\big(t, Y(t)\big)\,dt + B(t)\,dW(t), \quad 0 < t < T, \qquad Y(0) = 0.$$

By [33, Theorem 1.1] we know that there exists a unique solution Y to the above equation and it satisfies

$$Y(\cdot) \in L^\alpha\big([0,T]; V\big) \cap C\big([0,T]; H\big); \quad \mathbb{P}\text{-a.s.}$$

Let $u(t) = X(t) - Y(t)$. Then it is easy to see that $u(t)$ satisfies the following equation:

$$u'(t) = \tilde{A}\big(t, u(t)\big), \quad 0 < t < T, \qquad u(0) = u_0, \tag{2.13}$$

where (for fixed ω which we omit in the notation for simplicity)

$$\tilde{A}(t, v) := A_1\big(t, v + Y(t)\big) - A_1\big(t, Y(t)\big) + A_2\big(t, v + Y(t)\big), \quad v \in V.$$

It is easy to show that $\tilde{A}$ is a well defined operator from $[0,T] \times V$ to V^* since $Y(\cdot) \in L^\alpha([0,T]; V)$.

To obtain the existence and uniqueness of solutions to (2.13) we only need to show that $\tilde{A}$ satisfies all the assumptions of Theorem 1.1.

Since $Y(t)$ is measurable, $\tilde{A}(t,v)$ is $\mathcal{B}([0,T]) \otimes \mathcal{B}(V)$-measurable. It is also easy to show that $\tilde{A}$ is hemicontinuous since (H1) holds for both A_1 and A_2.

For $u, v \in V$ we have

$$\begin{aligned}
\big\langle \tilde{A}(t,u) - \tilde{A}(t,v), u - v \big\rangle_V &= \big\langle A_1\big(t, u + Y(t)\big) - A_1\big(t, v + Y(t)\big), u - v \big\rangle_V \\
&\quad + \big\langle A_2\big(t, u + Y(t)\big) - A_2\big(t, v + Y(t)\big), u - v \big\rangle_V \\
&\leqslant \big(f(t) + \eta\big(v + Y(t)\big)\big)\|u - v\|_H^2 \\
&\quad + \big(f(t) + \rho\big(v + Y(t)\big) + \eta\big(v + Y(t)\big)\big)\|u - v\|_H^2 \\
&\leqslant C\big[f(t) + \rho\big(Y(t)\big) + \eta\big(Y(t)\big) + \rho(v) + \eta(v)\big]\|u - v\|_H^2,
\end{aligned}$$

i.e. (H2) holds for $\tilde{A}$ with

$$\tilde{f}(t) = C\big[f(t) + \rho\big(Y(t)\big) + \eta\big(Y(t)\big)\big] \in L^1\big([0,T]\big).$$

Since A_2 satisfies (H3) and (H4), by Young's inequality we have

$$\begin{aligned}
2\big\langle A_2\big(t, v + Y(t)\big), v \big\rangle_V &= 2\big\langle A_2\big(t, v + Y(t)\big), v + Y(t) - Y(t) \big\rangle_V \\
&\leqslant -\delta \big\| v + Y(t) \big\|_V^\alpha + g\big(\big\| v + Y(t) \big\|_H^2\big) + f(t) - 2\big\langle A_2\big(t, v + Y(t)\big), Y(t) \big\rangle_V \\
&\leqslant -\delta \big\| v + Y(t) \big\|_V^\alpha + g\big(\big\| v + Y(t) \big\|_H^2\big) + f(t)
\end{aligned}$$

$$
\begin{aligned}
&+ C\big(f(t)^{\frac{\alpha-1}{\alpha}} + \|v+Y(t)\|_V^{\alpha-1}\big)\big(1+\|v+Y(t)\|_H^{\beta}\big)\|Y(t)\|_V \\
&\leqslant -\frac{\delta}{2}\|v+Y(t)\|_V^{\alpha} + g\big(\|v+Y(t)\|_H^2\big) + \left(1+\frac{\delta}{2}\right)f(t) \\
&\quad + C\|Y(t)\|_V^{\alpha}\big(1+\|v+Y(t)\|_H^{\alpha\beta}\big) \\
&\leqslant -\frac{\delta}{2}\big(2^{1-\alpha}\|v\|_V^{\alpha} - \|Y(t)\|_V^{\alpha}\big) + g\big(2\|v\|_H^2 + 2\|Y(t)\|_H^2\big) \\
&\quad + \left(1+\frac{\delta}{2}\right)f(t) + C\|Y(t)\|_V^{\alpha}\big(1+\|v\|_H^{\alpha\beta} + \|Y(t)\|_H^{\alpha\beta}\big) \\
&\leqslant -2^{-\alpha}\delta\|v\|_V^{\alpha} + g\big(2\|v\|_H^2 + 2\|Y(t)\|_H^2\big) + C\|Y(t)\|_V^{\alpha}\|v\|_H^{\alpha\beta} \\
&\quad + \left(1+\frac{\delta}{2}\right)f(t) + C\|Y(t)\|_V^{\alpha}\big(1+\|Y(t)\|_H^{\alpha\beta}\big), \quad v\in V,
\end{aligned}
$$

where C is some constant changing from line to line (but independent of t and ω).

Similarly, we have

$$
\begin{aligned}
&2\big\langle A_1\big(t, v+Y(t)\big) - A_1\big(t, Y(t)\big), v\big\rangle_V \\
&= 2\big\langle A_1\big(t, v+Y(t)\big), v+Y(t)-Y(t)\big\rangle_V - 2\big\langle A_1\big(t, Y(t)\big), v\big\rangle_V \\
&\leqslant -\delta\|v+Y(t)\|_V^{\alpha} + C\|v+Y(t)\|_H^2 + f(t) \\
&\quad + \|Y(t)\|_V\big(f(t)^{\frac{\alpha-1}{\alpha}} + C\|v+Y(t)\|_V^{\alpha-1}\big) + \|v\|_V\big\|A_1\big(t, Y(t)\big)\big\|_{V^*} \\
&\leqslant -\frac{\delta}{2}\|v+Y(t)\|_V^{\alpha} + C\|v+Y(t)\|_H^2 + \left(1+\frac{\delta}{2}\right)f(t) \\
&\quad + C\|Y(t)\|_V^{\alpha} + \|v\|_V\big\|A_1\big(t, Y(t)\big)\big\|_{V^*} \\
&\leqslant -\frac{\delta}{2}\big(2^{1-\alpha}\|v\|_V^{\alpha} - \|Y(t)\|_V^{\alpha}\big) + C\big(\|v\|_H^2 + \|Y(t)\|_H^2\big) \\
&\quad + C\big(f(t) + \|Y(t)\|_V^{\alpha}\big) + \|v\|_V\big(f(t)^{\frac{\alpha-1}{\alpha}} + C\|Y(t)\|_V^{\alpha-1}\big) \\
&\leqslant -2^{-\alpha-1}\delta\|v\|_V^{\alpha} + C\|v\|_H^2 + C\big(f(t) + \|Y(t)\|_V^{\alpha} + \|Y(t)\|_H^2\big), \quad v\in V.
\end{aligned}
$$

Since $Y(\cdot)\in L^{\alpha}([0,T];V)\cap C([0,T];H)$, we know that $\tilde{A}$ satisfies (H3) with

$$
\tilde{f}(t) = C\big(f(t) + \|Y(t)\|_V^{\alpha} + \|Y(t)\|_H^2 + \|Y(t)\|_V^{\alpha}\|Y(t)\|_H^{\alpha\beta}\big).
$$

The growth condition (H4) also holds for $\tilde{A}$ since

$$
\begin{aligned}
\big\|\tilde{A}(t,v)\big\|_{V^*} &= \big\|A_1\big(t, v+Y(t)\big)\big\|_{V^*} + \big\|A_1\big(t, Y(t)\big)\big\|_{V^*} + \big\|A_2\big(t, v+Y(t)\big)\big\|_{V^*} \\
&\leqslant C\big(f(t)^{\frac{\alpha-1}{\alpha}} + \|v+Y(t)\|_V^{\alpha-1}\big)\big(1+\|v+Y(t)\|_H^{\beta}\big) \\
&\quad + f(t)^{\frac{\alpha-1}{\alpha}} + C\|Y(t)\|_V^{\alpha-1} \\
&\leqslant \big(Cf(t)^{\frac{\alpha-1}{\alpha}} + C\|Y(t)\|_V^{\alpha-1} + C\|v\|_V^{\alpha-1}\big)\big(1+\|Y(t)\|_H^{\beta} + \|v\|_H^{\beta}\big) \\
&\leqslant \big(\tilde{f}(t)^{\frac{\alpha-1}{\alpha}} + C\|v\|_V^{\alpha-1}\big)\big(1+\|v\|_H^{\beta}\big).
\end{aligned}
$$

Therefore, according to Theorem 1.1, (2.13) has a unique local solution on $[0, T_0(\omega)]$ for $\mathbb{P}$-$a.s.$ ω. Define

$$X(t) := u(t) + Y(t),$$

then it is easy to show that $X(t)$ is the unique local solution to (1.4).

Now the proof is complete. □

3. Application to examples

Since Theorem 1.1 is a generalization of a classical result for monotone operators (cf. [1,30,49,53]) and of a recent result for locally monotone operators (cf. [32,33]), it can be applied to a large class of semilinear and quasilinear evolution equations such as reaction–diffusion equations, generalized Burgers equations, 2D Navier–Stokes equation, 2D magneto-hydrodynamic equations, 2D magnetic Bénard problem, 3D Leray-α model, porous medium equations and generalized p-Laplace equations with locally monotone perturbations (cf. [15,32,33,41]). In this section we will first apply our general results to some known cases (Section 3.1, 3.2 and 3.3), but which have not been covered by the more restricted framework in the above references. Subsequently, in Sections 3.4 and 3.5 we apply our results to cases, which are not covered in the existing literature, at least not in such generality. In this section we use C to denote a generic constant which may change from line to line.

3.1. 3D Navier–Stokes equation

As we mentioned in the introduction, the first example here is to apply Theorem 1.1 to the 3D Navier–Stokes equation, which is a classical model to describe the time evolution of an incompressible fluid, given as follows:

$$\begin{aligned} &\partial_t u(t) = \nu \Delta u(t) - \big(u(t)\cdot\nabla\big)u(t) + \nabla p(t) + f(t), \\ &\operatorname{div}(u) = 0, \qquad u|_{\partial\Lambda} = 0, \qquad u(0) = u_0, \end{aligned} \tag{3.1}$$

where $u(t,x) = (u^1(t,x), u^2(t,x), u^3(t,x))$ represents the velocity field of the fluid, ν is the viscosity constant, the pressure $p(t,x)$ is an unknown scalar function and f is a (known) external force field acting on the fluid. In the pioneering work [29] Leray proved the existence of a weak solution for the 3D Navier–Stokes equation in the whole space. However, up to now, the uniqueness and regularity of weak solutions are still open problems (cf. [31,50,51]).

Let Λ be a smooth bounded open domain in $\mathbb{R}^3$. Let $C_0^\infty(\Lambda, \mathbb{R}^3)$ denote the set of all smooth functions from Λ to $\mathbb{R}^3$ with compact support. For $p \geqslant 1$, let $L^p := L^p(\Lambda, \mathbb{R}^3)$ be the vector valued L^p-space in which the norm is denoted by $\|\cdot\|_{L^p}$. For any integer $m \geqslant 0$, let $W_0^{m,2}$ be the standard Sobolev space on Λ with values in $\mathbb{R}^3$, i.e. the closure of $C_0^\infty(\Lambda, \mathbb{R}^3)$ with respect to the Sobolev norm.

For the reader's convenience, we recall the following Gagliardo–Nirenberg interpolation inequality, which plays an essential role in the study of Navier–Stokes equations.

If $q \in [1, \infty]$ such that

$$\frac{1}{q} = \frac{1}{2} - \frac{m\alpha}{3}, \quad 0 \leqslant \alpha \leqslant 1,$$

then there exists a constant $C_{m,q} > 0$ such that

$$\|u\|_{L^q} \leqslant C_{m,q} \|u\|_{W_0^{m,2}}^{\alpha} \|u\|_{L^2}^{1-\alpha}, \quad u \in W_0^{m,2}. \tag{3.2}$$

Now we define

$$H^m := \{u \in W_0^{m,2}\colon \operatorname{div}(u) = 0\}.$$

The norm of $W_0^{m,2}$ restricted to H^m will be denoted by $\|\cdot\|_{H^m}$. We recall that H^0 is a closed linear subspace of the Hilbert space $L^2(\Lambda, \mathbb{R}^3)$. In the literature it is well known that one can use the Gelfand triple $H^1 \subseteq H^0 \subseteq (H^1)^*$ to analyze the Navier–Stokes equation and it works very well in the 2D case even with general stochastic perturbations (cf. [12,33,50] and the references therein). However, as pointed out in [32,33], the growth condition (H4) fails to hold on this triple for the 3D Navier–Stokes equation. We also refer to Section 3.5 below to see that one needs certain modification in order to verify (H4) on this triple.

Motivated by some recent papers on the (stochastic) tamed 3D Navier–Stokes equation (cf. [45–48]), we will use the following Gelfand triple in order to verify the growth condition (H4):

$$V := H^2 \subseteq H := H^1 \subseteq V^*.$$

The main reason is that we can use the following inequality in the 3D case (see e.g. [27]):

$$\sup_x |u(x)|^2 \leqslant C\|\Delta u\|_{H^0}\|\nabla u\|_{H^0}. \tag{3.3}$$

Let $\mathcal{P}$ be the orthogonal (Helmhotz–Leray) projection from $L^2(\Lambda, \mathbb{R}^3)$ to H^0 (cf. [50,31]). For any $u \in H^0$ and $v \in L^2(\Lambda, \mathbb{R}^3)$ we have

$$\langle u, v\rangle_{H^0} := \langle u, \mathcal{P}v\rangle_{H^0} = \langle u, v\rangle_{L^2}.$$

Then by means of the divergence free Hilbert spaces H^2, H^1 and the orthogonal projection $\mathcal{P}$, the classical 3D Navier–Stokes equation (3.1) can be reformulated in the following abstract form:

$$u' = Au + B(u) + F, \qquad u(0) = u_0 \in H^1, \tag{3.4}$$

where

$$\begin{gathered} A : H^2 \to V^*, \qquad Au = \nu\mathcal{P}\Delta u; \\ B : H^2 \times H^2 \to V^*, \qquad B(u, v) = -\mathcal{P}\big[(u \cdot \nabla)v\big], \qquad B(u) = B(u, u); \\ F : [0, T] \to H^0 \end{gathered}$$

are well defined.

Remark 3.1. (1) It is obvious that $H^0 \subseteq L^2(\Lambda, \mathbb{R}^3) \subseteq V^*$ and

$$\|u\|_{V^*} \leqslant \|u\|_{L^2} = \|u\|_{H^0}, \quad u \in H^0.$$

(2) It is well known that

$$\big\langle B(u, v), w\big\rangle_{L^2} = -\big\langle B(u, w), v\big\rangle_{L^2}, \qquad \big\langle B(u, v), v\big\rangle_{L^2} = 0, \quad u, v, w \in H^2.$$

However, one should note that

$$\langle B(u,v),v\rangle_{H^2} := {}_{H^0}\langle B(u,v),v\rangle_{H^2} = \langle B(u,v),(I-\Delta)v\rangle_{L^2}, \quad u,v,w\in H^2,$$

which might not be equal to 0 in general.

Therefore, it is not obvious whether the usual coercivity condition still holds on this new triple or not? In fact, this is one reason that we introduce a generalized coercivity condition in order to handle this nonlinear term using this new triple.

For simplicity we only apply Theorem 1.1 to the deterministic 3D Navier–Stokes equation and give a simple proof for this well known result. We refer to [31,50,51] for the historical remarks and references on the classical local existence and uniqueness result of 3D Navier–Stokes equation. Note that one can also add a general type additive noise to (3.4) and obtain the corresponding result in the stochastic case by applying Theorem 1.3 and Remark 1.3.

Example 3.1 *(3D Navier–Stokes equation). If $F\in L^2(0,T;H^0)$ and $u_0\in H^1$, then there exists a constant $T_0\in(0,T]$ such that (3.4) has a unique strong solution $u\in L^2([0,T_0];H^2)\cap C([0,T_0];H^1)$.*

In particular, it is enough to choose $T_0\in(0,T]$ such that the following property holds:

$$T_0<\frac{C}{\|u_0\|_{H^1}^2+\int_0^{T_0}(1+\|F(t)\|_{L^2}^2)\,dt},$$

where $C>0$ is some (given) constant only depending on the viscosity constant ν.

Proof. The hemicontinuity (H1) is easy to verify since B is a bilinear map.

By (3.3) and Young's inequality we have

$$\begin{aligned}\langle B(u)-B(v),u-v\rangle_V &= \langle B(u)-B(v),(I-\Delta)(u-v)\rangle_{L^2}\\ &\leqslant \|u-v\|_V\big\|(u\cdot\nabla)u-(v\cdot\nabla)v\big\|_{L^2}\\ &\leqslant \|u-v\|_V\big(\|u\|_{L^\infty}\|\nabla u-\nabla v\|_{L^2}+\|u-v\|_{L^\infty}\|\nabla v\|_{L^2}\big)\\ &\leqslant \|u-v\|_V\big(\|u\|_{L^\infty}\|u-v\|_H+C\|u-v\|_V^{1/2}\|u-v\|_H^{1/2}\|v\|_H\big)\\ &\leqslant \frac{\nu}{2}\|u-v\|_V^2+C\big(\|u\|_{L^\infty}^2+\|v\|_H^4\big)\|u-v\|_H^2, \quad u,v\in V, \end{aligned} \tag{3.5}$$

where $C>0$ is a constant only depending on ν.

Hence we have the following local monotonicity (H2):

$$\begin{aligned}&\langle Au+B(u)-Av-B(v),u-v\rangle_V\\ &\quad\leqslant -\frac{\nu}{2}\|u-v\|_V^2+\nu\|u-v\|_H^2+C\big(\|u\|_{L^\infty}^2+\|v\|_H^4\big)\|u-v\|_H^2, \quad u,v\in V.\end{aligned}$$

In particular, there exists a constant C such that (let $u=0$)

$$\langle Av+B(v),v\rangle_V\leqslant -\frac{\nu}{2}\|v\|_V^2+C\big(1+\|v\|_H^6\big), \quad v\in V.$$

Then it is easy to show that (H3) holds with $g(x)=Cx^3$:

$$\begin{aligned}\langle Av + B(v) + F, v\rangle_V &\leqslant -\frac{\nu}{2}\|v\|_V^2 + C\left(1 + \|v\|_H^6\right) + \|F\|_{V^*}\|v\|_V \\ &\leqslant -\frac{\nu}{4}\|v\|_V^2 + C\|v\|_H^6 + C\left(1 + \|F\|_{L^2}^2\right), \quad v \in V.\end{aligned}$$

Note that by (3.3) we have

$$\|B(v)\|_{V^*}^2 \leqslant \|(v\cdot\nabla)v\|_{L^2}^2 \leqslant \|v\|_{L^\infty}^2\|\nabla v\|_{L^2}^2 \leqslant C\|v\|_V\|v\|_H^3 \leqslant C\|v\|_V^2\|v\|_H^2, \quad v \in V. \tag{3.6}$$

Hence (H4) holds with $\beta = 1$.

Then the local existence and uniqueness of solutions to (3.4) follows from Theorem 1.1. □

Remark 3.2. Note that the solution here is a strong solution in the sense of PDE. It is obvious that we can also allow F in (3.4) to depend on the unknown solution u provided F satisfies some locally monotone condition (cf. [32]).

Remark 3.3. If we analyze (3.4) by using the following Gelfand triple

$$V := H^1 \subseteq H := H^0 \subseteq V^*,$$

then $\langle B(v), v\rangle_V = 0$ and we have the classical coercivity (i.e. (H3) with $g(x) = Cx$):

$$\begin{aligned}\langle Av + B(v) + F, v\rangle_V &\leqslant -\nu\|v\|_V^2 + \nu\|v\|_H^2 + \|F\|_{V^*}\|v\|_V \\ &\leqslant -\frac{\nu}{2}\|v\|_V^2 + \nu\|v\|_H^2 + \frac{1}{2\nu}\|F\|_{V^*}^2, \quad v \in V.\end{aligned}$$

By (3.3) and Young's inequality we have

$$\begin{aligned}\langle B(u) - B(v), u - v\rangle_V &= -\langle B(u, u-v), v\rangle_V + \langle B(v, u-v), v\rangle_V \\ &= -\langle B(u-v), v\rangle_V \\ &\leqslant C\|u-v\|_V^{3/2}\|u-v\|_H^{1/2}\|v\|_{L^6} \\ &\leqslant \frac{\nu}{2}\|u-v\|_V^2 + C\|v\|_{L^6}^4\|u-v\|_H^2, \quad u, v \in V.\end{aligned} \tag{3.7}$$

Hence we have the local monotonicity (H2):

$$\langle Au + B(u) - Av - B(v), u - v\rangle_V \leqslant -\frac{\nu}{2}\|u-v\|_V^2 + C\left(1 + \|v\|_{L^6}^4\right)\|u-v\|_H^2.$$

Concerning the growth condition we have

$$\|B(u)\|_{V^*}^2 \leqslant C\|u\|_{L^4}^4 \leqslant C\|u\|_V^3\|u\|_H, \quad u \in V. \tag{3.8}$$

However, this is not enough to verify (H4).

One may refer to Section 3.5 where (H4) is verified for a modified version of Navier–Stokes equation, i.e. equation of power law fluids.

3.2. Tamed 3D Navier–Stokes equation

In the case of the 3D Navier–Stokes equation we see that the generalized coercivity condition holds with $g(x) = Cx^3$, hence we only get local existence and uniqueness of solutions. In this part we consider a tamed version of the (stochastic) 3D Navier–Stokes equation, which was proposed recently in [47,48] (see also [46,45]). The main feature of this tamed equation is that if there is a bounded smooth solution to the classical 3D Navier–Stokes equation (3.1), then this smooth solution must also satisfy the following tamed equation (3.9) (for N large enough):

$$\begin{aligned} \partial_t u(t) &= \nu \Delta u(t) - \big(u(t)\cdot\nabla\big)u(t) + \nabla p(t) - g_N\big(\big|u(t)\big|^2\big)u(t) + F(t), \\ &\operatorname{div}(u) = 0, \qquad u|_{\partial\Lambda} = 0, \qquad u(0) = u_0, \end{aligned} \tag{3.9}$$

where the taming function $g_N : \mathbb{R}_+ \to \mathbb{R}_+$ is smooth and satisfies for some $N > 0$,

$$\begin{cases} g_N(r) = 0, & \text{if } r \leqslant N, \\ g_N(r) = (r-N)/\nu, & \text{if } r \geqslant N+1, \\ 0 \leqslant g_N'(r) \leqslant C, & r \geqslant 0. \end{cases}$$

Example 3.2 *(Tamed 3D Navier–Stokes equation). For $F \in L^2(0,T;H^0)$ and $u_0 \in H^1$, (3.9) has a unique strong solution $u \in L^2([0,T];H^2) \cap C([0,T];H^1)$.*

Proof. Without loss of generality we may assume $\nu = 1$ for simplicity.

Using the Gelfand triple

$$V := H^2 \subseteq H := H^1 \subseteq V^*,$$

(3.9) can be rewritten in the abstract form:

$$u' = Au + B(u) - \mathcal{P}\big[g_N\big(|u|^2\big)u\big] + F, \qquad u(0) = u_0 \in H^1.$$

We recall the following estimates for $v \in H^2$ (cf. [48, Lemma 2.3]):

$$\begin{aligned} \langle Av, v\rangle_V &= \big\langle \mathcal{P}\Delta v, (I-\Delta)v\big\rangle_{L^2} \leqslant -\|v\|_V^2 + \|v\|_H^2; \\ \big\langle B(v), v\big\rangle_V &= -\big\langle \mathcal{P}(v\cdot\nabla)v, (I-\Delta)v\big\rangle_{L^2} \leqslant \frac{1}{4}\|v\|_V^2 + \frac{1}{2}\big\||v|\cdot|\nabla v|\big\|_{L^2}^2; \\ -\big\langle \mathcal{P}\big[g_N\big(|v|^2\big)v\big], v\big\rangle_V &= -\big\langle \mathcal{P}\big[g_N\big(|v|^2\big)v\big], (I-\Delta)v\big\rangle_{L^2} \leqslant -\big\||v|\cdot|\nabla v|\big\|_{L^2}^2 + CN\|v\|_H^2. \end{aligned} \tag{3.10}$$

Then it is easy to get the following coercivity (H3) with $g(x) = C(N+1)x$:

$$\big\langle Av + B(v) - \mathcal{P}\big[g_N\big(|v|^2\big)v\big] + F, v\big\rangle_V \leqslant -\frac{1}{2}\|v\|_V^2 + C(N+1)\|v\|_H^2 + C\|F\|_{V^*}^2, \quad v \in V.$$

By (3.3) we have

$$\begin{aligned} &-\big\langle \mathcal{P}\big[g_N\big(|u|^2\big)u\big] - \mathcal{P}\big[g_N\big(|v|^2\big)v\big], u-v\big\rangle_V \\ &\quad = -\big\langle \mathcal{P}\big[g_N\big(|u|^2\big)u\big] - \mathcal{P}\big[g_N\big(|v|^2\big)v\big], (I-\Delta)(u-v)\big\rangle_{L^2} \\ &\quad \leqslant \|u-v\|_V \big\| g_N\big(|u|^2\big)u - g_N\big(|v|^2\big)v\big\|_{L^2} \end{aligned}$$

$$
\begin{aligned}
&\leqslant \|u-v\|_V \big\| \big(g_N\big(|u|^2\big) - g_N\big(|v|^2\big)\big)u - g_N\big(|v|^2\big)(u-v)\big\|_{L^2} \\
&\leqslant \|u-v\|_V \big(C\|u-v\|_{L^\infty} \big\| |u|^2 + |v|^2 \big\|_{L^2} + C\big\| |v|^2 \big\|_{L^2} \|u-v\|_{L^\infty}\big) \\
&\leqslant C\|u-v\|_V^{\frac{3}{2}} \|u-v\|_H^{\frac{1}{2}} \big(\|u\|_{L^4}^2 + \|v\|_{L^4}^2\big) \\
&\leqslant \frac{1}{4}\|u-v\|_V^2 + C\big(\|u\|_{L^4}^8 + \|v\|_{L^4}^8\big)\|u-v\|_H^2, \quad u, v \in V.
\end{aligned} \tag{3.11}
$$

Hence by (3.5) we have the following estimate (note that $\nu = 1$):

$$
\begin{aligned}
&\big\langle Au + B(u) - \mathcal{P}\big[g_N\big(|u|^2\big)u\big] - Av - B(v) + \mathcal{P}\big[g_N\big(|v|^2\big)v\big], u - v\big\rangle_V \\
&\quad \leqslant -\frac{1}{4}\|u-v\|_V^2 + C\big(1 + \|u\|_{L^\infty}^2 + \|u\|_{L^4}^8 + \|v\|_H^4 + \|v\|_{L^4}^8\big)\|u-v\|_H^2, \quad u, v \in V,
\end{aligned} \tag{3.12}
$$

i.e. (H2) holds with $\rho(u) = \|u\|_{L^\infty}^2 + \|u\|_{L^4}^8$ and $\eta(v) = \|v\|_H^4 + \|v\|_{L^4}^8$.

By (3.2) we have

$$
\big\|\mathcal{P}\big[g_N\big(|v|^2\big)v\big]\big\|_{V^*}^2 \leqslant C\|v\|_{L^6}^2 \leqslant C\|v\|_H^2, \quad v \in V.
$$

Then by (3.6) we obtain that (H4) holds with $\beta = 2$.

Since (1.2) also holds, the global existence and uniqueness of solutions to (3.9) follows from Theorem 1.1. □

Remark 3.4. One should note that if we use the Gelfand triple $V := H^1 \subseteq H := H^0 \subseteq V^*$ for the tamed 3D Navier–Stokes equation, then (H4) fails too because the additional term in (3.9) has no help to decrease the exponent appeared in the Growth condition of the quadratic term $(u \cdot \nabla)u$. This difficulty is overcomed by using the new Gelfand triple, however, the standard coercivity condition does not hold anymore under this new triple. Therefore, the role of the taming term $g_N(|u|^2)u$ is to compensate the coercivity property (see e.g. (3.10)) such that (H3) still holds in this case.

3.3. Cahn–Hilliard equation

The Cahn–Hilliard equation is a classical model to describe phase separation in a binary alloy and some other media, we refer to [39] for a survey on this model (see also [19,16] for the stochastic case). Let Λ be a bounded open domain in $\mathbb{R}^d$ $(d \leqslant 3)$ with smooth boundary. The Cahn–Hilliard equation has the following form:

$$
\begin{aligned}
&\partial_t u = -\Delta^2 u + \Delta\varphi(u), \qquad u(0) = u_0, \\
&\nabla u \cdot n = \nabla(\Delta u) \cdot n = 0 \quad \text{on } \partial\Lambda,
\end{aligned} \tag{3.13}
$$

where Δ is the Laplace operator, n is the outward unit normal vector on the boundary $\partial\Lambda$ and the nonlinear term φ is some polynomial function.

Now we consider the following Gelfand triple

$$
V \subseteq H := L^2(\Lambda) \subseteq V^*,
$$

where $V := \{u \in W^{2,2}(\Lambda): \ \nabla u \cdot n = \nabla(\Delta u) \cdot n = 0 \text{ on } \partial\Lambda\}$.

Then we get the following existence and uniqueness result for (3.13).

Example 3.3. *Suppose that $\varphi \in C^1(\mathbb{R})$ and there exist some positive constants C and $p \leqslant \frac{d+4}{d}$ such that*

$$\varphi'(x) \geqslant -C, \qquad \big|\varphi(x)\big| \leqslant C\big(1+|x|^p\big), \quad x \in \mathbb{R};$$
$$\big|\varphi(x)-\varphi(y)\big| \leqslant C\big(1+|x|^{p-1}+|y|^{p-1}\big)|x-y|, \quad x, y \in \mathbb{R}.$$

Then for any $u_0 \in L^2(\Lambda)$, there exists a unique solution to (3.13).

Proof. For any $u, v \in V$, we have

$$-\big\langle \Delta^2 u - \Delta^2 v, u-v\big\rangle_V = -\|u-v\|_V^2.$$

By the assumptions on φ and Young's inequality we get

$$\begin{aligned}\big\langle \Delta\varphi(u)-\Delta\varphi(v), u-v\big\rangle_V &\leqslant \|u-v\|_V \big\|\varphi(u)-\varphi(v)\big\|_{L^2} \\ &\leqslant \|u-v\|_V \cdot C\big(1+\|u\|_{L^\infty}^{p-1}+\|v\|_{L^\infty}^{p-1}\big)\|u-v\|_{L^2} \\ &\leqslant \frac{1}{2}\|u-v\|_V^2 + C\big(1+\|u\|_{L^\infty}^{2p-2}+\|v\|_{L^\infty}^{2p-2}\big)\|u-v\|_H^2, \quad u, v \in V.\end{aligned}$$

Hence (H2) holds with $\rho(u) = \eta(u) = C\|u\|_{L^\infty}^{2p-2}$.

Similarly, by the interpolation inequality we have for any $v \in V$,

$$\big\langle \Delta\varphi(v), v\big\rangle_V = -\int_\Lambda \varphi'(v)|\nabla v|^2\,dx \leqslant C\|v\|_{W^{1,2}}^2 \leqslant \frac{1}{2}\|v\|_V^2 + C\|v\|_H^2,$$

i.e. (H3) holds with $\alpha = 2$ and $g(x) = Cx$.

It is also easy to see that

$$\begin{aligned}\big\|\Delta\varphi(v)\big\|_{V^*} &\leqslant \big\|\varphi(v)\big\|_H \\ &\leqslant C\big(1+\|v\|_{L^{2p}}^p\big) \\ &\leqslant C\big(1+\|v\|_V^{\frac{(p-1)d}{4}}\|v\|_H^{\frac{d+(4-d)p}{4}}\big) \\ &\leqslant C\big(1+\|v\|_V^{\frac{(p-1)d}{4}}\big)\big(1+\|v\|_H^{\frac{d+(4-d)p}{4}}\big), \quad v \in V.\end{aligned}$$

Since $p \leqslant \frac{4}{d}+1$ (i.e. $\frac{(p-1)d}{4} \leqslant 1$) and $\|v\|_H \leqslant C\|v\|_V$, we have

$$\big\|\Delta\varphi(v)\big\|_{V^*} \leqslant C\big(1+\|v\|_V\big)\big(1+\|v\|_H^{p-1}\big), \quad v \in V,$$

i.e. (H4) holds with $\beta = p-1$.

Note that for any $v \in V$,

$$\rho(v) = C\|v\|_{L^\infty}^{2p-2} \leqslant C\|v\|_V^{\frac{(p-1)d}{2}}\|v\|_H^{\frac{(p-1)(4-d)}{2}},$$

i.e. (1.2) also holds.

Therefore, the conclusion follows directly from Theorem 1.1. □

3.4. Surface growth PDE with random noise

We consider a model which appears in the theory of growth of surfaces, which describes an amorphous material deposited on an initially flat surface in high vacuum (cf. [42,4] and the references therein). Taking account of random noises the equation is formulated on the interval $\Lambda := [0, L]$ as follows:

$$dX(t) = \left[-\partial_x^4 X(t) - \partial_x^2 X(t) + \partial_x^2 \left(\partial_x X(t)\right)^2\right] dt + B(t)\, dW(t),$$
$$X(t)\big|_{\partial\Lambda} = 0, \qquad X(0) = x_0, \tag{3.14}$$

where ∂_x, ∂_x^2, ∂_x^4 denote the first, second and fourth spatial derivatives respectively.

Recall that $W(t)$ is a U-valued cylindrical Wiener process. Using the following Gelfand triple

$$V := W_0^{4,2}\big([0, L]\big) \subseteq H := W^{2,2}\big([0, L]\big) \subseteq V^*$$

we can obtain the following local existence and uniqueness of strong solutions for (3.14).

Example 3.4. *Suppose that $B \in L^2([0, T]; L_2(U; H))$. For any $X_0 \in L^2(\Omega \to H; \mathcal{F}_0; \mathbb{P})$, there exists a unique local solution $\{X(t)\}_{t\in[0,\tau]}$ to (3.14) satisfying*

$$X(\cdot) \in L^2\big([0, \tau]; V\big) \cap C\big([0, \tau]; H^2\big), \quad \mathbb{P}\text{-a.s.}$$

Proof. It is sufficient to verify (H1)–(H4) for (3.14), then the conclusion follows from Theorem 1.3.

For $u, v \in V$, by standard interpolation inequalities and Young's inequality we have

$$\begin{aligned}
\big\langle \partial_x^2(\partial_x u)^2 - \partial_x^2(\partial_x v)^2, u - v\big\rangle_V &= \big\langle \partial_x^2(\partial_x u)^2 - \partial_x^2(\partial_x v)^2, \partial_x^4 u - \partial_x^4 v\big\rangle_{L^2} \\
&\leqslant \|u - v\|_V \big\|\partial_x^2(\partial_x u)^2 - \partial_x^2(\partial_x v)^2\big\|_{L^2} \\
&\leqslant \|u - v\|_V \Big[\big\|\big(\partial_x^2 u\big)^2 - \big(\partial_x^2 v\big)^2\big\|_{L^2} + \big\|\partial_x u \partial_x^3 u - \partial_x v \partial_x^3 v\big\|_{L^2}\Big] \\
&\leqslant \|u - v\|_V \Big[\big(\big\|\partial_x^2 u\big\|_{L^\infty} + \big\|\partial_x^2 v\big\|_{L^\infty}\big)\|u - v\|_H \\
&\quad + \|\partial_x u\|_{L^\infty} \big\|\partial_x^3 u - \partial_x^3 v\big\|_{L^2} + \big\|\partial_x^3 v\big\|_{L^2} \|\partial_x u - \partial_x v\|_{L^\infty}\Big] \\
&\leqslant \|u - v\|_V \Big[\big(\big\|\partial_x^2 u\big\|_{L^\infty} + \big\|\partial_x^2 v\big\|_{L^\infty}\big)\|u - v\|_H \\
&\quad + \|\partial_x u\|_{L^\infty} \|u - v\|_V^{\frac{1}{2}} \|u - v\|_H^{\frac{1}{2}} + \big\|\partial_x^3 v\big\|_{L^2} \|u - v\|_H\Big] \\
&\leqslant \frac{1}{4}\|u - v\|_V^2 + C\big(\|u\|_{W^{2,\infty}}^2 + \|u\|_{W^{1,\infty}}^4 + \|v\|_{W^{2,\infty}}^2 \\
&\quad + \|v\|_{W^{3,2}}^2\big)\|u - v\|_H^2,
\end{aligned}$$

where C is some constant.

Note that

$$\begin{aligned}
\big\langle -\partial_x^4 u - \partial_x^2 u + \partial_x^4 v + \partial_x^2 v, u - v\big\rangle_V &\leqslant -\|u - v\|_V^2 + \|u - v\|_V \|u - v\|_H \\
&\leqslant -\frac{3}{4}\|u - v\|_V^2 + \|u - v\|_H^2.
\end{aligned}$$

Hence we know that (H2) holds with

$$\rho(u) = \|u\|_{W^{2,\infty}}^2 + \|u\|_{W^{1,\infty}}^4, \qquad \eta(v) = \|v\|_{W^{2,\infty}}^2 + \|v\|_{W^{3,2}}^2.$$

Similarly,

$$\begin{aligned}\left\|\partial_x^2(\partial_x v)^2\right\|_{V^*} &\leqslant \left\|\left(\partial_x^2 v\right)^2 + \partial_x v \partial_x^3 v\right\|_{L^2} \\ &\leqslant \|v\|_{W^{2,4}}^2 + \|v\|_{W^{1,\infty}}\|v\|_{W^{3,2}} \\ &\leqslant C\|v\|_V^{\frac{1}{2}}\|v\|_H^{\frac{3}{2}}, \quad v \in V,\end{aligned}$$

i.e. (H4) holds with $\beta = 1$.

Moreover, this also implies that

$$2\left\langle\partial_x^2(\partial_x v)^2, v\right\rangle_V \leqslant 2\|v\|_V \left\|\partial_x^2(\partial_x v)^2\right\|_{V^*} \leqslant C\|v\|_V^{\frac{3}{2}}\|v\|_H^{\frac{3}{2}} \leqslant \frac{1}{2}\|v\|_V^2 + C\|v\|_H^6.$$

Since

$$2\left\langle -\partial_x^4 v - \partial_x^2 v, v\right\rangle_V \leqslant \frac{3}{2}\|v\|_V^2 + \|v\|_H^2,$$

we deduce that (H3) holds with $g(x) = Cx^3$.

Now the proof is complete. □

Remark 3.5. (1) It is known in the literature that the (1-dimension) surface growth model has some similar features of difficulty as the 3D Navier–Stokes equation, the uniqueness of weak solutions for this model is still an open problem in both the deterministic and stochastic cases. From the proof above one can see these similarities (e.g. (H2)–(H4)) very clearly between this model and the 3D Navier–Stokes equation (Example 3.1).

(2) The solution obtained here for the stochastic surface growth model is a strong solution in the sense of both PDE and SPDE. We should remark that for the space–time white noise case, the existence of a weak martingale solution was obtained by Blömker, Flandoli and Romito in [4] for this model, and the existence of a Markov selection and ergodicity properties were also proved there.

(3) We should also remark that in [5,6] Blömker and Romito established the local existence and uniqueness of solutions for surface growth model with more general initial conditions in the critical Hilbert space $H^{1/2}$ or some Besov space (the largest possible critical space where weak solutions make sense). They used the fixed point arguments and the technique introduced by Koch and Tataru for the Navier–Stokes equations (cf. [6] and more references therein).

3.5. Stochastic power law fluids

The next example of (S)PDE is a model which describes the velocity field of a viscous and incompressible non-Newtonian fluid subject to some random forcing. The deterministic model has been studied intensively in PDE theory (cf. [23,34] and the references therein). Let Λ be a bounded domain in $\mathbb{R}^d$ ($d \geqslant 2$) with sufficiently smooth boundary. For a vector field (e.g. the velocity field of the fluid) $u : \Lambda \to \mathbb{R}^d$, we denote the rate of strain tensor by

$$e(u) : \Lambda \to \mathbb{R}^d \otimes \mathbb{R}^d; \qquad e_{i,j}(u) = \frac{\partial_i u_j + \partial_j u_i}{2}, \quad i, j = 1, \ldots, d.$$

In this paper we consider the case that the extra stress tensor has the following polynomial form:

$$\tau(u) : \Lambda \to \mathbb{R}^d \otimes \mathbb{R}^d; \qquad \tau(u) = 2\nu\big(1 + \big|e(u)\big|\big)^{p-2} e(u),$$

where $\nu > 0$ is the kinematic viscosity and $p > 1$ is some constant.

In the case of deterministic forcing, the dynamics of power law fluids can be modeled by the following PDE:

$$\begin{aligned} \partial_t u &= \operatorname{div}\big(\tau(u)\big) - (u \cdot \nabla)u - \nabla p + f, \\ \operatorname{div}(u) &= 0, \end{aligned} \tag{3.15}$$

where $u = u(t,x) = (u_i(t,x))_{i=1}^d$ is the velocity field, p is the pressure, f is some external force and

$$u \cdot \nabla = \sum_{j=1}^d u_j \partial_j, \qquad \operatorname{div}\big(\tau(u)\big) = \left(\sum_{j=1}^d \partial_j \tau_{i,j}(u)\right)_{i=1}^d.$$

Remark 3.6. (1) Note that $p = 2$ describes the Newtonian fluids and (3.15) reduces to the classical Navier–Stokes equation (3.1).

(2) The shear shining fluids (i.e. $p \in (1,2)$) and the shear thickening fluids (i.e. $p \in (2,\infty)$) has been also widely studied in different fields of science and engineering (cf. [23,34]).

Now we consider the following Gelfand triple

$$V \subseteq H \subseteq V^*,$$

where

$$V = \big\{u \in W_0^{1,p}\big(\Lambda; \mathbb{R}^d\big) \colon \operatorname{div}(u) = 0\big\}; \qquad H = \big\{u \in L^2\big(\Lambda; \mathbb{R}^d\big) \colon \operatorname{div}(u) = 0\big\}.$$

Let $\mathcal{P}$ be the orthogonal (Helmhotz–Leray) projection from $L^2(\Lambda, \mathbb{R}^d)$ to H. It is well known that the following operators

$$A : W_0^{2,p}\big(\Lambda; \mathbb{R}^d\big) \cap V \to H, \qquad A(u) = \mathcal{P}\big[\operatorname{div}\big(\tau(u)\big)\big];$$
$$B : W_0^{2,p}\big(\Lambda; \mathbb{R}^d\big) \cap V \times W_0^{2,p}\big(\Lambda; \mathbb{R}^d\big) \cap V \to H; \qquad B(u,v) = -\mathcal{P}\big[(u \cdot \nabla)v\big], \qquad B(u) := B(u,u)$$

can be extended to the well defined operators:

$$A : V \to V^*; \qquad B : V \times V \to V^*.$$

In particular, one can show that

$$\big\langle A(u), v\big\rangle_V = -\int_\Lambda \sum_{i,j=1}^d \tau_{i,j}(u) e_{i,j}(v)\,dx; \quad u, v \in V;$$
$$\big\langle B(u,v), w\big\rangle_V = -\big\langle B(u,w), v\big\rangle_V, \qquad \big\langle B(u,v), v\big\rangle_V = 0, \quad u, v, w \in V.$$

Now (3.15) can be reformulated in the following variational form:

$$u'(t) = A\big(u(t)\big) + B\big(u(t)\big) + F(t), \qquad u(0) = u_0. \tag{3.16}$$

Example 3.5. *Suppose that $u_0 \in H$, $F \in L^2([0,T]; V^*)$ and $p \geqslant \frac{3d+2}{d+2}$, then (3.16) has a solution. Moreover, if $p \geqslant \frac{d+2}{2}$, then the solution of (3.16) is also unique.*

Proof. Without loss of generality we may assume $\nu = 1$.

We first recall the well known Korn's inequality for $p \in (1,\infty)$:

$$\int_\Lambda \left|e(u)\right|^p dx \geqslant C_p \|u\|_{1,p}, \quad u \in W_0^{1,p}\left(\Lambda; \mathbb{R}^d\right),$$

where $C_p > 0$ is some constant.

The following inequalities are also used very often in the study of power law fluids (cf. [34, p. 198, Lemma 1.19]):

$$\begin{aligned}
&\sum_{i,j=1}^d \tau_{i,j}(u) e_{i,j}(u) \geqslant C\left(\left|e(u)\right|^p - 1\right);\\
&\sum_{i,j=1}^d \left(\tau_{i,j}(u) - \tau_{i,j}(v)\right)\left(e_{i,j}(u) - e_{i,j}(v)\right) \geqslant C\left(\left|e(u) - e(v)\right|^2 + \left|e(u) - e(v)\right|^p\right);\\
&\left|\tau_{i,j}(u)\right| \leqslant C\left(1 + \left|e(u)\right|\right)^{p-1}, \quad i,j = 1,\dots,d.
\end{aligned} \tag{3.17}$$

Then by the interpolation inequality and Young's inequality one can show that

$$\begin{aligned}
\left\langle B(u) - B(v), u - v\right\rangle_V &= -\left\langle B(u-v), v\right\rangle_V\\
&= \left\langle B(u-v, v), u - v\right\rangle_V\\
&\leqslant C\|v\|_V \|u - v\|^2_{\frac{2p}{p-1}}\\
&\leqslant C\|v\|_V \|u - v\|_{1,2}^{\frac{d}{p}} \|u - v\|_H^{\frac{2p-d}{p}}\\
&\leqslant \varepsilon\|u - v\|_{1,2}^2 + C_\varepsilon \|v\|_V^{\frac{2p}{2p-d}} \|u - v\|_H^2, \quad u, v \in V.
\end{aligned}$$

By (3.17) and Korn's inequality we have

$$\begin{aligned}
\left\langle A(u) - A(v), u - v\right\rangle_V &= -\int_\Lambda \sum_{i,j=1}^d \left(\tau_{i,j}(u) - \tau_{i,j}(v)\right)\left(e_{i,j}(u) - e_{i,j}(v)\right) dx\\
&\leqslant -C\left\|e(u) - e(v)\right\|_H^2\\
&\leqslant -C\|u - v\|_{1,2}^2.
\end{aligned}$$

Hence we have the following estimate:

$$\left\langle Au + B(u) - Av - B(v), u - v\right\rangle_V \leqslant -(C - \varepsilon)\|u - v\|_{1,2}^2 + C_\varepsilon \|v\|_V^{\frac{2p}{2p-d}} \|u - v\|_H^2,$$

i.e. (H2) holds with $\rho(v) = C_\varepsilon \|v\|_V^{\frac{2p}{2p-d}}$.

It is also easy to verify (H3) with $\alpha = p$ as follows:

$$\langle A(v) + B(v), v\rangle_V \leqslant -C_1 \int_\Lambda \big|e(v)\big|^p \, dx + C_2 \leqslant -C_3 \|v\|_V^p + C_2.$$

Note that

$$\big|\langle B(v), u\rangle_V\big| = \big|\langle B(v,u), v\rangle_V\big| \leqslant \|u\|_V \|v\|^2_{\frac{2p}{p-1}}, \quad u, v \in V,$$

hence we have

$$\big\|B(v)\big\|_{V^*} \leqslant \|v\|^2_{\frac{2p}{p-1}}, \quad v \in V.$$

Then by the interpolation inequality and Sobolev's inequality we have

$$\|v\|_{\frac{2p}{p-1}} \leqslant \|v\|_q^\gamma \|v\|_2^{1-\gamma} \leqslant C\|v\|_V^\gamma \|v\|_H^{1-\gamma},$$

where $q = \frac{dp}{d-p}$ and $\gamma = \frac{d}{(d+2)p-2d}$.

Note that $2\gamma \leqslant p-1$ if $p \geqslant \frac{3d+2}{d+2}$, and it is also easy to see that

$$\big\|A(v)\big\|_{V^*} \leqslant C\big(1 + \|v\|_V^{p-1}\big), \quad v \in V.$$

Hence the growth condition (H4) also holds.

Then the existence of solutions to (3.16) follows from Theorem 1.1. Moreover, if $d \geqslant \frac{2+d}{2}$, then (1.2) holds and hence the solution of (3.16) is unique. □

Remark 3.7. We consider the power law fluids with state-dependent random forcing which can be described by the following SPDE:

$$dX(t) = \big(A\big(X(t)\big) + B\big(X(t)\big)\big)\,dt + Q\big(X(t)\big)\,dW(t), \qquad X(0) = X_0, \tag{3.18}$$

where $W(t)$ is a cylindrical Wiener process on a Hilbert space U w.r.t. the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

By applying [33, Theorem 1.1], we can show that for $p \geqslant \frac{2+d}{2}$, $X_0 \in L^4(\Omega \to H; \mathcal{F}_0, \mathbb{P})$ and Q is a Lipschitz map from V to $L_2(U; H)$, (3.18) has a unique strong solution $X \in L^4([0,T] \times \Omega, dt \times \mathbb{P}, V) \cap L^4(\Omega, \mathbb{P}, C([0,T]; H))$.

Remark 3.8. In [52] the authors established the existence and uniqueness of weak solutions for (3.16) with additive Wiener noise. They first considered the Galerkin approximation and showed the tightness of the distributions of the corresponding approximating solutions. Then they proved that the limit is a weak solution of (3.16) with additive Wiener noise.

Here we apply the main result (Theorem 1.1) directly to (3.16) and establish the existence and uniqueness of solutions in L^2-space of divergence free vector fields. Therefore, by Theorem 1.3 and Remark 1.3 we can obtain the existence and uniqueness of strong solutions (in the sense of SPDE) for (3.16) with general additive type noises. Moreover, as just mentioned in the previous remark, we can also prove the analogous result for (3.16) with multiplicative Wiener noise.

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