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NOTE

FREE UPPER REGULAR BANDS*

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Abstract. In universal domains, the retractions form a fundamental notion of 'data type'. We define a partial order on the retractions of Scott's domain $P\omega$ and completely describe the set of data types which can be generated by an arbitrary chain of such retractions. This set is precisely the free upper regular band over the chain, and is a lattice-ordered semigroup.

1. Motivation

In universal domains [2, 3, 5], the retractions (actually, their mappings) form a fundamental notion of 'data type'. In Scott's domain $P\omega$, an element, b, is an improvement over or better defined than another, a, if $a \subseteq b$. When a and b are retractions, $a \subseteq b$ indicates that b is a more widely defined data type than a.

If in addition $a = a \circ b \circ a$, the data type a is an invariant subtype of b. This indicates that a and b cannot be much different in structure. Furthermore, considered as data types, b includes an image of a, in much the same way as the reals include the integers. Smyth and Plotkin [4] suggest that the partial order

$$a \leqslant b \text{ iff } a \subseteq b \;\&\; a = a \circ b \circ a$$

is a reasonable method of ordering retractions.

Consider a finite (or countable) chain $a_1 \leqslant a_2 \leqslant a_3 \leqslant \cdots$ of retractions. What data types can be generated from the a_i? We give a complete answer to this question by demonstrating the free system of retractions over the generators.

2. Upper regular bands

Definition. Let S be an ordered semigroup. That is, $x \leqslant x'$ and $y \leqslant y'$ imply $xy \leqslant x'y'$ for some fixed partial order $\leqslant$. An element x is called *upper regular* if for all y in

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S such that $x \leqslant y$, $x = xyx$. S is called *upper regular* if all its elements are upper regular. If every element is also idempotent (i.e., $xx = x$) then S is an *upper regular band.*

Theorem 1. *Let A be a (possibly infinite) chain of elements in an arbitrary ordered semigroup, where for all $a, b \in A$, $a \leqslant b$ implies $aa = a = aba$. Also let $x_1, x_2, \ldots, x_n$ be arbitrary elements of A. Then the following hold:*

(i) $x_1 \leqslant x_2$ *implies* $x_1x_3 = x_1x_2x_3$ & $x_3x_1 = x_3x_2x_1$,

(ii) $x_1x_2 \ldots x_n = x_1x_{\min}x_n$, where $x_{\min} = \min(x_1, x_2, \ldots, x_n)$.

Proof. (i) We show $x_1x_3 \leqslant x_1x_2x_3 \leqslant x_1x_3$. As $x_1 \leqslant x_2$,

$$x_1x_3 = x_1x_1x_3 \leqslant x_1x_2x_3.$$

The relation $x_1x_2x_3 \leqslant x_1x_3$ is shown in three cases:

Case 1. $x_1 \leqslant x_2 \leqslant x_3$:

$$x_1x_2x_3 \leqslant x_1x_3x_3 = x_1x_3.$$

Case 2. $x_1 \leqslant x_3 \leqslant x_2$:

$$x_1x_2x_3 = x_1x_1x_2x_3 \leqslant x_1x_3x_2x_3 = x_1(x_3x_2x_3) = x_1x_3.$$

Case 3. $x_3 \leqslant x_1 \leqslant x_2$:

$$x_1x_2x_3 = x_1x_2x_3x_3 \leqslant x_1x_2x_1x_3 = (x_1x_2x_1)x_3 = x_1x_3.$$

This shows $x_1x_3 = x_1x_2x_3$. The other equality ($x_3x_1 = x_3x_2x_1$) follows similarly.

An induction on n provides the proof of (ii).

The second result required is

Theorem 2. *Let A be a (possibly infinite) chain of elements in an arbitrary ordered semigroup, where for all $a, b \in A$, $a \leqslant b$ implies $aa = a = aba$. Then the subsemigroup generated by A is an upper regular band.*

Proof. Let $x = x_1x_2 \cdots x_n$ and $y = y_1y_2 \cdots y_m$ be arbitrary compositions of a finite number of elements of A. We must show

(i) $xx = x$,

(ii) $x \leqslant y$ implies $x = xyx$.

From Theorem 1 we have $xx = x_1x_2 \cdots x_nx_1x_2 \cdots x_n = x_1x_{\min}x_n = x$. To show (ii), there are two cases. First, if $\min(x_1, x_2, \ldots, x_n) \leqslant \min(y_1, y_2, \ldots, y_m)$ then by Theorem 1.

$$xyx = x_1 \min(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots y_m)x_n = x_1x_{\min}x_n = x.$$

On the other hand, if $\min(y_1, y_2, \ldots, y_m) \leqslant \min(x_1, x_2, \ldots, x_n)$ then $x_nyx_1 = x_ny_{\min}x_1$

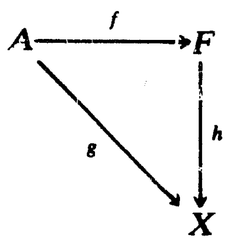
and

$$
\begin{aligned}
x &= xxx \\
&\leqslant xyx \\
&= x_1x_2\cdots(x_nyx_1)x_2\cdots x_n \\
&= x_1x_2\cdots(x_ny_{\min}x_1)x_2\cdots x_n \\
&= x_1x_2\cdots x_ny_{\min}^{n}x_1x_2\cdots x_n \\
&\leqslant x_1x_2\cdots x_nx_1x_2\cdots x_nx_1x_2\cdots x_n \\
&= xxx \\
&= x.
\end{aligned}
$$

3. The construction

We construct a free upper regular band over an arbitrary totally ordered set. In the next section, this construction is shown to be precisely the free system generated by a chain of retractions in Pω.

Let A be an arbitrary totally ordered set. By a *free upper regular band over A* we mean an upper regular band F together with a monotone function $f: A \to F$ such that for every monotone function $g: A \to X$, where X is an upper regular band, there exists a unique homomorphism $h: F \to X$ so that $hf = g$, as in the following diagram:



For any totally ordered set A, a free upper regular band is constructed as follows: Let F be the set of all ordered triples of A for which

$$(a, b, c) \in F \text{ iff } b \leqslant a \;\&\; b \leqslant c.$$

Now define a binary operation in F as

$$(a, b, c)\cdot(a', b', c') = (a, \min(b, b'), c'),$$

where $\min(b, b')$ is the smaller of b and b' in A. Also define an order on F by

$$(a, b, c) \leqslant (a', b', c') \text{ iff } a \leqslant a' \;\&\; b \leqslant b' \;\&\; c \leqslant c'.$$

F is an upper regular band. Finally define $f: A \to F$ so that for all $a \in A$, $f(a) = (a, a, a)$.

Theorem 3. *(F, f) is a free upper regular band over A.*

Proof. Let $g: A \to X$ be an arbitrary monotone function, where X is an upper regular band. Define a function $h: F \to X$ by

$$h(a, b, c) = g(a) \cdot g(b) \cdot g(c),$$

for every $(a, b, c) \in F$. Then h is the unique homomorphism preserving $hf = g$ by standard algebraic techniques.

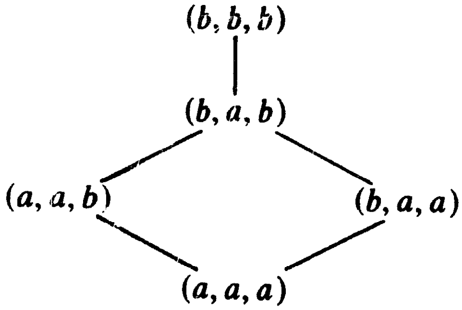
Theorem 4. *The order on any free upper regular band (F, f) is a lattice order, so that F is a lattice ordered semigroup.*

Proof. It suffices to consider (F, f) as constructed above. From the definition of $\leqslant$, least upper bounds can be shown to be

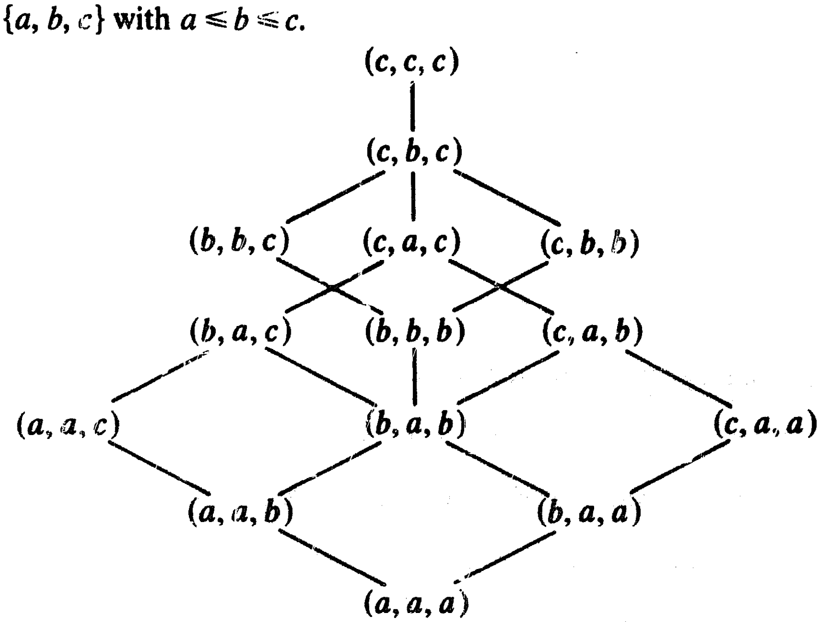
$$(a, b, c) \vee (a', b', c') = (\max(a, a'), \max(b, b'), \max(c, c')),$$

and similarly for greatest lower bounds.

Examples. (1) $A = \{a, b\}$ with $a \leqslant b$.



(2) $A = \{a, b, c\}$ with $a \leqslant b \leqslant c$.



The next theorem follows from the definition of F.

Theorem 5. *If A has n elements, for any integer n, then F has* $\sum_{i=1}^{n} i^2 = \sum(n^2) = n(n+1)(2n+1)/6$ *elements.*

4. Applications to Pω

The powerdomain of the integers is defined to be:

$$\mathrm{P}\omega = \{x \mid x \subseteq \{0, 1, 2, \ldots\}\}.$$

Scott [3] constructs a function, $\mathrm{fun}: \mathrm{P}\omega \to [\mathrm{P}\omega \to \mathrm{P}\omega]_c$, which associates a continuous function on Pω (i.e., an element of $[\mathrm{P}\omega \to \mathrm{P}\omega]_c$) with each element of Pω. A second function, $\mathrm{graph}: [\mathrm{P}\omega \to \mathrm{P}\omega]_c \to \mathrm{P}\omega$, associates an element of Pω with each continuous function on Pω. A binary operation can be defined in Pω by

$$a \circ b = \mathrm{graph}(\mathrm{fun}(a) \circ \mathrm{fun}(b)),$$

where $\mathrm{fun}(a) \circ \mathrm{fun}(b)$ is the functional composition of $\mathrm{fun}(a)$ and $\mathrm{fun}(b)$. Formally, $\mathrm{fun}(a) \circ \mathrm{fun}(b)$ is that function f, such that for all $x \in \mathrm{P}\omega$, $f(x) = \mathrm{fun}(a)[\mathrm{fun}(b)(x)]$.

With this operation and subset ordering, Pω forms an ordered semigroup. Of particular importance are those elements where $a \circ a = a$. Such an element is a retraction and its mapping a retract. The retracts in Pω form a fundamental notion of 'data type' [3, 5].

If $a \subseteq b$ and $a = a \circ b \circ a$, then a is said to be an *invariant subtype of b*. The constraint $a = a \circ b \circ a$ implies that some subset of the image of b is isomorphic with the image of a. Thus, considered as a data type, b 'includes' a. If a is an invariant subtype of b, we write $a \leqslant b$.

Theorem 6. $\leqslant$ *is a partial order on the retractions of* Pω.

Proof. Clearly $\leqslant$ is antisymmetric and reflexive. We must show that it is transitive. Let $a \leqslant b$ and $b \leqslant c$. Since $\subseteq$ is transitive, $a \subseteq c$. It remains to be shown that $a = a \circ c \circ a$. This follows from the fact that Pω is an ordered semigroup with order $\subseteq$ and operation $\circ$. Therefore,

$$a = a \circ a \circ a \subseteq a \circ c \circ a.$$

But also

$$\begin{aligned} a \circ c \circ a &= a \circ a \circ c \circ a \circ a \\ &\subseteq a \circ b \circ c \circ b \circ a \\ &= a \circ b \circ a \\ &= a \end{aligned}$$

Thus, $a = a \circ c \circ a$.

Now we can consider a (possibly infinite) chain $a_1 \leq a_2 \leq a_3 \leq \cdots$ of retractions. Call this chain A and note that for all $a, b \in A$, $a \leq b$ implies $a \circ a = a = a \circ b \circ a$. Following directly from Theorem 2 is

Theorem 7. *A chain of retractions in* Pω, *ordered by* $\leq$, *generates an upper regular band with the operation of* $\circ$.

We have completely characterized the data types that can by constructed from a chain of retractions, A. The free upper regular band over A gives the free system of retractions over the generators. Furthermore, if A has n elements, no more than $\sum (n^2)$ data types can be constructed.

The structure of a free upper regular band is particularly clear from Example 2, having $A = \{a, b, c\}$ and $a \leq b \leq c$. The illustrated lattice has three 'planes', corresponding to the three generators. The a-plane consists of the 9 triples with a at the second coordinate. The b-plane has 4 elements and the c-plane is just a point. A fourth generator would add another plane with $4^2 = 16$ elements, and so on. Each of these planes is an upper regular band, and, in fact, a free rectangular band. (In a rectangular band, for all x and y, $x = xyx$; see [1].)

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