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The Classification of the Simple Modular Lie Algebras
II. The Toral Structure

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Let L be a simple Lie algebra over an algebraically closed field of characteristic $p>7$ and T an optimal torus in some p-envelope L_p. We determine the action of T on the two-sections of L, which have been described in [St4]. We also give some new and noncomputational proofs to determine the conjugacy classes of the tori in $W(n; \mathbf{1})$ and of the Cartan subalgebras of $W(1; \mathbf{n})$.

I. PRELIMINARIES

This article is devoted to the investigation of the detailed structure of some Lie algebras occurring in the context of the classification of simple Lie algebras over algebraically closed fields of characteristic $p>7$. In particular, we deal with some Cartan-type Lie algebras of "small size." For the following notations and facts we refer to [SF, Section 4]. The graded Cartan-type Lie algebras can be described in terms of derivations of divided power algebras $A(m; \mathbf{n})$: $\mathbf{n}$ is an m-tuple of natural numbers, $\tau(\mathbf{n}) := (p^{n_1}-1, ..., p^{n_m}-1) \in \mathbb{N}^m$. For m-tuples a, b we write $a \leqslant b$ if $a_i \leqslant b_i$ for all i. $A(m; \mathbf{n})$ is the commutative and associative F-algebra of dimension $p^{\sum n_i}$ having a basis $(x^{(a)} \mid a \in \mathbb{N}^m,\ 0 \leqslant a \leqslant \tau(\mathbf{n}))$ and multiplication $x^{(a)}x^{(b)} := \binom{a+b}{a} x^{(a+b)}$. Let ε_i be the m-tuple with all 0's except in the ith slot which contains 1. Then $W(m; \mathbf{n}) := \{D \in \operatorname{Der} A(m; \mathbf{n}) \mid D(x^{(a)}) = \sum_{1 \leqslant i \leqslant m} x^{(a-\varepsilon_i)} D(x^{(\varepsilon_i)})\ \forall a \leqslant \tau(\mathbf{n})\}$ is the algebra of "special derivations" (Witt algebra). D_i denotes the "partial derivative" defined by the property $D_i(x^{(a)}) = x^{(a-\varepsilon_i)}$ for all (a). $A(m; \mathbf{n})$ carries a filtration by putting $A(m; \mathbf{n})_{(k)} := \operatorname{span}\{x^{(a)} \mid \sum a_i \geqslant k\}$. This inherits a filtration on $W(m; \mathbf{n})$ given by $W(m; \mathbf{n})_{(k)} := \operatorname{span}\{x^{(a)} D_i \mid \sum a_i \geqslant k+1\}$. $W(m; \mathbf{n})$ is restricted if and only if $\mathbf{n} = (1, ..., 1)$. For $\mathbf{n} = (1, ..., 1) =: \mathbb{1}$ one often prefers a different notation, since then $A(m; \mathbb{1}) \cong F[X_1, ..., X_m]/(X_1^p, ..., X_m^p)$ is isomorphic to the truncated polynomial ring in m generators $x_i := X_i + (X_1^p, ..., X_m^p)$, the

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isomorphism is given by $x^{(a)} \mapsto (a_1! \cdots a_m!)^{-1} x_1^{a_1} \cdots x_m^{a_m}$. In this case I prefer to write the monomials as $x_1^{a_1} \cdots x_m^{a_m}$ and the ordinary partial derivatives by ∂_i. Obviously, $W(m; \mathbb{1})$ is the full derivation algebra Der $A(m; \mathbb{1})$. For a description of the relevant subalgebras $S(m; \mathbf{n}; \Phi)$ (special algebras), $H(2r; \mathbf{n}; \Phi)$ (hamiltonian algebras), $K(2r+1; \mathbf{n}; \Phi)$ (contact algebras) we generally refer to [SF] or [BW2], and in particular to the parts of this article, where we deal with some of them in detail.

One of the very basic concepts in the theory of modular Lie algebras is the concept of a p-envelope. We recall the definition from [St2] or [SF, Section 2.5]:

DEFINITION. A triple $(G, [p], i)$ consisting of a restricted Lie algebra $(G, [p])$, and a Lie algebra homomorphism $i: L \to G$ is called a *p-envelope* of L if

(a) i is injective

(b) $i(L)_p = G$,

where $i(L)_p$ denotes the restricted subalgebra generated by $i(L)$ and $[p]$.

Using the concept of a p-envelope we introduced an invariant for any modular Lie algebra. Put $C(G) := \{x \in G \mid [x, G] = 0\}$ the center of G, and more generally the centralizer of a set S in G is denoted by $C_G(S) := \{x \in G \mid [x, S] = 0\}$.

DEFINITION [St3]. Let L be a Lie algebra and $(H, [p], i)$ a p-envelope of L. Suppose that G is a subalgebra of L and G_p is the restricted subalgebra of $(H, [p])$ generated by $i(G)$.

(1) $TR(G, L) := \max\{\dim T \mid T \text{ is a torus of } (G_p + C(H))/C(H)\}$ is called the *absolute toral rank of G in L*.

(2) $TR(L) := TR(L, L)$ is called the *absolute toral rank of L*.

Let L_p be a p-envelope of L containing L and $T \subset L_p$ a torus. Then L is an ideal of L_p and hence T acts on L. Decompose L into eigenspaces

$$L = \sum_{\alpha \in \Phi} L_\alpha(T).$$

In the classification theory of restricted algebras some distinguished tori ("optimal tori") play an important role. The corresponding definition in the general context is

DEFINITION (St4, p. 669]. Let L be a Lie algebra and T a torus in some p-envelope. A root $\alpha \neq 0$ (with respect to T) is called *proper*, if there is $i \neq 0$

such that $\alpha([L_{i\alpha}, L_{-i\alpha}])=0$. A torus is called *optimal*, if it has maximal absolute toral rank in L_p and if among all these tori the number of proper roots with respect to T is maximal.

If L is simple and $p>7$, then every root with respect to an optimal torus is proper [St4, (5.3)]. To explain this concept we consider tori in the restricted algebras $W(m; \mathbb{1})$. Demuskin ([D1]) showed that every maximal torus R is conjugate under an automorphism of $W(m; \mathbb{1})$ to one of the types $T_k := \sum_{1 \leqslant j \leqslant k} F(1+x_j)\,\partial_j \oplus \sum_{k+1 \leqslant j \leqslant m} Fx_j\,\partial_j$ ($k = 0, ..., m$). Here and in the following we want to treat several cases in common. We will use the expression t_i for either x_i or $1+x_i$. Observe that with this notation $t_i^p=0$ in the first case and $t_i^p=1$ in the second case. The root spaces $W(m; \mathbb{1})_\alpha$ for the torus $\sum Ft_i\,\partial_i$ are m-dimensional and given by $\sum_{1 \leqslant i \leqslant m} Ft^{a+\varepsilon_i}\,\partial_i$, where the exponent is meant to be an m-tuple of natural numbers between 0 and $p-1$ taken mod(p). The corresponding root α is given by $\alpha(t_j\,\partial_j)=a_j$. It is easy to check (in fact we shall do this for particular cases in subsequent sections) that T_0 is the only optimal torus among all these T_k. Thus $W(m; \mathbb{1})$ has exactly one conjugacy class of optimal tori.

For any subset $\mathscr{S} \subset \Phi = \Phi(T)$ of the set of roots with respect to some torus T let span $\mathscr{S}$ denote the $GF(p)$-vector space generated by $\mathscr{S}$. If

$$k := \dim_{GF(p)} \operatorname{span} \mathscr{S}$$

then

$$L(\mathscr{S}) := \sum_{\alpha \in \operatorname{span} \mathscr{S}} L_\alpha(T)$$

is called a *k-section with respect to T.*

It is known that a k-section with respect to a torus of maximal absolute toral rank has absolute toral rank $\leqslant k$ [St3, (2.6)]. As a consequence of this remark one can use results of R. L. Wilson ([W3]) to determine the structure of the one-sections (cf. [St3, (4.2)], [BOSt, (1.9)]).

THEOREM I.1. *Let L be a simple Lie algebra over an algebraically closed field of characteristic $p>7$. Let T be a torus of some p-envelope L_p of L of maximal absolute toral rank. Let $L(\alpha)$ be a one-section with respect to T. Then one of the following cases occurs*:

(1) *$L(\alpha)$ is solvable,*

(2) *$L(\alpha)/\operatorname{rad} L(\alpha) \cong \mathrm{sl}(2)$,*

(3) *$L(\alpha)/\operatorname{rad} L(\alpha) \cong W(1; \mathbb{1})$,*

(4) *$H(2; \mathbb{1})^{(2)} \subset L(\alpha)/\operatorname{rad} L(\alpha) \subset H(2; \mathbb{1})$.*

According to [BW2, (5.3.2)] we say that a root α is *solvable*, if $L(\alpha)$ is solvable, *classical* if $L(\alpha)/\text{rad}\, L(\alpha) \cong \text{sl}(2)$, *Witt* if $L(\alpha)/\text{rad}\, L(\alpha) \cong W(1; \mathbb{1})$, and *hamiltonian* if $H(2; \mathbb{1})^{(2)} \subset L(\alpha)/\text{rad}\, L(\alpha)$. A Lie algebra A is said to be *compositionally classical* ([BW2, (5.3.5)]), if every composition factor is abelian or classical.

THEOREM I.2 [St4, (1.10)]. *Let L be simple, T a torus of maximal absolute toral rank in some p-envelope L_p. Then every one-section of L and of L_p with respect to T contains a unique (not necessarily proper) subalgebra M of maximal dimension with codimension $\leqslant 2$, such that $M/\text{rad}\, M$ is 0 or isomorphic to* sl(2).

In coincidence with the notation of [BW2] and [BOSt] we write $Q(\mu) = Q(\mu, T)$, for this distinguished subalgebra of the one-section $L(\mu)$.

The aim of this note is threefold. At some point in the classification theory of the restricted simple Lie algebras knowledge on the behaviour of the one-sections within the two-sections is needed. It has turned out, that in the general classification theory the same is true in a much more serious meaning. It is essential to have very detailed information, how one-sections can occur in two-sections. Only then can one lift this information to results on the problem, how one-sections can sit in the whole algebra. The already known results are not satisfactory for our purposes. First, we therefore determine the action of a torus R on the semisimple quotient $K(\alpha, \beta) + R$ of a two-section $L(\alpha, \beta) + T$ of a simple algebra L with respect to an optimal torus T by a case-by-case analysis. In some cases we shall also give a more detailed description of the two-sections themselves (Sections IV, V, VII). In the course of pursuing this aim we secondly shall often step aside the prove some results in more generality than they are needed in the classification. We will, for instance, for arbitrary m give a new proof for the determination of the conjugacy classes of tori in the restricted Jacobson–Witt algebras $W(m; \mathbb{1})$ which needs no computations at all (Section IX) and we shall determine all Cartan subalgebras of $W(1; \mathbf{n})$ by noncomputational methods (Section V). This partially improves results of Demuskin [D1] and Brown [Br]. In doing all this, we thirdly hope that we will make the reader more acquainted with the concepts of an optimal torus and a section and thereby hopefully shall promote the understanding of the previous and forthcoming papers on the classification of simple modular Lie algebras.

I appreciate very much the very careful reading of this manuscript by the referee and her/his valuable comments.

II. Remarks on a Preceding Paper

Unfortunately it happened that some proofs in [St4] are incorrect. The results, however, are not seriously affected by this, as we shall see in this article.

Lemma 4.2 and in consequence Proposition 4.3 in [St4] are not valid. It is therefore not true (as claimed in Theorem 4.4) that we can exclude the algebras of type (d) from [BW2, (9.1.1)] in Theorem 4.4. In case (7) below $R(C_A(\pi(T)), A)$ denotes the maximal ideal of $C_A(\pi(T))$ which acts nilpotently on A. The revised proof of [St4, p. 667] yields immediately

Theorem [St4, (4.4) revisited]. *Let L be a simple Lie algebra over an algebraically closed field of characteristic $p>7$. Let T be a torus of some p-envelope L_p of L of maximal absolute toral rank. Consider the root space decomposition of L with respect to T,*

$$L=\sum_{\alpha\in\Phi} L_\alpha(T)=\sum_{\alpha\in\Phi} L_\alpha,$$

and the two-section

$$L_p(\alpha,\beta):=C_{L_p}(T)+\sum_{i,j\in GF(p)} L_{i\alpha+j\beta}.$$

Let π be the canonical homomorphism

$$\pi=\pi_{\alpha,\beta}\colon L_p(\alpha,\beta)\to L_p(\alpha,\beta)/\mathrm{rad}\, L_p(\alpha,\beta)=:A(\alpha,\beta)=:A.$$

Then one of the following cases can occur:

$TR(A)=0$:

(1) $A=0$;

$TR(A)=1$:

(2) $S\subset A\subset(\mathrm{Der}\, S)^{(1)}$,

$$\exists\gamma\in GF(p)\alpha+GF(p)\beta \text{ with } A=\pi(L_p(\gamma)),$$
$$S\in\{sl(2),\ W(1;\mathbb{1}),\ H(2;\mathbb{1})^{(2)}\};$$

$TR(A)=2$:

(3) $S_1\oplus S_2\subset A\subset(\mathrm{Der}\, S_1)^{(1)}\oplus(\mathrm{Der}\, S_2)^{(1)}$,

$$S_1, S_2\in\{\mathrm{sl}(2),\ W(1;\mathbb{1}),\ H(2;\mathbb{1})^{(2)}\};$$

(4) $H(2;\mathbb{1})^{(2)} \subset A \subset \operatorname{Der} H(2;\mathbb{1})^{(2)}$;

(5) $S \otimes A(n;\mathbb{1}) \subset A \subset \operatorname{Der}(S \otimes A(n;\mathbb{1}))$, $n \neq 0$,

$$\pi(T) \not\subset (\operatorname{Der} S) \otimes A(n;\mathbb{1}),$$

$$S \in \{\mathrm{sl}(2), W(1;\mathbb{1}), H(2;\mathbb{1})^{(2)}\};$$

(6) $S \subset A \subset \operatorname{Der} S$, $\pi(T) \subset S_p$,

$$S \in \{W(1;\mathbf{2}), H(2;(2,1))^{(2)}, H(2;\mathbb{1};\Phi(\tau))^{(1)}, H(2;\mathbb{1};\Delta)\};$$

(7) $A = S + R(C_A(\pi(T)), A)$,

$$S \in \{A_2, C_2, G_2, W(2;\mathbb{1}), S(3;\mathbb{1})^{(1)}, H(4;\mathbb{1})^{(2)}, K(3;\mathbb{1})\};$$

(8) (*added to the original version*)

$$S \otimes A(n;\mathbb{1}) \subset A \subset \operatorname{Der}(S \otimes A(n;\mathbb{1})),$$

$$\pi(T) \subset (S \otimes A(n;\mathbb{1}))_p, \qquad n > 0$$

$$S \cong H(2;\mathbb{1};\Phi(\tau))^{(1)},$$

$C_{S \otimes A(n;\mathbb{1})}(\pi(T))$ *acts nilpotently on* A.

This change of (4.4) has no impact on the proofs and the results of Section 5 in [St4], since there we only refer to [BW2, (9.1.1)]. In particular, Theorem 5.3 is valid and so all roots with respect to an optimal torus are proper. To salvage Theorem 6.3 we need the following:

LEMMA II.1. *Let L be simple, $p > 7$, and T an optimal torus in some p-envelope L_p. Put $H := C_L(T)$ and Φ the set of roots. If $\alpha \in \Phi$ is a solvable root, then every $x \in [L_\alpha, H]$ acts nilpotently on L.*

Proof. (1) If $\alpha(H) = 0$ then $L(\alpha)$ is nilpotent and hence triangulable.

(2) Suppose that $\alpha(H) \neq 0$ and that $x \in [L_\alpha, H]$ acts nonnilpotently. Then there is $\mu \in \Phi$ with $\mu(x) \neq 0$. $\Omega := \{\mu \in \Phi \mid \mu(x) \neq 0\}$ is a nonvoid set. The simplicity of L enforces $H = \sum_{\mu \in \Omega} [L_\mu, L_{-\mu}]$. Therefore there is $\beta \in \Omega$ with $\alpha([L_\beta, L_{-\beta}]) \neq 0$. We consider $A(\alpha, \beta)$ the semisimple quotient of $L_p(\alpha, \beta)$. [BW2, (10.2.1)] applies and yields that $A(\alpha, \beta)$ is one of the algebras listed in [BW2, (9.1.1)(e)–(h)]. Then [BW2, (11.2.1)] applies to prove that $\beta(x) = 0$, a contradiction. ∎

We are now in the position to prove a revised version of [St4, (6.3)]. We have to consider optimal tori instead of tori just of maximal absolute toral rank.

THEOREM II.2 [St4, (6.3) revisited]. *Let L be a simple Lie algebra over an algebraically closed field of characteristic $p > 7$. Let T be an optimal torus in some p-envelope L_p of L. Consider the root space decomposition of L with respect to T:*

$$L = \sum_{\alpha \in \Phi} L_\alpha(T) = \sum_{\alpha \in \Phi} L_\alpha.$$

Put

$$L(\alpha, \beta) := \sum_{i, j \in GF(p)} L_{i\alpha + j\beta}.$$

Let $I := I(\alpha, \beta)$ be the maximal solvable ideal of $L(\alpha, \beta) + T$ and consider

$$\sigma = \sigma_{\alpha,\beta}\colon L(\alpha, \beta) + T \to (L(\alpha, \beta) + T)/I(\alpha, \beta).$$

Put $K := K(\alpha, \beta) := \sigma(L(\alpha, \beta))$, $H := \sigma(C_L(T))$, and $R := \sigma(T)$. Then only one of the following cases can occur:

(1) $K = 0$;

(2) $S \subset K + R \subset (\mathrm{Der}\, S)^{(1)}$,

$$\exists \gamma \in GF(p)\alpha + GF(p)\beta \quad \text{with} \quad K = \sigma(L(\gamma)),$$
$$S \in \{\mathrm{sl}(2), W(1; \mathbb{1}), H(2; \mathbb{1})^{(2)}\};$$

(3) $S_1 \oplus S_2 \subset K + R \subset (\mathrm{Der}\, S_1)^{(1)} \oplus (\mathrm{Der}\, S_2)^{(1)}$,

$$S_1, S_2 \in \{\mathrm{sl}(2), W(1; \mathbb{1}), H(2; \mathbb{1})^{(2)}\};$$

(4) $H(2; \mathbb{1})^{(2)} \subset K + R \subset \mathrm{Der}(H(2; \mathbb{1})^{(2)})$, $R \not\subset H(2; \mathbb{1})$;

(5) $S \otimes A(n; \mathbb{1}) \subset K + R \subset \mathrm{Der}(S \otimes A(n; \mathbb{1}))$, $n = 1, 2$,

$$\gamma(C_L(T)) \neq 0 \qquad \forall \gamma \in GF(p)\alpha + GF(p)\beta - \{0\},$$
$$S \in \{\mathrm{sl}(2), W(1; \mathbb{1}), H(2; \mathbb{1})^{(2)}\};$$

(6) $S \otimes A(1; \mathbb{1}) \subset K + R \subset \mathrm{Der}(S \otimes A(1; \mathbb{1}))$,

$$(S \otimes A(1; \mathbb{1})) \cap (\mathrm{rad}\, K) = S \otimes xA(1; \mathbb{1}),$$
$$S \in \{\mathrm{sl}(2), W(1; \mathbb{1}), H(2; \mathbb{1})^{(2)}\};$$

(7) $S \subset K + R \subset \mathrm{Der}\, S$,

$$S \in \{W(1; \mathbf{2}), H(2; (2, 1))^{(2)}, H(2; \mathbb{1}; \Phi(\tau))^{(1)}, H(2; \mathbb{1}; \Delta)\};$$

(8) $K = S + C_K(R)$,

$$S \in \{A_2, C_2, G_2, W(2; \mathbb{1}), S(3; \mathbb{1})^{(1)}, H(4; \mathbb{1})^{(2)}, K(3; \mathbb{1})\}.$$

In all cases except (6), *K is semisimple, i.e.*, rad $L(\alpha, \beta) = I(\alpha, \beta) \cap L(\alpha, \beta)$.

Proof. (a) The first part of the proof remains unchanged (cf. [St4, p. 672]): Consider the canonical mapping

$$\pi: L_p(\alpha, \beta) \to A := L_p(\alpha, \beta)/\text{rad } L_p(\alpha, \beta).$$

There exists a homomorphism μ from $A^{(1)}$ into K with solvable kernel. Let C denote the socle of A,

$$C \subset A^{(1)} \subset \pi(L_p(\alpha, \beta) \subset A \subset \text{Der } C.$$

(b) We only have, in addition to the proof given in [St4], to consider the case that A is of type (8) of the revised version of [St4, (4.4)]

$$C = S \otimes A(n; \mathbb{1}) \subset A \subset \text{Der}(S \otimes A(n; \mathbb{1})), \pi(T) \subset (S \otimes A(n; \mathbb{1}))_p,$$

$$n > 0, \quad S \cong H(2; \mathbb{1}; \Phi(\tau))^{(1)}, \quad C_{S \otimes A(n; \mathbb{1})}\pi(T) \text{ acts nilpotently on } A.$$

Put $H' := C_L(T)$. As a consequence of the assumptions "$S \otimes A(n; \mathbb{1}) \subset A^{(1)} \subset \pi(L(\alpha, \beta))$" and "$\pi(T) \subset (S \otimes A(n; \mathbb{1})_p)$" we have

$$\pi(L(\alpha, \beta)) = S \otimes A(n; \mathbb{1}) + \pi(H'),$$

and since $C_{S \otimes A(n; \mathbb{1})}\pi(T)$ acts nilpotently, every one-section of $S \otimes A(n; \mathbb{1})$ with respect to $\pi(T)$ is solvable. Then every one-section of $\pi(L(\alpha, \beta))$ is solvable as well, proving that the same is true for $L(\alpha, \beta)$. According to (II.1) $[L_\mu, H']$ acts nilpotently on L for all $\mu \in GF(p)\alpha + GF(p)\beta$.

Let $J := \text{rad}\{S \otimes A(n; \mathbb{1})\}$ denote the maximal ideal of $S \otimes A(n; \mathbb{1})$. Observe that this is invariant under $\pi(T)$. $J' := J + [J, H']$ is also an ideal of $S \otimes A(n; \mathbb{1})$. The maximality of J enforces $J' = S \otimes A(n; \mathbb{1})$ or $J' = J$. In the first case we obtain

$$S \otimes A(n; \mathbb{1}) \subset J + \sum_{\mu \in GF(p)\alpha + GF(p)\beta} \pi([L_\mu, H']).$$

The Engel–Jacobson Theorem now implies that $S \otimes A(n; \mathbb{1})$ is nilpotent, a contradiction. Therefore the second case is true, showing that J is invariant under $\pi(H')$, and hence is a solvable ideal of $\pi(L(\alpha, \beta))$. Thus there is a homomorphism with solvable kernel, which maps $\pi(L(\alpha, \beta))$ into $\text{Der}(H(2; \mathbb{1}; \Phi(\tau))^{(1)})$. We are now in case (7) of Theorem 6.3. ▮

It has been claimed in [St4, (7.2)] that in case (5) of that theorem the ideal $S \otimes \sum x_i A(n; \mathbb{1})$ is invariant under R and that the arguments given by Block and Wilson in [BW2] to settle the corresponding case for restricted algebras remains valid in general. This last statement on the proof seems to be incorrect. We shall salvage the result by offering further arguments, which need more information on simple Cartan-type Lie algebras of toral rank two and their root space decompositions. We therefore shall postpone the treatment of this case to the end of this article.

This revision does not at all affect [St5], since all tori used there are supposed to be optimal.

III. Cases (2)–(4) of Theorem II.2

The algebras K mentioned in cases (2)–(4) of (II.2) are treated in this section. We determine the conjugacy classes of tori in general and optimal tori in particular, and the root space decompositions with respect to such tori. This is done for the sake of completeness and elucidation, although the results are essentially known (but partially not published yet).

(A) *Case* 2

We consider algebras

$$S \subset K + R \subset (\mathrm{Der}\, S)^{(1)} \text{ for } S \in \{\mathrm{sl}(2), W(1; \mathbb{1}), H(2; \mathbb{1})^{(2)}\},$$

R any torus.

$S = \mathrm{sl}(2)$. In this case $\mathrm{Der}\, S = S$ and therefore $K + R = S$ is true. There is a standard basis (e, h, f) of $\mathrm{sl}(2)$ with $R = Fh$. Since every nonzero root α has a multiple $i\alpha$ which is not a root, every torus R of $\mathrm{sl}(2)$ is optimal.

$S = W(1; \mathbb{1})$. We have $\mathrm{Der}\, S = S$ and therefore $K + R = S$. In the case under consideration R is conjugate (cf. the remark in Section I) to a torus T_0 or T_1 which we write without index $Ft\partial$. The root spaces with respect to the torus $Ft\partial$ are one-dimensional and of the form $Ft^a\partial$, the corresponding eigenvalue being $\alpha(t\partial) = a - 1$. As it was pointed out in Section I, R is not an optimal torus if $t = 1 + x$, while for $t = x$ it is optimal.

$S = H(2; \mathbb{1})^{(2)}$. Put $\tau := \tau(\mathbb{1}) = (p-1, p-1) \in \mathbb{N} \times \mathbb{N}$ and

$$D_H(x_1^{a_1} x_2^{a_2}) := a_1 x_1^{a_1 - 1} x_2^{a_2}\, \partial_2 - a_2 x_1^{a_1} x_2^{a_2 - 1}\, \partial_1,$$

as well as

$$H(2; \mathbb{1}) := \{f_1\, \partial_1 + f_2\, \partial_2 \mid f_1, f_2 \in A(2; \mathbb{1}), \partial_1(f_1) + \partial_2(f_2) = 0\}$$

$$H(2; \mathbb{1})^{(1)} = \mathrm{span}\{D_H(x_1^{a_1} x_2^{a_2}) \mid (0, 0) < (a_1, a_2) \leqslant \tau\}$$

$$H(2; \mathbb{1})^{(2)} = \mathrm{span}\{D_H(x_1^{a_1} x_2^{a_2}) \mid (0, 0) < (a_1, a_2) < \tau\}.$$

These graded Cartan-type Lie algebras are described in [SF, Section 4] (where we have used a slightly different notation). The following can easily be derived from the results presented there. Some of the material is also treated in [BW1] and [BW2].

$$H(2;\mathbb{1})^{(2)} = \bigoplus_{(0,0)<(a_1,a_2)<\tau} F(a_1 t_1^{a_1-1} t_2^{a_2}\,\partial_2 - a_2 t_1^{a_1} t_2^{a_2-1}\,\partial_1)$$

$$H(2;\mathbb{1})^{(1)} = H(2;\mathbb{1})^{(2)} \oplus F(t_1^{p-2} t_2^{p-1}\,\partial_2 - t_1^{p-1} t_2^{p-2}\,\partial_1)$$

$$H(2;\mathbb{1}) = H(2;\mathbb{1})^{(1)} \oplus F t_1^{p-1}\,\partial_2 \oplus F t_2^{p-1}\,\partial_1$$

$$\operatorname{Der}(H(2;\mathbb{1})^{(2)}) \cong H(2;\mathbb{1}) \oplus F(t_1\partial_1 + t_2\partial_2) \subset W(2;\mathbb{1}) \text{ (if } p>3).$$

We shall always consider $\operatorname{Der}(H(2;\mathbb{1})^{(2)})$ as a subalgebra of $W(2;\mathbb{1})$.

PROPOSITION III.1.

(1) $\dim \operatorname{Der}(H(2;\mathbb{1})^{(2)})/H(2;\mathbb{1})^{(2)} = 4$, $\dim \operatorname{Der}(H(2;\mathbb{1})^{(2)}) = p^2+2$.

(2) *Every one of the four algebras listed above is a restricted subalgebra of* $W(2;\mathbb{1})$.

(3) (a) $\operatorname{Der}(H(2;\mathbb{1})^{(2)})/H(2;\mathbb{1})$ *is a one-dimensional torus.*

(b) $H(2;\mathbb{1})/H(2;\mathbb{1})^{(2)}$ *is p-nilpotent with* $H(2;\mathbb{1})^p \subset H(2;\mathbb{1})^{(2)}$.

(c) *Every maximal torus of* $\operatorname{Der}(H(2;\mathbb{1})^{(2)})$ *has dimension* 2. *Its intersection with* $H(2;\mathbb{1})^{(2)}$ *is one-dimensional.*

(4) $TR(\operatorname{Der}(H(2;\mathbb{1})^{(2)}))=2$, $TR(H(2;\mathbb{1}))=1$, $TR(H(2;\mathbb{1})^{(2)})=1$.

(5) *Every maximal torus of* $\operatorname{Der}(H(2;\mathbb{1})^{(2)})$ *is conjugate under an automorphism of* $W(2;\mathbb{1})$, *which stabilizes* $H(2;\mathbb{1})$, $H(2;\mathbb{1})^{(2)}$, *and* $\operatorname{Der}(H(2;\mathbb{1})^{(2)})$, *to one of*

$$T_0 = Fx_1\,\partial_1 \oplus Fx_2\,\partial_2,$$

$$T_1 = F(1+x_1)\,\partial_1 \oplus Fx_2\,\partial_2,$$

$$T_2 = F(1+x_1)\,\partial_1 \oplus F(1+x_2)\,\partial_2.$$

Proof. (1) is an obvious consequence of the above mentioned facts on the algebras under consideration.

(2) Let L denote one of the algebras $H(2;\mathbb{1})^{(2)}$, $H(2;\mathbb{1})^{(1)}$, $H(2;\mathbb{1})$, $\operatorname{Der}(H(2;\mathbb{1})^{(2)})$. A direct computation shows that L is invariant under the torus $T_0 := Fx_1\,\partial_1 + Fx_2\,\partial_2$ of $W(2;\mathbb{1})$. Thus L decomposes into root spaces with respect to T_0

$$L = \bigoplus_\alpha L \cap W(2;\mathbb{1})_\alpha(T_0).$$

[SF, (4.2.7) and the succeeding remark] yields

$$(L \cap W(2;\mathbb{1})_\alpha)^p = 0 \qquad \text{if} \quad \alpha \neq 0,$$

while $L \cap W(2;\mathbb{1})_0(T_0) = L \cap T_0$ (which is either $F(x_1\,\partial_1 - x_2\,\partial_2)$ or T_0) is in any case closed under pth powers. Thus [SF, (2.2.3)] implies that L is a restricted subalgebra of $W(2;\mathbb{1})$.

(3) (a) $H(2;\mathbb{1})$ is a restricted ideal of $\mathrm{Der}(H(2;\mathbb{1})^{(2)})$ and

$$\mathrm{Der}(H(2;\mathbb{1})^{(2)}) = H(2;\mathbb{1}) \oplus F(x_1\,\partial_1 + x_2\,\partial_2).$$

$x_1\,\partial_1 + x_2\,\partial_2$ is a toral element.

(b) $G := Fx_1^{p-1}\,\partial_2 \oplus Fx_2^{p-1}\,\partial_1 \oplus F(x_1^{p-2}x_2^{p-1}\,\partial_2 - x_1^{p-1}x_2^{p-2}\,\partial_1)$ is a three-dimensional restricted subalgebra (isomorphic to the Heisenberg algebra) with $G^p = 0$. The equation $H(2;\mathbb{1}) = H(2;\mathbb{1})^{(2)} \oplus G$ yields $H(2;\mathbb{1})^p \subset H(2;\mathbb{1})^{(2)}$.

(c) Let T be a maximal torus of $\mathrm{Der}(H(2;\mathbb{1})^{(2)})$ and consider the restricted homomorphism

$$\varphi\colon \mathrm{Der}(H(2;\mathbb{1})^{(2)}) \to \mathrm{Der}(H(2;\mathbb{1})^{(2)})/H(2;\mathbb{1})^{(2)} =: L.$$

$\varphi(T)$ is a maximal torus of L [SF, (2.4.5)]. Since $L = \varphi(F(x_1\,\partial_1 + x_2\,\partial_2)) \oplus \varphi(G)$ and $\varphi(G)$ is a p-nilpotent ideal of L, we obtain $\dim \varphi(T) = 1$. Assume that $T \cap \ker \varphi = 0$. Then $\dim T = 1$, $T \cap H(2;\mathbb{1})^{(2)} = 0$, and $H(2;\mathbb{1})^{(2)}$ decomposes with respect to T into eigenspaces

$$H(2;\mathbb{1})^{(2)} = \bigoplus_{i \in GF(p)} H(2;\mathbb{1})^{(2)}_{i\alpha}(T).$$

Take $u \in H(2;\mathbb{1})^{(2)}_\alpha(T)$. Then some p-power $t = u^{p^r}$ is semisimple and (as $H(2;\mathbb{1})^{(2)}$ is restricted) is contained in $H(2;\mathbb{1})^{(2)}_0(T)$, i.e., in $T \cap H(2;\mathbb{1})^{(2)} = 0$. Applying the Engel–Jacobson Theorem we obtain that $H(2;\mathbb{1})^{(2)}$ is nilpotent, a contradiction. Hence $T \cap \ker \varphi \neq 0$. As $\mathrm{Der}(H(2;\mathbb{1})^{(2)}) \subset W(2;\mathbb{1})$, and $TR(W(2;\mathbb{1})) = 2$ we obtain $\dim T = 2$ and $\dim T \cap H(2;\mathbb{1})^{(2)} = 1$.

(4) We conclude from (3) that $TR(\mathrm{Der}(H(2;\mathbb{1})^{(2)})) = 2$. Let T denote a maximal torus of $H(2;\mathbb{1})^{(2)}$. If $\dim T \geqslant 2$, then T is a maximal torus of $\mathrm{Der}(H(2;\mathbb{1})^{(2)})$ and hence $T = T \cap H(2;\mathbb{1})^{(2)}$ is one-dimensional according to (3). This contradiction shows that $\dim T = 1$ and $TR(H(2;\mathbb{1})^{(2)}) = 1$. [St3, Chap. 2] then yields

$$\begin{aligned} 1 = TR(H(2;\mathbb{1})^{(2)}) &\leqslant TR(H(2;\mathbb{1})) \\ &\leqslant TR(H(2;\mathbb{1})^{(2)}) + TR(H(2;\mathbb{1})/H(2;\mathbb{1})^{(2)}) = 1. \end{aligned}$$

(5) Every automorphism of $H(2;\mathbb{1})$ defines in a natural way an automorphism of $H(2;\mathbb{1})^{(2)}$ and of $\mathrm{Der}(H(2;\mathbb{1})^{(2)})$. Therefore (5) results from [BW1, (1.18.4)]. ▮

We now return to case (2) of II.2. Since every torus R of $(\mathrm{Der}(H(2;\mathbb{1})^{(2)}))^{(1)} \subset H(2;\mathbb{1})$ is contained in a maximal torus of $\mathrm{Der}(H(2;\mathbb{1})^{(2)})$, the preceding theorem applies to the case under consideration. R is conjugate to a subtorus of T_0, T_1, or T_2. The definition of $H(2;\mathbb{1})$ shows, that R is conjugate to either $F(x_1\,\partial_1 - x_2\,\partial_2)$ or $F((1+x_1)\,\partial_1 - x_2\,\partial_2)$ or $F((1+x_1)\,\partial_1 - (1+x_2)\,\partial_2)$. In the last case we substitute x_2 by $x_1 + x_2$, and show by this that R is even conjugate to one of

$$F(t_1\,\partial_1 - x_2\,\partial_2), \qquad t_1 \text{ being } (1+x_1) \text{ or } x_1.$$

THEOREM III.2. *Assume that $S \subset K \subset (\mathrm{Der}\, S)^{(1)}$ with $S = H(2;\mathbb{1})^{(2)}$ and let R be a maximal torus. Then*

(1) $H(2;\mathbb{1})^{(2)} \subset K + R \subset H(2;\mathbb{1})$.

(2) *If R is optimal, then there is $\psi \in \mathrm{Aut}\, W(2;\mathbb{1})$, which stabilizes $H(2;\mathbb{1})^{(2)}$ and $H(2;\mathbb{1})$, such that $R = \psi(F(x_1\,\partial_1 - x_2\,\partial_2))$.*

(3) *If $\alpha \neq 0$ then*

$$K_\alpha = \psi\left\{\sum_{a-b\equiv\alpha(t_1\partial_1 - x_2\partial_2)} F(bt_1^a x_2^{b-1}\,\partial_1 - at_1^{a-1}x_2^b\,\partial_2)\right\}.$$

If $\alpha = 0$ then

$$K_\alpha = K \cap \psi\left\{\sum_{0\leqslant a\leqslant p-1} F(t_1^a x_2^{a-1}\,\partial_1 - t_1^{a-1}x_2^a\,\partial_2) + Ft_1^{p-1}\,\partial_2 + Fx_1^{p-1}\,\partial_1\right\}.$$

Proof. (1) $(\mathrm{Der}(H(2;\mathbb{1})^{(2)}))^{(1)} \subset H(2;\mathbb{1})^{(1)} + [H(2;\mathbb{1}), x_1\,\partial_1 + x_2\,\partial_2] = H(2;\mathbb{1})$. (2), (3) The above deliberations show that there is $\psi \in \mathrm{Aut}\, W(2;\mathbb{1})$, which stabilizes $H(2;\mathbb{1})^{(2)}$ and $H(2;\mathbb{1})$, such that $R = \psi(F(t_1\,\partial_1 - x_2\,\partial_2))$ with $t_1 = 1 + x_1$ or $= x_1$. The following spaces are contained in the root spaces $H(2;\mathbb{1})_\alpha$ with respect to $F(t_1\,\partial_1 - x_2\,\partial_2)$:

$$\sum_{a-b\equiv\alpha(t_1\partial_1 - x_2\partial_2)} F(bt_1^a x_2^{b-1}\,\partial_1 - at_1^{a-1}x_2^b\,\partial_2) \qquad \alpha \neq 0$$

$$\sum_{1\leqslant a\leqslant p-1} F(t_1^a x_2^{a-1}\,\partial_1 - t_1^{a-1}x_2^a\,\partial_2) + Ft_1^{p-1}\,\partial_2 + Fx_2^{p-1}\,\partial_1 \qquad \alpha = 0.$$

These spaces span $H(2;\mathbb{1})$, so they constitute the full root spaces. Since for $\alpha \neq 0$ we have $H(2;\mathbb{1})_\alpha \subset K_\alpha$, this gives (3). Take $t_1 = (1+x_1)$. Choose $\alpha \neq 0$

and $a \in \mathbb{N}$ with $a-1 \equiv \alpha(t_1\partial_1 - x_2\partial_2)$, $0 < a-1 < p$. Then $[K_\alpha, K_{-\alpha}] = [H(2;\mathbb{1})_\alpha, H(2;\mathbb{1})_{-\alpha}]$ contains

$$\begin{aligned}&[t_1^a\partial_1 - at_1^{a-1}x_2\partial_2, t_1^{p-a+2}\partial_1 - (2-a)\, t_1^{p-a+1}x_2\partial_2]\\&\quad = 2(1-a)(t_1^{p+1}\partial_1 - t_1^p x_2\partial_2) = 2(1-a)(t_1\partial_1 - x_2\partial_2) \neq 0.\end{aligned}$$

This means, that no root is proper and this is not an optimal torus. ∎

(B) *Case* 3

We consider algebras $K+R$ with

$$S = S_1 \oplus S_2, \qquad S_i \in \{\mathrm{sl}(2), W(1;\mathbb{1}), H(2;\mathbb{1})^{(2)}\},$$
$$S \subset K+R \subset (\mathrm{Der}\, S_1)^{(1)} + (\mathrm{Der}\, S_2)^{(1)}, \qquad R \text{ any torus.}$$

Observe that

$$2 \geqslant TR(K+R) \geqslant TR(S) = TR(S_1) + TR(S_2) = 2.$$

We identify $K+R$ with $\mathrm{ad}_S(K+R) \subset \mathrm{Der}\, S$. Let $K_p + R$, S_p denote the p-envelopes in $\mathrm{Der}\, S$ of $K+R$ and S, respectively. Note that S is restricted. Therefore $S_p = S$ is a restricted subalgebra of $\mathrm{Der}\, S$ under this identification. The above equation in combination with [St4, (1.3.(5))] proves that $R \subset S + C(K_p + R) \subset \mathrm{Der}\, S$. The above identification, however, implies $C(K_p + R) \subset C_{K_p+R}(S) = 0$ and therefore $R \subset S$. Put $H_i := C_{S_i}(R)$ and $H := C_K(R)$. We obtain that $R \subset H \cap S = H_1 \oplus H_2$ and $K+R = S+H$. The root space decomposition with respect to R is given by the action of H_i on S_i. We have now reduced this case to the former one.

(C) *Case* 4

Suppose that $K+R$ satisfies

$$H(2;\mathbb{1})^{(2)} \subset K+R \subset \mathrm{Der}(H(2;\mathbb{1})^{(2)}),\ R \not\subset H(2;\mathbb{1}),$$
$$R \text{ any two-dimensional torus.}$$

According to Proposition III.1, R is conjugate to one of T_0, T_1, T_2. We are interested in the one-sections with respect to R.

PROPOSITION III.3. *Let T denote one of the tori T_0, T_1, T_2 and write $t_i := 1 + x_i$ or $t_i := x_i$. Let α be a root with respect to T. Choose a, b such that* $0 \leqslant a, b \leqslant p-1$, $a \equiv \alpha(t_1\partial_1) + 1$, $b \equiv \alpha(t_2\partial_2) + 1 \mod(p)$. *Then* $(\mathrm{Der}(H(2;\mathbb{1})^{(2)}))_\alpha$ *coincides with*

$$\begin{array}{ll} F(at_1^{a-1}t_2^b\,\partial_2 - bt_1^a t_2^{b-1}\,\partial_1) & \text{if } (a,b)\neq(0,0),(1,1) \\ T & \text{if } a=b=1 \\ Ft_1^{p-1}\,\partial_2 \oplus Ft_2^{p-1}\,\partial_1 & \text{if } a=b=0. \end{array}$$

Proof. The displayed vector spaces are in the appropriate root spaces. Since they span a (p^2+2)-dimensional vector space (which is the dimension of $\mathrm{Der}(H(2;\mathbb{1})^{(2)})$), equality holds. ▌

THEOREM III.4. *With the notation as in* (III.3) *the following is true*:

(1) *If* $\alpha(t_1\,\partial_1)\neq\alpha(t_2\,\partial_2)$ *choose* a_i, b_i $(1\leqslant i\leqslant p-1)$, *such that* $0\leqslant a_i$, $b_i\leqslant p-1$, $a_i\equiv i\alpha(t_1\,\partial_1)+1$, $b_i\equiv i\alpha(t_2\,\partial_2)+1 \bmod(p)$. *Then*

$$(\mathrm{Der}(H(2;\mathbb{1})^{(2)}))(\alpha)=T\oplus\sum_{i\neq 0}F(a_i t_1^{a_i-1}t_2^{b_i}\,\partial_2-b_i t_1^{a_i}t_2^{b_i-1}\,\partial_1).$$

(2) *If* $\alpha(t_1\,\partial_1)=\alpha(t_2\,\partial_2)\neq 0$ *then*

$$(\mathrm{Der}(H(2;\mathbb{1})^{(2)}))(\alpha)=T\oplus Ft_1^{p-1}\,\partial_2\oplus Ft_2^{p-1}\,\partial_1 \oplus\sum_{2\leqslant a\leqslant p-1}F(t_1^{a-1}t_2^a\,\partial_2-t_1^a t_2^{a-1}\,\partial_1).$$

Proof. (1) If $\alpha(t_1\,\partial_1)\neq\alpha(t_2\,\partial_2)$ then $a_i\neq b_i$ $\forall i\neq 0$. The root space $(\mathrm{Der}\,H(2;\mathbb{1})^{(2)}))_{i\alpha}$ $(i\neq 0)$ is given by the first case of Proposition III.3.

(2) If $\alpha(t_1\,\partial_1)=\alpha(t_2\,\partial_2)$ then $a_i=b_i$ $\forall i\neq 0$. Since $\alpha\neq 0$ we have $\alpha(t_1\,\partial_1)\neq 0$ and hence $\{a_i\mid i\neq 0\}=\{a\in GF(p)\mid a\neq 1\}$. Proposition III.3 yields the result. ▌

THEOREM III.5. *Assume that* $H(2;\mathbb{1})^{(2)}\subset K\subset\mathrm{Der}(H(2;\mathbb{1})^{(2)})$. *Let* R *be an optimal torus in* $K+R$ *with* $R\not\subset H(2;\mathbb{1})$ *and* α *be a nonzero root.*

(1) *There exists* $\psi\in\mathrm{Aut}\,W(2;\mathbb{1})$, *which stabilizes* $H(2;\mathbb{1})^{(2)}$ *and* $H(2;\mathbb{1})$, *such that* $R=\psi(T_0)$.

(2) *If* $\alpha(\psi(x_1\,\partial_1))=0$, *then*

$$K(\alpha)=\psi\left\{\sum_{0\leqslant i\leqslant p-1}F(x_2^i\,\partial_2-ix_1x_2^{i-1}\,\partial_1)\right\}+R\cap K.$$

$\{K(\alpha)+R\}/R\cap(\ker\alpha)$ *is isomorphic to* $W(1;\mathbb{1})$, *the isomorphism is induced by*

$$\psi(x_2^i\,\partial_2-ix_1x_2^{i-1}\,\partial_1)\mapsto x_2^i\,\partial_2.$$

(3) *If* $\alpha(\psi(x_1\partial_1+x_2\partial_2))=0$, *then* $K(\alpha)\subset W(2;\mathbb{1})_{(0)}$ *and*

$$\{K(\alpha)+R\}/\{K(\alpha)\cap W(2;\mathbb{1})_{(1)}+R\cap(\ker\alpha)\}\cong \mathrm{sl}(2).$$

(4) *If* $\alpha(\psi(x_1\partial_1))\neq 0$, $\alpha(\psi(x_2\partial_2))\neq 0$, $\alpha(\psi(x_1\partial_1+x_2\partial_2))\neq 0$, *then* $K(\alpha)\subset R+W(2;\mathbb{1})_{(1)}$ *and* $K(\alpha)$ *is solvable.*

Proof. The optimality of R in conjunction with the assumption "$R\not\subset H(2;\mathbb{1})$" ensures that $\dim R=2$. (III.1.(5)) yields that there is $\psi\in \operatorname{Aut} W(2;\mathbb{1})$, which stabilizes $H(2;\mathbb{1})^{(2)}$ and $H(2;\mathbb{1})$, such that $\psi^{-1}(R)=:T$ is one of the tori described in (III.1). Consider the root space decomposition with respect to T.

(1) Take any nonzero root α with $\alpha(t_1\partial_1)=0$. Then (as $\alpha\neq 0$ and hence $\alpha(t_2\partial_2)\neq 0$) we are in case (1) of the preceding theorem. Adjust α, such that $\alpha(t_2\partial_2)=1$. Observe that $a_i=1\ \forall i$ and $\{b_i\mid i\neq 0\}=\{b\in GF(p)\mid b\neq 1\}$. Note that according to (III.3) $[K_{(i-1)\alpha},K_{(j-1)\alpha}]$ contains

$$\begin{aligned}&[t_2^i\partial_2-it_1t_2^{i-1}\partial_1,\, t_2^j-jt_1t_2^{j-1}\partial_1]\\ &\quad=(j-i)(t_2^{i+j-1}\partial_2-(i+j-1)\,t_1t_2^{i+j-2}\partial_1)\end{aligned}$$

for $0\leqslant i,j\leqslant p-1$. Since R and $T=\psi^{-1}(R)$ are optimal and hence every root is proper, this implies that $t_2^p=0$, i.e., $t_2=x_2$. Similarly, $t_1=x_1$ and $T=T_0$. This proves (1).

(2) Case (1) of (III.4) yields that

$$K(\alpha)=\psi\left\{\sum_{0\leqslant j\leqslant p-1,\, j\neq 1} F(x_2^j\partial_2-jx_1x_2^{j-1}\partial_1)\right\}+R\cap K.$$

As $R\cap K\subset\psi(F(x_2\partial_2-x_1\partial_1))+C(K(\alpha))$ a short computation proves (2).

(3) We are in case (1) of the preceding theorem with

$$a_i+b_i\equiv i(\alpha(x_1\partial_1)+\alpha(x_2\partial_2))+2\equiv 2 \bmod (p).$$

In particular, $a_i+b_i\geqslant 2$ and therefore the one-section is contained in the zero part of the filtration: $\psi^{-1}(K(\alpha))\subset W(2;\mathbb{1})_{(0)}$. As ψ is an automorphism of $W(2;\mathbb{1})$ it stabilizes $W(2;\mathbb{1})_{(0)}$. Hence $K(\alpha)\subset W(2;\mathbb{1})_{(0)}$.

Since $\alpha(x_1\partial_1)\neq 0$, $\alpha(x_2\partial_2)\neq 0$ there are i, j such that $a_i=2$, $b_i=0$ and $a_j=0$, $b_j=2$. The corresponding monomials lie in $K(\alpha)$. Put $J:=K(\alpha)\cap W(2;\mathbb{1})_{(1)}$. Then J is a solvable ideal of $K(\alpha)+R$ and

$$K(\alpha)+R=J+R+\psi\{Fx_2\partial_1+Fx_1\partial_2\}.$$

Then $\{K(\alpha)+R\}/\{J+R\cap(\ker\alpha)\}\cong\mathrm{sl}(2)$.

(4) If $a_j + b_j \geqslant 3$ then the corresponding root space is contained in $W(2; \mathbb{1})_{(1)}$. If $a_j = 1$ for some $j \neq 0$ then $\alpha(x_1 \partial_1) = 0$, which is not this case. Hence $a_i, b_i \neq 1$ for $i \neq 0$. If there is some $j \neq 0$ such that $a_j = 0 = b_j$, then the corresponding root space is contained in $W(2; \mathbb{1})_{(1)}$ (III.3). If there is some $j \neq 0$ such that $a_j = 0$, $b_j = 2$ then $j\alpha(x_1 \partial_1) = -1$, $j\alpha(x_2 \partial_2) = 1$ and hence $\alpha(x_1 \partial_1 + x_2 \partial_2) = 0$, which cannot happen in this case. ∎

The theorem yields that there is only one conjugacy class of optimal tori. For such a torus we have determined all one-sections modulo the radical: two of them are Witt algebras, one of them is isomorphic to sl(2), all other are 0.

Let $-\alpha_1$, $-\alpha_2$ denote the nonclassical roots, which stick out of $K \cap W(2; \mathbb{1})_{(0)}$. Then

$$-\delta_{ij} = [x_i \partial_i, \partial_j] = -\alpha_j(x_i \partial_i)$$

and

$$(\alpha_1 - \alpha_2)(x_1 \partial_1 + x_2 \partial_2) = 0.$$

Thus $K(\alpha_1 - \alpha_2)$ is the classical nonsolvable one-section. The root spaces of $K \cap W(2; \mathbb{1})_{(0)}$ sticking out of $K \cap W(2; \mathbb{1})_{(1)}$ are represented by $x_1 \partial_2, x_2 \partial_1, R \cap K$. The corresponding roots are $\alpha_1 - \alpha_2, \alpha_2 - \alpha_1, 0$.

IV. Case (6) of Theorem II.2

We start with some general observations.

Lemma IV.1. *Let G be a Lie algebra and put $K := G \otimes A(1; \mathbb{1})$. Consider a torus $T \subset \operatorname{Der} K$, put $T_0 := T \cap (\operatorname{Der} G) \otimes A(1; \mathbb{1})$. Assume that every T-invariant ideal I of K decomposes $I = I_0 \otimes A(1; \mathbb{1})$, $I_0 \subset G$.*

(1) *There is $\omega \in \operatorname{Der} K$, such that*

(a) $T = T \cap ((\operatorname{Der} G) \otimes A(1; \mathbb{1})) \oplus F(\mathrm{id} \otimes \partial + \omega)$

(b) $\omega(G \otimes A(1; \mathbb{1})_{(j)}) \subset G \otimes A(1; \mathbb{1})_{(j)}$ *for all j.*

(2) *T_0 acts on G and $G \otimes F$ via the isomorphism $G \cong G \otimes F \cong G \otimes A(1; \mathbb{1})/G \otimes A(1; \mathbb{1})_{(1)}$. Decompose $G \otimes F = \sum_\mu (G \otimes F)_\mu$ with respect to T_0,*

$$(G \otimes F)_\mu := \{u \in G \otimes F \mid t(u) - \mu(t)u \in G \otimes A(1; \mathbb{1})_{(1)}\ \forall t \in T_0\}.$$

For any $e \in (G \otimes F)_\mu$ *and each* $j \in GF(p)$ *there exists* $u \in G \otimes A(1; \mathbb{1})$ *with the properties*

(1) $u \equiv e \mod G \otimes A(1; \mathbb{1})_{(1)}$

(2) $t(u) = \mu(t) u \quad \forall t \in T_0$

(3) $(\mathrm{id} \otimes \partial + \omega)(u) = ju.$

Proof. (1) According to a well known result of R. E. Block,

$$\mathrm{Der}(G \otimes A(1; \mathbb{1})) = (\mathrm{Der}\, G) \otimes A(1; \mathbb{1}) + F\, \mathrm{id} \otimes W(1; \mathbb{1}),$$

and thus there is a restricted homomorphism

$$\psi\colon \mathrm{Der}(G \otimes A(1; \mathbb{1})) \to \mathrm{Der}(G \otimes A(1; \mathbb{1}))/(\mathrm{Der}\, G) \otimes A(1, \mathbb{1}) \cong W(1; \mathbb{1}).$$

Put $U := \psi^{-1}(W(1; \mathbb{1})_{(0)})$. Note that for all j, $G \otimes A(1; \mathbb{1})_{(j)}$ is a U-invariant subspace. If $T \subset U$, then $G \otimes A(1; \mathbb{1})_{(1)}$ is a T-invariant ideal, which has trivial intersection with $G \otimes F$. This contradiction shows that $T \not\subset U$. $\psi(T)$ is a torus of $W(1; \mathbb{1})$ and thus conjugate to $F(1+x)\partial$. Then $T = T \cap \{(\mathrm{Der}\, G) \otimes A(1; \mathbb{1})\} \oplus FD$, where D is mapped onto $(1+x)\partial$. Hence $D \equiv \mathrm{id} \otimes \partial \mod U$.

(2) We will first construct inductively elements $u_0, ..., u_{p-1}$ with the properties

(a) $u_k \equiv e \mod G \otimes A(1; \mathbb{1})_{(1)}$

(b) $D(u_k) - ju_k \in G \otimes A(1; \mathbb{1})_{(k)}$.

For $k = 0$ we put $u_0 = e$ and observe, that (b) is true since $A(1; \mathbb{1})_{(0)} = A(1; \mathbb{1})$. Assume that we have constructed u_k for some $k \leqslant p-1$. Write

$$D(u_k) - ju_k \equiv w \otimes x^k \mod G \otimes A(1; \mathbb{1})_{(k+1)}.$$

If $k < p-1$ put

$$u_{k+1} := u_k - (k+1)^{-1} w \otimes x^{k+1}.$$

Then

$$u_{k+1} \equiv u_k \equiv e \mod G \otimes A(1; \mathbb{1})_{(1)}$$

and

$$D(u_{k+1}) = D(u_k) - D((k+1)^{-1} w \otimes x^{k+1}) \equiv ju_k \equiv ju_{k+1} \mod G \otimes A(1; \mathbb{1})_{(k+1)}.$$

Consider the case $k = p-1$. Since D is toral and $j \in GF(p)$, we have

$(D-j\,\mathrm{id})^p = D^p - j^p\,\mathrm{id} = D - j\,\mathrm{id}$. Hence computing $\bmod G\otimes A(1;\mathbb{1})_{(1)}$ we obtain

$$\begin{aligned} w\otimes x^{p-1} &= (D-j\,\mathrm{id})(u_{p-1}) = (D-j\,\mathrm{id})^p(u_{p-1}) \\ &= (D-j\,\mathrm{id})^{p-1}(w\otimes x^{p-1}) \\ &\equiv \mathrm{id}\otimes\partial^{p-1}(w\otimes x^{p-1}) = (p-1)!\, w\otimes 1. \end{aligned}$$

Thus $w=0$. We end up with an element u_{p-1} satisfying (1) and (3). Consider $V := \{v\in G\otimes A(1;\mathbb{1}) \mid D(v) = jv\}$. V is invariant under T_0 and decomposes into a direct sum of eigenspaces with respect to T_0. Then $V + G\otimes A(1;\mathbb{1})_{(1)}$ is also invariant. As $u_{p-1}\in V$, we have according to (a) $e\in V + G\otimes A(1;\mathbb{1})_{(1)}$. Moreover, e is mapped onto an eigenvector in $\{V + G\otimes A(1;\mathbb{1})_{(1)}\}/\{G\otimes A(1;\mathbb{1})_{(1)}\} \cong V$ with respect to T_0. Therefore there is some eigenvector $u\in V$, such that $u\equiv e \bmod G\otimes A(1;\mathbb{1})_{(1)}$. This is the desired element. ▮

COROLLARY IV.2. *Let G be a Lie algebra and put $K := G\otimes A(m;\mathbb{1})$. Consider a torus $T\subset \mathrm{Der}\, K$. Assume that every T-invariant ideal I of K decomposes $I = I_0\otimes A(m;\mathbb{1})$, $I_0\subset G$. Decompose $T = T_0\oplus T_1$ into a direct sum of subtori with $T_0 := T\cap\{(\mathrm{Der}\, G)\otimes A(m;\mathbb{1})\}$. T_0 acts on G via the isomorphism $G\cong G\otimes A(m;\mathbb{1})/G\otimes A(m;\mathbb{1})_{(1)}$. Then*:

(1) $m = \dim T_1$.

(2) *For any root $\alpha\in T_0^*$ on G and any $\beta\in T_1^*$ there is a root $\alpha\oplus\beta$ of T on K. The homomorphism $\pi\colon K\to K/G\otimes A(1;\mathbb{1})_{(1)}\cong G$ maps $K_{\alpha\oplus\beta}$ bijectively onto G_α.*

Proof. Put $G' := G\otimes A(m-1;\mathbb{1})$. Then $K\cong G'\otimes A(1;\mathbb{1})$ and the assumption if (IV.1) is fulfilled. We now proceed by induction on m. (1) and the first part of (2) are direct consequences of (IV.1). In order to prove bijectivity we first observe that according to (IV.1) $\pi(K_{\alpha\oplus\beta}) = G_\alpha$. Put $I := (\ker\pi)\cap K_{\alpha\oplus\beta}$. I is T-invariant since every $t\in T$ acts as $(\alpha(t) + \beta(t))\,\mathrm{id}$ on $K_{\alpha\oplus\beta}$. Then $\sum_{n\geqslant 0} (\mathrm{ad}\, K)^n(I)\subset\ker\pi$ is a T-invariant ideal of K. By assumption this has to vanish. Hence $\pi\,|\,K_{\alpha\oplus\beta}$ is injective. ▮

The following are applications of (IV.2).

THEOREM IV.3. *Let G be a Lie algebra and put $K := G\otimes A(m;\mathbb{1})$. Consider a torus $T\subset\mathrm{Der}\, K$. Assume that every T-invariant ideal I of K decomposes*

$$I = I_0\otimes A(m;\mathbb{1}), \qquad I_0\subset G.$$

(1) *There are roots* $\beta_1, ..., \beta_t$, *such that* $G \cong K(\beta_1, ..., \beta_t)$.

(2) $K \cong K(\beta_1, ..., \beta_t) \otimes A(m; \mathbb{1})$.

(3) *T is conjugate under an automorphism ψ of K to some torus* $\{T' \otimes F\} \oplus \{\bigoplus_{1 \leqslant i \leqslant m} \mathrm{id} \otimes F(1+x_i)\,\partial_i\}$, *where T' is a torus of* Der G *and* $T' \otimes F = \psi \circ (T \cap \{(\mathrm{Der}\, G) \otimes A(1; \mathbb{1})\}) \circ \psi^{-1}$.

Proof. (1) Put according (IV.2) $T = T_0 \oplus T_1$, $T_0 = T \cap \{(\mathrm{Der}\, G) \otimes A(m; \mathbb{1})\}$, and $t := \dim T_0$. Choose $\alpha_1, ..., \alpha_t \in T_0^*$ roots of T_0 on G which span the root lattice of T_0 on G and let β_i denote the extensions of α_i by putting $\beta_i(T_1) = 0$. (IV.2) shows that the homomorphism

$$\pi\colon K \to G \otimes A(m; \mathbb{1}) / G \otimes A(m; \mathbb{1})_{(1)} \cong G$$

maps $K(\beta_1, ..., \beta_t)$ bijectively onto G. Thus $\pi|_{K(\beta_1, ..., \beta_t)}$ is an isomorphism of Lie algebras.

(2) Use multiindex notation for monomials x^a of $A(m; \mathbb{1})$. Apply the Lie algebra homomorphism $\varphi\colon K \otimes A(m; \mathbb{1}) \to K$, $\varphi(u \otimes x^a \otimes x^b) = u \otimes x^{a+b}$ to the subalgebra $K(\beta_1, ..., \beta_t) \otimes A(m; \mathbb{1})$. For $g \in G$ there are (cf. (1)) $g_i \in G$, $f_i \in A(m; \mathbb{1})_{(1)}$ such that $g' := g \otimes 1 + \sum g_i \otimes f_i \in K(\beta_1, ..., \beta_t)$. Then $\varphi(g' \otimes x^a) = g \otimes x^a + \sum g_i \otimes f_i x^a$, where $f_i x^a$ is of higher order than x^a. Inductively we see, that $\varphi(K(\beta_1, ..., \beta_t) \otimes A(m; \mathbb{1})) = K$. Checking dimensions (with use of (1)) we obtain that this restriction of φ is an isomorphism of Lie algebras, which maps $K(\beta_1, ..., \beta_t) \otimes F$ onto $K(\beta_1, ..., \beta_t)$.

(3) φ induces an isomorphism

$$\psi'\colon \mathrm{Der}(K(\beta_1, ..., \beta_t) \otimes A(m; \mathbb{1})) \xrightarrow{\sim} \mathrm{Der}\, K.$$

Put $\tilde{T}_i := (\psi')^{-1}(T_i)$ $(i = 0, 1)$. The property $\beta_i(T_1) = 0$ implies that $\tilde{T}_1 \subset \mathrm{id} \otimes W(m; \mathbb{1})$. Demuskin's result then shows that $\tilde{T}_1$ is conjugate to some torus $\sum_{1 \leqslant i \leqslant s} \mathrm{id} \otimes F(1+x_i)\,\partial_i + \sum_{s+1 \leqslant i \leqslant m} \mathrm{id} \otimes F x_i\,\partial_i$. Our assumption on the ideal structure of K enforces $s = m$.

Since $[\tilde{T}_1, \tilde{T}_0] = 0$ the preceding result yields that $\tilde{T}_0 \subset (\mathrm{Der}\, K(\beta_1, ..., \beta_t)) \otimes F$, i.e., $\tilde{T}_0 = \tilde{T}' \otimes F$, $\tilde{T}'$ a torus in $\mathrm{Der}\, K(\beta_1, ..., \beta_t)$. Next the isomorphism obtained in (1) $\pi'\colon K(\beta_1, ..., \beta_t) \to G$ extends to $\pi' \otimes \mathrm{id}\colon K(\beta_1, ..., \beta_t) \otimes A(m; \mathbb{1}) \to G \otimes A(m; \mathbb{1}) = K$. Putting isomorphisms together we obtain that T is conjugate to some $\{T' \otimes F\} \oplus \{\bigoplus_{1 \leqslant i \leqslant m} \mathrm{id} \otimes F(1+x_i)\,\partial_i\}$. ∎

THEOREM IV.4. *With the assumptions and notations of* (II.2) *let* $K + R$ *be as in case* (6).

(1) *There is a root* $\beta \neq 0$ *with* $\beta(H)=0$. $K(\beta)$ *is nilpotent.*

(2) *There is a root* α *with* $\alpha(H)\neq 0$.

(3) *For every root* α *with* $\alpha(H)\neq 0$, *and any* $j \in GF(p)$, $\alpha + j\beta$ *is a root.*

(4) *For every root* α *with* $\alpha(H)\neq 0$, $\bigcap_{n \geqslant 0} K(\alpha)^{(n)} \cong S$.

(5) $K(\beta)$ *is the only solvable one-section.*

Proof. We identify K with $\mathrm{ad}_{S \otimes A(1;\mathbb{1})} K$. As in the case under consideration S is restricted, $S \otimes A(1, \mathbb{1})$ is considered as a restricted subalgebra of $\mathrm{Der}(S \otimes A(1;\mathbb{1}))$. Then $R \cap (S \otimes A(1;\mathbb{1}))$ is one-dimensional. Put $R_0 := R \cap (S \otimes A(1;\mathbb{1})) =: Fr_1$ with some toral element r_1.

$S \otimes A(1;\mathbb{1})_{(1)}$ is the unique maximal ideal of $S \otimes A(1;\mathbb{1})$. It is invariant under K, but not under $K+R$. Then $S \otimes A(1;\mathbb{1})$ is R-simple and therefore meets the assumptions of (IV.3).

Since $S \otimes A(1;\mathbb{1})_{(1)}$ is not $(K+R)$-invariant but K-invariant, we conclude that $R \not\subset K_p$ and therefore $R \cap K_p = R_0$ as well.

(1) Let β be a root with $\beta(r_1)=0$. If $\beta(H)\neq 0$, then $TR(H, K)=2$. As a consequence of [St4, (1.3.(5))], $R \subset K_p + C(K_p + R) = K_p$. This would contradict the above observations. Thus $\beta(H)=0$ and $K(\beta)$ is a nilpotent one-section.

(2) Since $r_1 \in H$ there is a root α with $\alpha(H)\neq 0$.

(3) As $\alpha(H)\neq 0$ we have $\alpha(Fr_1)=\alpha(R \cap H)\neq 0$ and $K(\alpha) \subset [r_1, K(\alpha)] + H \subset S \otimes A(1;\mathbb{1}) + H$. (IV.2) proves that $\alpha + j\beta$ is a root for each j.

(4) Let $\alpha \in (Fr_1)^*$ be a nonzero root of Fr_1 on S and extend α to a root α' of R on K. (IV.2) yields that $\bigcap K(\alpha')^{(n)} = K(\alpha') \cap S \otimes A(1;\ \mathbb{1}) \cong S$.

(5) Consider any root $\mu = i\alpha + j\beta$, $i \neq 0$. Then $\mu(H)\neq 0$. Apply (4). ∎

THEOREM IV.5. *With the assumptions and notations of case* (6) *of* (II.2) *the following are true*:

(1) R *is conjugate under an automorphism of* $S \otimes A(1;\mathbb{1})$ *to* $\{R_0 \otimes F\} \oplus \{\mathrm{id} \otimes F(1+x)\partial\}$, R_0 *an optimal torus of* S.

(2) $K \subset (\mathrm{Der}\, S)^{(1)} \otimes A(1;\mathbb{1})$.

(3) *For every* $j \in GF(p)$, $K_{j\beta}$ *contains an element* x_j *such that* $\alpha(x)\neq 0$.

Proof. (1) As before $S \otimes A(1;\mathbb{1})$ is R-simple and (IV.3) applies. Then R is conjugate under an automorphism of $S \otimes A(1;\mathbb{1})$ to $\{R_0 \otimes F\} \oplus F(1+x)\partial$, R_0 a torus of $\mathrm{Der}\, S$. R_0 is one-dimensional. The optimality of R ensures that R_0 is a maximal torus in $S + R_0$. Hence R_0 is an optimal torus in S. We suppress the notion of this automorphism.

(2) Consider the restricted homomorphism

$$\begin{aligned}\psi\colon \mathrm{Der}(S\otimes A(1;\mathbb{1})) &= (\mathrm{Der}\, S)\otimes A(1;\mathbb{1}) + F\,\mathrm{id}\otimes W(1;\mathbb{1})\\ &\to \mathrm{Der}(S\otimes A(1;\mathbb{1}))/(\mathrm{Der}\, S)\otimes A(1;\mathbb{1}) \cong W(1;\mathbb{1}).\end{aligned}$$

According to (1) there is $r_1 \in R$, such that $\psi(r_1) = (1+x)\partial$. As $S\otimes A(1;\mathbb{1})_{(1)}$ is an ideal of K, we have $\psi(K) \subset W(1;\mathbb{1})_{(0)}$. Moreover, $\psi(K)$ is invariant under $(1+x)\partial$, which is now only possible if $\psi(K)=0$. Hence $K \subset \ker\psi = (\mathrm{Der}\, S)\otimes A(1;\mathbb{1})$.

Consider the isomorphism $\rho\colon (\mathrm{Der}\, S)\otimes A(1;\mathbb{1})/(\mathrm{Der}\, S)\otimes A(1;\mathbb{1})_{(1)} \cong \mathrm{Der}\, S$. As $TR(\rho(K)) \leqslant TR(K) = TR(S) = 1$, we have $\rho(K) = S \in \{\mathrm{sl}(2), W(1;\mathbb{1})\}$ or $S = H(2;\mathbb{1})^{(2)} \subset \rho(K) \subset H(2;\mathbb{1}) = (\mathrm{Der}\, S)^{(1)}$. This shows that $K \subset (\mathrm{Der}\, S)^{(1)}\otimes A(1;\mathbb{1}) + (\mathrm{Der}\, S)\otimes A(1;\mathbb{1})_{(1)}$. Since the k-fold application of $r_1 \in R$ defines a surjective mapping $(\mathrm{Der}\, S)\otimes A(1;\mathbb{1})_{(k)}/(\mathrm{Der}\, S)\otimes A(1;\mathbb{1})_{(k+1)}$ onto $(\mathrm{Der}\, S)\otimes A(1;\mathbb{1})/(\mathrm{Der}\, S)\otimes A(1;\mathbb{1})_{(1)}$ we obtain that $K \subset (\mathrm{Der}\, S)^{(1)}\otimes A(1;\mathbb{1})$.

(3) Choose $r \in R_0 \subset S$. Then $r\otimes(1+x)^i \in K_{i\beta}$ acts nonnilpotently. ∎

V. The Zassenhaus Algebra $W(1;\mathbf{n})$

We will consider $W(1;\mathbf{n})$ in more detail. Since we only have one "indeterminate," we omit indices. $W(1;\mathbf{n})$ has dimension p^n. The derivation algebra is given by $\mathrm{Der}\, W(1;\mathbf{n}) = \sum_{1\leqslant i\leqslant n-1} FD^{p^i} \oplus W(1;\mathbf{n})$. Since $\mathrm{Der}\, W(1;\mathbf{n})$ contains a p-envelope of $W(1;\mathbf{n})$ and every p-envelope in $\mathrm{Der}\, W(1;\mathbf{n})$ has to contain $\sum_{1\leqslant i\leqslant n-1} FD^{p^i}$, $\mathrm{Der}\, W(1;\mathbf{n})$ is also a p-envelope of $W(1;\mathbf{n})$.

Some general remark might be helpful. There is a canonical injection of $A(m;\mathbf{n})$ into $A(n_1+\cdots+n_m;\mathbb{1})$ induced by $x^{(p^i\varepsilon_j)} \to x^{(\varepsilon_k)}$, where $k = n_1+\cdots+n_{j-1}+i+1$ $(1\leqslant j\leqslant m,\ 0\leqslant i\leqslant n_j-1)$. This mapping gives rise to an injection $W(m;\mathbf{n}) \to W(n_1+\cdots+n_m;\mathbb{1})$. In the present case we have an injection $W(1;\mathbf{n})$ into $W(n;\mathbb{1})$. As a consequence, $TR(W(1;\mathbf{n})) \leqslant TR(W(n;\mathbb{1})) = n$.

Lemma V.1. (1) *$W(1;\mathbf{n})_{(0)}$ is closed under pth powers.*

(2) *Let $t := D^{p^r} + \sum_{0\leqslant i\leqslant r-1} \gamma_i D^{p^i} + u$, $u \in W(1;\mathbf{n})_{(0)}$, $r \geqslant 0$, be a semisimple element, T the torus generated by t, and $W(1;\mathbf{n})_\alpha$ a root space with respect to T. Then*

(a) $\dim T/T\cap W(1;\mathbf{n})_{(0)} \geqslant n-r$.

(b) $W(1;\mathbf{n})_\alpha \cap W(1;\mathbf{n})_{(p^r-1)} = 0$, $\dim W(1;\mathbf{n})_\alpha \leqslant p^r$.

(c) *If* $\dim W(1;\mathbf{n})_\alpha = p^r$, *then*

$$W(1;\mathbf{n})_\alpha \oplus W(1;\mathbf{n})_{(p^r-1)} = W(1;\mathbf{n}).$$

Proof. (1) Let g be an element of $W(1;\mathbf{n})_{(0)}$. Then g^p is a derivation of $W(1;\mathbf{n})$ which does not lower the degree of a monomial. As $\operatorname{Der} W(1;\mathbf{n}) = \sum_{0\leqslant i\leqslant n-1} FD^{p^i} + W(1;\mathbf{n})_{(0)}$ this shows $g^p \in W(1;\mathbf{n})_{(0)}$.

(2) (a) We consider pth powers of t: $t^{p^i} = D^{p^{i+r}} + \sum_{j<i+r} \beta_j D^{p^j} + u_i$, $u_i \in W(1;\mathbf{n})_{(0)}$. For $i=0, ..., n-r-1$ these are linearly independent mod $W(1;\mathbf{n})_{(0)}$.

(b) Let $w = \sum_{j\geqslant k} \gamma_j x^{(j)}D$, $\gamma_k \neq 0$, be an eigenvector with respect to t. If $k > p^r - 1$ then, as $w \in W(1;\mathbf{n})_{(k-1)}$

$$\gamma w = [t, w] \equiv D^{p^r}(\gamma_k x^{(k)})D \equiv \gamma_k x^{(k-p^r)}D \not\equiv 0 \bmod W(1;\mathbf{n})_{(k-p^r)},$$

a contradiction. Hence $w \notin W(1;\mathbf{n})_{(p^r-1)}$ and the mapping $W(1;\mathbf{n}) \to W(1;\mathbf{n})/W(1;\mathbf{n})_{(p^r-1)}$ is injective on every root space. Therefore every root space has dimension at most p^r.

(c) If the dimension of a root space is exactly p^r, then this mapping is bijective. ∎

THEOREM V.2. *Let T be a maximal torus of $W(1;\mathbf{n})_p$ and assume that T contains an element of the form $D+u$, $u \in W(1;\mathbf{n})_{(0)}$. Then*

(1) $\dim T = n$; *T is generated as a restricted subalgebra by $D+u$.*

(2) *Every root space of $W(1;\mathbf{n})$ with respect to T is one-dimensional.*

(3) *$W(1;\mathbf{n})$ has a basis $(u_\vartheta)_{\vartheta\in\Theta\subset F}$ of root vectors with multiplication* $[u_\vartheta, u_\rho] = (\rho - \vartheta)u_{\vartheta+\rho}$.

(4) *Every one-section is isomorphic to $W(1;\mathbb{1})$.*

Proof. Put $t := D + u \in T$. Thus we may apply the lemma with $r=0$. The torus generated by t has dimension at least n. As $W(1;\mathbf{n})$ may considered a subalgebra of $W(n;\mathbb{1})$ every torus has dimension at most n. This proves (1). The lemma also implies, that every root space is one-dimensional. Write $t = D + \alpha x^{(1)}D + u'$, $u' \in W(1;\mathbf{n})_{(1)}$. According to the lemma, every root space has zero intersection with $W(1;\mathbf{n})_{(0)}$. Let $v := D + \vartheta x^{(1)}D + v'$, $v' \in W(1;\mathbf{n})_{(1)}$ be a root vector with respect to t. The corresponding eigenvalue r is given by

$$r(D + \vartheta x^{(1)}D + v') = rv = [t, v] = [D + \alpha x^{(1)}D + u', D + \vartheta x^{(1)}D + v']$$
$$\equiv (\vartheta - \alpha)D \mod W(1;\mathbf{n})_{(0)}.$$

Thus $r = \vartheta - \alpha$ and $W(1;\mathbf{n})$ has an eigenvector basis (u_ρ) of the form

$$u_\rho = D + (\rho+\alpha)\, x^{(1)} D + u'_\rho, \qquad u'_\rho \in W(1;\mathbf{n})_{(1)}, \qquad [t, u_\rho] = \rho u_\rho.$$

Next we determine the product of two of these vectors. Considering the eigenvalues we see that $[u_\vartheta, u_\rho] \in F u_{\vartheta+\rho}$.

$$\begin{aligned}[u_\vartheta, u_\rho] &= [D + (\vartheta+\alpha)\, x^{(1)} D + u'_\vartheta, D + (\rho+\alpha)\, x^{(1)} D + u'_\rho] \\ &\equiv (\rho - \vartheta) D \equiv (\rho - \vartheta)\, u_{\vartheta+\rho} \mod W(1;\mathbf{n})_{(0)}.\end{aligned}$$

Then $[u_\vartheta, u_\rho] = (\rho-\vartheta)\, u_{\vartheta+\rho}$.

Every one-section $W(1;\mathbf{n})(\vartheta)$ is of the form $\sum_{i\in GF(p)} F u_{i\vartheta}$, $\vartheta \neq 0$. The mapping $u_{i\vartheta} \mapsto \vartheta^{-1}(1+x)^i\, \partial$ establishes an isomorphism onto $W(1;\mathbb{1})$. ∎

G. M. Benkart and J. M. Osborn call a basis of this type a "group basis" and they say that the roots are "dependent." No root is proper.

THEOREM V.3. *Let T denote a maximal torus of $W(1;\mathbf{n})_p$ with* $\dim T < n$. *Then*

(1) $0 \neq T \cap W(1;\mathbf{n}) \subset W(1;\mathbf{n})_{(0)}$;

(2) $C_{W(1;\mathbf{n})}(T\cap W(1;\mathbf{n})) =: H$ *is a CSA of* $W(1;\mathbf{n})$ *and is contained in* $W(1;\mathbf{n})_{(0)}$;

(3) $C_{W(1;\mathbf{n})_p}(T) =: \tilde{H}$ *is a CSA of* $W(1;\mathbf{n})_p$. $\tilde{H}^{(1)}$ *does not act nilpotently on* $W(1;\mathbf{n})$.

Proof. (1) $T\cap W(1;\mathbf{n}) \subset W(1;\mathbf{n})_{(0)}$ is true since $\dim T < n$ (apply (V.2.(1))). Choose $r \geq 1$ maximal such that $T \cap (\sum_{0\leq i\leq r-1} F D^{p^i} + W(1;\mathbf{n})_{(0)}) \subset W(1;\mathbf{n})_{(0)}$. Then T contains an element $t_1 := D^{p^r} + \sum_{0\leq i<r} \gamma_i D^{p^i} + u$, $u \in W(1;\mathbf{n})_{(0)}$. t_1 generates a torus T'. Then $\dim T' \geq n-r$ (V.1.(2a)). Assume that $T\cap W(1;\mathbf{n}) = 0$. Then $T' = T$, T' has dimension $n-r$, and therefore there are at most p^{n-r} different roots. Adding dimensions, (V.1.(2b)) in combination with $\dim W(1;\mathbf{n}) = p^n$ yields $\dim W(1;\mathbf{n})_\alpha = p^r$ for all roots. Applying the lemma for $\alpha = 0$ we obtain

$$\tilde{H} \cap W(1;\mathbf{n}) + W(1;\mathbf{n})_{(p^r-1)} = W(1;\mathbf{n}).$$

In particular, $\tilde{H} \cap W(1;\mathbf{n})_{(0)} \not\subset W(1;\mathbf{n})_{(1)}$. Since $W(1;\mathbf{n})_{(0)}$ and $\tilde{H}$ are closed under pth powers, the intersection of these contains a nonzero semisimple element. This element lies in $T\cap W(1;\mathbf{n})_{(0)}$, a contradiction.

(2) Since $T \cap W(1;\mathbf{n})_{(1)} = 0$, there is an element t of the form

$t=x^{(1)}D+u$, $u\in W(1;\mathbf{n})_{(1)}$ and $T\cap W(1;\mathbf{n})=Ft$. Then t acts on the quotients $W(1;\mathbf{n})_{(k)}/W(1;\mathbf{n})_{(k+1)}$ with eigenvalue k. Hence

$$H=C_{W(1;\mathbf{n})}(T\cap W(1;\mathbf{n}))=C_{W(1;\mathbf{n})}(Ft)=H\cap W(1;\mathbf{n})_{(1)}+Ft$$

and H is a CSA.

(3) $\tilde{H}$ is a CSA since T is a maximal torus. Since $W(1;\mathbf{n})_p^{(1)}\subset W(1;\mathbf{n})$, $W(1;\mathbf{n})$ contains every root space for any nonzero root. Hence we have $W(1;\mathbf{n})_p=W(1;\mathbf{n})+\tilde{H}$ and so there is $u\in W(1;\mathbf{n})$ such that $D^p+u\in\tilde{H}$.

T acts on H and $\tilde{H}\cap W(1;\mathbf{n})\subset H$. In (2) we have described H to some extent

$$H=\bigoplus_{0\leqslant i\leqslant p^{n-1}-1} F(x^{(ip+1)}D+u_i),\text{ with some } u_i\in W(1;\mathbf{n})_{(ip+1)}.$$

Recall the definition of T' from (1) and note that $T=T'+Ft$. Let α' be any root with respect to T'. According to the lemma we have

$$W(1;\mathbf{n})_{\alpha'}\cap W(1;\mathbf{n})_{(p^r-1)}=0,$$

hence $\dim W(1;\mathbf{n})_{\alpha'}\cap H\leqslant p^{r-1}$ for all roots α' on T'. Then $\dim W(1;\mathbf{n})_{\alpha}\cap H\leqslant p^{r-1}$ for all roots α on T with $\alpha(t)=0$. There are at most p^{n-r} roots of T vanishing on $T\cap W(1;\mathbf{n})_{(0)}$ (since $\dim T/T\cap W(1;\mathbf{n})_{(0)}=n-r$). Since $\dim H=p^{n-1}$, and $H=\sum_{\alpha(t)=0}H\cap W(1;\mathbf{n})_\alpha$ this dimension argument yields $\dim W(1;\mathbf{n})_\alpha\cap H=p^{r-1}$ for all roots with $\alpha(t)=0$. In particular, $\dim\tilde{H}\cap H=p^{r-1}$. Observe that $n>\dim T=(n-r)+1$, i.e., $r\geqslant 2$. Therefore there are $i>0$ and $v\in W(1;\mathbf{n})_{(ip+1)}$ such that $x^{(ip+1)}D+v\in\tilde{H}$. Then

$$\operatorname{ad}^i(D^p+u)(x^{(ip+1)}D+v)\equiv x^{(1)}D \mod W(1;\mathbf{n})_{(1)}$$

and lies in $\tilde{H}^{(1)}$. Hence $\tilde{H}^{(1)}$ contains an element of the form $h=x^{(1)}D+w$, $w\in W(1;\mathbf{n})_{(1)}$. h does not act nilpotently on $W(1;\mathbf{n})$. ∎

The torus $Fx^{(1)}D$ is an example for this type of torus. In the particular case of $n=2$, $\dim T=1$ we obtain the result $T\subset W(1;\mathbf{2})_{(0)}$. A torus of type (V.2) is not optimal, since no root is proper. A torus of type (V.3) is not optimal, since its dimension is less than $n=TR(W(1;\mathbf{n}))$. Thus there is a remaining class of (optimal) tori.

THEOREM V.4. *Let T denote a maximal torus of $W(1;\mathbf{n})_p$. Assume that T is none of the types described in* (V.2) *and* (V.3). *Then*

(1) $\dim T=n$, $\dim T\cap W(1;\mathbf{n})=1$, $T\cap W(1;\mathbf{n})=T\cap W(1;\mathbf{n})_{(0)}$.

(2) *Every root space is one-dimensional. For every root* α *there is* i, $0 \leqslant i \leqslant p-1$, *and* $w_{i,\alpha} \in W(1;\mathbf{n})_{(i)}$ *such that* $W(1;\mathbf{n})_\alpha = F(x^{(i)}D + w_{i,\alpha})$.

(3) *If* α *is a root such that* $\alpha(T \cap W(1;\mathbf{n})) \neq 0$ *then* $W(1;\mathbf{n})(\alpha) \cong W(1;\mathbb{1})$. *The isomorphism is given by* $x^{(i)}D + w_{i,\alpha} \mapsto (i!)^{-1} x^i \partial$.

(4) *If* α *is a root such that* $\alpha(T \cap W(1;\mathbf{n})) = 0$ *then* $W(1;\mathbf{n})(\alpha)$ *is abelian and every nonzero root vector of* $W(1;\mathbf{n})(\alpha)$ *acts nonnilpotently on* $W(1;\mathbf{n})$.

(5) T *is an optimal torus.*

Proof. (1) We are not in case (V.3) and therefore $\dim T = n$. As $\dim W(1;\mathbf{n})_p / W(1;\mathbf{n}) = n-1$, this yields $T \cap W(1;\mathbf{n}) \neq 0$. Since we are not in case (V.2), $T \cap W(1;\mathbf{n}) = T \cap W(1;\mathbf{n})_{(0)}$. As $T \cap W(1;\mathbf{n})_{(1)} = 0$, we have $\dim T \cap W(1;\mathbf{n}) = 1$.

(2) There is a nonzero $t \in T \cap W(1;\mathbf{n})_{(0)}$ and we may choose $t = x^{(1)}D + u$, $u \in W(1;\mathbf{n})_{(1)}$. In addition, since $\dim T/T \cap W(1;\mathbf{n}) = n-1 = \dim W(1;\mathbf{n})_p / W(1;\mathbf{n})$ T contains an element $D^p + v \in T$, $v \in W(1;\mathbf{n})$. Let w be an eigenvector with respect to T for a root $\alpha \in T^*$. Write

$$w = \sum_{j \geqslant k} r_j x^{(j)} D, \qquad r_j \in F, \qquad r_k \neq 0.$$

According to (V.1.(2b)) we have $k < p$.

$$\alpha(t) w = [x^{(1)}D + u, w] \equiv (k-1)\, r_k x^{(k)} D \mod W(1;\mathbf{n})_{(k)}.$$

Choose i such that

$$0 \leqslant i \leqslant p-1, \qquad i \equiv 1 + \alpha(t) \mod (p).$$

The above shows that if the eigenvalue with respect to t is $\alpha(t)$ then w is of the form

$$w = r x^{(i)} D + w_i, \qquad w_i \in W(1;\mathbf{n})_{(i)}, \qquad r \in F, r \neq 0.$$

This implies that every root space of $W(1;\mathbf{n})$ is one-dimensional and by this proves (2).

(3) Consider a root α with $\alpha(t) \neq 0$. $W(1;\mathbf{n})(\alpha)$ is p-dimensional and it has a basis of the form

$$(x^{(i)}D + w_{i,\alpha}), \qquad w_{i,\alpha} \in W(1;\mathbf{n})_{(i)}, \qquad 0 \leqslant i \leqslant p-1.$$

The mapping $W(1;\mathbf{n})(\alpha) \to W(1;\mathbb{1})$ given by $x^{(i)}D + w_{i,\alpha} \mapsto (i!)^{-1} x^i \partial$ is an isomorphism from $W(1;\mathbf{n})(\alpha)$ onto $W(1;\mathbb{1})$.

(4) Consider α, such that $\alpha(t)=0$. $W(1;\mathbf{n})(\alpha)$ is p-dimensional and it has a basis of root vectors of the form

$$(x^{(1)}D+w_i), \qquad w_i \in W(1;\mathbf{n})_{(1)}, \qquad 0 \leqslant i \leqslant p-1.$$

The product of any two of these vectors is an element of $W(1;\mathbf{n})_{(1)}$. Since every nonzero element has to have a nonvanishing summand $rx^{(1)}D$, all these products vanish. Every element of type $x^{(1)}D+w_i$ acts nonnilpotently on $W(1;\mathbf{n})$.

(5) According to the preceding results, every root is proper and $\dim T = TR(W(1;\mathbf{n}))$. ▮

Remark. Let H denote any CSA of $W(1;\mathbf{n})$ and R the maximal torus of the p-envelope of H in a p-envelope of $W(1;\mathbf{n})$. Let T be a maximal torus of $W(1;\mathbf{n})_p$ containing R. According to the preceding theorems there are essentially two different situations: If T is a torus described in theorem V.2, then $R \subset T = F(D+u)_p$. H is a section with respect to T, since R is a subtorus of T. Since H is nilpotent, and according to (V.2.(3)) no one-section is nilpotent, H must be a zero-section, i.e., $H = C_{W(1;\mathbf{n})}(T) = F(D+u)$. The multiplication of the root vectors with respect to H is given by (V.2.(3)). If T is a torus described by theorems V.3 and V.4 then $H \supset T \cap W(1;\mathbf{n}) = F(x^{(1)}D+u_1)$ and $H = C_{W(1;\mathbf{n})}(x^{(1)}D+u_1)$, as the latter is a Cartan subalgebra. In this case the root spaces with respect to H are p^{n-1}-dimensional. We obtain some of the main results of Brown [Br] with only very few computations just by using the concept of a p-envelope.

We are going to apply these results to the situation of case (7) in (II.2).

COROLLARY V.5. *With the assumptions and notations of* (II.2) *let* $K+R$ *be as in case* (7) *with* $S = W(1;\mathbf{2})$. *Then* $K = W(1;\mathbf{2})$ *and* R *is a torus of type* (V.4).

Proof. It is clear from the above, that R is a torus as described in (V.4). There is a root α with $\alpha(R \cap W(1;\mathbf{2}))=0$. α is a solvable root. If $R \subset K$, then $R \subset H$ and H would be a Cartan subalgebra on which no root vanishes. In particular, $\alpha(C_L(T)) \neq 0$. Lemma II.1 shows that L_α acts nilpotently on L, yielding a contradiction to (V.4.(4)). Thus $R \not\subset K$ and hence

$$W(1;\mathbf{2}) \subset K \neq \operatorname{Der} W(1;\mathbf{2}) = W(1;\mathbf{2}) + FD^p.$$

This gives the result. ▮

VI. The Hamiltonian Algebra $H(2;(2,1))^{(2)}$

The algebra $H(2;(2,1))$ is defined analogously to $H(2;\mathbb{1})$ in Section III. Put $\tau := \tau(2,1) := (p^2-1, p-1)$ and

$$D_H(x^{(a)}) := x^{(a-\varepsilon_1)}D_2 - x^{(a-\varepsilon_2)}D_1$$

as well as

$$\begin{aligned} H(2;(2,1)) &:= \{f_1D_1 + f_2D_2 \mid f_1, f_2 \in A(2;(2,1)), D_1(f_1)+D_2(f_2)=0\} \\ &= \operatorname{span}\{D_H(x^{(a)}) \mid 0 < a \leqslant \tau\} \cup \{D_H(x^{(p^2\varepsilon_1)}), D_H(x^{(p\varepsilon_2)})\}, \\ H(2;(2,1))^{(1)} &= \operatorname{span}\{D_H(x^{(a)}) \mid 0 < a \leqslant \tau\}, \\ H(2;(2,1))^{(2)} &= \operatorname{span}\{D_H(x^{(a)}) \mid 0 < a < \tau\}. \end{aligned}$$

Recall that $A(2;\mathbb{1})$ is a subalgebra of $A(2;(2,1))$, isomorphic to the truncated polynomial ring $F[x_1, x_2]$. It is mentioned in [BW2, (10.1.1)] that $TR(H(2;(2,1))^{(2)})=2$. Put as an abbreviation $G := H(2;(2,1))$ and consider any optimal torus $T \subset G_p = H(2;(2,1)) \oplus FD_1^p$. The following proposition is a consequence of [BW2, (10.1.1)].

Proposition VI.1. *Let T be an optimal torus in G_p. Then* $\dim T = 2$, *$T \subset H(2;(2,1))_p^{(2)}$ and $T \cap G = T \cap G_{(0)}$ is one-dimensional. Thus $T = Ft_1 \oplus Ft_2$ with toral elements t_i, and $t_1 = D_1^p + u$, $u \in H(2;(2,1))^{(2)}$, $t_2 = r + v$, $r = \sum \alpha_{ij} x^{(\varepsilon_i)} D_j$, $\alpha_{ij} \in F$, $v \in H(2;(2,1))_{(1)}^{(2)}$.*

Proof. As $H(2;(2,1))^{(2)}$ is an ideal of G_p and $G_p/H(2;(2,1))_p^{(2)}$ is p-nilpotent, we have $TR(G_p) = TR(H(2;(2,1))^{(2)}) = 2$. Since $C(G_p)=0$ [St4, (1.3.(5))] implies that $T \subset H(2;(2,1))_p^{(2)}$. Put in [BW2, (10.1.1)] $A := G_p$. We obtain by (d) of this theorem that $T \cap H(2;(2,1))_{(0)}^{(2)} \neq 0$. If $T \subset G$ then T would be contained in $G \cap H(2;(2,1))_p^{(2)} = H(2;(2,1))^{(2)}$. Part (f) of that theorem shows that this is impossible. ∎

We are now going to discuss the root space decomposition of G with respec to T. Put

$$U := \operatorname{span}\{D_H(x^{(a)}) \mid (a_1 \geqslant p+1) \vee (a_1 = p, a_2 \neq 0)\}.$$

Note that $G = H(2;(2,1)) = U \oplus H(2;\mathbb{1})$ and U has codimension p^2+1 in G.

Theorem VI.2. *Put $G := H(2;(2,1))$ and let $T \subset G_p$ be an optimal torus. Let $G = \bigoplus G_\alpha$ be the root space decomposition with respect to T.*

(1) *$G_\alpha \cap U = (0)$ for all roots α.*

(2) *$\dim G_\alpha = p + \delta_{\alpha,0}$ for all α.*

(3) *Any root vector $u_\alpha \in G_\alpha$ can be written as*

$$u_\alpha = u_{\alpha,k} + w_{\alpha,k}, \qquad u_{\alpha,k} \in G_{(k)} - G_{(k+1)}, \qquad w_{\alpha,k} \in G_{(k+1)} + U \cap G_{(k)};$$

$u_{\alpha,k} \in H(2;\mathbb{1})$ is a root vector with respect to r corresponding to the eigenvalue $\alpha(t_2)$ and homogeneous of degree k.

(4) *Given any homogeneous root vector $u_{\alpha,k}$ of $H(2;\mathbb{1})$ with respect to r corresponding to the eigenvalue $\alpha(t_2)$ and of degree k, there is $w_{\alpha,k} \in G_{(k+1)} + U \cap G_{(k)}$ such that $u_{\alpha,k} + w_{\alpha,k} \in G_\alpha$.*

Proof. Take any $w \in G_\alpha$ and write

$$w = \sum_{a_1 \geqslant s} \kappa(a)\, D_H(x^{(a)}), \qquad \kappa(a) \in F.$$

If $w \in G_{(k+1)} + U \cap G_{(k)}$, but $w \notin G_{(k+1)}$ then there occur monomials in w with $a_1 \geqslant p$ and

$$\alpha(D_1^p + u)\,w = [D_1^p + u, w] = \sum \kappa(a)\, D_H(x^{(a - p\varepsilon_1)}) + \sum \kappa'(b)\, D_H(x^{(b)}),$$

where in the right hand side sum there occur monomials $D_H(x^{(c)})$ with $D_H(x^{(c)}) \notin G_{(k+1-p)}$. This contradiction shows that

$$G_\alpha \cap (U \cap G_{(k)} + G_{(k+1)}) = G_\alpha \cap G_{(k+1)}.$$

This proves (1). We now consider the graded space

$$\operatorname{gr} G_\alpha := \bigoplus_k G_\alpha \cap G_{(k)}/G_\alpha \cap G_{(k+1)}$$
$$= \bigoplus_k G_\alpha \cap G_{(k)}/G_\alpha \cap (U \cap G_{(k)} + G_{(k+1)})$$

which embeds canonically into $H(2;\mathbb{1})$. t_2 acts on $\operatorname{gr} G_\alpha$ via the action of r on $H(2;\mathbb{1})$. As a consequence, $\dim G_\alpha \leqslant p + \delta_{\alpha,0}$. In addition, the above reasoning proves (3). We now count dimensions: there are at most p^2 roots on G, each of dimension at most $p + \delta_{\alpha,0}$. As $\dim G = p^3 + 1$, we obtain $\dim G_\alpha = p + \delta_{\alpha,0}$ and $\operatorname{gr} G_\alpha$ is the full corresponding root space in $H(2;\mathbb{1})$. This proves (4). ∎

COROLLARY VI.3. *Suppose that $\alpha(t_2) \neq 0$. $G(\alpha)$ is filtered by $G(\alpha)_{(k)} = G(\alpha) \cap G_{(k)}$. There is an isomorphism of graded algebras μ: $\operatorname{gr} G(\alpha) \xrightarrow{\sim} H(2;\mathbb{1})$ with $\mu(t_2) \in F(x_1\,\partial_1 - x_2\,\partial_2)$.*

Proof. Every $G(\alpha)_{(k)}$ is invariant under t_2, and as $T = Ft_2 + (\ker \alpha)$, it is invariant under T. Therefore the results of (VI.2.(1), (3)) yield

$$\operatorname{gr} G(\alpha) := \bigoplus_k G(\alpha)_{(k)}/G(\alpha)_{(k+1)}$$

$$= \bigoplus_{i \in GF(p)} \bigoplus_k G_{i\alpha} \cap G_{(k)}/G_{i\alpha} \cap G_{(k+1)}.$$

Since $\alpha(t_2) \neq 0$ the isomorphic image of the right hand side in $H(2; \mathbb{1})$ runs through all root spaces with respect to r. Hence we have a surjective mapping $\operatorname{gr} G(\alpha) \to H(2; \mathbb{1})$. Note that in every homogeneous space $G(\alpha)_{(k)}/G(\alpha)_{(k+1)}$ the root spaces are at most one-dimensional. Therefore this mapping yields an isomorphism of algebras.

Clearly this isomorphism maps t_2 onto some toral element contained in $H(2; \mathbb{1})_{(0)}$. Demuskin's result now gives rise to an automorphism of $H(2; \mathbb{1})$, which maps this element into $F(x_1 \partial_1 - x_2 \partial_2)$. ▌

The preceding result says, that there is a choice of a root vector basis $\{u_{\alpha,k} + w_{\alpha,k}\}_{\alpha,k}$ such that $\mu(u_{\alpha,k} + w_{\alpha,k}) = D_H(x_1^i x_2^j)$ with $i + j = k + 2$, $i - j \equiv \alpha(r_2)$.

We now apply the results to the situation which occurs in the context of the classification theory.

THEOREM VI.4. *With the assumptions and notations of* (II.2) *let* $K + R$ *be as in case* (7) *with* $S = H(2; (2, 1))^{(2)}$. *Then*

(1) $R \subset H(2; (2, 1))_p^{(2)}$.

(2) $K \subset H(2; (2, 1))$.

(3) *There is a root* β *with* $\beta(H) = 0$; $GF(p)\beta$ *is the set of all solvable roots; for every* $i \neq 0$ *there is a root vector* $e_{i\beta} \in K_{i\beta}$ *which acts nonnilpotently on* K.

Proof. The assumption ensures that $TR(K) = 2$. It is known that $\operatorname{Der} H(2; (2, 1))^{(2)} = H(2; (2, 1)) + FD_1^p + F(x^{(\varepsilon_1)}D_1 + x^{(\varepsilon_2)}D_2)$. From $TR(H(2; (2, 1))^{(2)}) = 2$ we conclude that $K_p/H(2; (2, 1))_p^{(2)}$ is p-nilpotent. As a result, K cannot meet $F(x^{(\varepsilon_1)}D_1 + x^{(\varepsilon_2)}D_2)$, showing that $K \subset H(2; (2, 1)) + FD_1^p$. Similarly, $R \subset H(2; (2, 1))_p^{(2)}$.

We now apply the preceding results. After the application of a suitable automorphism the root spaces are of the form given in (VI.2). Let $\beta \neq 0$ be a root with $\beta(t_2) = 0$. As $i\beta \neq 0$ for $i \neq 0$, $K_{i\beta} \subset H(2; (2, 1))^{(2)}$. (VI.2.(4)) shows that $K_{i\beta}$ contains some element $e_{i\beta} := r + w_i$, $w_i \in H(2; (2, 1))_{(1)}$. This element acts nonnilpotently.

Suppose $K \not\subset H(2; (2, 1))$. Then $R \subset H(2; (2, 1))_p^{(2)} \subset H(2; (2, 1)) + FD_1^p = K + H(2; (2, 1)) = K + \operatorname{span}\{D_H(x^{(p^2\varepsilon_1)}),\ D_H(x^{(p\varepsilon_2)}),\ D_H(x^{(\tau)})\}$. In this case no root vanishes on $H = C_K(R)$. According to Lemma II.1 all root vectors e_μ, $\mu \in GF(p)\alpha + GF(p)\beta$, $\mu \neq 0$ act nilpotently. This contradiction proves the assertion. ∎

VII. The Hamiltonian Algebra $H(2; \mathbb{1}; \Phi(\tau))^{(1)}$

We will introduce a description of some of the hamiltonian type algebras, which is more appropriate than the description as derivations of a truncated polynomial ring or a divided power ring. We follow R. D. Schafer [Sch]:

Let $F[x_1, ..., x_{2r}]$, $x_i^p = 0$, denote the truncated polynomial ring in $2r$ generators and let $c_{ij} \in F[x_1, ..., x_{2r}]$, $1 \leqslant i, j \leqslant 2r$, be arbitrary elements satisfying

(i) $c_{ij} = -c_{ji}$

(ii) $\sum_{1 \leqslant t \leqslant 2r} (\partial_t c_{ij})\, c_{tk} + (\partial_t c_{jk})\, c_{ti} + (\partial_t c_{ki})\, c_{tj} = 0$

(iii) one of the c_{ij} has nonzero constant term.

Define a "Poisson bracket" on $F[x_1, ..., x_{2r}]$ by use of (c_{ij})

$$\{f, g\} := \sum_{1 \leqslant i, j \leqslant 2r} (\partial_i f)(\partial_j g)\, c_{ij}.$$

Then $(F[x_1, ..., x_{2r}], \{\ ,\ \}) = (F[x_1, ..., x_{2r}], \{\ ,\ \}, (c_{ij}))$ is a Lie algebra and $F[x_1, ..., x_{2r}]^{(1)}/F1 \cap F[x_1, ..., x_{2r}]^{(1)}$ is a simple Lie algebra of Cartan type. Its dimension is $p^{2r} - 1$ or $p^{2r} - 2$. [Sch, St1]. We call these algebras *Poisson Lie algebras* (PLA). All Lie algebras of type $L(G, \delta, f)$ of R. E. Block [B] are PLAs [Sch], every PLA is of hamiltonian type [W1].

Theorem VII.1. *There is exactly one isomorphism class of PLAs of dimension $p^2 - 1$. Every such algebra can be realized as a PLA on a truncated polynomial ring $F[x_1, x_2]$ with generators x_1, x_2 and a Poisson bracket $\{\ ,\ \}$ such that*

$$(1) \quad \{x_1, x_2\} = 1 - x_1^{p-1} x_2^{p-1}.$$

It also can be realized as a PLA on a truncated polynomial ring $F[y_1, y_2]$ with generators y_1, y_2 such that

$$(2) \quad \{y_1, y_2\} = (1 + y_1)(1 + y_2).$$

Proof. Let L be a PLA with dimension $p^2 - 1$. Then it has generators x_1, x_2 of the form [Sch]

$$\{x_1, x_2\} = 1 + \alpha x_1^{p-1} x_2^{p-1}, \qquad \alpha \in F, \qquad \alpha \neq 0.$$

Due to [O] this algebra is isomorphic to a PLA $F[x'_1, x'_2]$ with generators x'_1, x'_2 such that $\{x'_1, x'_2\} = 1 - x_1'^{p-1} x_2'^{p-1}$. Hence every such algebra can be described by a Poisson bracket of type (1) and therefore there is exactly one isomorphism class.

Define as in [Sch] on the truncated polynomial ring $F[y_1, y_2]$, $y_i^p = 0$, a Poisson bracket by

$$\{y_1, y_2\} = (1 + y_1)(1 + y_2).$$

Then conditions (i)–(iii) above are satisfied and therefore this also defines a PLA of dimension $p^2 - 1$. Since there is only one isomorphism class of these algebras, it is isomorphic to L. This means that L can be realized this way. ∎

THEOREM VII.2. *$H(2; \mathbb{1}; \Phi(\tau))^{(1)}$ is a PLA of dimension $p^2 - 1$.*

Proof. $H(2; \mathbb{1}; \Phi(\tau))^{(1)}$ has been described in detail in [BW2, Section 2.1], where they use the notation of divided power algebras:

In that paper, $\gamma(\mathbb{1}) = (p-1)\varepsilon_1 + (p-1)\varepsilon_2 = \tau(\mathbb{1}) =: \tau$, $a := J(\Phi(\mathbb{1})) = 1 + x^{\gamma(\mathbb{1})}$, $a_i := a^{-1} D_i(a)$ and (cf. [BW2, Definition 2.1.2, (2.1.3)])

$$D_a : A(2; \mathbb{1}) \to H(2; \mathbb{1}; \Phi(\tau))$$

is given by

$$\begin{aligned} D_a(f) &= (D_2 + a_2)(f)\, D_1 - (D_1 + a_1)(f)\, D_2 \\ &= a^{-1} D_2(af)\, D_1 - a^{-1} D_1(af)\, D_2. \end{aligned}$$

[BW2, (2.1.5)] yields

$$\begin{aligned} &[D_a(f), D_a(g)] \\ &\quad = D_a\{(D_1 + a_1)(g)(D_2 + a_2)(f) - (D_1 + a_1)(f)(D_2 + a_2)(g)\} \\ &\quad = D_a\{a^{-1} D_1(ag)\, a^{-1} D_2(af) - a^{-1} D_1(af)\, a^{-1} D_2(ag)\}. \end{aligned}$$

We will transpose that notation into one using the truncated polynomial ring $F[x_1, x_2]$ which is canonically isomorphic to the divided power algebra $A(2; \mathbb{1})$ under the isomorphism given by $\Psi(x_1^{a_1} x_2^{a_2}) := a_1!\, a_2!\, x^{(a_1\varepsilon_1 + a_2\varepsilon_2)}$. Then

$$a = 1 + x^{((p-1)\varepsilon_1 + (p-1)\varepsilon_2)} = 1 + \Psi(x_1^{p-1} x_2^{p-1}), \qquad a^{-1} = \Psi(1 - x_1^{p-1} x_2^{p-1}).$$

Since $\Psi^{-1} \circ D_i \circ \Psi(x_j) = \Psi^{-1} \circ D_i(x^{(\varepsilon_j)}) = \delta_{ij}$, we obtain $\Psi^{-1} \circ D_i \circ \Psi = \partial_i$.

Define the Poisson bracket on $F[x_1, x_2]$ by $\{x_1, x_2\} := 1 - x_1^{p-1}x_2^{p-1}$ and put

$$\varphi\colon (F[x_1, x_2], \{\ ,\ \}) \to H(2; \mathbb{1}; \Phi(\tau)), \qquad \varphi(f) := D_a(-a^{-1}\Psi(f)).$$

Then the above equation yields

$$\begin{aligned}
&[\varphi(f), \varphi(g)] \\
&\quad = D_a\{a^{-2}(D_1\Psi(g))(D_2\Psi(f)) - a^{-2}(D_1\Psi(f))(D_2\Psi(g))\} \\
&\quad = -\varphi\Psi^{-1}\{a^{-1}(D_1\Psi(g))(D_2\Psi(f)) - a^{-1}(D_1\Psi(f))(D_2\Psi(g))\} \\
&\quad = \varphi((1 - x_1^{p-1}x_2^{p-1})(\partial_1(f)\,\partial_2(g) - \partial_1(g)\,\partial_2(f)) = \varphi(\{f, g\}).
\end{aligned}$$

Hence φ is an isomorphism from $(F[x_1, x_2], \{\})$ onto $H(2; \mathbb{1}; \Phi(\tau))$. This proves the result. ∎

It is well known, that, if the dimension of a PLA is $p^2 - 1$, then $F1 \cap F[x_1, x_2]^{(1)} = 0$. This implies that

$$H(2; \mathbb{1}; \Phi(\tau))^{(1)} \cong (F[x_1, x_2], \{\})^{(1)} \cong (F[x_1, x_2], \{\})/F1.$$

It is also known (cf. [Sch]), that the mapping $\{f, ?\}\colon F[x_1, x_2] \to F[x_1, x_2]$ is a derivation of the truncated polynomial ring. As such it can be described in the realization (1) of theorem (VII.1) as

$$\{f, ?\} = c_{12}\,\partial_1(f)\,\partial_2 - c_{21}\,\partial_2(f)\,\partial_1 = (1 - x_1^{p-1}x_2^{p-1})(\partial_1(f)\,\partial_2 - \partial_2(f)\,\partial_1).$$

Let Φ denote any automorphism of the truncated polynomial ring $F[x_1, x_2]$ and put $y_i := \Phi(x_i)$, $i = 1, 2$. y_1, y_2 are generators for the truncated polynomial ring and are polynomials in x_1, x_2

$$y_i = \alpha_{1i}x_1 + \alpha_{2i}x_2 + f_i, \qquad \deg f_i > 1, \qquad \alpha_{ij} \in F, \qquad \det(\alpha_{ij}) \neq 0.$$

One can express the Poisson brackets in terms of y_1, y_2 according to the chain rule,

$$\{f, g\} = d_{12}(\partial f/\partial y_1)(\partial g/\partial y_2) - d_{12}(\partial g/\partial y_1)(\partial f/\partial y_2),$$

with $d_{12} = \{y_1, y_2\} = \det(\alpha_{ij})1 + g$, $\deg g \geqslant 1$.

It is known (cf. [Sch]), that $\dim \operatorname{Der} H(2; \mathbb{1}; \Phi(\tau))^{(1)} = p^2 + 1$. Thus it is easy to check that

$$\begin{aligned}
&\operatorname{Der} H(2; \mathbb{1}; \Phi(\tau))^{(1)} \\
&\quad = \{(1 - x_1^{p-1}x_2^{p-1})(\partial_1(f)\,\partial_2 - \partial_2(f)\,\partial_1) \mid f \in F[x_1, x_2]\} \\
&\qquad \oplus Fx_1^{p-1}\,\partial_2 \oplus Fx_2^{p-1}\,\partial_1.
\end{aligned}$$

THEOREM VII.3. *Let T be a torus of* $\operatorname{Der} H(2; \mathbb{1}; \Phi(\tau))^{(1)}$ *of maximal toral rank. Then*

(1) $\dim T = 2$, $\operatorname{Der} H(2; \mathbb{1}; \Phi(\tau))^{(1)}$ *is a p-envelope of* $H(2; \mathbb{1}; \Phi(\tau))^{(1)}$.

(2) *There is* $\psi \in \operatorname{Aut} F[x_1, x_2]$, *such that for the generators* $y_i := \psi(x_i)$, $i = 1, 2$, *the Poisson bracket has the form*

$$\{y_1, y_2\} = (1 + y_1)(1 + y_2)$$

and

$$T = F(1 + y_1)\partial/\partial y_1 \oplus F(1 + y_2)\partial/\partial y_2.$$

(3) *Every root space of* $H(2; \mathbb{1}; \Phi(\tau))^{(1)}$ *for a nonzero root with respect to T is one-dimensional, while* $C_{H(2;\mathbb{1};\Phi(\tau))^{(1)}}(T) = 0$. *Every root vector acts nonnilpotently on* $H(2; \mathbb{1}; \Phi(\tau))^{(1)}$.

(4) *Every one-section is abelian.*

(5) $H(2; \mathbb{1}; \Phi(\tau))^{(1)}$ *has an eigenvector basis* (u_μ) *and a biadditive form f, such that* $[u_\lambda, u_\mu] = f(\lambda, \mu)\, u_{\lambda+\mu}$.

Proof. (1) We realize $H(2; \mathbb{1}; \Phi(\tau))^{(1)}$ as a PLA with generators y_1, y_2 and $\{y_1, y_2\} = (1 + y_1)(1 + y_2)$. Then the p-fold Lie multiplication with y_1 is an element of $W(2; \mathbb{1})$ and hence is of the form $f_1\partial/\partial y_1 + f_2\partial/\partial y_2$. Application of this derivation to y_i yields

$$\begin{aligned} f_1 &= f_1\, \partial/\partial y_1(y_1) = (\operatorname{ad} y_1)^p(y_1) = 0, \\ f_2 &= f_2\partial/\partial y_2(y_2) = (\operatorname{ad} y_1)^p(y_2) = \{y_1, \dots, \{y_1, y_2\}\dots\} \\ &= (1 + y_1)^p(1 + y_2) = (1 + y_2). \end{aligned}$$

Therefore the p-envelope of $H(2; \mathbb{1}: \Phi(\tau))^{(1)}$ contains the torus $F(1 + y_1)\, \partial/\partial y_1 \oplus F(1 + y_2)\partial/\partial y_2$ of dimension 2. This proves $TR(H(2; \mathbb{1}; \Phi(\tau))^{(1)}) = 2$. It is straightforward to prove that this torus has trivial intersection with $H(2; \mathbb{1}; \Phi(\tau))^{(1)}$. A dimension argument yields that $H(2; \mathbb{1}; \Phi(\tau))^{(1)} \oplus F(1 + y_1)\, \partial/\partial y_1 \oplus F(1 + y_2)\, \partial/\partial y_2$ coincides with $\operatorname{Der} H(2; \mathbb{1}; \Phi(\tau))^{(1)}$. Thus the latter is a p-envelope of $H(2; \mathbb{1}; \Phi(\tau))^{(1)}$.

(2) Choose $\psi \in \operatorname{Aut} F[x_1, x_2]$ such that $T' := \psi^{-1} \circ T \circ \psi$ is one of the tori T_0, T_1, or T_2. There are $\delta_i \in \{0, 1\}$ for which $t_i := \delta_i + x_i$ are eigenvectors with respect to T'. Put $y_i := \psi(x_i)$. Then

$$\psi \circ (\delta_j + x_j)\, \partial_j \circ \psi^{-1}(y_i) = \delta_{ij}\psi(\delta_j + x_j) = \delta_{ij}(\delta_j + y_j).$$

This means that $T = F(\delta_1 + y_1)\, \partial/\partial y_1 + F(\delta_2 + y_2)\, \partial/\partial y_2$. We have to compute the Poisson bracket $\{y_1, y_2\} =: d$.

$$\begin{aligned}(\delta_1 + y_1)\, &\partial/\partial y_1(d)\\ &= (\delta_1 + y_1)\, \partial/\partial y_1(\{y_1, y_2\})\\ &= \{(\delta_1 + y_1)\, \partial/\partial y_1(y_1), y_2\} + \{y_1, (\delta_1 + y_1)\, \partial/\partial y_1(y_2)\} = d.\end{aligned}$$

Thus writing $d = \sum \beta_{ij}(\delta_1 + y_1)^i(\delta_2 + y_2)^j$ we obtain

$$d = (\delta_1 + y_1)\, \partial/\partial y_1(d) = \sum i\beta_{ij}(\delta_1 + y_1)^i(\delta_2 + y_2)^j.$$

This shows $\beta_{ij} = 0$ for $i \neq 1$. By symmetry we have $\beta_{ij} = 0$ for $j \neq 1$ and $d = (\delta_1 + y_1)(\delta_2 + y_2)$. The constant term of this polynomial is $\delta_1 \delta_2$. It has been mentioned earlier that this constant term is nonzero. As a result, $\delta_1 = \delta_2 = 1$. This proves (2).

(3) Every monomial $(1 + y_1)^a(1 + y_2)^b$ is an eigenvector with respect to T with root α, the root is given by $\alpha((1 + y_1)\, \partial/\partial y_1) \equiv a$, $\alpha((1 + y_2)\, \partial/\partial y_2) \equiv b \bmod(p)$. Different exponents yield different roots. Hence every root space in $H(2; \mathbb{1}; \Phi(\tau))$ is one-dimensional. Since $F1$ is the zero root space and is not contained in $H(2; \mathbb{1}; \Phi(\tau))^{(1)}$, this yields the assertion about the root spaces. Recall that

$$\{(1 + y_1)^a(1 + y_2)^b, (1 + y_1)^r(1 + y_2)^s\} = (as - br)(1 + y_1)^{a+r}(1 + y_2)^{b+s}.$$

This vanishes only if $\det\begin{pmatrix} a & r \\ b & s\end{pmatrix} = 0$. Therefore $(1 + y_1)^a(1 + y_2)^b$ does not act nilpotently on $H(2; \mathbb{1}; \Phi(\tau))^{(1)}$ if $(a, b) \neq (0, 0)$.

(4) Every one-section of $H(2; \mathbb{1}; \Phi(\tau))^{(1)}$ is of the form $\sum_{1 \leqslant i \leqslant p-1} F(1 + y_1)^{ia}(1 + y_2)^{ib}$ for fixed a, b. Hence it is abelian.

(5) The eigenvector $(1 + y_1)^a(1 + y_2)^b$ corresponds to the root (a, b). Then

$$\{(1 + y_1)^a(1 + y_2)^b, (1 + y_1)^r(1 + y_2)^s\} = (as - br)(1 + y_1)^{a+r}(1 + y_2)^{b+s},$$

and we put $f((a, b), (r, s)) := (as - br)$. ∎

THEOREM VII.4. *With the assumptions and notations of* (II.2) *let* $K + R$ *be as in case* (7) *with* $S = H(2; \mathbb{1}; \Phi(\tau))^{(1)}$. *Then* $K \cong H(2; \mathbb{1}; \Phi(\tau))^{(1)}$.

Proof. Let α be any nonzero root. There is, due the preceding theorem, $x \in K_\alpha$, which acts nonnilpotently on K. (II.1) shows that $\alpha(H) = 0$. Hence $(R \cap K) \subset (\bigcap_\alpha \ker \alpha) = 0$ and therefore K has codimension at least 2 in Der $H(2; \mathbb{1}; \Phi(\tau))^{(1)}$. As $H(2; \mathbb{1}; \Phi(\tau))^{(1)}$, which is contained in K also has codimension 2, we have equality. ∎

VIII. The Hamiltonian Algebra $H(2; \mathbb{1}; \Delta)$

The algebra $H(2; \mathbb{1}; \Delta)$ is described in [BW2, (2.1.8)] as follows: Put

$$D_\Delta(f) := D_2(f)\, D_1 - (D_1(f) + x^{((p-1)\varepsilon_1)} f)\, D_2$$
$$H(2; \mathbb{1}; \Delta) := \{D_\Delta(f) \mid f \in A(2; \mathbb{1})\}.$$

$H(2; \mathbb{1}; \Delta)$ is a simple algebra of dimension p^2 and $\operatorname{Der} H(2; \mathbb{1}; \Delta) = H(2; \mathbb{1}; \Delta) \oplus F(x^{(\varepsilon_1)} D_1 + x^{(\varepsilon_2)} D_2) = H(2; \mathbb{1}; \Delta)_p$ is the (p^2+1)-dimensional p-envelope. $H(2; \mathbb{1}; \Delta)$ carries the filtration inherited by the canonical filtration of $W(2; \mathbb{1})$.

Theorem VIII.1. *Assume that* $H(2; \mathbb{1}; \Delta) \subset K + R \subset \operatorname{Der} H(2; \mathbb{1}; \Delta)$. *Let R be a two-dimensional torus such that all roots with respect to R are proper. Then*

(1) $K + R = H(2; \mathbb{1}; \Delta) \oplus F(x^{(\varepsilon_1)} D_1 + x^{(\varepsilon_2)} D_2)$

(2) $R \subset (K + R) \cap W(2; \mathbb{1})_{(0)}$

(3) *Every root space of* $H(2; \mathbb{1}; \Delta)$ *is one-dimensional.*

Proof. Put in [BW2, (11.1.3)] $A := H(2; \mathbb{1}; \Delta) + F(x^{(\varepsilon_1)} D_1 + x^{(\varepsilon_2)} D_2)$ and $T := R$. Then it is mentioned in the proof there, that $R \not\subset H(2; \mathbb{1}; \Delta)$ (but $R \subset W(2; \mathbb{1})_{(0)}$). Thus by a dimension argument we have that $K + R = A$. A has $p^2 - 1$ nonzero roots of multiplicity one [BW2, (11.1.3)]. All these root spaces lie in $A^{(1)} \subset K$. As $\dim A = p^2 + 1$, the 0-root space has dimension two in A and hence has dimension 1 in $H(2; \mathbb{1}; \Delta)$. ∎

Corollary VIII.2. *With the assumptions and notations of* (II.2) *let* $K + R$ *be as in case* (7) *with* $S = H(2; \mathbb{1}; \Delta)$.

(1) *There are exactly two roots* α, β *such that* $K_{-\alpha}, K_{-\beta} \not\subset H(2; \mathbb{1}; \Delta)_{(0)}$. *For these holds* $(\mu = \alpha, \beta)$

$$\{K(\mu) + R\}/R \cap (\ker \mu) \cong W(1; \mathbb{1}).$$

The isomorphism is filtration preserving.

(2) $K(\alpha - \beta) \subset H(2; \mathbb{1}; \Delta)_{(0)} + R$; $\operatorname{rad} K(\alpha - \beta) \subset H(2; \mathbb{1}; \Delta)_{(1)} + R$;

$$K(\alpha - \beta)/\operatorname{rad} K(\alpha - \beta) \cong \mathrm{sl}(2).$$

(3) *Every root* $\gamma \notin GF(p)\alpha \cup GF(p)\beta \cup GF(p)(\alpha - \beta)$ *is solvable with* $K(\gamma) \subset R + H(2; \mathbb{1}; \Delta)_{(1)}$.

Proof. The graded algebra associated with the filtration of $H(2; \mathbb{1}; \Delta)$ determined by the maximal subalgebra

$$H(2; \mathbb{1}; \Delta)_{(0)} = H(2; \mathbb{1}; \Delta) \cap W(2; \mathbb{1})_{(0)}$$

is isomorphic to $H(2;\mathbb{1})^{(1)}+Fx_1^{p-1}\partial_2$. Since R acts on every subspace $H(2;\mathbb{1};\Delta)_{(j)}$, it acts as a two-dimensional torus on $H(2;\mathbb{1})^{(1)}+Fx_1^{p-1}\partial_2$ as well and the root spaces correspond. We now apply Theorem III.5 and the remark following it. ▮

IX. Restricted Cartan-Type Algebras

The following is a result of Demuskin [D1]. We shall give a noncomputational proof for it.

Theorem IX.1. *For every maximal torus T of $W(m;\mathbb{1})$ there is $\psi\in\operatorname{Aut}F[x_1,...,x_m]$ such that for some r, $0\leqslant r\leqslant m$,*

$$\psi^{-1}\circ T\circ\psi=T_r:=\sum_{1\leqslant i\leqslant r}F(1+x_i)\,\partial_i+\sum_{r+1\leqslant i\leqslant m}Fx_i\,\partial_i.$$

r is uniquely determined by T.

Proof. As T is a torus, it acts on the truncated polynomial ring $F[x_1,...,x_m]$ by semisimple endomorphisms. Consequently, the latter is the direct sum of eigenspaces with respect to T. Let V denote the space of polynomials without linear term and consider the canonical linear mapping

$$\pi\colon F[x_1,...,x_m]\to F[x_1,...,x_m]/V\cong\bigoplus_i Fx_i.$$

Thus there are eigenvectors $u_1,...,u_m$ such that $\bigoplus_{1\leqslant i\leqslant m}Fx_i=\bigoplus_{1\leqslant j\leqslant m}F\pi(u_j)$. Adjusting these eigenvectors by a scalar and permuting the indices if necessary we find polynomials $y_1,...,y_m$ in the generators $x_1,...,x_m$ such that

(a) $y_1,...,y_m$ have constant term 0,

(b) for some $r\in\{0,...,m\}$ $(1+y_1),...,(1+y_r)$, $y_{r+1},...,y_m$ are eigenvectors,

(c) $\pi(y_1),...,\pi(y_m)$ span $\bigoplus_{1\leqslant i\leqslant m}Fx_i$.

Note that condition (c) means, that $y_1,...,y_m$ generate $F[x_1,...,x_m]$ as an algebra.

Let ψ denote the automorphism of the truncated polynomial ring $F[x_1,...,x_m]$ defined by $\psi(x_i):=y_i$, $1\leqslant i\leqslant m$, and $\Psi\in\operatorname{Aut}W(m;\mathbb{1})$ given by $\Psi(D):=\psi^{-1}\circ D\circ\psi$. Put $\delta_i:=1$ for $1\leqslant i\leqslant r$ and $\delta_i:=0$ for $r+1\leqslant i\leqslant m$ and define $\alpha_i\in T^*$ by $D(\delta_i+y_i)=\alpha_i(D)(\delta_i+y_i)\;\forall D\in T$. Then

$$\begin{aligned}\Psi(D)(\delta_i+x_i)&=\psi^{-1}(D(\psi(\delta_i+x_i)))=\psi^{-1}(D(\delta_i+y_i))\\&=\psi^{-1}(\alpha_i(D)(\delta_i+y_i))=\alpha_i(D)(\delta_i+x_i).\end{aligned}$$

Hence $\Psi(D) = \sum \alpha_i(D)(\delta_i + x_i)\,\partial_i$, proving $\Psi(T) \subset T_r$. The maximality of T implies $T = \Psi^{-1}(T_r)$.

Note that $r = \dim T/T \cap W(m; \mathbb{1})_{(0)}$. ▮

Let S denote one of the algebras $W(2; \mathbb{1})$, $S(3; \mathbb{1})^{(1)}$, $H(4; \mathbb{1})^{(2)}$, $K(3; \mathbb{1})$. It is checked in [BW2], that the only optimal torus in S is conjugate to a torus $S \cap (\sum_i Fx_i\,\partial_i)$ under an automorphism of S. Every such automorphism of S can naturally considered an automorphism of Der S and $(\mathrm{Der}\, S)^{(k)}$ for all k.

LEMMA IX.2. *Let J be a restricted ideal of* Der S, *such that* $(\mathrm{Der}\, S)/J$ *is a torus, and $S \subset J$. Suppose that K is a subalgebra satisfying $S \subset K \subset \mathrm{Der}\, S$ and $TR(K) = 2$. Then $K \subset J$.*

Proof. Let K_p denote the p-envelope of K in Der S'. [St 4, (1.3.5)] yields that every torus of K_p is contained in $J \cap K_p + C'(K_p) \subset J$. Then $J + K_p/J$ is $[p]$-nilpotent and a torus. ▮

We now shall describe the one-sections in detail.

THEOREM IX.3. *With the assumptions and notations of* (II.2) *let $K + R$ as in case* (8) *with $S = W(2; \mathbb{1})$. Let α be a nonzero root.*

(1) *$K = W(2; \mathbb{1})$; there exists $\psi \in \mathrm{Aut}\, W(2; \mathbb{1})$ such that*

$$R = \psi\{Fx_1\,\partial_1 \oplus Fx_2\,\partial_2\}.$$

(2) *If $\alpha(\psi(x_1\,\partial_1)) = 0$, then*

$$K(\alpha) = \psi\left\{\sum_{0 \leq i \leq p-1} Fx_1x_2^i\,\partial_1 + \sum_{0 \leq i \leq p-1} Fx_2^i\,\partial_2\right\},$$

$$\mathrm{rad}\, K(\alpha) = \psi\left\{\sum_{0 \leq i \leq p-1} Fx_1x_2^i\,\partial_1\right\} \subset W(2; \mathbb{1})_{(1)} + F\psi(x_1\,\partial_1),$$

$$K(\alpha)/\mathrm{rad}\, K(\alpha) \cong \sum_{0 \leq i \leq p-1} Fx_2^i\,\partial_2 \cong W(1; \mathbb{1}).$$

(3) *If $\alpha(\psi(x_1\,\partial_1 + x_2\,\partial_2)) = 0$, then*

$$K(\alpha) \subset W(2; \mathbb{1})_{(0)},\ \mathrm{rad}\, K(\alpha) \subset W(2; \mathbb{1})_{(1)} + F\psi(x_1\,\partial_1 + x_2\,\partial_2),$$
$$K(\alpha)/\mathrm{rad}\, K(\alpha) \cong \mathrm{sl}(2).$$

(4) *If $\alpha(\psi(x_1\,\partial_1)) \neq 0$, $\alpha(\psi(x_2\,\partial_2)) \neq 0$, $\alpha(\psi(x_1\,\partial_1 + x_2\,\partial_2)) \neq 0$, then $K(\alpha) \subset R + W(2; \mathbb{1})_{(1)}$ is solvable.*

Proof. (1) follows from (IX.1).

(2)–(4) It is easy to check that for any root α

$$\psi^{-1}(K_\alpha) = Fx_1^a x_2^b \partial_1 + Fx_1^c x_2^d \partial_2,$$

with

$$a - 1 \equiv c \equiv \alpha(x_1 \partial_1), \qquad b \equiv d - 1 \equiv \alpha(x_2 \partial_2).$$

In particular, choosing α as claimed, we obtain the assertions. ∎

Let $-\alpha_1$, $-\alpha_2$ denote the nonclassical roots, which stick out of $K \cap W(2; \mathbb{1})_{(0)}$. Then

$$-\delta_{ij} = [x_i \partial_i, \partial_j] = -\alpha_j(x_i \partial_i)$$

and

$$(\alpha_1 - \alpha_2)(x_1 \partial_1 + x_2 \partial_2) = 0.$$

The root spaces of $K \cap W(2; \mathbb{1})_{(0)}$ sticking out of $W(2; \mathbb{1})_{(1)}$ are represented by $x_1 \partial_2$, $x_2 \partial_1$, R. The corresponding roots are $\pm(\alpha_1 - \alpha_2)$. $K(\alpha_1 - \alpha_2)$ is the classical nonsolvable one-section.

The special algebra $S(n; \mathbb{1})^{(1)}$ $(n \geqslant 3)$ is given in the following way. Put

$$D_{ij}(x^a) := Q_j x^{a - \varepsilon_j} \partial_i - a_i x^{a - \varepsilon_i} \partial_j,$$

$$S(n; \mathbb{1}) := \left\{ \sum f_i \partial_i \,\middle|\, \sum \partial_i(f_i) = 0 \right\}.$$

Then $S(n; \mathbb{1})^{(1)} = \operatorname{span}\{D_{ij}(x^a) \mid 1 \leqslant i < j \leqslant n,\ 0 < a \leqslant \tau(\mathbb{1})\}$ is a simple algebra of dimension $(n-1)(p^n - 1)$ [SF, (4.3.5), (4.3.7)]. In addition,

$$S(n; \mathbb{1}) = S(n; \mathbb{1})^{(1)} + \sum_{1 \leqslant j \leqslant n} \prod_{i \neq j} x_i^{p-1} \partial_j,$$

$$\operatorname{Der} S(n; \mathbb{1})^{(1)} = S(n; \mathbb{1}) \oplus Fx_1 \partial_1$$

[SF, (4.8.6), proof of (4.3.7)].

THEOREM IX.4. *With the assumptions and notations of* (II.2) *let* $K + R$ *be as in case* (8) *with* $S = S(3; \mathbb{1})^{(1)}$. *Let* α *be a nonzero root. Put* $V := Fx_1^{p-1}x_2^{p-1}\partial_3 \oplus Fx_1^{p-1}x_3^{p-1}\partial_2 \oplus Fx_2^{p-1}x_3^{p-1}\partial_1$.

(1) $S(3; \mathbb{1})^{(1)} \subset K \subset S(3; \mathbb{1})$; *there exists* $\psi \in \operatorname{Aut} S(3; \mathbb{1})$ *such that*

$$R = \psi\{F(x_1 \partial_1 - Fx_2 \partial_2) + F(x_1 \partial_1 - Fx_3 \partial_3)\}.$$

(2) *If* $\alpha(\psi(x_1\partial_1 - x_2\partial_2)) = 0$, *then*

$$K(\alpha) = K \cap \operatorname{span} \psi(\{D_{ij}(x^a) \mid a_1 - \delta_{1i} - \delta_{1j} - a_2 + \delta_{2i} + \delta_{2j} = 0\}) + K \cap \psi(V),$$

$$\operatorname{rad} K(\alpha) \subset K(\alpha) \cap \operatorname{span} \psi(\{D_{ij}(x^a) \mid a_1 + a_2 \geqslant 2\}) + K \cap \psi(V) \subset S(3; \mathbb{1})_{(1)} + R,$$

$$K(\alpha) = \operatorname{rad} K(\alpha) \oplus \psi\left\{\sum_{0 \leqslant i \leqslant p-1} F(ix_2 x_3^{i-1}\partial_2 - x_3^i\partial_3)\right\},$$

$$K(\alpha)/\operatorname{rad} K(\alpha) \cong W(1; \mathbb{1}).$$

(3) *If* $\alpha(\psi(2x_1\partial_1 - x_2\partial_2 - x_3\partial_3)) = 0$, *then*

$$K(\alpha) \subset S(3; \mathbb{1})_{(0)}, \operatorname{rad} K(\alpha) \subset S(3; \mathbb{1})_{(1)} + R,$$

$$K(\alpha)/\operatorname{rad} K(\alpha) \cong \operatorname{sl}(2).$$

(4) *If* $\alpha(\psi(x_i\partial_i - x_j\partial_j)) \neq 0$ *for* i, j, *and* $\alpha(\psi(2x_i\partial_i - x_j\partial_j - x_k\partial_k)) \neq 0$ *for all* i, j, k, *then* $K(\alpha) \subset R + S(3; \mathbb{1})_{(1)}$ *is solvable.*

Proof. (1) [SF, (4.8.6)] shows that $\operatorname{Der} S(3; \mathbb{1})^{(1)} = S(3; \mathbb{1}) + Fx_1\partial_1$. (IX.2) yields $K \subset S(3; \mathbb{1})$. (1) is then true according to earlier remarks.

(2) Note that $D_{ij}(x^a)$ is a root vector with respect to $\psi^{-1}(R)$. Let μ denote the corresponding root. $\mu(x_1\partial_1 - x_2\partial_2) = 0$ if and only if $a_1 - \delta_{1i} - \delta_{1j} - a_2 + \delta_{2i} + \delta_{2j} = 0$ [SF, 4.3.4)]. Thus

$$K(\alpha) = \psi(\operatorname{span}\{D_{ij}(x^a) \mid a_1 - \delta_{1i} - \delta_{1j} - a_2 + \delta_{2i} + \delta_{2j} = 0\}) + K \cap \psi(V).$$

Put $U := K(\alpha) \cap \operatorname{span} \psi(\{D_{ij}(x^a) \mid a_1 + a_2 \geqslant 2\}) + K \cap \psi(V)$. Note that $U \subset R + S(3; \mathbb{1})_{(1)}$. Moreover, a detailed but direct computation shows, that

$$K(\alpha) = U \oplus \psi\left\{\sum_{0 \leqslant i \leqslant p-1} F(ix_2 x_3^{i-1}\partial_2 - x_3^i\partial_3)\right\}.$$

We observe that $\psi(\partial_3) \in K(\alpha) \subset \psi(F\partial_3) + S(3; \mathbb{1})_{(0)}$ and $[\psi(\partial_3), U] \subset U \subset S(3; \mathbb{1})_{(0)}$. It is then clear, that the ideal of $K(\alpha)$ generated by U is contained in $S(3; \mathbb{1})_{(0)}$, which then has to be U itself.

(3) The only roots sticking out of $S(3; \mathbb{1})_{(0)}$ are of the type discussed in (2). Therefore $K(\alpha) \subset S(3; \mathbb{1})_{(0)}$ in the present case. More exactly, $K(\alpha) \subset \psi(Fx_2\partial_3 + Fx_3\partial_2) + R + S(3; \mathbb{1})_{(1)}$. Then $\operatorname{rad} K(\alpha) \subset S(3; \mathbb{1})_{(1)} + R$ and $K(\alpha)/\operatorname{rad} K(\alpha) \cong \operatorname{sl}(2)$.

(4) The only nonzero roots sticking out of $S(3;\mathbb{1})_{(1)}$ are of types discussed in (2), (3). Thus $K(\alpha)\subset S(3;\mathbb{1})_{(1)}+R$. ∎

Define roots α, β according to $\alpha(x_1\partial_1-x_2\partial_2)=1$, $\alpha(x_1\partial_1-x_3\partial_3)=0$, $\beta(x_1\partial_1-x_2\partial_2)=0$, $\beta(x_1\partial_1-x_3\partial_3)=1$. Then the roots sticking out of $S(3;\mathbb{1})_{(0)}$ are represented by ∂_1, ∂_2, ∂_3, and hence are $-\alpha-\beta$, α, β. These roots are of Witt type. Similarly, the roots of $S(3;\mathbb{1})_{(0)}$ sticking out of $S(3;\mathbb{1})_{(1)}$ are 0, $\pm(2\alpha+\beta)$, $\pm(2\beta+\alpha)$, $\pm(\alpha-\beta)$. The corresponding one-sections are classical.

THEOREM IX.5. *With the assumptions and notations of* (II.2) *let* $K+R$ *be as in case* (8) *with* $S=H(4;\mathbb{1})^{(2)}$.

(1) $H(4;\mathbb{1})^{(2)}\subset K\subset H(4;\mathbb{1})$; *there is* $\psi\in\operatorname{Aut}H(4;\mathbb{1})$ *such that*

$$R=\psi\{F(x_1\partial_1-x_3\partial_3)\oplus F(x_2\partial_2-x_4\partial_4)\};$$

$$H=K\cap\operatorname{span}\psi(\{D_H(x_1^a x_3^a x_2^b x_4^b)\mid 0\leqslant a,b\leqslant p-1\}).$$

(2) *If* $\alpha(\psi(x_1\partial_1-x_3\partial_3))=0$, $\alpha\neq 0$, *then*

$$K(\alpha)=\operatorname{span}\psi(\{D_H(x_1^a x_3^a x_2^i x_4^j)\mid 0\leqslant a,i,j\leqslant p-1, i\neq j\})+H,$$

$$\operatorname{rad}K(\alpha)\subset\operatorname{span}\psi(\{D_H(x_1^a x_3^a x_2^i x_4^j)\mid i\neq j, 0<a\})+H$$
$$\subset H(4;\mathbb{1})_{(1)}+R,$$

$K(\alpha)/\operatorname{rad}K(\alpha)$ *is of hamiltonian type.*

(3) *Let* α,β *be the roots given by* $\alpha(\psi(x_1\partial_1-x_3\partial_3))=1$, $\alpha(\psi(x_2\partial_2-x_4\partial_4))=0$, $\beta(\psi(x_1\partial_1-x_3\partial_3)=0$, $\beta(\psi(x_2\partial_2-x_4\partial_4))=1$. *Then* $K(\alpha+\beta)$, $K(\alpha-\beta)$ *are contained in* $H(4;\mathbb{1})_{(0)}$, $\operatorname{rad}K(\alpha\pm\beta)\subset H(4;\mathbb{1})_{(1)}+R$, *and* $K(\alpha\pm\beta)/\operatorname{rad}K(\alpha\pm\beta)\cong\mathrm{sl}(2)$.

(4) *All one-sections different from* $K(\alpha)$, $K(\beta)$, $K(\alpha\pm\beta)$ *are contained in* $H(4;\mathbb{1})_{(1)}+R$ *and hence are solvable.*

Proof. (1) [SF, (4.8.7)] shows that $\operatorname{Der}H(4;\mathbb{1})^{(2)}=H(4;\mathbb{1})+F(\sum x_i\partial_i)$. (IX.2) yields $K\subset H(4;\mathbb{1})$. (1) is then true according to earlier remarks and simple computations.

(2) It is clear, that the right hand side vector space is exactly the eigenspace of $(x_1\partial_1-x_3\partial_3)$ for the eigenvalue 0. This proves the first part. Then $J:=\operatorname{span}\psi(\{D_H(x_1^a x_3^a x_2^i x_4^j)\mid 0<a\})$ is invariant under $K(\alpha)$, and contained in $H(4;\mathbb{1})_{(0)}+R$. Thus J is solvable and $K(\alpha)/J\cap K(\alpha)$ is a nonsolvable homomorphic image of span $\psi(\{D_H(x_2^i x_4^j)\mid i\neq j\})+H$. Then $K(\alpha)/J$ is of Hamiltonian type.

(3) We observe, that $G:=FD_H(x_1x_2)\oplus FD_H(x_1x_3+x_2x_4)\oplus FD_H(x_3x_4)$ is a subalgebra, isomorphic to sl(2). It is a simple computation,

that $\psi(G) \subset K(\alpha+\beta)$, $\operatorname{rad} K(\alpha+\beta) \subset H(4; \mathbb{1})_{(1)} + R$. Hence $K(\alpha+\beta)/\operatorname{rad}(\alpha+\beta) \cong \mathrm{sl}(2)$. The assertion on $\alpha-\beta$ follows by symmetry.

(4) The only roots sticking out of $H(4; 1)_{(0)}$ are represented by $D_H(x_1), ..., D_H(x_4)$ and hence are $\pm\alpha$, $\pm\beta$. Let γ be a root with $K_\gamma \subset H(4; \mathbb{1})_{(0)}$, $K_\gamma \not\subset H(4; \mathbb{1})_{(1)}$. Then $[K, K_\gamma] \not\subset H(4; \mathbb{1})_{(0)}$ and therefore there are roots μ, $\delta \in \{\pm\alpha, \pm\beta\}$ with $[K_\delta, K_\gamma] \subset K_\mu$. Therefore $\gamma \in \{0, \pm 2\alpha, \pm 2\beta, \pm(\alpha \pm \beta)\}$. ∎

Let $\pm\alpha$, $\pm\beta$ denote the nonclassical roots, which stick out of $H(4; \mathbb{1})_{(0)}$. Then we obtain after a suitable adjustment

$$\partial_1 \in K_\alpha, \qquad \partial_3 \in K_{-\alpha}, \; \partial_2 \in K_\beta, \qquad \partial_4 \in K_{-\beta},$$
$$x_1 \partial_3 \in K_{-2\alpha}, \qquad x_3 \partial_1 \in K_{2\alpha}, \; x_2 \partial_4 \in K_{-2\beta}, \qquad x_4 \partial_2 \in K_{2\beta}$$
$$x_1 \partial_2 \in K_{\beta-\alpha}, \qquad x_2 \partial_1 \in K_{\alpha-\beta}, \; x_1 \partial_4 \in K_{-\alpha-\beta}, \qquad x_4 \partial_1 \in K_{\alpha+\beta}.$$

Thus the roots sticking out of $H(4; \mathbb{1})_{(1)}$ are $\{0, \pm\alpha, \pm\beta, \pm 2\alpha, \pm 2\beta, \pm(\alpha \pm \beta)\}$.

The contact algebra $K(2r+1; \mathbb{1})^{(1)}$ is given in the following way. Define on $A(2r+1; \mathbb{1})$ a Lie bracket by means of

$$\langle x^{(a)}, x^{(b)} \rangle := \sum_{1 \leqslant i \leqslant r} \left\{ \binom{a+b-\varepsilon_i-\varepsilon_{i+r}}{b} - \binom{a+b-\varepsilon_i-\varepsilon_{i+r}}{a} \right\} x^{(a+b-\varepsilon_i-\varepsilon_{i+r})}$$
$$+ \left\{ \|b\| \binom{a+b-\varepsilon_{2r+1}}{b} - \|a\| \binom{a+b-\varepsilon_{2r+1}}{a} \right\} x^{(a+b-\varepsilon_{2r+1})},$$

where $\|a\| := \sum_{0 \leqslant i \leqslant 2r} a_i + 2a_{2r+1} - 2$. If $2r+4 \not\equiv 0 \bmod (p)$ then $K(2r+1; \mathbb{1})$ is simple of dimension p^{2r+1}, while otherwise $K(2r+1; \mathbb{1})^{(1)}$ is simple of dimension $p^{2r+1} - 1$.

We have for $2r+1=3$ [SF, p. 173(iii), (iv)]

$$\langle x^{(\varepsilon_1+\varepsilon_2)}, x^{(a)} \rangle = (a_2 - a_1)\, x^{(a)}$$
$$\langle x^{(\varepsilon_3)}, x^{(a)} \rangle = \|a\|\, x^{(a)}.$$

THEOREM IX.6. *With the assumptions and notations of* (II.2) *let* $K+R$ *be as in case* (8) *with* $S = K(3; \mathbb{1})$.

(1) $K = K(3; \mathbb{1})$; *there is* $\psi \in \operatorname{Aut} K(3; \mathbb{1})$ *such that* $R = \psi(Fx^{(\varepsilon_1+\varepsilon_2)} \oplus Fx^{(\varepsilon_3)})$. *Define* α, β *by*

$$\alpha(x^{(\varepsilon_1+\varepsilon_2)}) = 1, \qquad \alpha(x^{(\varepsilon_3)}) = 0,$$
$$\beta(x^{(\varepsilon_1+\varepsilon_2)}) = 0, \qquad \beta(x^{(\varepsilon_3)}) = 1.$$

(2) $K(\beta)=\psi\ \{\sum Fx^{(a)}\,|\,a_1=a_2\}\subset F\psi(x^{(0)})+R+K(3;\mathbb{1})_{(1)}$,

$$K(-\alpha-\beta)=\psi\left\{\sum Fx^{(a)}\,|\,a_1+a_3\equiv 1 \bmod(p)\right\}$$
$$\subset F\psi(x^{(\varepsilon_1)})+R+K(3;\mathbb{1})_{(1)}.$$
$$K(\alpha-\beta)=\psi\left\{\sum Fx^{(a)}\,|\,a_2+a_3\equiv 1 \bmod(p)\right\}$$
$$\subset F\psi(x^{(\varepsilon_2)})+R+K(3;\mathbb{1})_{(1)}.$$

The solvable radical of each of these one-sections is contained in $R+K(3;\mathbb{1})_{(1)}$. *These one-sections are if Witt type.*

(3) $K(\alpha)=\psi\ \{\sum Fx^{(a)}\,|\,\|a\|=0\}\subset K(3;\mathbb{1})_{(0)}$,

$$\operatorname{rad} K(\alpha)\subset K(3;\mathbb{1})_{(1)}+R,$$
$$K(\alpha)/\operatorname{rad} K(\alpha)\cong \mathrm{sl}(2).$$

(4) *All one-sections different from* $K(\beta)$, $K(-\alpha-\beta)$, $K(\alpha-\beta)$, $K(\alpha)$ *are contained in* $K(3;\mathbb{1})_{(1)}+R$ *and hence are solvable.*

Proof. (1) Der $K(3;\mathbb{1})\cong K(3;\mathbb{1})$ [SF, (4.8.8)]. Earlier remarks prove (1).

(2)–(4) $x^{(a)}$ is a root vector for $Fx^{(\varepsilon_1+\varepsilon_2)}\oplus Fx^{(\varepsilon_3)}$, the root being $i\alpha+j\beta$ with

$$(a_2-a_1)\,x^{(a)}=\langle x^{(\varepsilon_1+\varepsilon_2)}, x^{(a)}\rangle = ix^{(a)},$$
$$\|a\|\,x^{(a)}=\langle x^{(\varepsilon_3)}, x^{(a)}\rangle = jx^{(a)}.$$

Hence $x^{(a)}\in K(\beta)$ if and only if $a_2=a_1$, and $x^{(a)}\in K(-\alpha-\beta)$ if and only if $a_2-a_1\equiv\|a\| \bmod(p)$. The latter means $a_2-a_1\equiv a_1+a_2+2a_3-2 \bmod(p)$, i.e., $a_1+a_3\equiv 1$. The other one-section are treated similarly. ▌

With the adjustments of the theorem we have

$$x^{(0)}\in K_{-2\beta},\qquad x^{(\varepsilon_1)}\in K_{-\alpha-\beta},\qquad x^{(\varepsilon_2)}\in K_{\alpha-\beta},$$
$$x^{(2\varepsilon_2)}\in K_{2\alpha},\qquad x^{(2\varepsilon_1)}\in K_{-2\alpha}.$$

Thus the roots sticking out of $K(3;\mathbb{1})_{(1)}$ are $\{0, \pm 2\alpha, -2\beta, \pm\alpha-\beta\}$.

X. Case (5) of Theorem II.2

As we have pointed out in Section II, we need some additional arguments to settle case (5) of (II.2). They go along the lines of

[BW2, (9.1.1.b] and [BOSt, (2.3)]. We use the notion of extended roots (cf. [St4]): if T is a maximal torus and α, β are roots, let $x_s \in T$ denote the semisimple part of a root vector $x \in L_\alpha$. Put $\beta(x) := \beta(x_s)$.

LEMMA X.1. *With the assumptions and notations of* (II.2) *let* γ, δ *be roots such that* $K(\gamma, \delta) + R$ *is of types* (7) *or* (8). *Suppose that* $\delta(C_L(T)) \neq 0$.

(1) *If* $w \in L_\delta \cap \operatorname{rad} L(\delta)$ *then* $\gamma(w) = 0$.

(2) *Let* $Q(\delta)$ *denote the distinguished maximal compositionally classical subalgebra of* $L(\delta)$ *and suppose that* $w \in [L_{-\delta} \cap Q(\delta), L_\delta \cap \operatorname{rad} L(\delta)]$. *Then* $\gamma(w) = 0$.

Proof. Consider the homomorphism $\sigma: L(\gamma, \delta) \to K(\gamma, \delta) =: K$ mentioned in (II.2). As $\sigma(\operatorname{rad} L(\delta)) = \operatorname{rad} K(\delta)$, we may argue in K. Since $\delta(H) \neq 0$, we have $\operatorname{rad} K(\delta) = 0$ if K is of type (7) with $S \cong W(1; \mathbf{2})$ (Theorem V.4.(3)) or $S \cong H(2; (2, 1))^{(2)}$ (VI.3). Type (7) with $S \cong H(2; \mathbb{1}; \Phi(\tau))^{(1)}$ does not occur, since in that case $\mu(H) = 0$ for all μ. For type (7) with $S \cong H(2; \mathbb{1}; \Delta)$ and type (8) we might consider K as a subalgebra of some suitable $W(m; \mathbb{1})$. We observe (Theorems (VIII.2), (IX.3)–(IX.6)), that $\operatorname{rad} K(\delta) \subset W(m; \mathbb{1})_{(1)} + R$ and $\sigma(Q(\delta)) \subset W(m; \mathbb{1})_{(0)}$ in all these cases. Thus $\sigma(w) \in W(m; \mathbb{1})_{(1)}$.

As a conclusion, $\sigma(w)$ acts nilpotently on K in all cases. Moreover, there is a root $\mu = i\gamma + j\delta$, $(i \neq 0)$ with $K_\mu \neq 0$. So $\mu(\sigma(w)) = 0$ and therefore $0 = \mu(w) = i\gamma(x)$, i.e. $\gamma(w) = 0$. ∎

LEMMA X.2. *With the assumptions and notations of* (II.2) *let* $K + R$ *be as in case* (5). *Let* M *denote the restricted subalgebra of* $\operatorname{Der}(S \otimes A(n; \mathbb{1}))$ *generated by* $K + R$ *and* $H' := C_M(R)$. *Then*

(1) $R \cap \{S \otimes A(n; \mathbb{1})\}$ *is one-dimensional.*

(2) *There is a root* β *such that*

(a) $\beta(R \cap \{S \otimes A(n; \mathbb{1})\}) = 0$,

(b) $K \subset S \otimes A(n; \mathbb{1}) + K(\beta)$,

(c) $M(\beta) \cap \{S \otimes A(n; \mathbb{1})\} \subset \operatorname{rad} K(\beta)$.

(3)(a) β *is nonclassical.*

(b) $[Q(\beta) + H', S \otimes A(n, \mathbb{1})_{(1)}] \subset S \otimes A(n; \mathbb{1})_{(1)}$.

Proof. Put $G := S \otimes A(n; \mathbb{1})$, $J := S \otimes A(n; \mathbb{1})_{(1)}$. We identify G with $\operatorname{ad}_G G \subset \operatorname{Der} G$. The associative pth power mapping turns $\operatorname{Der} G$ into a restricted algebra. The $[p]$-mapping on $R \subset \operatorname{Der} G$ then coincides with the p-structure on $\operatorname{Der} G$. In the present situation S is a restricted algebra, so

G is restricted and again the $[p]$-structure coincides with the p-structure on Der G. Thus we may consider G and R as restricted subalgebras of M.

(1) Note that $\dim R \cap G \leqslant TR(G) = TR(S) = 1$. Since every pth power of a root vector of G is contained in $C_G(R)$, and $R \cap C_G(R)$ is a maximal torus of $C_G(R)$ we obtain $R \cap G \neq 0$. Let t be a toral element with $Ft = R \cap G$.

(2) If μ is any root with $\mu(t) \neq 0$, then $K_\mu \subset G$. Choose a root $\beta \neq 0$ with $\beta(t) = 0$. Then $K = G + K(\beta)$. Consider the homomorphism

$$\pi: G \to S \otimes A(n; \mathbb{1})/S \otimes A(n; \mathbb{1})_{(1)} \cong S.$$

$F\pi(t)$ is a torus of S, so it is a maximal torus. As $\pi(M(\beta) \cap G) \subset C_S(\pi(t))$ we have that $\pi(M(\beta) \cap G)$ is a triangulable algebra. $\ker \pi$ is a nilpotent ideal, and therefore $M(\beta) \cap G$ is a solvable ideal of $K(\beta)$. Thus $M(\beta) \cap G \subset \operatorname{rad} K(\beta)$.

(3) Take any root vector $y \in M_\mu$ for some root $\mu \in GF(p)\alpha + GF(p)\beta$ (including 0). Consider $I_y := J + [y, J]$. This is an ideal of G containing J. If $I_y = J$ for all $y \in \bigcup_\mu M_\mu$, then J would be a nilpotent ideal of M, which is impossible. Thus there is μ and $y \in M_\mu$ such that $I_y \neq J$. Since J is a maximal ideal of G we obtain $I_y = G$. It follows that

$$G = J + [y, J] = J + [y, G].$$

Recall that $t \in R \cap G$ is toral and that $\beta(t) = 0$. The above equation is only possible if $y \notin G$ and hence $\mu \in GF(p)\beta$, $\mu(t) = 0$, $[t, y] = 0$. Write

$$t = u + \sum_\lambda [y, v_\lambda], \qquad u \in J, \qquad v_\lambda \in G_\lambda.$$

Then

$$\sum_\lambda [y, \lambda(t) v_\lambda] = \sum_\lambda [y, [t, v_\lambda]] = -[t, u] \in J.$$

Hence we may assume that $\lambda(t) = 0$ for every λ occurring in the above sum, i.e., $t \in J + [y, G \cap K(\beta)] \subset J + G \cap K(\beta)$.

Suppose that all $[y, v_\lambda]$ occurring in the presentation of t act nilpotently on K. Then we recall from (2) that $\pi(K(\beta) \cap G)^{(1)}$ acts nilpotently on S, and so $\pi(t)$ would act nilpotently on S. This contradiction shows that there is $v_\lambda \in G \cap K(\beta)$ such that $[y, v_\lambda]$ acts nonnilpotently.

We lift this information to L: Let M' be the restricted subalgebra of L_p generated by $L(\alpha, \beta) + T$. Then $I(\alpha, \beta) = \operatorname{rad}\{L(\alpha, \beta) + T\} = \{\operatorname{rad} M'\} \cap \{L(\alpha, \beta) + T\}$ and therefore $K + R$ embeds canonically into

$M'/\text{rad}\, M' =: \bar{M}$. In fact, $\bar{M}$ is a semisimple p-envelope of $K+R$. As M is also a semisimple p-envelope, they are both minimal p-envelopes and hence canonically isomorphic (even as restricted algebras, since they are centerless). Then there are root vectors $y' \in M'_\mu$, $v'_\lambda \in L_\lambda$, such that $w := [y', v'_\lambda] \in \text{rad}\, L(\beta)$ acts nonnilpotently on L. Since $C_{L_p}(T)$ acts triangulably, we obtain as a first consequence that $\mu \neq 0$ or $\lambda \neq 0$. As a further observation, $\Omega := \{\rho \in \Phi \mid \rho(w) \neq 0\}$ is nonvoid, and the simplicity of L implies $L = \sum_{\rho \in \Omega} L_\rho + \sum_{\rho, \rho' \in \Omega} [L_\rho, L_{\rho'}]$. In the present case we have $\beta(C_L(T)) \neq 0$. Then there exists $\gamma \in \Omega$ with $\beta([L_\gamma, L_{-\gamma}]) \neq 0$. The two-section $L(\beta, \gamma)$ is necessarily of type (II.2.(7)) or (II.2.(8)) (cf. [BW2, (10.2.1)]. In case $\lambda + \mu \neq 0$ we would have that $w \in L_{\lambda+\mu} \cap \text{rad}\, L(\lambda+\mu)$, $(\lambda+\mu)(C_L(T)) \neq 0$, $\gamma(w) \neq 0$. This contradicts (X.1). So $\lambda+\mu = 0$, $\mu \neq 0$, $GF(p)\mu = GF(p)\beta$ and $y' \in M'_\mu \subset L$. The assumption $y' \in Q(\beta)$ would imply $w \in [L_\mu \cap Q(\mu), L_{-\mu} \cap \text{rad}\, L(\mu)]$. This again contradicts (X.1).

Consequently, $y' \in L$ and $y' \notin Q(\beta)$. Therefore β cannot be classical. Moreover, as $\mu \neq 0$, we have $y \notin \sigma(Q(\beta)) + H'$ and the choice of y in $K(\alpha, \beta)$ yields $[\sigma(Q(\beta)) + H', J] \subset J$. ∎

Lemma X.3. *With the assumptions and notations of* (X.2) *the following are true*:

(1) *R stabilizes $S \otimes A(n; \mathbb{1})_{(1)}$.*

(2) *$K(\beta) \cap \{(\text{Der}\, S) \otimes A(n; \mathbb{1})\} \subset \text{rad}\, K(\beta)$.*

(3) *If $n=1$ then β is Witt, if $n=2$ then β is hamiltonian.*

Proof. (1) As $R \subset H'$, this is a direct consequence of (X.2.3.b).

(2) Since $(\text{Der}\, S) \otimes A(n; \mathbb{1})/S \otimes A(n; \mathbb{1})$ is solvable, (X.2.(2.c)) yields the result.

(3) Consider the homomorphism

$$\pi\colon \text{Der}\{S \otimes A(n; \mathbb{1})\} \to \text{Der}\{S \otimes A(n; \mathbb{1})\}/(\text{Der}\, S) \otimes A(n; \mathbb{1}) \cong W(n; \mathbb{1}).$$

$\pi(K(\beta))$ is a nonclassical algebra. A dimension argument ensures that for $n=1$ β is Witt.

Suppose that $n=2$ and β is Witt. Then there is $x \in L_{-\beta}$ with $L(\beta) = Q(\beta) + Fx$. Recall that $S \otimes A(2; \mathbb{1})$ has ideals $S \otimes A(2; \mathbb{1})_{(k)} = \sum_{i+j \geqslant k} S \otimes Fx_1^i x_2^j$. This is the kth power of $S \otimes A(2; \mathbb{1})_{(1)}$. Since $\sigma(Q(\beta)) + H'$ stabilizes $S \otimes A(2; \mathbb{1})_{(1)}$, and $\sigma(x)^{[p]} \in H'$, we have that

$$\sum_{0 \leqslant i \leqslant p-1} (\text{ad}\, \sigma(x))^i (S \otimes A(2; \mathbb{1})_{(2p-1)}) \subset S \otimes A(2; \mathbb{1})_{(p)}$$

is a nonzero solvable ideal of $K+R$, a contradiction. ∎

We are now able to normalize the torus R.

THEOREM X.4. *With the assumptions and notations of* (II.2) *let* $K+R$ *be as in case* (5). *If* $S\cong \mathrm{sl}(2)$ *take a canonical basis* (e,h,f), *if* $S\cong W(1;\mathbb{1})$ *write* $S=\sum_{0\leqslant i\leqslant p-1} Fy^iD$, *if* $S\cong H(2;\mathbb{1})^{(2)}$ *write* $S=\sum_{(0,0)<(a,b)<(p-1,p-1)} F(ay_1^{a-1}y_2^bD_2-by_1^ay_2^{b-1}D_1)$. *Then there is* $\psi\in\mathrm{Aut\,Der}(S\otimes A(n;\mathbb{1}))$, *which stabilizes* $S\otimes A(n;\mathbb{1})$, *such that* $\psi^{-1}(R)$ *is one of the following*:

$\underline{n=1}$	$S\cong \mathrm{sl}(2)$	$Fh\otimes 1+F\,\mathrm{id}\otimes x\,\partial$
	$S\cong W(1;\mathbb{1})$	$FyD\otimes 1+F\,\mathrm{id}\otimes x\,\partial$
	$S\cong H(2;\mathbb{1})^{(2)}$	$F(y_1D_1-y_2D_2)\otimes 1$ $+F\{m(y_1D_1+y_2D_2)\otimes 1+\mathrm{id}\otimes x\,\partial\},\ m\in GF(p)$
$\underline{n=2}$	$S\cong \mathrm{sl}(2)$	$Fh\otimes 1+F\,\mathrm{id}\otimes(x_1\,\partial_1-x_2\,\partial_2)$
	$S\cong W(1;\mathbb{1})$	$FyD\otimes 1+F\,\mathrm{id}\otimes(x_1\,\partial_1-x_2\,\partial_2)$
	$S\cong H(2;\mathbb{1})^{(2)}$	$F(y_1D_1-y_2D_2)\otimes 1$ $+F\{m(y_1D_1+y_2D_2)\otimes 1+\mathrm{id}\otimes(x_1\,\partial_1-x_2\,\partial_2)\}$, $m\in GF(p)$.

Proof. (1) According to (X.2) $R\cap(S\otimes A(n;\mathbb{1}))$ contains some toral element r_1. Decompose r_1 into its homogeneous components $r_1=d_1\otimes 1+\sum_{k\geqslant l}\omega_k$, $\omega_k=\sum_{|i|=k} d_{(i)}\otimes x^{(i)}$, $l>0$.

$$r_1=r_1^{[p]}\equiv d_1^{[p]}\otimes 1+\sum_{|i|=l}\{\mathrm{ad}\,d_1\}^{p-1}(d_{(i)})\otimes x^{(i)}$$
$$\mathrm{mod}\ S\otimes A(n;\mathbb{1})_{(l+1)}.$$

Then d_1 is toral and $\{\mathrm{ad}\,d_1\}^{p-1}(d_{(i)})=d_{(i)}$ for all $d_{(i)}$ with $|i|=l$. We therefore may assume that every $d_{(i)}$, $(|i|=l)$ is an eigenvector for d_1 with nonzero eigenvalue.

Note that as $\{x^{(i)}\}^p=0$ for every monomial with $|i|\neq 0$, the mapping

$$\exp\{\mathrm{ad}\,(\lambda d_{(i)}\otimes x^{(i)})\}=\sum_{0\leqslant j\leqslant p-1}(j!)^{-1}\lambda^j\{\mathrm{ad}\,d_{(i)}\}^j\otimes\{x^{(i)}\}^j$$

is an automorphism of $S\otimes A(n;\mathbb{1})$. A successive application of suitable automorphisms of this type reduces the number of monomials of degree l occurring in r_1, and so eventually raises the degree itself. Hence there is an automorphism $\psi_1\in\mathrm{Aut}\{S\otimes A(n;\mathbb{1})\}$ with $\psi_1(r_1)=d_1$. Next we suppress the notion of ψ_1 and assume that $r_1=d_1\otimes 1$, d_1 toral in S. Choose β as in (X.2), $\beta(r_1)=0$. There is a toral element $r_2 \in R$ with $\beta(r_2) \neq 0$. Recall, that

$$\mathrm{Der}(S \otimes A(n; \mathbb{1})) = (\mathrm{Der}\, S) \otimes A(n; \mathbb{1}) + F\,\mathrm{id} \otimes W(n; \mathbb{1}).$$

Write accordingly

$$r_2 = \tilde{d}_1 \otimes 1 + \sum_{|i| \geq l} d_{(i)} \otimes x^{(i)} + id \otimes d_2$$

and observe that $d_2 \in W(n; \mathbb{1})_{(0)}$, since r_2 stabilizes $S \otimes A(n; \mathbb{1})_{(1)}$. Put $D := \tilde{d}_1 \otimes 1 + id \otimes d_2$. As above

$$r_2 = r_2^{[p]} \equiv \tilde{d}_1^{[p]} \otimes 1 + id \otimes d_2^{[p]} + D^{p-1}\left(\sum_{|i|=l} d_{(i)} \otimes x^{(i)}\right)$$

$$\mathrm{mod}(\mathrm{Der}\, S) \otimes A(n; \mathbb{1})_{(l+1)}.$$

We conclude that $\tilde{d}_1$, d_2 are toral. As $d_2 \in W(n; \mathbb{1})_{(0)}$ there is $\psi_2 \in \mathrm{Aut}\, A(n; \mathbb{1})$, such that $\psi_2(x_1), ..., \psi_2(x_n)$ are eigenvectors with respect to d_2 (see also the proof of (IX.1)). ψ_2 induces an automorphism $id \otimes \psi_2$ of $S \otimes A(n; \mathbb{1})$. Note that $(\mathrm{id} \otimes \psi_2)(r_1) = r_1$.

Thus we may assume that every $d_{(i)} \otimes x^{(i)}$ $(|i| = l)$ is an eigenvector for D with nonzero eigenvalue. Applying as above successively automorphisms of type $\exp\{\mathrm{ad}(\lambda d_{(i)} \otimes x^{(i)})\}$ we find $\psi_3 \in \mathrm{Aut}\{S \otimes A(n; \mathbb{1})\}$ with $r_2 = \psi_3 \circ (\tilde{d}_1 \otimes 1 + id \otimes d_2) \circ \psi_3^{-1}$. Moreover, we have $0 = [r_1, r_2] = [d_1, \tilde{d}_1] \otimes 1 + \sum [d_1, d_{(i)}] \otimes x^{(i)}$, and hence $[d_1, \tilde{d}_1] = 0 = [d_1, d_{(i)}]$, which yields $\psi_3 \circ r_1 \circ \psi_3^{-1} = r_1$. d_1, $\tilde{d}_1$ are toral elements in Der S. In case that R acts on S as a one-dimensional torus then $d_1, \tilde{d}_1$ are linearly dependent. So we may assume $\tilde{d}_1 = 0$ and a multiple of d_1 is conjugate under an automorphism ψ_4 of S to h $(S \cong \mathrm{sl}(2))$, yD $(S \cong W(1; \underline{1}))$, $y_1 D_1 - y_2 D_2$ $(S \cong H(2, \mathbb{1})^{(2)})$. In the last case $S \cong H(2; \mathbb{1})^{(2)}$ the assertion is true for $m = 0$.

Otherwise $S \cong H(2; \mathbb{1})^{(2)}$, R acts on S as a two-dimensional torus and so $S + R$ is of type (II.2.(4)). Then we find $\psi_4 \in \mathrm{Aut}\, S$ and r, $s \in F$, so that $\psi_4(rd_1) = y_1 D_1 - y_2 D_2$, $\psi_4(rd_1 + s\tilde{d}_1) = y_1 D_1 + y_2 D_2$.

Since d_2 stabilizes $A(n; \mathbb{1})_{(1)}$ we find $\psi_5 \in \mathrm{Aut}\, A(n; \mathbb{1})$ such that $t := \psi_5 \circ d_2 \circ \psi_5^{-1} \in Fx\,\partial$ if $n = 1$, and $t \in Fx_1\,\partial_1 + Fx_2\,\partial_2$ if $n = 2$. Consider the case that $n = 2$. We want show that $id \otimes t$ is mapped under the homomorphism

$$\eta\colon \mathrm{Der}(S \otimes A(2; \mathbb{1})) \to \mathrm{Der}(S \otimes A(2; \mathbb{1}))/(\mathrm{Der}\, S) \otimes A(2; \mathbb{1})$$

$$\cong W(2; \mathbb{1})$$

into $F(x_1\partial_1 - x_2\partial_2)$. (X.3) yields that $\eta(K(\beta))$ is of Hamiltonian type. As $\eta(K(\beta)) \cap W(2;\mathbb{1})_{(0)}$ is a compositionally classical subalgebra of codimension $\leqslant 2$, it has to be the distinguished maximal compositionally classical subalgebra in $\eta(K(\beta))$. Choose $h \in H \cap \sigma(Q(\beta))^{(1)}$, $\beta(h) \neq 0$. Then some $r_2 := ah^{(p)^k} + bd_1 \otimes 1$ is toral with $\beta(r_2) \neq 0$. Using this element we construct d_2, t as above. Every automorphism of $S \otimes A(2;\mathbb{1})$ induces via the extension to Der $S \otimes A(n;\mathbb{1})$ and composition with η an automorphism of $W(2;\mathbb{1})$, and as such it stabilizes $W(2;\mathbb{1})_{(0)}$. Thus h is mapped into $W(2;\mathbb{1})^{(1)}_{(0)}$. As the latter is closed under the $[p]$-mapping, we obtain

$$\eta(id \otimes t) \in \{Fx_1\partial_1 + Fx_2\partial_2\} \cap W(2;\mathbb{1})^{(1)}_{(0)} = F(x_1\partial_1 - x_2\partial_2).$$

Thus we may adjust r_2, so that $\psi_4(r'd_1 + s'\tilde{d}_1) = m(y_1 D_1 + y_2 D_2)$ and $\psi_5 \circ d_2 \circ \psi_5^{-1} = x\partial$ (if $n=1$) and $= x_1\partial_1 - x_2\partial_2$ (if $n=2$). To have r_2 toral means $m \in GF(p)$. ψ_5 extends to an automorphism of Der $S \otimes A(n;\mathbb{1})$ which leaves d_1, $\tilde{d}_1$ unchanged. Putting these automorphisms together we obtain $\psi \in$ Aut Der$(S \otimes A(n;\mathbb{1}))$, so that $\psi^{-1}(R)$ is of the claimed form. All automorphisms occurring in this context are induced by automorphisms of $S \otimes A(n;\mathbb{1})$. Hence $S \otimes (n;\mathbb{1})$ is invariant under ψ. ▮

We apply these results to determine root spaces (see also [BOSt, (2.7)]).

THEOREM X.5. *With the assumptions and notations of* (II.2) *let* $K+R$ *be as in case* (5). *Let* ψ *denote the automorphism constructed in* (X.4). *Put*

$$\begin{aligned}
S_{(0)} &:= 0 && \text{if } S \cong \mathrm{sl}(2),\\
&:= \sum_{i>0} Fx^i\partial && \text{if } S \cong W(1;\mathbb{1}),\\
&:= H(2;\mathbb{1})^{(2)} \cap W(2;\mathbb{1})_{(0)} && \text{if } S \cong H(2;\mathbb{1})^{(2)},\\
K_{(0)} &:= \mathrm{Nor}_K(S_{(0)} \otimes A(n;\mathbb{1}) + S \otimes A(n;\mathbb{1})_{(1)})\\
K_{(i+1)} &:= \{u \in K_{(i)} \mid [u, K] \subset K_{(i)}\}, && i \geqslant 0.
\end{aligned}$$

Then

(1) $S_{(0)} \otimes A(n;\mathbb{1}) + S \otimes A(n;\mathbb{1})_{(1)}$ *is invariant under* ψ.

(2) $K_{(i)}$ *is invariant under* R *for all* $i \geqslant 0$.

(3) *The only roots sticking out of* $K_{(0)}$ *are* (*after a suitable adjustment*)

	$n=1$	$n=2$
$S \cong \mathrm{sl}(2)$	$-\beta$	$\pm\beta$
$S \cong W(1;\mathbb{1})$	$-\alpha$, $-\beta$	$-\alpha$, $\pm\beta$
$S \cong H(2;\mathbb{1})^{(2)}$	$-\alpha$, $\alpha - 2m\beta$, $-\beta$	$-\alpha$, $\alpha - 2m\beta$, $\pm\beta$

(4) *$K_{(0)} \cap K(\mu)$ is the distinguished maximal compositionally classical subalgebra of $K(\mu)$, for all $\mu \in FG(p)\alpha + GF(p)\beta$.*

(5) *The only roots of $K_{(0)}$ sticking out of $K_{(1)}$ are (after the adjustment of (3))*

	$n=1$	$n=2$
$S \cong \mathrm{sl}(2)$	$0, \pm\alpha$	$0, \pm\alpha, \pm 2\beta$
$S \cong W(1;\mathbb{1})$	$0, -\alpha+\beta$	$0, -\alpha \pm \beta, \pm 2\beta$
$S \cong H(2;\mathbb{1})^{(2)}$	$0, -\alpha+\beta, \alpha - 2m\beta + \beta,$ $\pm 2(\alpha - m\beta)$	$0, -\alpha \pm \beta, \alpha - 2m\beta \pm \beta,$ $\pm 2(\alpha - mb), \pm 2\beta$

(6) *$K_{(1)}$ is solvable.*

Proof. (1) ψ stabilizes $S \otimes A(n;\mathbb{1})$. Since $S \otimes A(n;\mathbb{1})_{(1)}$ is the unique maximal ideal of $S \otimes A(n;\mathbb{1})$, ψ stabilizes this ideal. Hence it induces an automorphism on $S \otimes A(n;\mathbb{1})/S \otimes A(n;\mathbb{1})_{(1)} \cong S$. As $S_{(0)}$ is invariant under all automorphisms we obtain the result.

(2) In all cases of (X.4) $\psi^{-1}(R)$ maps $S_{(0)} \otimes A(n;\mathbb{1}) + S \otimes A(n;\mathbb{1})_{(1)}$ into itself. According to (1) this subalgebra is therefore invariant under R. We now conclude by induction that $K_{(i)}$ is invariant under R.

(3) Applying (X.4) we see that the only root vectors with respect to $\psi^{-1}(R)$ sticking out of $K_{(0)}$ are represented by the following:

	$n=1$	$n=2$
$S \cong \mathrm{sl}(2)$	$\mathrm{id} \otimes \partial$	$\mathrm{id} \otimes \partial_1, \mathrm{id} \otimes \partial_2$
$S \cong W(1;\mathbb{1})$	$D \otimes 1, \mathrm{id} \otimes \partial$	$D \otimes 1, \mathrm{id} \otimes \partial_1, \mathrm{id} \otimes \partial_2$
$S \cong H(2;\mathbb{1})^{(2)}$	$D_1 \otimes 1, D_2 \otimes 1, \mathrm{id} \otimes \partial$	$D_1 \otimes 1, D_2 \otimes 1, \mathrm{id} \otimes \partial_1,$ $\mathrm{id} \otimes \partial_2.$

If $S \cong \mathrm{sl}(2)$ or $\cong W(1;\mathbb{1})$ the roots are obviously as claimed. We only have to have a closer look if $S \cong H(2;\mathbb{1})^{(2)}$.

$n=1$: Let $D_1 \otimes 1$, $D_2 \otimes 1$, $\mathrm{id} \otimes \partial$ be root vectors for the roots $-\alpha_1$, $-\alpha_2$, $-\beta$. Put $t_1 := (y_1 D_1 - y_2 D_2) \otimes 1$, $t_2 := m(y_1 D_1 + y_2 D_2) \otimes 1 + \mathrm{id} \otimes x\partial$. Then

$$-\beta(t_1)\,\mathrm{id} \otimes \partial = [t_1, \mathrm{id} \otimes \partial] = 0,$$

$$-\beta(t_2)\,\mathrm{id} \otimes \partial = [t_2, \mathrm{id} \otimes \partial] = -\mathrm{id} \otimes \partial,$$

$$-\alpha_i(t_1)\, D_i \otimes 1 = [t_1, D_i \otimes 1] = (-1)^i D_i \otimes 1,$$

$$-\alpha_i(t_2)\, D_i \otimes 1 = [t_2, D_i \otimes 1] = -m D_i \otimes 1.$$

Thus $\beta(t_1)=0$, $\beta(t_2)=1$, $\alpha_i(t_1)=(-1)^i$, $\alpha_i(t_2)=m$ and hence $\alpha_1+\alpha_2=2m\beta$. Put $\alpha:=\alpha_1$ to obtain the result.

For $n=2$ a similar computation proves the claim.

(4) $K_{(0)}$ is invariant under R and hence decomposes into root spaces. (3) shows that all classical or solvable one-sections are contained in $K_{(0)}$. Moreover, if μ is a nonclassical root $K_{(0)}\cap K_{(\mu)}$ is a subalgebra of codimension 1 (if μ is Witt) or 2 (μ hamiltonian), which contains rad $K(\mu)$.

(5) A computation similar to that in (3) yields the result. We do that explicitly only for the case $S\cong H(2;\mathbb{1})^{(2)}$, $n=2$. A basis of $K_{(0)}/K_{(1)}$ is represented by

$$R,\ D_i\otimes x_j,\ y_iD_j\otimes 1\ (i\neq j),\ \mathrm{id}\otimes x_i\,\partial_j\ (i\neq j) \qquad \text{for} \quad i,j=1,2.$$

The corresponding roots are 0, $-\alpha\pm\beta$, $\alpha-2m\beta\pm\beta$, $\pm 2(\alpha-m\beta)$, $\pm 2\beta$.

(6) We proved in the course of (3) and (5) that $K_{(1)}\subset(\mathrm{Der}\,S)_{(1)}\otimes A(n;\mathbb{1})+(\mathrm{Der}\,S)_{(0)}\otimes A(n;\mathbb{1})_{(1)}+(\mathrm{Der}\,S)\otimes A(n;\mathbb{1})_{(2)}+F\,\mathrm{id}\otimes W(n;\mathbb{1})_{(1)}$. The latter is solvable. ▮

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