Some Further Generalizations of the
Hyers–Ulam–Rassias Stability of Functional Equations

Wang Jian

Department of Mathematics, Nankai University, TianJin 300071, China, and
Department of Mathematics, Fujian Teachers University, Fuzhou 350007, China
E-mail: wjmath@eyou.com, wjmath@263.net

Submitted by Themistocles M. Rassias

Received May 11, 2000

In this paper we study the Hyers–Ulam–Rassias stability theory by considering
the cases where the approximate remainder \( \phi \) is defined by

\[
\begin{align*}
    f(x + y) - f(x) - f(y) &= \phi(x, y) \quad (\forall x, y \in G), \quad (1) \\
    f(x + y) - g(x) - h(y) &= \phi(x, y) \quad (\forall x, y \in G), \quad (2) \\
    2f((x + y)^{1/2}) - f(x) - f(y) &= \phi(x, y) \quad (\forall x, y \in G), \quad (3)
\end{align*}
\]

where \((G, *)\) is a certain kind of algebraic system, \(E\) is a real or complex
Hausdorff topological vector space, and \(f, g, h\) are mappings from \(G\) into \(E\). We
prove theorems for the Hyers–Ulam–Rassias stability of the above three kinds of
functional equations and obtain the corresponding error formulas.

1. INTRODUCTION

Throughout this paper, we denote by \(G\) a certain kind of algebraic
system and by \(E\) a real or complex Hausdorff topological vector space. By
\(N\) and \(Z\) we denote the sets of positive integers and of integers, respectively. \(e\) stands for the unit (which satisfies \(x * e = e * x = x\) for all \(x \in G\))

\(^1\)Supported by the National Science Foundation of China (19971046), the Doctoral Pro-
gramme Foundation of Institution of Higher Education, and the Foundation of Fujian
Educational Committee (JA99154).
of $G$ (if it exists), while it is $\theta$ instead of $e$ if $G$ is an abelian group. A mapping $T: G' \to E$ ($G'$ with the property that $G' \subseteq G$ such that $x * y \in G'$ for all $x, y \in G'$) is called additive on $G'$ if $T(x * y) = T(x) + T(y)$ for all $x, y \in G'$. Let $f, g, h$ be mappings from $G$ into $E$. We refer to the equations

$$f(x * y) - f(x) - f(y) = \theta \quad (\forall x, y \in G),$$

$$f(x * y) - g(x) - h(y) = \theta \quad (\forall x, y \in G),$$

$$2f((x * y)^{1/2}) - f(x) - f(y) = \theta \quad (\forall x, y \in G)$$

as a Cauchy equation, a Pexider equation, and a Jensen equation, respectively. $\phi$ in (1)–(3) is called the approximate remainder of the corresponding functional equation. The stability of these functional equations is called Hyers–Ulam–Rassias stability.

In 1940, S. M. Ulam [18] proposed the following problem for the stability of Cauchy equations:

Let $G$ be a group and let $E$ be a metric group with the metry $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h: G \to E$ satisfies the inequality $d(h(x), h(x^y)) < \delta$ for all $x, y \in G$, then there exists a homomorphism $H: G \to E$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G$?

In 1941, D. H. Hyers [3] answered this question in the affirmative when $G$ and $E$ are Banach spaces. In 1978, Rassias [12] generalized the result of Hyers. The result of the stability of Cauchy equations was further generalized by Rassias [13], Rassias and Šemrl [14], Gavruta [2], and Jung [7].

J. Rätz [11] considered the stability of Cauchy equations under the assumption that $G$ and $E$ are a power-associative groupoid and a sequentially complete topological vector space, respectively. The case of the stability of Pexider equations was generalized by J. Chmieliński and Tabor [1] and Kil-Woung Jun et al. [6]. The stability problems of Jensen equations can be found in [8–10]. For more theories of the Hyers–Ulam–Rassias stability of all kinds of functional equations, we refer the reader to [15–17].

In this paper, still using the direct method, we obtain some generalizations of the above theorems by considering the approximate remainders $\phi$.

### 2. STABILITY OF CAUCHY EQUATIONS

Throughout this section let $G$ be a power-associative groupoid and let $f$ satisfy (1). A set $G$ is called a power-associative groupoid if $G$ is a nonempty set with a binary relation $x * y \in G$ such that the left powers satisfy $x^{m+n} = x^m * x^n$ for all $m, n \in \mathbb{N}$ and all $x \in G$. Left powers are defined by $x^1 = x, x^{m+1} = x * x^m, m \in \mathbb{N}$.
A subset $B$ of $E$ is called ideally convex if for any bounded sequence 
$x_n \subseteq B$ and sequence $\{\lambda_n\} \subseteq (0, +\infty)$ with $\sum_{n=1}^{\infty} \lambda_n = 1$, the series 
$\sum_{n=1}^{\infty} \lambda_n x_n$ either converges to an element of $B$ or does not converge at all.

**Theorem 1.** If there exists $p \in \mathbb{N} \setminus \{1\}$ such that $f$ satisfies (1) and

$$f((x \ast y)^{p^n}) = f(x^{p^n} \ast y^{p^n}) \quad (\forall x, y \in G),$$

then

$$\lim_{n \to \infty} \frac{\phi(x^{p^n}, y^{p^n})}{p^n} = \theta \quad (\forall x, y \in G),$$

$$\lim_{n \to \infty} \frac{1}{p^n} \sum_{k=1}^{p^n-1} \phi(x^k, x) \quad \text{or} \quad \lim_{n \to \infty} \frac{1}{p^n} \sum_{k=1}^{p^n-1} \phi(x, x^k) \in E \quad (\forall x \in G),$$

(T.1.3)

if and only if the limit $T(x) = \lim_{n \to \infty} f(x^{p^n})/p^n$ exists for any $x \in G$, and $T$

is additive. In this case, the equality

$$T(x) = f(x) + \lim_{n \to \infty} \frac{1}{p^n} \sum_{k=1}^{p^n-1} \phi(x^k, x) = f(x) + \lim_{n \to \infty} \frac{1}{p^n} \sum_{k=1}^{p^n-1} \phi(x, x^k)$$

(T.1.4)

holds for all $x \in G$.

**Proof.** (Necessity). For every $k \in \mathbb{N}$, putting $y = x^k$ in (1), we obtain that

$$f(x^{k+1}) - f(x) - f(x^k) = \phi(x, x^k) \quad (\forall x \in G).$$

(T.1.5)

Adding the $n$ formulas together in (T.1.5) from 1 to $n$, we conclude that

$$f(x^{n+1}) - (n + 1) f(x) = \sum_{k=1}^{n} \phi(x, x^k) \quad (\forall x \in G).$$

(T.1.6)

Since $\phi(x, y) = \phi(y, x)$ for all $x, y \in G$, $\phi(x, x^k) = \phi(x^k, x)$ for all $x \in G$ and all $k \in \mathbb{N}$. This yields two limit formulas in (T.1.3) that converge or diverge simultaneously, and they are equal when they converge. Thus, by (T.1.3), we may assume that

$$\lim_{n \to \infty} \frac{1}{p^n} \sum_{k=1}^{p^n-1} \phi(x, x^k) = \lim_{n \to \infty} \frac{1}{p^n} \sum_{k=1}^{p^n-1} \phi(x^k, x) = \eta(x) \quad (\forall x \in G).$$
With \( p^n \) in place of \( n + 1 \) in (T.1.6), and dividing by \( p^n \), we have
\[
\frac{f(x^{p^n})}{p^n} - f(x) = \frac{1}{p^n} \sum_{k=1}^{p^{n-1}} \phi(x^k, x) \quad (\forall x \in G, n \in \mathbb{N}). \quad (T.1.7)
\]

Let \( n \to \infty \) in (T.1.7) to obtain
\[
T(x) = \lim_{n \to \infty} \frac{f(x^n)}{p^n} = f(x) + \eta(x) \quad (\forall x \in G).
\]

To show that \( T \) is additive, replace \( x \) with \( x^{p^n} \) and \( y \) with \( y^{p^n} \) in (1), then divide by \( p^n \) to obtain, by (T.1.1),
\[
\frac{f((x \ast y)^{p^n})}{p^n} - \frac{f(x^{p^n})}{p^n} - \frac{f(y^{p^n})}{p^n} = \frac{\phi(x^{p^n}, y^{p^n})}{p^n} \quad (\forall x, y \in G, n \in \mathbb{N}).
\]

Consequently, the left side of the above equality tends to \( \theta \) as \( n \to \infty \) by (T.1.2). Thus it follows that
\[
T(x \ast y) = T(x) + T(y) \quad (\forall x, y \in G).
\]

The proof of the sufficiency is straightforward. It leads to the asserted result. \( \blacksquare \)

**Corollary 1.** Let \( E \) be sequentially complete. Set
\[
B(x) = \text{co}\{(0) \cup \{\phi(x^i, x^j)\}_{i,j=1}^\infty\} \quad (\forall x \in G).
\]

If there exists \( p \in \mathbb{N} \setminus \{1\} \) such that \( f \) satisfies (1) and (T.1.1), \( \phi \) satisfies (T.1.2), and \( B(x) \) are bounded for all \( x \in G \), then there exists a unique additive mapping \( T: G \to E \) such that
\[
T(x) - f(x) \in B^p(x) \quad (\forall x \in G), \quad (C.1.1)
\]
where \( \text{co}(A) \) is the convex hull of \( A \), and \( \bar{A} \) denotes the sequential closure of \( A \).

In particular, if \( E \) is locally convex, then the boundedness of \( B(x) \) can be replaced by the boundedness of \( \{\phi(x^i, x^j)\}_{i,j=1}^\infty \).

**Proof.** First, we show that (T.1.3) holds. Let \( x \in G \). By the definition of \( B(x) \), \( B(x^n) \subseteq B(x) \) for all \( n \in \mathbb{N} \) and all \( x \in G \). Since \( B(x) \) is convex, which contains \( \theta \), \( B(x) \) is a starlike set (i.e., \( tB(x) \subseteq B(x) \) for any \( t \in (0, 1] \)). It is easy to see that \( \lim_{n \to \infty} \frac{1}{p^n} \sum_{k=1}^{p^n} \phi(x^k, x) \) exists for every \( x \in G \).
Indeed, from (T.1.7), we conclude that for any \( m \geq n(m, n \in \mathbb{N}) \) and any \( x \in G \),

\[
\frac{f(x^{p^n})}{p^m} - \frac{f(x^{p^n})}{p^n} = \frac{1}{p^n} \left( \frac{f(x^{p^n})}{p^m - n} - f(x^{p^n}) \right) = \frac{1}{p^n} \left( \frac{p^{m-n}}{p^m - n} \sum_{k=1}^{p^{m-n}-1} \phi(x^{p^n})^k \right) = \frac{1}{p^n} \left( \phi(x^{p^n})^k \right) \]

Evidently, \( \frac{1}{p^n} B(x^{p^n}) \subseteq \frac{1}{p^n} B(x) \).

By the hypothesis of boundedness of \( B(x) \), \( \{f(x^{p^n})/p^n\} \) is a Cauchy sequence of \( E \). Since \( E \) is sequentially complete, \( \{f(x^{p^n})/p^n\} \) converges to an element of \( E \). This implies that

\[
\lim_{n \to \infty} \frac{1}{p^n} \sum_{k=1}^{p^{n-1}} \phi(x^k, x) = \lim_{n \to \infty} \frac{1}{p^n} \sum_{k=1}^{p^{n-1}} \phi(x, x^k)
\]

exists for all \( x \in G \).

We shall show that \( \overline{B}(x) \) is ideally convex. In fact, first, we can claim that \( \overline{B}(x) \) is convex by [19, Theorem 4-2-12]. Furthermore, we note that any closed convex set is ideally convex. This implies that \( \overline{B}(x) \) is ideally convex. Since \( B(x) \) is a starlike convex set,

\[
\frac{1}{p^n} \sum_{k=1}^{p^{n-1}} \phi(x^k, x) = \frac{1}{p^n - 1} \sum_{k=1}^{p^{n-1}} \frac{p^n - 1}{p^n} \phi(x, x^k) \in B(x) \quad (\forall x \in G).
\]

Moreover,

\[
\lim_{n \to \infty} \frac{1}{p^n} \sum_{k=1}^{p^{n-1}} \phi(x^k, x) = \lim_{n \to \infty} \frac{1}{p^n - 1} \sum_{k=1}^{p^{n-1}} \frac{p^n - 1}{p^n} \phi(x, x^k) \in \overline{B}(x)
\]

(\forall x \in G).

Clearly,

\[
\lim_{n \to \infty} \frac{1}{p^n} \sum_{k=1}^{p^{n-1}} \phi(x^k, x) = \lim_{n \to \infty} \frac{1}{p^n} \sum_{k=1}^{p^{n-1}} \phi(x, x^k) \in \overline{B}(x) \quad (\forall x \in G).
\]

Hence (C.1.1) holds by Theorem 1.
Now we show the uniqueness of $T$. Suppose that $U: G \to E$ is another additive mapping that satisfies $U(x) - f(x) \in \overline{B}(x)$ for all $x \in G$. Then it follows from (C.1.1) that

\[
U(x) - T(x) = \frac{1}{n} \left( U(x^n) - T(x^n) \right)
\]
\[
= \frac{1}{n} \left( U(x^n) - f(x^n) + f(x^n) - T(x^n) \right)
\]
\[
\leq \frac{1}{n} \left( \overline{B}(x^n) - \overline{B}(x^n) \right)
\]
\[
\leq \frac{1}{n} \left( \overline{B}(x) - \overline{B}(x) \right) \quad (\forall x \in G).
\]

But $\overline{B}(x)$ is bounded by the boundedness of $B(x)$. Consequently, we conclude that $U(x) - T(x) \to \theta$ as $n \to \infty$. Thus it follows that $U(x) = T(x)$ for all $x \in G$.

Finally, note that if $E$ is also locally convex, then the boundedness of $B(x)$ as a sequel to $\{ \phi(x^i, x^j) \}_{i,j=1}^n$ is bounded. This concludes the proof. □

**Theorem 2.** If (T.1.1) holds for $p = 2$, then

\[
\lim_{n \to \infty} \frac{\phi(x^{2^n}, y^{2^n})}{2^n} = \theta \quad (\forall x, y \in G),
\]

(T.2.1)

\[
\sum_{k=1}^{\infty} \frac{\phi(x^{2k-1}, x^{2k-1})}{2^k} = \eta(x) \in E \quad (\forall x \in G),
\]

(T.2.2)

if the only if the limit $T(x) = \lim_{n \to \infty} f(x^{2^n})/2^n$ exists for any $x \in G$, and $T$ is additive. In this case, we have $T(x) = f(x) + \eta(x)$ for any $x \in G$.

**Proof.** We need only show the necessity. Put $y = x$ in (1) to obtain

\[
\frac{1}{2}f(x^2) - f(x) = \frac{1}{2} \phi(x, x) \quad (\forall x \in G).
\]

(T.2.3)

Assume that

\[
\frac{1}{2^n}f(x^{2^n}) - f(x) = \sum_{k=1}^{n} \frac{\phi(x^{2k-1}, x^{2k-1})}{2^k} \quad (\forall x \in G) \quad (T.2.4)
\]
WANG JIAN

holds for a certain \( n \in \mathbb{N} \). Then for all \( x \in G \),

\[
\frac{1}{2^{n+1}} f(x^{2^{n+1}}) - f(x) = \frac{1}{2} \left[ f \left( \frac{(x^2)^{2^n}}{2^n} - f(x^2) \right) \right] + \frac{1}{2} f(x^2) - f(x)
\]

\[
= \frac{1}{2} \sum_{k=1}^{n} \phi \left( \frac{(x^2)^{2^{k-1}}, (x^2)^{2^{k-1}}}{2^k} \right) + \frac{1}{2} \phi(x, x)
\]

\[
= \sum_{k=1}^{n+1} \phi \left( \frac{x^{2^{k-1}}, x^{2^{k-1}}}{2^k} \right).
\]

and so (T.2.4) holds for any \( n \in \mathbb{N} \) and any \( x \in G \) by induction.

In the same way as in the proof Theorem 1, we achieve the result. □

**COROLLARY 2.** Let \( E \) be sequentially complete. If (T.1.1) and (T.2.1) hold, and \( B(x) = \text{co}(\{0\} \cup \{\phi(x^{2^i}, x^{2^i})\}_{i=1}^{n}) \) are bounded for all \( x \in G \), then there exists a unique additive mapping \( T: G \rightarrow E \) such that

\[ T(x) - f(x) \in \overline{B}^0(x) \quad (\forall x \in G). \]  \hspace{1cm} (C.2.1)

When \( E \) is also locally convex, the boundedness of \( B(x) \) can be replaced by the boundedness of \( \{\phi(x^{2^i}, x^{2^i})\}_{i=1}^{n} \).

Remark 1. Corollary 1 is a generalization of the result of J. Rätz [11], and Theorem 2 is just a generalization of the result of Găvrută [2].

### 3. STABILITY OF PEXIDER EQUATIONS

Throughout this section let \( G \) be a power-associative groupoid with a unit \( e \). \( f \) satisfies (2) and

\[ f \left( (x \ast y)^{2^n} \right) = f(x^{2^n} \ast y^{2^n}) \quad (\forall x, y \in G, n \in \mathbb{N}). \] \hspace{1cm} (T.3.1)

Let \( \phi_{(g, h)} \) be the approximate reminder of the Pexider equation with respect to \( g, h: G \rightarrow E \).

**THEOREM 3.** \( \phi \) satisfies

\[
\lim_{n \to \infty} \frac{\phi(x^{2^n}, y^{2^n})}{2^n} = \theta \quad (\forall x, y \in G) \] \hspace{1cm} (T.3.2)
and
\[
\sum_{k=1}^{\infty} \frac{\phi(x^{2^k}, x^{2^{k-1}}) - \phi(x^{2^{k-1}}, e) - \phi(e, x^{2^{k-1}})}{2^k} = \eta(x) \in E
\]

\[(\forall x \in G) \quad \text{(T.3.3)}\]

if and only if the limit \( T(x) = \lim_{n \to \infty} f(x^{2^n})/2^n \) exists for any \( x \in G \), and \( T \) is additive. In addition, we have
\[
T(x) - f(x) = \eta(x) - g(e) - h(e) \quad (\forall x \in G), \quad \text{(T.3.4)}
\]
\[
T(x) - g(x) = \eta(x) - g(e) + \phi(x, e) \quad (\forall x \in G), \quad \text{(T.3.5)}
\]
\[
T(x) - h(x) = \eta(x) - h(e) + \phi(e, x) \quad (\forall x \in G). \quad \text{(T.3.6)}
\]

Moreover, \( T \) is independent of \( g, h \), whose \( \phi \) satisfies (T.3.2) and (T.3.3).

Proof. We have only to show the necessity. In (2), set \( y = x \) to obtain
\[
f(x^2) - g(x) - h(x) = \phi(x, x) \quad (\forall x \in G). \quad \text{(T.3.7)}
\]

Put \( y = e \) in (2) to obtain
\[
f(x) - g(x) = h(e) + \phi(x, e) \quad (\forall x \in G). \quad \text{(T.3.8)}
\]

Putting \( x = e \) with \( x \) in place of \( y \) in (2), we have
\[
f(x) - h(x) = g(e) + \phi(e, x) \quad (\forall x \in G). \quad \text{(T.3.9)}
\]

By induction, we can show that
\[
\frac{f(x^{2^n})}{2^n} - f(x) = \sum_{k=1}^{n} \frac{\psi(x^{2^{k-1}})}{2^k} \quad (\forall x \in G, \forall n \in \mathbb{N}), \quad \text{(T.3.10)}
\]

where \( \psi(x) = -g(e) - h(e) + \phi(x, x) \neq 0 \).

Indeed, for \( n = 1 \), it follows from (T.3.7)–(T.3.9) that for all \( x \in G \),
\[
\frac{1}{2} f(x^2) - f(x) = \frac{1}{2} \left[ f(x^2) - g(x) - h(x) \right] + \frac{1}{2} \left[ g(x) - f(x) \right] \\
+ \frac{1}{2} \left[ h(x) - f(x) \right] \\
= \frac{1}{2} \left[ \psi(x) \right].
\]
Assume that (T.3.10) holds for a certain $n$. Then
\[
\frac{1}{2n+1} f(x^{2^n+1}) - f(x) = \frac{1}{2} \left[ \frac{f((x^2)^{2^n})}{2^n} - f(x^2) \right] + \frac{1}{2} f(x^2) - f(x)
\]
\[
= \frac{1}{2} \sum_{k=1}^{n} \frac{\psi((x^2)^{2^{k-1}})}{2^k} + \frac{1}{2} \psi(x)
\]
\[
= \sum_{k=1}^{n} \frac{\psi(x^{2^k})}{2^k} + \frac{1}{2} \psi(x)
\]
\[
= \sum_{k=1}^{n+1} \frac{\psi(x^{2^{k-1}})}{2^k} \quad (\forall x \in G),
\]
and so (T.3.10) holds for $n + 1$.

Moreover, we conclude from (T.3.3) that
\[
\sum_{k=1}^{\infty} \frac{\psi(x^{2^{k-1}})}{2^k}
\]
\[
= \sum_{k=1}^{\infty} \frac{-g(e) - h(e) + \phi(x^{2^{k-1}}, x^{2^{k-1}}) - \phi(x^{2^{k-1}}, e) - \phi(e, x^{2^{k-1}})}{2^k}
\]
\[
= -g(e) - h(e) + \eta(x) \quad (\forall x \in G).
\]

Letting $n \to \infty$ in (T.3.10), for all $x \in G$ we achieve
\[
T(x) = \lim_{n \to \infty} \frac{f(x^{2^n})}{2^n} = f(x) + \sum_{k=1}^{\infty} \frac{\psi(x^{2^{k-1}})}{2^k}
\]
\[
= f(x) - g(e) - h(e) + \eta(x).
\]
This implies that (T.3.4) holds.

It easily follows from (T.3.8) and (T.3.9) that
\[
\lim_{n \to \infty} \frac{g(x^{2^n})}{2^n} = \lim_{n \to \infty} \frac{h(x^{2^n})}{2^n} = \lim_{n \to \infty} \frac{f(x^{2^n})}{2^n} = T(x) \quad (\forall x \in G)
\]
by (T.3.2). Replacing $x$ with $x^{2^n}$ and $y$ with $y^{2^n}$ in (2) and then dividing by $2^n$, by (T.3.1) we get for any $x \in G$ and any $n \in \mathbb{N}$
\[
\frac{f((x \ast y)^{2^n})}{2^n} - \frac{g(x^{2^n})}{2^n} - \frac{h(y^{2^n})}{2^n} = \frac{\phi(x^{2^n}, y^{2^n})}{2^n}.
\]
Letting $n \to \infty$, by (T.3.2) we conclude that $T(x * y) = T(x) + T(y)$ $(\forall x, y \in G)$. From (T.3.8), (T.3.9), and (T.3.4), we get that (T.3.5) and (T.3.6) hold.

Let $T$ and $T'$ be additive mappings with respect to $(g, h)$ and $(g', h')$, which satisfy (T.3.4)–(T.3.6), respectively. To show that $T = T'$, we observe that for any $x \in G$ and all $n \in \mathbb{N}$,

$$T(x) - T'(x) = \frac{1}{2^n} \left[ T(x^{2^k}) - T'(x^{2^k}) \right]$$

$$= \frac{1}{2^n} \left[ T(x^{2^k}) - f(x^{2^k}) + g(e) + h(e) - g(e) - h(e) + f(x^{2^k}) - T'(x^{2^k}) - g'(e) - h'(e) + g'(e) + h'(e) \right]$$

$$= \frac{1}{2^n} \left[ \eta_{g, h}(x^{2^k}) - \eta_{g', h'}(x^{2^k}) + C \right],$$

where $C = -g(e) - h(e) + g'(e) + h'(e)$.

We need only show $\lim_{n \to \infty} \eta(x^{2^k})/2^n = \theta$ for all $x \in G$. Indeed, by (T.3.3) we obtain

$$\lim_{n \to \infty} \frac{\eta(x^{2^k})}{2^n} = \lim_{n \to \infty} \frac{1}{2^n} \sum_{k=1}^{\infty} \phi \left( (x^{2^{k-1}}, x^{2^{k-1}}) - \phi \left( (x^{2^{k-1}}, e) - \phi \left( e, (x^{2^{k-1}})^{2^k} \right) \right) \right)$$

$$= \lim_{n \to \infty} \frac{1}{2^n} \sum_{k=1}^{\infty} \frac{\phi(x^{2^{k-1}}, x^{2^{k-1}}) - \phi(x^{2^{k-1}}, e) - \phi(e, x^{2^{k-1}})}{2^{k+2}}$$

$$= \lim_{n \to \infty} \frac{1}{2^n} \sum_{k=n+1}^{\infty} \frac{\phi(x^{2^{k-1}}, x^{2^{k-1}}) - \phi(x^{2^{k-1}}, e) - \phi(e, x^{2^{k-1}})}{2^{k}} = \theta \quad (\forall x \in G).$$

This completes the proof.

For abbreviation we set

$$A(x, y) = A_{(g, h)}(x, y) = \left\{ \phi_{(g, h)}(x^{2^j}, y^{2^j}) \right\}_{j=1}^{\infty} \quad (\forall x, y \in G),$$

$$B(x, y) = B_{(g, h)}(x, y) = \text{co} \left( \{0\} \cup A_{(g, h)}(x, y) \right) \quad (\forall x, y \in G).$$

$\phi_{(g, h)}$ is said to have property (C.B) if for any $x \in G B_{(g, h)}(x, x)$. $B_{(g, h)}(x, e)$ and $B_{(g, h)}(e, x)$ are bounded, and (T.3.2) holds. We denote $\mathcal{F} = \{(g, h), \phi_{(g, h)} \}$ has property (C.B).

If $E$ is also locally convex, the boundedness of $B_{(g, h)}(x, y)$ can be replaced by the boundedness of $A_{(g, h)}(x, y)$. 

HYERS–ULAM–RASSIAS STABILITY 415
COROLLARY 3. Let $E$ be sequentially complete. If $(g, h) \in \mathcal{F}$, then there exists a unique additive mapping $T: G \to E$ such that

\[
T(x) - f(x) + g(e) + h(e) \in \overline{B}^s(x, x) - \overline{B}^s(e, e) - \overline{B}^s(e, x)
(\forall x \in G), \quad (C.3.1)
\]

\[
T(x) - g(x) + g(e) \in \overline{B}^s(x, x) - \overline{B}^s(e, e) + B(x, e)
(\forall x \in G), \quad (C.3.2)
\]

\[
T(x) - h(x) + h(e) \in \overline{B}^s(x, x) - \overline{B}^s(e, e) + B(e, x)
(\forall x \in G). \quad (C.3.3)
\]

Proof. As in the proof Corollary 1, we see that $\overline{B}(x, x)$, $\overline{B}(x, e)$, and $\overline{B}(e, x)$ are ideally convex for any $x \in G$. It follows that for all $x \in G$,

\[
\eta(x) = \sum_{k=1}^{\infty} \frac{\phi(x^{2^k-1}, x^{2^k-1}) - \phi(x^{2^k-1}, e) - \phi(e, x^{2^k-1})}{2^k}
\]

\[
\in \overline{B}^s(x, x) - \overline{B}^s(x, e) - \overline{B}^s(e, x).
\]

By the definition of $B(x, y)$, we have $B(x^{2^n}, y^{2^n}) \subseteq B(x, y)(\forall x, y \in G, \forall n \in \mathbb{N})$.

In a manner analogous to that of Corollary 1, we can complete the proof.

COROLLARY 4. If $E$ is sequentially complete, then there exists a unique additive mapping $T: G \to E$ such that for any $x \in G$ and any $(g, h) \in \mathcal{F}$

\[
T(x) - f(x) + g(e) + h(e) \in B_{(g, h)}(x), \quad (C.4.1)
\]

\[
T(x) - g(x) + g(e) \in B_{(g, h)}(x) + B_{(g, h)}(x, e), \quad (C.4.2)
\]

\[
T(x) - h(x) + h(e) \in B_{(g, h)}(x) + B_{(g, h)}(e, x), \quad (C.4.3)
\]

where $B_{(g, h)}(x) \overset{\text{def}}{=} \overline{B}_{(g, h)}^s(x, x) - \overline{B}_{(g, h)}^s(x, e) - \overline{B}_{(g, h)}^s(e, x)$.

Proof. Corollary 3 asserts that there is a unique additive mapping $T$ that satisfies (C.4.1)–(C.4.3) for any $(g, h) \in \mathcal{F}$.

We shall show the uniqueness of $T$. Let $T$ and $T'$ be additive mappings with respect to $(g, h)$ and $(g', h')$ in $\mathcal{F}$, respectively. To show that $T = T'$,
we observe that for any \( x \in G \) and all \( n \in \mathbb{N} \),

\[
T(x) - T'(x) = \frac{1}{2^n} \left[ T(x^{2^n}) - T'(x^{2^n}) \right]
= \frac{1}{2^n} \left[ T(x^{2^n}) - f(x^{2^n}) + g(e) + h(e) - g(e) - h(e) \\
+ f(x^{2^n}) - T'(x^{2^n}) - g'(e) - h'(e) + g'(e) + h'(e) \right]
\leq \frac{1}{2^n} \left[ B_{(g, h)}(x^{2^n}) - B_{(g', h')}(x^{2^n}) + C \right]
\leq \frac{1}{2^n} \left( B_{(g, h)}(x) - B_{(g', h')}(x) + C \right),
\]

where \( C = -g(e) - h(e) + g'(e) + h'(e) \). Because \( B_{(g, h)}(x) \) and \( B_{(g', h')}(x) \) are bounded, letting \( n \to \infty \), we conclude that \( T(x) = T'(x) \) for all \( x \in G \).

**Corollary 5.** Let \( B \) be a bounded convex subset of \( E \) which contains \( \theta \), where \( E \) is sequentially complete. Then there is a unique additive mapping \( T: G \to E \) such that for any \( x \in G \) and any \( (g, h) \in \mathcal{F} \)

\[
T(x) - f(x) + g(e) + h(e) \in \overline{B}^s - 2\overline{B}^s, \tag{C.5.1}
T(x) - g(x) + h(e) \in 2\overline{B}^s - 2\overline{B}^s, \tag{C.5.2}
T(x) - h(x) + g(e) \in 2\overline{B}^s - 2\overline{B}^s, \tag{C.5.3}
\]

where \( \mathcal{F}_B = \{(g, h): \phi_{(g, h)}(x, y) \in B \text{ for all } x, y \in G\} \).

If \( E \) is also locally convex and \( A \) is a bounded set of \( E \), then (C.5.1)–(C.5.3) hold for all \( (g, h) \in \mathcal{F}_A \) and \( x \in G \), where \( B = \text{co}(\{\theta \} \cup A) \).

**Remark 2.** Theorem 3 shows that we have succeeded here in giving a generalization of the result of [6].

### 4. Stability of Jensen Equations

Throughout this section let \( (G, \ast) \) be a power-associative groupoid with a unit element \( e \) and inverse elements. By \( x^{-1} \) we denote the inverse element of \( x \) in \( G \) (which satisfies \( x^{-1} \ast x = x \ast x^{-1} = e \)). Let \( G_0 \) be a subset of \( G \) such that \( x^n \in G_0 \) for any \( n \in \mathbb{Z} \) and all \( x \in G_0 \). We assume
that if \( x \in G_0 \setminus \{e\} \), then \( x^2 \neq e \), \( x^3 \neq e \). If there exists \( z \in G_0 \) such that \( x \ast y = z^2 \) for \( x, y \in G_0 \), then we can state that symbolically \((x \ast y)^{1/2} = z\).

In particular, \( x \ast y = [(x \ast y)^2]^{1/2} \) for all \( x, y \in G_0 \) with \( x \ast y \in G_0 \).

Let \( f : G_0 \to E \) satisfy (3)

\[
 f\left(\left[(x \ast y)^{1/2}\right]^{3n}\right) = f\left(\left(x^{3n} \ast y^{3n}\right)^{1/2}\right) \tag{T.4.1}
\]

and

\[
 f\left(\left[(x \ast y)^2\right]^{1/2}\right) = f\left(\left(x^2 \ast y^2\right)^{1/2}\right) \tag{T.4.2}
\]

for any \( x, y \in G_0 \setminus \{e\} \) with \((x \ast y)^{1/2} \in G_0\).

**Theorem 4.** \( \phi : G_0 \setminus \{e\} \times G_0 \setminus \{e\} \to E \) satisfies

\[
 \lim_{n \to \infty} \frac{\phi(x^{3n}, y^{3n})}{3^n} = \theta \quad (\forall x, y \in G_0 \setminus \{e\}), \tag{T.4.3}
\]

\[
 \sum_{k=1}^{\infty} \frac{\phi(x^{3k-1}, x^{-3k-1}) - \phi(x^{-3k-1}, x^{3k})}{3^k} = \eta(x) \in E \quad (\forall x \in G_0 \setminus \{e\}), \tag{T.4.4}
\]

if and only if the limit \( T(x) = \lim_{n \to \infty} f(x^{3n})/3^n \) exists for all \( x \in G_0 \setminus \{e\} \), and \( T \) is additive in the sense that \( T(x \ast y) = T(x) + T(y) \) for all \( x, y \in G_0 \setminus \{e\} \). In addition, we have

\[
 T(x) - f(x) = \eta(x) - f(e) \quad (\forall x \in G_0 \setminus \{e\}). \tag{T.4.5}
\]

**Proof.** We shall show only the necessity. In (3), take \( y = x^{-1} \) to get

\[
 2f(e) - f(x) - f(x^{-1}) = \phi(x, x^{-1}) \quad (\forall x \in G_0 \setminus \{e\}). \tag{T.4.6}
\]

Replacing \( x \) with \( x^{-1} \) and \( y \) with \( x^3 \), we obtain

\[
 2f(x) - f(x^{-1}) - f(x^3) = \phi(x^{-1}, x^3) \quad (\forall x \in G_0 \setminus \{e\}). \tag{T.4.7}
\]

By (T.4.6) and (T.4.7), we conclude that for all \( x \in G_0 \setminus \{e\},

\[
 \frac{1}{2}f(x^3) - f(x) = \frac{1}{2}\left[f(x^3) + f(x^{-1}) - 2f(x) - f(x^{-1}) - f(x) + 2f(e) - 2f(e)\right]
\]

\[
 = \frac{1}{2}\left[-\phi(x^{-1}, x^3) + \phi(x, x^{-1}) - 2f(e)\right] = \frac{1}{2}u(x),
\]
where \( u(x) = \phi(x, x^{-1}) - \phi(x^{-1}, x^3) - 2f(e) \). If

\[
\frac{1}{3^n} f(x^{3^n}) - f(x) = \frac{1}{3^n} \sum_{k=1}^{n} u(x^{3^{k-1}}) \quad (\forall x \in G_0 \setminus \{e\}) \quad (T.4.8)
\]

holds for a certain \( n \), then for every \( x \in G_0 \setminus \{e\} \),

\[
\frac{1}{3^{n+1}} f(x^{3^{n+1}}) - f(x) = \frac{1}{3^n} \sum_{k=1}^{n} \frac{u(x^{3^{k-1}})}{3^k} + \frac{1}{3^k} f(x^3) - f(x)
\]

\[
= \frac{1}{3} \sum_{k=1}^{n} \frac{u(x^{3^k})}{3^{k+1}} + \frac{1}{3} u(x) = \sum_{k=1}^{n+1} \frac{u(x^{3^k})}{3^k},
\]

and so (T.4.8) holds for any \( n \in \mathbb{N} \) and any \( x \in G_0 \setminus \{e\} \) by induction.

From (T.4.4) we compute that for every \( x \in G_0 \setminus \{e\} \),

\[
\sum_{k=1}^{\infty} \frac{u(x^{3^k})}{3^k} = \sum_{k=1}^{\infty} \frac{\phi(x^{3^k}, x^{-3^{k-1}}) - \phi(x^{-3^{k-1}}, x^3)}{3^k} - \sum_{k=1}^{\infty} \frac{2f(e)}{3^k} = \eta(x) - f(e).
\]

This implies that \( \{f(x^{3^n})/3^n\} \) converges in \( E \). That is, \( T(x) = \lim_{n \to \infty} \{f(x^{3^n})/3^n\} \) exists for any \( x \in G_0 \setminus \{e\} \). Moreover, \( T(x) - f(x) = \eta(x) - f(e) \) holds for any \( x \in G_0 \setminus \{e\} \) by (T.4.8).

Now we show that \( T \) is additive in several steps.

**Step 1.** From (T.4.1) and (3), we have

\[
\frac{2f(\left[(x \ast y)^{1/2}\right]^{3^n})}{3^n} - \frac{f(x^{3^n})}{3^n} - \frac{f(y^{3^n})}{3^n} = \frac{\phi(x^{3^n}, y^{3^n})}{3^n}
\]

for any \( x, y \in G_0 \setminus \{e\} \) with \( (x \ast y)^{1/2} \in G_0 \).

Letting \( n \to \infty \), by (T.4.3), we obtain

\[
2T((x \ast y)^{1/2}) = T(x) + T(y) \quad (T.4.9)
\]

for any \( x, y \in G_0 \setminus \{e\} \) with \( (x \ast y)^{1/2} \in G_0 \).

**Step 2.** The definition of \( T \) implies that for any \( x \in G_0 \setminus \{e\} \) and any \( n \in \mathbb{N} \),

\[
T(x^{3^n}) = \lim_{m \to \infty} \frac{f((x^{3^n})^{3^m})}{3^m} = \lim_{m \to \infty} \frac{f(x^{3^{n+m}})}{3^{n+m}} = 3^n T(x).
\]
Step 3. For any $x \in G_0 \setminus \{e\}$, we obtain by (T.4.4), that

$$
\lim_{n \to \infty} \frac{\eta(x^{3^n})}{3^n} = \lim_{n \to \infty} \frac{1}{3^n} \sum_{k=1}^{\infty} \phi\left((x^{3^n})^{3^{k-1}}, (x^{3^n})^{-3^{k-1}}\right) - \phi\left((x^{3^n})^{-3^{k-1}}, (x^{3^n})^{3^k}\right) \\
= \lim_{n \to \infty} \sum_{k=1}^{\infty} \frac{\phi(x^{3^{n+k-1}}, x^{-3^{n+k-1}}) - \phi(x^{-3^{n+k-1}}, x^{3^{n+k}})}{3^{n+k}} \\
= \lim_{n \to \infty} \sum_{k=n+1}^{\infty} \frac{\phi(x^{3^{k-1}}, x^{-3^{k-1}}) - \phi(x^{-3^{k-1}}, x^{3^k})}{3^k} = \theta.
$$

Step 4. For any $x \in G_0 \setminus \{e\}$, we compute, by (T.4.1), (T.4.5), and (T.4.10), that

$$
2T(x^2) - 4T(x) = 2T(x^2) - T(x^3) - T(x) \\
= \frac{1}{3^n} \left[2T((x^2)^{3^n}) - T((x^3)^{3^n}) - T(x^{3^n})\right] \\
= \frac{1}{3^n} \left[2T((x^2)^{3^n}) - 2f((x^2)^{3^n}) \\
- T((x^3)^{3^n}) + f((x^3)^{3^n}) \\
- T(x^{3^n}) + f(x^{3^n}) + 2f\left(\left((x^3 \cdot x)^{1/2}\right)^{3^n}\right) \\
- f((x^3)^{3^n}) - f(x^{3^n})\right] \\
= \frac{1}{3^n} \left[2\eta((x^2)^{3^n}) - 2f(e) - \eta((x^3)^{3^n}) \\
+ f(e) - \eta(x^{3^n}) + f(e) + \phi(x^{3^n+1}, x^{3^n})\right] \\
= \frac{1}{3^n} \left[2\eta((x^2)^{3^n}) - \eta((x^3)^{3^n}) \\
- \eta(x^{3^n}) + \phi(x^{3^n+1}, x^{3^n})\right].
$$
Letting \( n \to \infty \), we claim, by Step 3 and (T.4.3), that
\[
T(x^2) = 2T(x) \quad (\forall x \in G_0 \setminus \{e\}). \tag{T.4.11}
\]

Finally, for any \( x, y \in G_0 \setminus \{e\} \) with \( x \star y \in G_0 \), we obtain, by (T.4.2), (T.4.9), and (T.4.11), that
\[
T(x \star y) = T\left(\left((x \star y)^2\right)^{1/2}\right) = T\left(\left((x^2 \star y^2)\right)^{1/2}\right) = 1/2(T(x^2) + T(y^2)) = T(x) + T(y).
\]

For abbreviation we denote
\[
B(x^{-1}, x) = \text{co}\left(\{\theta\} \cup \left\{\phi(x^{-3^i}, x^{3^i+1})\right\}_{i=1}^\infty\right) \quad (\forall x \in G_0 \setminus \{e\}),
\]
\[
B(x, x^{-1}) = \text{co}\left(\{\theta\} \cup \left\{\phi(x^{3^i}, x^{-3^i})\right\}_{i=1}^\infty\right) \quad (\forall x \in G_0 \setminus \{e\}).
\]

**COROLLARY 6.** Suppose that \( E \) is sequentially complete and (T.A.3) holds. If \( B(x^{-1}, x) \) and \( B(x, x^{-1}) \) are bounded for any \( x \in G_0 \setminus \{e\} \), then there exists a unique additive mapping \( T : G_0 \to E \) such that
\[
T(x) - f(x) + f(e) \in \overline{B}^i(x^{-1}, x) - \overline{B}^i(x, x^{-1}) \quad (\forall x \in G_0 \setminus \{e\}). \tag{C.6.1}
\]

If \( E \) is also locally convex, then the boundedness of \( \{\phi(x^{-3^i}, x^{3^i+1})\}_{i=1}^\infty \) and \( \{\phi(x^{3^i}, x^{-3^i})\}_{i=1}^\infty \) ensures the boundedness of \( B(x^{-1}, x) \) and \( B(x, x^{-1}) \), respectively.

**Proof.** Note that
\[
\sum_{k=1}^\infty \frac{\phi(x^{-3^k-1}, x^{3^k})}{3^k} = \sum_{k=1}^\infty \frac{(2/3)^k \phi(x^{-3^k-1}, x^{3^k})}{2^k} \quad (\forall x \in G_0 \setminus \{e\}),
\]
\[
\sum_{k=1}^\infty \frac{\phi(x^{-3^k-1}, x^{-3^k-1})}{3^k} = \sum_{k=1}^\infty \frac{(2/3)^k \phi(x^{3^k-1}, x^{-3^k-1})}{2^k} \quad (\forall x \in G_0 \setminus \{e\}).
\]

In the same way as in the proof Corollary 1, we may show the result.

**Remark 3.** If \( G \) is an abelian group and \( E \) is a Banach space, then Theorem 4 is a generalization of the result of [9].

Now we localize some conditions by the following theorem.
THEOREM 5. Let $G$ be a real topological vector space. If mapping $T: G \to E$ satisfies that there exists a $\theta$-neighborhood $U$ such that

$$T(x + y) = T(x) + T(y)$$

(T.5.1)

whenever $y - x \in U$ for all $x, y \in G$, then $T$ is additive.

Proof. We first may show by induction that

$$T\left(\frac{x}{2^n}\right) = \frac{1}{2^n}T(x) \quad (\forall x \in G \text{ and } \forall n \in \mathbb{N}).$$

(T.5.2)

Next let $x, y \in G$. We have only to consider the situation when $y - x \not\in U$ for all $x, y \in G$. By the absorptance of the neighborhood of zero, there exists $N \in \mathbb{N}$ such that $(y - x)/2^N \in U$. From (T.5.1) and (T.5.2) we get

$$T(x + y) = 2^NT\left(\frac{x + y}{2^N}\right) = 2^NT\left(\frac{x}{2^N}\right) + 2^NT\left(\frac{y}{2^N}\right) = T(x) + T(y)$$

for all $x, y \in G$ with $y - x \not\in U$.

Finally, for all $x, y \in G$, (T.5.1) holds (i.e., $T$ is additive).  \(\blacksquare\)

Remark 4. If $G$ is a real topological vector space, then, by Theorem 5, the conditions (T.1.1), (T.1.2), (T.2.1), (T.3.1), (T.3.2), and (T.4.1)–(T.4.3) hold as long as $y - x \in U$ for some neighborhood of zero $U$ in $G$ and any $x, y \in G$. In this case, the operation $*$ is a usual addition $+$.

ACKNOWLEDGMENTS

I express my deep gratitude to my Ph.D. advisor, Professor Ding Guanggui, for his advice. In addition, I express my appreciation to Professor Themistocles M. Rassias, whose research introduced me to this field of mathematics.

REFERENCES