A Geometric Characterization of Certain Groups of Lie Type

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1. Introduction

Let $\Gamma$ be an undirected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$ and let $G$ be a subgroup of $\text{Aut}(\Gamma)$. For each $\{x, y\} \in E(\Gamma)$, we denote by $G(\{x, y\})$ the stabilizer of $\{x, y\}$ in $G$ (which may or may not contain elements exchanging the vertices $x$ and $y$). For each $x \in V(\Gamma)$ we denote by $\Gamma(x)$ the set of vertices adjacent to $x$, by $G(x)$ the stabilizer of $x$ in $G$ and, for each $i \in \mathbb{N}$, by $G_i(x)$ the subgroup $\{a \in G | a \in G(u) \text{ for all } u \in V(\Gamma) \text{ with } \delta(x, u) \leq i\}$ where $\delta(x, u)$ denotes the distance from $x$ to $u$. An $s$-path (for $s \in \mathbb{N}$) is an $(s+1)$-tuple $(x_0, x_1, \ldots, x_s)$ of vertices such that $x_i \in \Gamma(x_{i-1})$ if $1 \leq i \leq s$ and $x_i \neq x_{i-2}$ if $2 \leq i \leq s$. Let $G(x_0, \ldots, x_s) = G(x_0) \cap \cdots \cap G(x_s)$ and $G_i(x_0, \ldots, x_s) = G_i(x_0) \cap \cdots \cap G_i(x_s)$ for each $s$-path $(x_0, \ldots, x_s)$ and each $i \in \mathbb{N}$. If $H$ is any group acting on a set $X$, we denote by $H^X$ the permutation group on $X$ induced by $H$.

In [12] we proved the following result.

THEOREM 1. Let $n \in \mathbb{N}$, $n \geq 2$. Let $\Gamma$ be an undirected connected graph with $|\Gamma(x)| \geq 3$ for every $x \in V(\Gamma)$ and let $G$ be a subgroup of $\text{Aut}(\Gamma)$ such that for each $n$-path $(x_0, \ldots, x_n)$

(i) $G_i(x_1, \ldots, x_{n-1})$ acts transitively on $\Gamma(x_n) - \{x_{n-1}\}$ and

(ii) $G_i(x_0, x_1) \cap G(x_0, \ldots, x_n) = 1$.

Then $n = 2, 3, 4, 6$ or 8.

We will call a graph fulfilling the hypotheses of Theorem 1 Moufang or, more precisely, $(G, n)$-Moufang. We confine our attention here to the class of finite Moufang graphs.

Let $L$ be a group of Lie type of rank 2 and let $\Gamma$ be the associated generalized $n$-gon. The graph $\Gamma$ is $(G, n)$-Moufang for any subgroup $G$ of $\text{Aut}(\Gamma)$ containing $L$ (which we may identify with a subgroup of $\text{Aut}(\Gamma)$). These are the Moufang polygons. They are by no means the only Moufang graphs, as we pointed out in [12]. It is tempting, however, to conjecture that the Moufang polygons, perhaps with certain sporadic exceptions, are in some geometrical or group theoretical sense the "primitive" objects in the class of all Moufang graphs. We prove two results in this direction.

THEOREM 2. Let $\Gamma$ be a finite $(G, n)$-Moufang graph with $n \geq 3$ having no vertices $x$ such that $G(x)_{\Gamma(x)} = 1$

(a) $L_2(2)$ or, if $n \geq 4$, $U_3(2)$,

(b) $L_2(3)$ or $U_3(3)$ if $n \geq 4$,

(c) $L_2(4)$ or $U_3(4)$ if $n = 6$,

(d) $L_2(5)$, $U_3(5)$, $L_2(9)$ or $U_3(9)$ if $n = 4$ or

(e) $L_2(7)$ or $U_3(7)$ if $n = 6$.

Let $\{x, y\}$ be an arbitrary edge of $\Gamma$ and let $\Gamma'$ denote the subgraph of $\Gamma$ induced by the set of vertices at a distance of at most $n - 1$ from $x$ or $y$. Then there is a group $L$ of Lie type of rank 2 (more precisely, $L = A_2(q)$ if $n = 3$, $B_2(q)^2$, $A_3(q)$ or $2A_4(q)$ if $n = 4$, $G_2(q)$ or $3D_4(q)$ if $n = 6$ and $2F_4(q)$ if $n = 8$ for some prime power $q$), which we consider as a subgroup of the automorphism group of its associated generalized $n$-gon $\Delta$, such that from some subgroup $D$ of $\text{Aut}(\Delta)$ containing $L$ and any edge $\{u, v\}$ of $\Delta$, $G(\{x, y\}) = D(\{u, v\})$,
$G(y) \cong D(v)$ or vice versa. Moreover, these isomorphisms are induced by an embedding of $\Gamma'$ in $\Delta$.

**Theorem 3.** Suppose $\Gamma$, $G$, $n$, $L$ and $\Delta$ are as in Theorem 2. Suppose that

(iii) For some $(n-2)$-path $(x_1, \ldots, x_{n-1})$ of $\Gamma$, each non-trivial element of $G_1(x_1, \ldots, x_{n-1})$ fixes only vertices contained in $\Gamma(x_1) \cup \Gamma(x_2)$ if $n = 3$ or in $\{u | \delta(x_{n/2}, u) \leq n/2\}$ if $n \geq 4$. Suppose too that

(f) the characteristic of $L$ is 2.

Then $\Gamma \cong \Delta$ and $L \cong G \cong \text{Aut}(L)$.

To prove Theorem 2, we construct the graph $\Delta$ and then use [2] to identify it. First, though (Lemma 1 below), we require the results of [4], [5], [7] and [8] to obtain information about the possible subconstituents (i.e., the permutation groups $G(x)^{\Gamma(x)}$ for $x \in V(\Gamma)$). The remainder of the proof, however, is largely self-contained except that we expect the reader to be at least partly familiar with [12]. Theorem 3 is little more than a corollary to Theorem 2.

Theorem 2 is closely related to and in fact a generalization of [13, (1.2)]. (We do not, however, expect the reader to be familiar with this result.) Theorem 3 is related to the results in [9]; note, however, that for $n \geq 4$, condition (iii) does not imply that $G_1(x_1, \ldots, x_{n-1})$ is a TI-subgroup.

I do not know to what extent the conclusions of Theorems 2 and 3 continue to hold without conditions (a), (b), (c), (d), (e) and (f). We mention three examples.

(1) As was pointed out in [13], there is a subgroup $S$ of $\text{Aut}(B_2(2))$ such that the generalized 4-gon $\Sigma$ associated with $B_2(2)$ is $(S, 3)$-Moufang without, however, the stabilizer in $S$ of an edge of $\Sigma$ being isomorphic to a subgroup of the stabilizer in $\text{Aut}(A_2(2))$ of an edge of the generalized 3-gon associated to $A_2(2)$.

(2) There is a trivalent $(G, 3)$-Moufang graph with $G = L_2(17)$ fulfilling condition (iii) of Theorem 3 (see [15]).

(3) If $\Gamma$ is the graph with automorphism group $J_3$ described in [1, part II], then $\Gamma$ is $(J_3, 3)$-Moufang (with $L = A_2(4)$), but condition (iii) does not hold (which follows, for instance, from the fact that five divides the order of the centralizer of an arbitrary involution in $J_3$).

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### 2. Preliminaries

Suppose that $\Gamma$ and $G$ fulfill the hypotheses of Theorem 2. For each edge $\{x, y\}$, we set $U(x, y) = \langle G_1(w, x), G_1(y, z) | w \in \Gamma(x)$ and $z \in \Gamma(y) \rangle$. The group $U(x, y)$ is normal in $G(x, y)$ and, as we have seen in [12], acts regularly on $\Gamma(x) - \{y\}$. Thus [8, Theorem 1.1] applies: For each $x \in V(\Gamma)$, $G(x)^{\Gamma(x)} \cong L_2(q), U_3(q), Sz(q)$ or $R(q)$ for some prime power $q$ (where $R(q)$ denotes a group of Ree type) or $G(x)^{\Gamma(x)}$ contains a regular normal subgroup. To proceed, though, we need to restrict the possibilities further.

**Lemma 1**

(a) There is a prime $p$ such that $U(x, y)$ is a $p$-group for each $\{x, y\} \in E(\Gamma)$.

(b) If $n = 3$, there is a power $q$ of $p$ such that $G(x)^{\Gamma(x)} \cong \text{PGL}(2, q)$ for every $x \in V(\Gamma)$; if $p \neq 2$, then for every $x \in V(\Gamma)$, $G_1(x)$ is not a $p$-group.

(c) If $n \geq 4$, then for each $x \in V(\Gamma)$ there is a power $q$ of $p$ (depending on $x$) with $q > 3$ such that $G(x)^{\Gamma(x)} \cong L_2(q), U_3(q), Sz(q)$ or $R(q)$ (also depending on $x$).
We define $L_1$ to be the graph with vertex set $X$. Section 6, but since it is crucial, we repeat it here. For each vertex $x$, $G_1(x)$ is not a $p$-group also when $p = 2$ (unless $\Gamma$ is trivalent).

It is interesting to note that we do not need to "pair parabolics" as in [2] (except to the extent that such information is contained in Lemma 1) but can proceed now directly to the construction of $\Delta$.

3. The Proof of Theorem 2 (continued)

For each vertex $x$, let $\overline{G}(x)$ denote the largest subgroup of $G(x)$ such that $\overline{G}(x)^{\Gamma(x)} \subseteq PGL(2, q)$, $PGU(3, q^2)$, $Sz(q)$ or $R(q)$ for some $q$. For any two neighbors $w$ and $y$ of $x$, $G(w, x, y) \cap \overline{G}(x)$ acts without fixed points (from now on: f.p.f.) and cyclicly on $\Gamma(x) - \{2, y\}$; let $f(x)$ denote the index of $G_1(x)$ in this group. We call an $s$-path $W = (x_0, \ldots, x_s)$ good if the index of $G(W) \cap G_1(x_i)$ in $G(W) \cap \overline{G}(x_i)$ is $f(x_i)$ for $0 < i < s$ where $G(W) = G(x_0, \ldots, x_s)$. Note that if $\Gamma$ is a Moufang $n$-gon, the good paths are simply those which proceed around a 2n-circuit.

Lemma 2. Let $W = (x_0, \ldots, x_s)$ be an arbitrary path. If $s \leq n + 1$, then $W$ is good. If $s \geq n + 1$ and $W$ is good, then there exists a unique vertex $x_{s+1} \in \Gamma(x_s)$ such that $(x_0, \ldots, x_s, x_{s+1})$ is a good path.

Proof (as in [13, (2.1)-(2.2)]). We first prove the uniqueness claim in the second statement. Suppose that $x'_{s+1}$ is a second such vertex. Choose elements $a \in G(x_0, \ldots, x_s, x'_{s+1}) \cap \overline{G}(x_s)$ and $b \in G(x_0, \ldots, x_s, x_{s+1}) \cap \overline{G}(x_s)$ both inducing permutations of order $f(x_s)$ on $\Gamma(x_s)$. Then the commutator subgroup $\langle a, b \rangle'$ of $\langle a, b \rangle$ induces a non-trivial $p$-group on $\Gamma(x_s)$. Since $\langle a, b \rangle \leq G(x_0, \ldots, x_s)$, we have $\langle a, b \rangle' \leq \overline{G}(x_s) \cap \overline{G}(x_{s+1})$. Thus, if $P$ is a $p$-Sylow subgroup of $\langle a, b \rangle'$, then $1 \neq P \leq \overline{G}_1(x_0, \ldots, x_{s+1})$. By condition (ii) in the definition of a Moufang graph (i.e. in the statement of Theorem 1 above), $s \leq n$.

Since every 1- (and every 2-) path is automatically good and $G(W)$ acts transitively on $\Gamma(x_s) - \{x_{s+1}\}$ if $s \leq n$, to prove Lemma 2 it now suffices to show that for every $s$, there exists at least one $(s + 1)$-path extending $W$. We can assume that $s \geq 2$ and, by induction, that there exists an $x_{s+1} \in \Gamma(x_s)$ such that $(x_1, \ldots, x_s, x_{s+1})$ is a good $s$-path. Choose a $p'$-element $b_s \in G(x_0, \ldots, x_{s+1}) \cap \overline{G}(x_s)$ inducing a permutation of order $f(x_s)$ on $\Gamma(x_s)$. For $0 < i < s$, choose a $p'$-element $b_i \in G(W) \cap \overline{G}(x_i)$ inducing a permutation of order $f(x_i)$ on $\Gamma(x_i)$. Since $b_i$ is a $p'$-element for $0 < i \leq s$, $b_i$ has at least one fixed point in $\Gamma(x_i) - \{x_2\}$ and $b_i$ (for $0 < i < s$) at least one in $\Gamma(x_s) - \{x_{s+1}\}$. It is easily checked that $\langle b_1, b_s \rangle$ contains an element $c$ such that $b_s^c \in G(x_0)$ (where $b_s^c$ denotes the element $cb_sc^{-1}$). Let $d_i = b_i^c$ and $x'_{i+1} = c(x_{i+1})$. For each $i$ with $0 < i < s$ there exists an element $c_i$ in $\langle b_n, a_i \rangle$ such that $b_i^{c_i} \in G(x'_{i+1})$; let $a_i = b_i^{c_i}$. Then for $0 < i < s + 1$, $a_i \in G(x_0, \ldots, x_s, x_{s+1}) \cap \overline{G}(x_i)$ and $a_i$ induces a permutation of order $f(x_i)$ on $\Gamma(x_i)$.

We are now in a position to construct the graph $\Delta$. The construction is the same as in [13, Section 6], but since it is crucial, we repeat it here. For each vertex $x$ and each $i \in \mathbb{N}$, let $\Gamma_i(x) = \{u \in V(\Gamma) | \delta(x, u) = i\}$. Note that condition (i) in the definition of a Moufang graph implies easily that $\Gamma$ contains no circuits of length $\leq 2n$. Let $\{x, y\}$ be an arbitrary edge of $\Gamma$. We define $\Delta$ to be the graph with vertex set $\Gamma_{n-1}(x) \cup \Gamma_{n-1}(y)$ where $u$ and $v$ are defined to be adjacent if and only if either

(a) $u$ or $v$ (or both) lies in $\Gamma_{n-2}(x) \cup \Gamma_{n-2}(y)$ and $u \in \Gamma(v)$ or

...
(b) there exists a good \((2n - 1)-\)path \((x_0, \ldots, x_{2n-1})\) with \(x_0 = u, x_{2n-1} = v\) and either \(x_{n-1} = x\) and \(x_n = y\) or vice versa.

Let \(a\) be any element in \(G(x)\) not in \(G(y)\). We define a permutation \(\hat{a}\) of \(V(\Delta)\) as follows: If \(u \in \Gamma_{n-1}(x)\), we set \(\hat{a}(u) = a(u)\). Otherwise there exists a unique good \(2n\)-path \((x_0, \ldots, x_{2n})\) with \(x_0 = u, x_{n-1} = y, x_n = x\) and \(x_{n+1} = a^{-1}(y)\) and we set \(\hat{a}(u) = a(x_{2n})\). It is straightforward to check using the following lemma, whose proof we postpone for a moment, that \(\hat{a}\) is in \(\text{Aut}(\Delta)\).

**Lemma 3.** Let \((x_0, \ldots, x_{2n})\) be a good \(2n\)-path. Let \(x_{n+1}' \in \Gamma(x_n) - \{x_{n-1}, x_{n+1}\}\) be arbitrary. Let \((x_0, \ldots, x_n, x_{n+1}', \ldots, x_{2n-1})\) be the unique good \((2n - 1)\)-path extending \((x_0, \ldots, x_n, x_{n+1}, \ldots, x_{2n})\). Then \((x_2, \ldots, x_{n+1}' + 2, x_{n+1}, x_{n+1}, \ldots, x_{2n} + 2)\) is also a good \((2n - 1)\)-path.

(We note that Lemma 3 is easily seen to hold if \(\Gamma\) is a Moufang \(n\)-gon, in fact with \(2n - 1\) replaced by \(2n\); see the remark following the definition of a good path.)

If \(a\) is contained in \(G(y)\) but not in \(G(x)\), we can define an element \(\hat{a}\) in \(\text{Aut}(\Delta)\) agreeing with \(a\) on \(\Gamma_{n-1}(y)\) as above. If \(a \in G(x, y)\), we define \(\hat{a}\) to be simply the restriction of \(a\) to \(V(\Delta)\); clearly \(\hat{a}\) is in \(\text{Aut}(\Delta)\) too. Finally, let \(D\) denote the subgroup of \(\text{Aut}(\Delta)\) generated by the elements \(\hat{a}\) for \(a \in G(x) \cup G(y) \cup G(x, y)\). Since \(D(a)^{\hat{a}(u)}\) is transitive for \(u = x\) and \(y\), \(D\) acts transitively on \(E(\Delta)\). In particular, \(\Delta\) is a generalized \(n\)-gon since it is bipartite and \(\{x, y\}\) does not lie on any circuit of length \(<2n\). We claim that \(\Delta\) is \((D, n)\)-Moufang.

Condition (ii) holds by [12, Theorem 2]. If \((x_0, \ldots, x_n)\) is any \(n\)-path with \(\{x_0, x_1\} = \{x, y\}\), then \(G_1(x_1, \ldots, x_{n-1})\) induces a subgroup of \(D_1(x_1, \ldots, x_{n-1})\) acting transitively on \(\Delta(x_0) - \{x_1\}\). Since \(D_{E(\Delta)}\) is transitive, condition (i) holds and so \(\Delta\) is in fact \((D, n)\)-Moufang. In particular, \(D\) fulfills the hypotheses of [2] (see Lemma 1 (a), above). We conclude that \(\Delta\) is isomorphic to the generalized \(n\)-gon associated with a group \(L\) of Lie type of rank 2 (which we identify with a subgroup of \(\text{Aut}(\Delta)\)) and that \(\Delta \leq D\). Note that for \(u = x\) and \(y\), \(G(u)\) acts faithfully on \(\Gamma_{n-1}(u)\), so we may identify \(G(u)\) with a subgroup of \(D(u)\) (containing all the \(p\)-Sylow subgroups of \(L(u)\)). It is a property of the groups of Lie type of rank 2 that then \(L(x, y) \leq G(x, y)\). Thus \(D\) is the smallest edge-transitive subgroup of \(\text{Aut}(\Delta)\) containing \(G(x, y)\) and, in particular, \(D(x, y) = G(x, y)\) and \(D(u) = G(u)\) for \(u = x\) and \(y\).

Thus to complete the proof of Theorem 2, it suffices to prove Lemma 3. To do this, we assume first the following result.

**Lemma 4.** Suppose that \(p \neq 2\) if \(n = 3\) (i.e., \(pn \neq 6\)). Let \(W = (x_0, \ldots, x_{n+1})\) be an arbitrary \((n + 1)\)-path. Then \(G(W) \cap G_1(x_i)\) acts f.p.f. on \(\Gamma(x_n) - \{x_{n-1}, x_{n+1}\}\) for \(0 < i < n\).

**Proof of Lemma 3.** Suppose first that \(pn = 6\). By condition (i), \((G_1(x_3, x_4), G_1(x_4', x_5'))\) contains an element \(a\) mapping \((x_1, x_2)\) to \((x_5, x_4)\). By Lemma 1, part (b), \(G_1(x, y)\) is elementary abelian for every \(\{x, y\}\). Since \([G_1(x_3, x_4), G_1(x_4', x_5')] \leq G_1(x_3, x_4', x_5') = 1\), \(a\) is an involution. Thus \(a(x_1, \ldots, x_5) = (x_5, \ldots, x_1)\). Since each 4-path has only one extension to a good 5-path, \(a\) must exchange \(x_0\) and \(x_6\). Since \(a \in G_1(x_4')\), we have \((a(x_0, x_3, x_4', x_5') = (x_6, \ldots, x_3, x_4', x_5').\)

Thus we may assume that \(pn \neq 6\). There exist unique vertices \(x_{n+2}', \ldots, x_{2n-1}'\) such that \((x_0, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{2n-1})\) is a good \((2n - 1)\)-path. From Lemma 4 we know that \(G(x_1, \ldots, x_n, x_{n+1}, x_{n+2}) \cap G(x_n)\) acts f.p.f. on \(\Gamma(x_{n+1}) - \{x_n, x_{n+2}\}\). By Lemma 2, \(G(x_1, \ldots, x_n, x_{n+1}, x_{n+2}) \leq G(x_0)\) and thus \(G(x_1, \ldots, x_n, x_{n+1}, x_{n+2}) \cap G(x_{n+1}) \leq G(x_0, \ldots, x_n, x_{n+1}) \cap G(x_{n+1}) \leq G(x_{2n}, \ldots, x_n, x_{n+1}) \leq G(x_{n+2}).\) It follows that \(x_{n+2} = x_{n+2}\). Repeated application of this argument yields the desired result.
To prove Lemma 4, we assume in turn one further result. A new definition is required: For each vertex $x$, let $e(x) = (q-1)/(q-1, 2)$ if $G(x)^{\Gamma(x)} \cong L_2(q)$, $e(x) = (q^2-1)/(q+1, 3)$ if $G(x)^{\Gamma(x)} \cong U_3(q)$ and $e(x) = q-1$ if $G(x)^{\Gamma(x)} \cong S_2(q)$ or $R(q)$. (Thus $f(x)/e(x)$ divides $(q-1, 2)$ if $G(x)^{\Gamma(x)} \cong L_2(q)$, $f(x)/e(x)$ divides $(q+1, 3)$ if $G(x)^{\Gamma(x)} \cong U_3(q)$ and $e(x) = f(x)$ otherwise.)

**Lemma 5.** Suppose $n \geq 4$. Let $m = n/2$. Then there exists a vertex $u$ such that $G_1(u)$ is not a $p$-group and a vertex $x$ such that $G_m(x) \neq 1$. Moreover, if $G_m(x) \neq 1$ and

(a) if $n = 4$ or $8$ and $G(x)^{\Gamma(x)} \cong L_2(q)$, for some $q$, then $G_1(x)$ is not a $p$-group;
(b) if $n = 6$ and $G(y)^{\Gamma(y)} \cong L_2(q)$ for some $y \in \Gamma(x)$ and some $q$, then $G_1(y)$ contains an element inducing a permutation on $\Gamma(x)$ of order $e(x)$.

**Proof of Lemma 4.** The reader should compare what follows with the “principle of induction” formulated in [10, part I, Lemma 8]. It clearly suffices to prove that for any $(n+1)$-path $(x_0, \ldots, x_{n+1})$, $G_1(x_0) \cap G(x_1, \ldots, x_{n+1})$ acts f.p.f. on $\Gamma(x_i) - \{x_{i-1}, x_{i+1}\}$ for $0 < i < n$. We prove this claim first under the assumption that $G_1(x_0)$ is not a $p$-group and then for arbitrary $(n+1)$-paths.

By Lemmas 1 and 5, we can choose a good path $W = (x_0, \ldots, x_t)$ with $t > 0$ and $x_0 = x_t$ (we will read the subscripts modulo $i$) such that $G_1(x_0)$ is not a $p$-group. Let $H = G_1(x_0) \cap G(x_0, \ldots, x_{n+1}) = G_1(x_0) \cap G(W)$. Since for every vertex $x$, $|\Gamma(x)| - 1$ is a power of $p$, clearly $H \neq 1$. We have $[H, U(x_0, \ldots, x_n)] \leq G_1(x_0) \cap U(x_0, \ldots, x_n) = 1$ (where $U(x_0, \ldots, x_n)$ denotes $G_1(x_1, \ldots, x_{n+1})$; see [12]) which implies that $H \leq G_1(x_n)$. Let $d = n$ be the smallest positive integer such that $H$ has at least three fixed points in $\Gamma(x_d)$. Out object is to show that $d = n$. We will show first that $d$ divides $n$.

By the choice of $d$, the subgroup of $U(x_0, \ldots, x_{n+d})$ centralized by $H$, i.e., $C_{U(x_0, \ldots, x_{n+d})}(H)$, is non-trivial. Therefore $H$ has at least three fixed points in $\Gamma(x_{n+d})$. Since $H \leq G_1(x_n)$, we have $[H, U(x_n, \ldots, x_{2n})] = 1$. Let $a$ be a non-trivial element in $U(x_n, \ldots, x_{2n})$ and $x'_d = a(x_d)$ for $d \leq i \leq n-1$. Then $H \leq G(x'_d, \ldots, x'_n, x_n, x_{n+d}, \ldots, x_{n+d})$ and $C_{U(x_n, \ldots, x_{n+d})}(H) \neq 1$ so that $H$ fixes at least three vertices in $\Gamma(x_d)$ too. Thus $C_{U(x'_d, \ldots, x'_n, x_n, x_{n+d}, \ldots, x_{n+d})}(H) \neq 1$ so that $H$ fixes at least three vertices in $\Gamma(x_{n-d})$. Thus we can find a non-trivial element $b$ in $U(x_{n-d}, \ldots, x_{2n-d})$ with $[b, H] = 1$ and so $H \leq G(b(x_0), \ldots, b(x_{n-d}), x_{n-d}, x_{n-d}, \ldots, x_n)$. At this point, it should be clear that $H$ fixes at least three vertices in (in fact, all of) $\Gamma(b(x_0))$ and hence in $\Gamma(x_{n-2d})$ too. Continuing in this fashion, we conclude that $d$ in fact divides $n$. Moreover, we have $|C_{U(x_{n-d}, \ldots, x_{n+d})}(H)| = |C_{U(x_{n-d}, \ldots, x_{n+d})}(H)|$ if $i-j$ is even. In particular, $H \leq G_1(x_{2d}) \cap G(W)$ and (since both groups have the same order, being conjugate in $G$) $H = G_1(x_{2d}) \cap G(W)$.

Let $\tilde{\Sigma}$ denote the graph whose vertices are the vertices $u$ of $\Gamma$ such that $H$ fixes $u$ and at least three neighbors of $u$ and whose edges are those pairs $\{u, v\}$ of such vertices with $\delta(u, v) = d$. Let $\Sigma$ be the connected component of $\tilde{\Sigma}$ containing the vertex $x_0$ and let $S$ be the subgroup of $C_G(H)/H$ mapping $\Sigma$ to itself. We claim that if $d < n$, then $\Sigma$ is $(S, n/d)$-Moufang. Since $C_{U(x_0, \ldots, x_n)}(H)$ is contained in $S$ (up to isomorphism) for all multiples $i$ of $d$, condition (i) holds. The pre-image $S_1(x_0, x_d) \cap S(x_0, x_d, \ldots, x_n)$ in $G$ is contained in $G_1(x_0) \cap G(W) = H$, i.e., $S_1(x_0, x_d) \cap S(x_0, x_d, \ldots, x_n) = 1$. Similarly, $S_1(x_d, x_{2d}) \cap S(x_d, x_{2d}, \ldots, x_{n+1}) = 1$ (see the last comment in the previous paragraph) so that condition (ii) holds too.

Suppose $H \neq G_1(x_d)$. Then we have $H \cap G(x_d) \neq G_1(x_d)$ since $H$ is normal in $G(W)$ and $G(W) \cap G(x_d)$ contains an element inducing a permutation of order $f(x_d)$ on $\Gamma(x_d)$. Since $H$ has at least three fixed points in $\Gamma(x_d)$, we must have $G(x_d)^{\Gamma(x_d)} \cong U_3(q)$ or $R(q)$ for some prime power $q$ and $S(x_d)^{\Gamma(x_d)} \cong L_2(q)$. By Lemma 1 there are no vertices $x$ such that $G(x)^{\Gamma(x)} \cong R(3)$. Since in conditions (a)–(e) for every $L_2(q)$ excluded, $U_3(q)$ is excluded too.
(except for $U_3(2)$ when $n = 3$, but Lemma 1, part (b), takes care of this case), $\Sigma$ and $S$ fulfill conditions (a)-(e).

By condition (ii), we know that $H \cong G_1(x_1)$. Thus if $d = 1$, $|\Sigma(x_1)| < \Gamma(x_1)$). It follows therefore by induction on $n$ and on the valencies of the vertices that if $\Sigma$ and $S$ fulfill the hypotheses of Theorem 2 (i.e., if $n/d = 3$, then $S_1(x_0) \cap S(x_0, x_{2d}, \ldots, x_0)$ acts f.p.f. on $\Sigma(x_0)-\{x_0, x_{2d}\}$ (condition (c) is needed here) which implies that $H$ acts f.p.f. on $\Gamma(x_0)-\{x_{d-1}, x_{d+1}\}$ after all, a contradiction.

Suppose therefore that $n/d = 2$. Let $A = (S_1(w)|w \in \Sigma(x_0))$. We have $S_1(w), S_1(x_0) = S_1(w, x_0) = 1$ for every $w \in \Sigma(x_0)$ and thus $[A, S_1(x_0)] = 1$. The group $A \cap S(x_0) \cap S(x_{2d})$ contains a $p'$-element $b$ acting f.p.f. on $\Sigma(x_0)-\{x_0, x_{2d}\}$. The element $b$ must fix some vertex in $\Sigma(x_0)-\{x_0\}$. Since $[b, S_1(x_0)] = [A, S_1(x_0)] = 1$ and $S_1(x_0)$ acts transitively on $\Sigma(x_0)-\{x_0\}$, we have $b \in S_1(x_0)$. It follows again that $H$ acts f.p.f. on $\Gamma(x_0)-\{x_{d-1}, x_{d+1}\}$ after all. With this contradiction, we conclude that $n/d = 1$, i.e., $d = n$, as claimed.

It remains to show that $G_1(u_0) \cap G(u_0, \ldots, u_{n+1})$ acts f.p.f. on $\Gamma(u_0)-\{u_{i-1}, u_{i+1}\}$ for $0 < i < n, (u_0, \ldots, u_{n+1})$ an arbitrary $(n + 1)$-path. It suffices to show that $G_1(x_1)$ (and hence $G_1(u)$ for arbitrary $u$) is not a $p'$-group since we can then repeat the argument just used to show that $d = n$. By Lemma 1, we may suppose that $n \geq 4$. Choose a $p'$-element $c$ contained in $H$ but not in $G_1(x_2)$. Let $a$ be an element in $G_1(x_1)$ exchanging $x_0$ and $x_2$. If $c \in G_1(x_1)$ then, since $G_1(x_1) \cap G(x_0) \cap G(x_2)$ acts cyclically on $\Gamma(x_1)$, there exists an integer $j$ relatively prime to the order of $c$ such that $aca^{-1}c^j \in G_1(x_1)$. Since $aca^{-1} \in G_1(x_2)$, $aca^{-1}c^j \in G_1(x_2)$. Thus $G_1(x_1)$ is not a $p'$-group.

We are thus reduced to the situation that $H \cap G_1(x_1)$ induces a $p'$-group on $\Gamma(x_2)$. Since $H \cap G_1(x_1) \leq G_1(u)$ and $G(W) \cap G_1(x_2)$ contains an element inducing a permutation of order $f(x_2)$ on $\Gamma(x_2)$, we have $H \cap G_1(x_1) \leq G_1(x_2)$. Let $h$ be a $p'$-element in $G(W) \cap G_1(x_1)$ inducing a permutation of order $f(x_1)$ on $\Gamma(x_1)$. In $G(x_2, x_3, x_4) \Gamma(x_5)$ there is no element of order greater than $f(x_3)$ (which equals $f(x_1)$) and every element of order $f(x_3)$ is in $G(x_3) \Gamma(x_5)$. We may assume that $h$ induces a permutation of order $f(x_3)$ on $\Gamma(x_3)$ since otherwise some non-trivial power of $h$ lies in $G_1(x_3)$. Since $[h, H \cap G_1(x_1)] \leq G(W) \cap G_1(x_0, x_1) = 1$, we conclude that $H \cap G_1(x_1) \leq G_1(x_3)$. It follows that $H \cap G_1(x_1) = G_1(x_1)$ for every even $i$ and $H \cap G_i(x_1) \leq G_1(x_i)$ for every odd $i$.

Now let $\tilde{\Sigma}$ denote the graph whose vertices are the vertices $u$ of $\Gamma$ such that $H \cap G_1(x_1) = G_1(u)$ and whose edges are those pairs $(u, v)$ of such vertices with $\delta(u, v) = 2$. Let $\Sigma$ be the connected component of $\tilde{\Sigma}$ containing the vertex $x_0$, let $\tilde{S}$ be the subgroup of $\text{Aut}(\tilde{\Sigma})$ induced by $C_G(H \cap G_1(x_1))$ and let $S$ be the subgroup of $\tilde{S}$ mapping $\Sigma$ to itself. We claim that $\Sigma$ is $(S, m)$-Moufang (where, as before, $m = n/2$). Since $U(x_n, \ldots, x_{n+n})$ is contained in $S$ (up to isomorphism) for even $i$, condition (i) holds. To verify condition (ii), it suffices to show that $G_1(x_0) \cap G_1(x_2) \cap G(W) \leq G_1(x_4)$. Since $G_1(u_0) \cap G(u_0, \ldots, u_n) \leq G_1(u_n)$ for every $n$-path $(u_0, \ldots, u_n)$, this certainly holds if $n = 4$. Suppose $n \geq 6$ and let $K = G_1(x_0) \cap G(x_2) \cap G(W)$. Let $u$ be an arbitrary vertex in $\Gamma(x_2)-\{x_1, x_3\}$ and let $v$ be the uniquely determined vertex such that $(u, u, x_2, x_3, \ldots, x_{n+2})$ is a good $(n + 2)$-path. Since $K \leq G(u_2, x_2, x_3, \ldots, x_{n+2})$, we have $K \leq G(v)$. Thus $K \leq G_1(x_4) \cap G(x_n, x_{n-1}, \ldots, x_2, u, v) \leq G_1(v)$ and so $K \leq G_1(v) \cap G(u, u_2, x_2, x_3, \ldots, x_{n-4}) \leq G_1(x_{n-4})$. Hence $K \leq G_1(x_{n-4}) \cap G(x_{n-4}, x_{n-5}, \ldots, x_4) \leq G_1(x_4)$. Thus $\Sigma$ is in fact $(S, m)$-Moufang.

The group $S$ acts transitively on $V(\Sigma)$. By induction, it follows that if $m \geq 3$, i.e. if $n \geq 6$, then $S(u)_{2^{[2(u)]}} \cong L_2(q)$ for some $q$ and every $u \in V(\Sigma)$ (and thus $G(x_0)_{\Gamma(x_0)} \cong L_2(q)$). If $1 < i < m$ then $S_1(x_0)$ contains an element inducing a permutation on $\Sigma(x_2)$ of order $k$ where $k = e(x_0)$ if $n = 4$ (see the argument used in the case $n/d = 2$ above), $k = (q - 1)/(q - 1, 3)$ if $n = 6$ and $k = q - 1$ if $n = 8$. Hence $\Sigma$ contains an element $a$ inducing a permutation on $\Gamma(x_2)$ of order $k$. Let $|\Gamma(x_2)| = 1 + p^N$. Since $H \cap G_1(x_1) \leq G_1(x_2)$, $k$ divides $|H|/H \cap G_1(x_1)$ and so $k$ divides $N$. Let $q = p^m$ where $G(x_0)_{\Gamma(x_0)} \cong L_2(q)$, $U_3(q)$, $Sz(q)$ or $R(q)$. 


Suppose that $n = 4$ or 8. If $n = 8$, then $G(x_0)^{G(x_0)} \cong L_2(q)$ and we may assume by Lemma 5 that $G_4(x_0) \neq 1$. We set $V_1 = \langle G_4(u) \mid u \in \Gamma(x_1) \rangle$ if $n = 8$ and $V_1 = \{a \in O_p G(x_1) \mid [a, O_p G(x_1) \leq G_2(x_1)] \}$ (where $O_p G(x_1) = O_p G_1(x_1) = U(x_0, x_1, x_2)$) if $n = 4$. In both cases, $|V_1|$ divides $p^{2M+N}$. Let $N = V_1/\Phi V_1$ and let $K$ be the centralizer of $V$ in $G(x_1)$. If $K < G_1(x_1)$, then $K$ contains a $p'$-element $b$ not contained in $G(x_0)$. By [6, (5.1.4)], $b$ centralizes $V_1$. If $n = 8$, we have $G_4(x_0) \leq U(x_0, x_1, b(x_0), \ldots, b(x_n)) = 1$, a contradiction. If $n = 4$, then either $G_1(x_0, x_1)$ or $G_1(x_0, x_1)^{G(x_0)}$ is contained in $V_1$ but not in $G_1(b(x_0))$, again a contradiction. We conclude that $K \leq G_1(x_1)$ and so $|G_1(x_1)^{G(x_1)}|$ and in particular $1 + p^N$ divides the order of the group induced by $G_1(x_1)$ on $V$ and hence the order of the general linear group on $V$ over $GF(p)$. By [16, p. 283], we have $|V| \geq p^{2N}$ unless perhaps $p^N = 8$. It follows that $2N \leq 2M + N$, i.e., $N \leq 2M$. Since $k \leq N$, we have $k \leq 2M$. This is possible only if (condition (d) is needed here) $q = 4$, $k = 3$ and since $k|N$; $p^N = 8$ after all.

Let $a$ be an element of $H$ inducing a permutation on $\Gamma(x_{n-2})$ of order $k$. From $k = 3$ it follows that $G(x_1)^{G(x_1)} \cong L_2(q)$ and that the element $a$ has at least three fixed points in $\Gamma(x_1)$. Choose vertices $u_0, \ldots, u_{n-2}$ with $u_{n-2} \notin \{x_1, x_2 \}$ such that $(u_0, \ldots, u_{n-2}, x_0, x_1, \ldots, x_n)$ is a good $(2n-1)$-path. Since $a$ fixes $(u_{n-2}, x_0, x_1, \ldots, x_n)$, $a$ fixes $(u_0, \ldots, u_{n-2}, x_0, x_1)$. Since $a$ has at least three fixed points in $\Gamma(x_1)$, the centralizer of $a$ in $U(u_0, \ldots, u_{n-2}, x_0, x_1)$ is non-trivial. It follows that $a$ has at least three fixed points in $\Gamma(x_{n-2})$ and hence at least three fixed points in $\Gamma(x_{n-1})$. Thus there exists a non-trivial element $b \in U(x_{n-1}, \ldots, x_{n-3})$ commuting with $a$. Let $c$ be an arbitrary non-trivial element in $U(x_0, \ldots, x_n)$. Since $a$ commutes with $b$ and $c$, $a$ commutes with $[b, c]$ which lies in $U(x_{n-2}, x_{n-1})$ but not in $G_1(x_{n-2})$. Let $d$ be the unique element in $U(x_{n-2}, \ldots, x_{n-1})$ such that $[b, c]dG_1(x_{n-2})$. Then $[b, c]^d = ([b, c]^d)^a = G_1(x_{n-2})a = G_1(x_{n-2})$. Since $a$ normalizes $U(x_{n-2}, \ldots, x_{n-1})$, $a$ centralizes $d$ which contradicts the fact that $a$, inducing a permutation of order $k$ on $\Gamma(x_{n-2})$, acts f.p.f. on $\Gamma(x_{n-2}) - \{x_{n-3}, x_{n-1}\}$.

Consider finally the case $n = 6$. Again, let $a$ be an element of $H$ inducing a permutation on $\Gamma(x_{n-2}) = \Gamma(x_4)$ of order $k$ where now $k = (q - 1)/(q - 1, 3)$ (and, we recall, $G(x_0)^{G(x_0)} \cong L_2(q)$). By Lemma 5, we may assume that $H$ contains an element $b$ inducing a permutation of order $e(x_1)$ on $\Gamma(x_1)$. We must have $b \notin G(x_1)$ and so $b \in G_1(x_4)$. If there exists an $i$ such that $ab^i$ has at least three fixed points in $\Gamma(x_1)$, we obtain a contradiction as in the previous paragraph. Thus we may assume that there is no such $i$. It follows that $G(x_1)^{G(x_1)} \cong L_2(r)$ for some odd $r$. But in this case, there exists a $j$ such that $a^j b^i$ has at least three fixed points in $\Gamma(x_1)$. Again we obtain a contradiction since, by conditions (c) and (e), $k > 2$.

To complete the proof of Lemma 4, we need only prove Lemma 5. We do this now. Suppose first that $G_m(x) \neq 1$ for some vertex $x$. Let $(x_0, \ldots, x_m)$ be an arbitrary $m$-path with $x_0 = x$. Since $G_m(x) \cong U(x_0, x_1)$, we have $G_m(x) \cap ZU(x_0, x_1) \neq 1$ (where $ZU(x_0, x_1)$ denotes the center of $U(x_0, x_1)$). By condition (ii), $G_m(x)$ acts faithfully on $\Gamma(x_m)$. Since $G_m(x) \cap ZU(x_0, x_1) \cong G_m(x_0, x_1)$ and $G_m(x_0, x_1) \cap G_m(x) \cong ZU(x_0, x_1)$ contains an element inducing a permutation of order $e(x_m)$ on $\Gamma(x_m)$, we conclude that $(G_m(x) \cap ZU(x_0, x_1))^e(x_m) = ZU(x_{m-1}, x_m)^e(x_m)$. Let $(y_0, y_1, \ldots, y_m)$ be an arbitrary $m$-path with $y_0 = x$ but $y_1 \neq x$. Since $[G_m(x) \cap ZU(x_0, x_1), U(y_0, \ldots, y_{m-1}, x_0, \ldots, x_m)] \leq U(y_{m-1}, y_0, \ldots, y_1, x_0, \ldots, x_m) \cap G_m(x) = 1$, we have $(G_m(x) \cap ZU(x_0, x_1))^e(y_m) \leq ZU(y_{m-1}, y_m)^e(y_m)$. It follows that $G_m(x) \cap ZU(x_0, x_1) \leq G_m(x) \cap ZU(y_0, x_1)$. Thus $G_m(x) \cap ZU(x_0, x_1) \leq ZU(y_0, x_1)$ for every vertex $y \in \Gamma(x)$. In $\langle U(x, y) \rangle \cap G(x_0, \ldots, x_m)$ we can find a $p'$-element $a$ inducing a permutation of order $e(x)$ on $\Gamma(x)$. Since $a$ is centralized by $G_m(x) \cap ZU(x_0, x_1)$ and fixes some vertex in $\Gamma(x_m) - \{x_m\}$, we have $a \in G_1(x_m)$ if $G_m(x_1)^e(x_m) \cong L_2(q)$ or $S_2(q)$, $a^q \in G_1(x_m)$ if $G(x_m)^e(x_m) \cong U_3(q)$ and $a^q \in G_1(x_m)$ if $G_m(x_1)^e(x_m) \cong R(q)$ for some $q$. (In particular, $a \in G_1(x_m)$ if $G(x_m)^e(x_m) \cong L_2(q)$ and so $e(x)$ divides the order of the permutation which $a$
induces on \( \Gamma(x_{m-i}) \) since \( G_1(x_{m-i}, x_m) \) is a p-group; this proves parts (a) and (b).) Since 
\( e(x) > 1 \) and \( e(x) > 2 \) if \( p = 3 \), we conclude that \( G_1(x_m) \) contains some non-trivial power of a except perhaps when \( G(x_m) \Gamma(x_m) \cong U_q(q) \) and \( e(x) \) divides \( q + 1 \). By conditions (a) and (b), it follows that \( f(x_0) < f(x_m) \). (We note that the case \( q = 3 \) and \( G(x_7) \Gamma(x_0) \cong L_2(9) \), which is ruled out by condition (b), could also be eliminated by considering the action of \( G(x_0) \) on \( G_2(x_1) \).) But \( G(x_0, \ldots, x_m) \) contains a \( p' \)-element inducing a permutation of order \( f(x_m) \) on \( \Gamma(x_m) \), some non-trivial power of which thus lies in \( G_1(x_0) \).

To conclude the proof of Lemma 5, we need to show that in fact \( G_m(x) \neq 1 \) for some vertex \( x \). For \( n = 8 \), this was shown in [12, p. 264]. For \( n = 6 \), we borrow an idea from an unpublished manuscript of J. Tits. Let \((x_0, x_1, x_2, \ldots)\) be an arbitrary path in \( \Gamma \). We claim first that \( Z \Gamma(x_0, x_1, x_2, x_3) \cap U(x_0, x_1, x_2, x_3) \neq 1 \) for some \( i \). To show this, we proceed as in the proof of [11, (3.16)]. Choose \( i, j \) and \( k \) with \( j < i < k < j + 6 \) such that \( X = \Gamma(x_i, \ldots, x_{i+6}) \cap U(x_i, \ldots, x_{i+6}) \) is non-trivial and such that \( k - j \) is maximal under this condition. (Since \( U(x_1, \ldots, x_9) \) is a p-group, \( Z \Gamma(x_1, \ldots, x_9) \neq 1 \). Since \( \Gamma(x_1, \ldots, x_9), U(x_0, \ldots, x_9) \leq U(x_0, \ldots, x_9) = 1 \) and \( U(x_1, \ldots, x_9), U(x_2, \ldots, x_9) = 1 \), such \( i, j \) and \( k \) actually exist.) We must show that \( i - j \) and \( k - i \) are both \( \geq 2 \). Suppose first that \( i - j = 1 \). Let \( Y = [X, U(x_{i-1}, \ldots, x_{i+5})] \). We have \( Y \neq 1 \) since otherwise we could replace \( j \) by \( j - 1 \). Moreover, \( Y = [X, U(x_{i-1}, \ldots, x_{i+4})] \leq [U(x_i, \ldots, x_{i+6}), U(x_{i-2}, \ldots, x_{i+4})] \leq U(x_{i-1}, \ldots, x_{i+5}) \). In particular, \( Y \) centralizes \( U(x_{i-1}, \ldots, x_{i+5}) \). Since \( U(x_{i+1}, \ldots, x_{i+5}) \) normalizes and \( X \) centralizes \( U(x_i, \ldots, x_{i+6}) \), \( Y \) centralizes \( U(x_i, \ldots, x_{i+6}) \). Thus \( U(x_{i-1}, \ldots, x_{i+6}) = U(x_i, \ldots, x_{i+6}) \), \( 1 \neq Y \leq Z \Gamma(x_i, \ldots, x_{i+6}) \cap G(x_{i-1}, \ldots, x_{i+6}) \) which contradicts the choice of \( i, j \) and \( k \). We obtain the same contradiction if \( k - i = 1 \) and so the claim is proven.

Thus we may suppose, say, that \( Z \Gamma(x_3, x_5, x_6, x_7) \cap U(x_3, \ldots, x_9) \neq 1 \); we denote this subgroup by \( B \). We have \( B \leq G_2(x_5) \cap G_2(x_7) \) since \( [B, U(x_5, x_6, x_7)] = 1 \). If \( [B, U(x_5, x_6, x_7)] \) (which we denote by \( C \)) is trivial, we conclude that \( B \leq G_3(x_8) \). Thus we may suppose that \( C \neq 1 \). We have \( C \leq [U(x_3, \ldots, x_9), U(x_0, \ldots, x_9)] \leq U(x_5, \ldots, x_9) \). Moreover, \( U(x_0, \ldots, x_9) \) normalizes \( U(x_5, \ldots, x_9) \) and so \( C \leq Z \Gamma(x_5, x_6, x_7) \). Thus \( C \leq G_2(x_8) \) since \( [C, U(x_3, \ldots, x_8)] = 1 \) and so \( C \leq G_2(u) \) for every \( u \in \Gamma(x_3) \). Let \( (u_0, \ldots, u_6) \) be an arbitrary 6-path with \( u_5 = x_6 \). If \( a \in Z \Gamma(x_5, x_6, x_7) \), then \( 1 \neq U(u_0, \ldots, u_6) \leq U(u_0, \ldots, u_4, a, u_3, \ldots, a, u_0) \). By condition (ii), \( (u_0, \ldots, u_4, a, u_3, \ldots, a, u_0) \) cannot be an 8-path, i.e., \( a = u_3 \). It follows that \( Z \Gamma(x_3, x_4, x_5) \subset G_2(x_8) \). Therefore \( C \leq G_2(x_8) \) and so \( C \leq G_2(x_7) \).

Finally, suppose that \( n = 4 \). Let \( (x_0, \ldots, x_6) \) be an arbitrary 6-path. We show first that \( [U(x_2, \ldots, x_6), U(x_2, \ldots, x_0)] \leq G_2(x_4) \), mimicking the proof of [3, (11.4.d)]. Let \( a \) and \( b \) be any two elements in \( U(x_2, \ldots, x_6) \) and let \( c \) be an arbitrary element of \( U(x_0, \ldots, x_4) \). Let \( h = c b^{-1} c^{-1} b \in U(x_1, \ldots, x_3) \). Then \( [a, h] \in [U(x_2, \ldots, x_6), U(x_2, \ldots, x_0)] = 1 \) and so \( [a, b] = [a, b^h] = ab^{-1} a^{-1} (b^h) = [a, b^h] \in [G(c(x_5)), G_1(c(x_5))] \leq G_1(c(x_5)) \). Since \( c \) is an arbitrary element of \( U(x_0, \ldots, x_4) \) and \( U(x_0, \ldots, x_4) \) acts transitively on \( \Gamma(x_4) - \{x_3\} \), it follows that in fact \( [a, b] \in G_2(x_4) \).

Now suppose that \( G_2(x) = 1 \) for every vertex \( x \). It follows by the previous paragraph that \( U(u_0, \ldots, u_4) \) is abelian for every 4-path \((u_0, \ldots, u_4)\). In particular, Lemma 1 implies that for each vertex \( x \), \( G(x) \Gamma(x) \cong L_2(q) \) (where \( q \) may depend on \( x \)). Again, let \( (x_0, \ldots, x_6) \) be our arbitrary 6-path. We know that \( Z \Gamma(x, y) \leq G_1(x, y) \) for every edge \( \{x, y\} \) (see the proof of [12, Lemma 5]). If \( Z \Gamma(x_3, x_4) \neq G_1(x_3, x_4) \), then \( Z \Gamma(x_3, x_4) \) acts transitively on \( \Gamma(x_2) - \{x_3\} \) since \( Z \Gamma(x_3, x_4) \cong G_2(x_3, x_4) \) and \( G(x_1, x_2, x_4) \cap G_2(x_2) \) contains an element inducing a permutation of order \( f(x_2) \) on \( \Gamma(x_2) \). Since \( [U(x_0, \ldots, x_4), Z \Gamma(x_3, x_4)] = 1 \), \( U(x_0, \ldots, x_4) \leq G_2(x_3) = 1 \), a contradiction. It follows that \( Z \Gamma(x_3, x_4) \leq G_2(x_2) \). Since \( Z \Gamma(x_3, x_4) \leq G_2(x_2) \) and \( G(x_3, x_4) \) acts transitively on \( \Gamma(x_3) - \{x_4\} \), we have \( Z \Gamma(x_3, x_4) \leq G_2(x_3) = 1 \). With this contradiction, the proof of Lemma 5 is concluded.
4. The Proof of Lemma 1

To conclude the proof of Theorem 2, it remains only to prove Lemma 1. We begin with part (a). (We note that part (a) would be an easy consequence of \[8, \text{Theorem 1.1}\] if it weren't for the possibility that some subconstituent contains a regular normal subgroup; in our proof of part (a), which proceeds along the lines used in \[14\], we do not require \[8, \text{Theorem 1.1}\].)

Let \(\{x, y\}\) be an arbitrary edge. Since, by \[12, \text{Lemma 4}\], \(\bigcup(x, y)\) is non-trivial, there exists a prime \(p\) such that \(\bigcup(x, y)\) is non-trivial. We will show that \(G_1(x, y)\) is a \(p\)-group from which it follows that \(U(x, y)\) is one too. Suppose that \(\bigcup G(x, y) \subseteq G_1(u)\) for \(u = x\) or \(y\). Since \(G_1(u) \subseteq G_1\) and \(\bigcup G(x, y) \cap G_1(u) = \bigcup G_1(u)\), we have \(\bigcup G(x, y) = \bigcup G_1(u) \subseteq G_1(u)\). Since \(\Gamma\) is connected, \(\langle \bigcup(x, y) \rangle\) acts transitively on \(E(\Gamma)\). We conclude that \(\bigcup G(x, y) \subseteq G_1(v)\) for \(v = x\) or \(y\).

Choose a vertex \(w \in \Gamma(v)\) and an element \(a \in \bigcup G(x, y)\) not in \(G(w)\). Let \(M = \bigcup G_1(v, w)\) and \(N = M^a\). Then \(N = G_1(v)\) and \(MN\) is a subgroup of \(G_1(v) \cap \bigcup G(x, y)\). Thus \(\bigcup MN\) is a subgroup of \(G_1(v) \cap \bigcup G(x, y)\). Since \(\bigcup MN\) divides \(|\bigcup G(x, y)| = |N| \cdot r\) for some power \(r\) of \(p\), it follows that \(\bigcup MN = \bigcup M\). Since \(\bigcup M = M\), we conclude that \(\bigcup MN = \bigcup M\). Therefore \(\bigcup MN = \bigcup M\) and \(\bigcup MN = \bigcup M\).

We may suppose that \(v = x\). Since \(\bigcup G_1(x, y) \subseteq G_1(w, x, y)\), we have \(\bigcup G_1(x, y) = 1\) if \(n = 3\). We may thus suppose that \(n \geq 4\). Since \(G_1(x, y)\) acts regularly on \(\Gamma(w)\) and \(\bigcup G_1(x, y) \subseteq G_1(w)\), we have that \(|\Gamma(w)| = q + 1\) and \(|\bigcup G(x, y)| = q\). Then \(\Gamma(x)\) is an arbitrary \(n\)-path with \(x = y\) and \(y = x\). Since \(U(x, \ldots, x, x) \subseteq U(\ldots, x, x) \subseteq \cdots \subseteq U(\ldots, x, x) \subseteq \cdots \subseteq U(\ldots, x, x) \subseteq \cdots \subseteq U(\ldots, x, x)\), we have \(U(x, \ldots, x, x) \subseteq \bigcup G(x, y)\). Therefore \(\bigcup G(x, y) \not\subseteq G(x)\). Repeating the argument of the previous paragraph, we conclude that \(\bigcup G(x, y) \subseteq \langle \bigcup(x, y) \rangle\) and so \(\bigcup G(x, y) = 1\).

We turn now to the proof of part (b), proceeding as in \[10, \text{part I, Lemma 10(i)}\]. We have \(n = 3\). Let \((x_0, \ldots, x_6)\) be an arbitrary \(6\)-path. Since \(\Gamma(x_0) = 1\) \(\bigcup U(x_0, x_1) = |\Gamma(x_2)| = \Gamma(x_2)\) and every vertex lies in the same \(G\)-orbit as \(x_0\) or \(x_3\) (or both), \(\Gamma(x)\) is independent of the vertex \(x\). Let \(a_2\) be an arbitrary non-trivial element in \(U(x_2, \ldots, x_3)\). The map which sends \(a\) to \([a_2, a]\) is an isomorphism from \(U(x_0, x_3)\) to \(U(x_1, \ldots, x_4)\). Let \(a\) be an arbitrary non-trivial element in \(U(x_0, \ldots, x_3)\). There exist elements \(b\) and \(c\) in \(U(x_3, \ldots, x_6)\) such that \(ab(x_2) = x_4\) and \(ca(x_4) = x_2\); let \(d = cab\). Since \(d\) maps \((x_2, x_3)\) to \((x_4, x_3)\), we have \(U(x_1, \ldots, x_4)^d = U(x_2, \ldots, x_3)\). Since \(b(x_3) = x_5\) and \([a_2, c] \in \bigcup(x_2, \ldots, x_5)\), \(U(x_2, \ldots, x_5)\), \(x_6\) \(= 1\), \(\Gamma(x_3)\) is fixed by both \(b\) and \(a_2\). \(\bigcup x_3\) and \(\bigcup x_5\) are an element of \(U(x_1, \ldots, x_4)^d = U(x_2, \ldots, x_3)\). Since \(a_2 = [a_2, a] \in \bigcup(x_5, x_6) = 1\), \(a = [a_2, a] \in \bigcup(x_5, x_6)\). Since \(a = [a_2, a] \in \bigcup(x_5, x_6)\), we conclude that \(G(x_5, x_6)\) acts transitively on \((x_1, \ldots, x_4)^d\), the set of non-trivial elements in \((U(x_1, \ldots, x_4)^d\). Hence \(G(x_4)^d = \langle U(x_1, \ldots, x_4)^d\rangle\) is 3-transitive which, by \[8, \text{Theorem 1.1}\], implies that \(G(x_4)^d = \bigcup PGL(2, q)\) for any power \(q\) of \(p\). Thus \(\Gamma(x)^d \subseteq PGL(2, q)\) for every vertex \(x\).

Let \(p\) be odd. We may assume, replacing \((x_0, \ldots, x_6)\) by another \(6\)-path if necessary, that \(G(x_0, \ldots, x_6) = \bigcup G(x_0)\) contains an involution \(h\) (where \(G(x)\) is defined as before). If \(h \in G_1(x_0)\) then \([h, U(x_0, \ldots, x_3)] = 1\), so \(h \in G_1(x_3)\) too which implies that for every \(x \in V(\Gamma)\), \(\bigcup G_1(x)\) is even. Suppose, therefore, that \(h \in G_1(x_0)\). We have \(h \in \bigcup G(x_0)\) since \(h\) acts f.p.f. on \((U(x_0, \ldots, x_3)^d)\). If \(h \notin \bigcup G(x)\) for \(i = 1\) and \(2\), then we can find non-trivial elements \(b \in G(x_2, \ldots, x_3)\) and \(c \in U(x_1, \ldots, x_4)\) commuting with \(h\). Moreover, there is an element \(a \in U(x_0, \ldots, x_3)\) such that \([a, b] = c\). We have \(c = c^k = [a, b]^k = [a^k, b] = [a, b] = c^{-1}\) which is impossible. Thus \(h \in \bigcup G(x_i)\) for \(i = 1\) or \(2\). If \(h \in \bigcup G(x_1)\) then
We obtain a contradiction now using \((4, \text{(1.4)})\): Let
\[
[a, b]^h = [a^h, b^h] = [a^{-1}, b^{-1}] = [a, b]
\]
for every \(a \in U(x_1, \ldots, x_4)\) and every \(b \in U(x_3, \ldots, x_6)\) which implies that \(h \in G_1(x_2)\) and hence \(h \in G_1(x_5)\) too. If \(h \in G_1(x_2)\), we have \(h \in G_1(x_1) \cap G_1(x_4)\) analogously.

We turn finally to part (c) where we have \(n \geq 4\). We suppose first that for some vertex \(x, G(x)^{(x)}\) contains a regular normal subgroup. By part (a), \(I(x, y)\) is either a Fermat prime, one greater than a Mersenne prime or 9; in any event, \(G(x)^{(x)}\) is solvable. For each \(y \in I(x, y)\), we have \(G(x) \cong G_1(x) \cong G_1(x, y)\) where \(G_1(x, y)\) is a p-group and \(G_1(x)/G_1(x, y)\) isomorphic to a subgroup of \(G(x, y)^{(x)}\) which is solvable in every case. Thus \(G(x)\) is solvable. Let \(V = Op G(x) / Op G(x)\) (we point out that \(Op G(x)^{(x)} = 1\) and so \(Op G(x) = Op G_1(x)\) and let \(K\) be the kernel of the action of \(G(x)\) on \(V\). We have \(Op G(x) \leq K \leq G(x)\).

If \(K \not\cong G_1(x)\), then \(K\) contains a \(p'\)-element \(a\) moving some vertex \(y \in I(x, y)\). By \([6, (5.1.4)]\), \(a\) acts trivially on \(Op G(x)\). Since \(G_1(x, y) \leq Op G(x)\), we have \(G_1(x, y) = G_1(x, a(y)) \leq G_1(a(y))\) which contradicts condition (i). It follows that \(K \cong G_1(x)\).

Let \((x_0, \ldots, x_{2n})\) be an arbitrary 2n-path with \(x_n = x\). Let \(y = x_{n+1}\). If \(Op G(x) \not\cong K\), we have \(K_t^{(x)} \neq U(x, y)^{(x)}\) and so \(K\) acts non-trivially on \(Op G(x) \mod (G_1(x, y) \Phi U_1(x_1, \ldots, x_{n+1}))\). This implies, however, that \(K\) acts non-trivially on \(V\) which contradicts the definition of \(K\). It follows that \(K = Op G(x)\).

In particular, \(Op G(x) / K = 1\).

Thus \([7, \text{Theorem B}]\) applies. Recall that \(|U(x_0, \ldots, x_n)|\) is either a Mersenne prime, one less than a Fermat prime or eight; in all three cases, \(U(x_0, \ldots, x_n)\) is cyclic (see condition (a)). Let \(a\) be a generator. Since \(K \cong G_1(x)\), \(a\) acts faithfully on \(V\). Let \((X - 1)^t\) be the minimal polynomial of \(a\) on \(V\). We have \([a, Op G(x)] \leq [U(x_0, \ldots, x_n), U(x_{n-1}, \ldots, x_{n+1})] \leq U(x_{n-2}, \ldots, x_{n+1})\) and \([U(x_0, \ldots, x_n), U(x_{n+1}, \ldots, x_{n+1})] = U(x_{1-1}, \ldots, x_{n+1})\) for \(1 \leq i \leq n - 2\). It follows that \(r \leq n - 1\). Hence, by \([7, \text{Theorem B}]\) (and conditions (a) and (b)), we have either

(A) \(n = 4, \quad |U(x_0, \ldots, x_4)| = 4\) and \(r = 3\).

(B) \(n = 8, \quad |U(x_0, \ldots, x_8)| = 8\) and \(6 \leq r \leq 7\).

(C) \(n = 8, \quad |U(x_0, \ldots, x_8)| = 7\) and \(r = 7\) or

(D) \(n \geq 6, \quad |U(x_0, \ldots, x_n)| = 4\) and \(3 \leq r \leq 4\).

Moreover, in cases (A) and (B), a 3- or 7-Sylow subgroup of \(G(x)\) must be non-abelian. Since in both cases, \(G_1(x, y)\) is a 2-group, this is possible only if \(G_1(x, y) = L_2(q), U_3(q)\) or \(S_2(q)\) for some power \(q\) of two, \(q > 2\). In particular, \(G(x, y, x_{n+2})\) acts f.p.f. on \(G(x, y) - \{x, x_{n+2}\}\). The proof of Lemma 2 above shows easily that we may assume, substituting the 2n-path \((x_0, \ldots, x_{2n})\) by another if necessary, that \(G(x_0, \ldots, x_{2n})\) contains an element \(d_i\) acting f.p.f. on \(G(x) = \{x_i - 1, x_i + 1\}\) for \(i\) odd, \(1 \leq i \leq 2n - 1\). We have \(d_i \in G_1(x_i)\) for \(i\) even, \(2 \leq i \leq 2n - 2\). It follows that \(d_{n+1}\) centralizes \([U(x_0, \ldots, x_n), U(x_2, \ldots, x_{n+2})]\) which is contained in \(U(x_1, \ldots, x_{n+1})\). Since \(d_{n+1}\) acts f.p.f. on \(G(x_{n+1}) = \{x_n, x_{n+2}\}\), we conclude that \([U(x, \ldots, x_n), U(x_2, \ldots, x_{n+2})] = 1\) and hence \([U(x_0, \ldots, x_n), U(x_2, \ldots, x_{n+1})] = 1\) since \([U(x_0, \ldots, x_n), U(x_1, \ldots, x_{n+1})] = 1\) in any event. This implies that \(r \leq n - 2\). Thus we have case (B) with \(r = 6\). But then for every \(a \in U(x_0, \ldots, x_8)\) and every \(b \in U(x_4, \ldots, x_1)\) there exists a unique element \(c \in U(x_3, \ldots, x_1)\) such that \([a, b]c \in U(x_2, \ldots, x_9)\). Since \(d_3\) centralizes \(a\) and \(b\) and normalizes \(U(x_3, \ldots, x_1)\) and \(U(x_2, \ldots, x_9)\), \(d_3\) must centralize \(c\) too which implies that \(c = 1\). Hence \([U(x_0, \ldots, x_8), U(x_4, \ldots, x_9)] \leq U(x_2, \ldots, x_9)\). It follows that \(r \leq 5\), a contradiction.

We suppose next that case (C) holds. Suppose \(G(x, y, x_{10})\) (and hence, we may assume, \(G(x_0, \ldots, x_{10})\)) contains a 7-element \(d\) such that \(d^3\) acts f.p.f. on \(G(x, y) - \{x_1, y_1\}\). Since \(d^3\) lies in \(G_1(x_8) \cap G_1(x_{10})\), we conclude as in the previous paragraph that \([U(x_0, \ldots, x_8), U(x_2, \ldots, x_{10})] = 1\). Hence \(r \leq 6\), a contradiction. It follows that either \(G(x)^{(x)} = L_2(7)\) or \(G(x)^{(x)}\) contains a regular normal subgroup of order 8 too; in neither case does 16 divide \(|G(x)|\) so that \(SL(2, 7)\) is not involved in \(G(x)\) (nor in \(G(x)\)). We obtain a contradiction now using \([4, (1.4)]\): Let \(a\) be an arbitrary element in \(O_7G(x)\).
For each \( w \in \Gamma(x) \), we have \( G_1(w, x) = G_1(a(w), x) \) since \([O_7G(x), G_1(x, w)] \leq [O_7G(x), O_7G(x)] = 1 \) and thus \( a \in G(w) \) by condition (i). Hence \( a \in G_1(x) \). Now let \( w \) be an arbitrary vertex in \( \Gamma(x) \) - \( \{ y \} \). Since \( a \) centralizes \( G_1(w, x) \) and \( G_1(w, x) \Gamma(y) \) is self-centralizing in \( G(x, y) \Gamma(y) \), we have \( a \in G_1(y) \). It follows that \( O_7G(x) = 1 \). Thus by [4, (1.4)], we have \( ZJO_7G(x, y) \equiv G(x) \). The same argument yields, however, that \( ZJO_7G(x, y) \equiv G(y) \) too. Thus \( ZJO_7G(x, y) = 1 \) although \( 1 \neq G_1(x, y) \equiv O_7G(x, y) \), a contradiction.

We conclude that case (D) holds, i.e., that \( G(x) \Gamma(x) \equiv Sz(2) \). For every such vertex \( u \), we set \( \bar{G}(u) = G(u) \) and \( f(u) = 1 \); for other vertices \( u \), we define \( \bar{G}(u) \) and \( f(u) \) as before. We can then define a good path as before and the proof of Lemma 2 is easily modified to show that an arbitrary good \( s \)-path \((x_0, \ldots, x_s)\) can be extended to a good \((s+1)\)-path \((x_0, \ldots, x_{s+1})\) and that \( G(x_0, \ldots, x_s) \leq G(x_0, \ldots, x_n, x_{n+1}) \) for every good \((s+1)\)-path \((x_0, \ldots, x_s, x_{s+1})\) with \( s \geq n+1 \). Thus, setting \( H = G(x_0, \ldots, x_{n+1}) = G_1(x_0) \cap G(x_0, \ldots, x_{n+1}) \), we can define \( d, S \) and \( \Sigma \) just as in the proof of Lemma 4 above. We have \( S(u)E(u) \equiv Sz(2) \) for every \( u \in V(\Sigma) \) and \( d \leq 2 \), i.e., \( n/d \geq 3 \). Let \( \{ u, v \} \) be an arbitrary edge of \( \Sigma \). Since the normalizer of \( S_1(u, v) \) in \( G(w) \), \( w = u \) or \( v \), is \( S(u, v) \), the centralizer of \( S_1(w) \) is contained in \( S_1(w) \). Thus [5, Theorem B] yields three non-trivial characteristic subgroups \( A_1, A_2 \) and \( A_3 \) of \( S(u, v) \) such that \( S(w) = N_{S(w)}(A_i) \cdot N_{S(w)}(A_j) \) for \( w = u \) and \( v \) and \( 1 \leq i < j \leq 3 \). This yields a contradiction since none of the \( A_i \) can be normal in both \( S(u) \) and \( S(v) \).

It remains only to consider the case that \( n \geq 4 \) and for some vertex \( x \), \( G(x) \Gamma(x) \equiv R(3) \). For every such vertex \( x \), we set \( \bar{G}(x) = G(x) \) and \( f(x) = 2 \); for other vertices \( x \), we define \( \bar{G}(x) \) and \( f(x) \) as before. Setting \( H = G(x_0, \ldots, x_{n+1}), (x_0, \ldots, x_{n+1}) \) and an arbitrary \((n+1)\)-path, we can again define \( d, S \) and \( \Sigma \) just as in the proof of Lemma 4 above. We have \( S(u)E(u) \equiv L_2(3) \) for every \( u \in V(\Sigma) \) and \( d \leq 2 \). Exactly as before, we conclude that \( S_1(u, v) = 1 \) for every \( \{ u, v \} \in E(\Sigma) \) using [4, (1.4)]. But \( \Sigma \) is \((S, n/d)\)-Moufang so \( n/d = 2 \) and thus \( n = 4 \).

To prove the conclusion of Lemma 1, we show that for any \( 4 \)-path \((x_0, \ldots, x_4) \) (and therefore \( U(x_0, x_1) \Gamma(x_0) \)) is of class at most 2 which contradicts the assumption that \( G(x) \Gamma(x) \equiv R(3) \) for some vertex \( x \). As we showed at the end of the proof of Lemma 5 above, \([U(x_0, \ldots, x_4), U(x_0, \ldots, x_4)] \leq G_2(x_2) \). Thus it suffices to show that \([G_2(x_2), U(x_0, \ldots, x_4)] = 1 \). We follow the proof of [3, (11.4.c)]. Extend \((x_0, \ldots, x_4) \) to a \( 7 \)-path \((x_0, \ldots, x_7) \). Let \( a \in G_2(x_2) \) and \( b \in U(x_0, \ldots, x_4) \) be arbitrary. Let \( c \) be any non-trivial element in \( U(x_3, x_7) \). We have \( b^c \in U(c(x_0), c(x_1), c(x_2), x_4, x_3) \leq G(x_2) \) and thus \([a, b^c] \leq G_2(x_2) \cap G_1(c(x_2)) = 1 \). Let \( d = b^cb^{-1} = [c, b] \); we have \( d \in [U(x_0, \ldots, x_4), U(x_3, x_7)] \leq U(x_2, \ldots, x_5) \). It follows that \([a, b] = 1 \) since \( 1 = [a, b^c] = [a, db] = [a, d] \cdot [a, b]^d \) and \([a, d] \in [G_2(x_2), U(x_2, \ldots, x_5)] \leq G_2(x_2) \cap G_1(x_4) = 1 \).

5. The Proof of Theorem 3

Let \( \Gamma \) and \( G \) fulfill the hypotheses of Theorem 3. For each \((n+1)\)-path \((x_0, \ldots, x_{n+1}) \) there is a unique shortest good continuation \((x_0, \ldots, x_n, x_{n+2}, \ldots, x_t) \) with \( x_0 = x_t \). The number \( t \) does not, of course, depend on the choice of \((x_0, \ldots, x_{n+1}) \). Let \( H = G_1(x_0) \cap G(x_1, \ldots, x_{n+1}) \). We now know (by Theorem 2) that \( H \leq G_1(x_n) \) but that \( H \) acts f.p.f. on \( \Gamma(x_i) - \{x_{i-1}, x_{i+1}\} \) for \( 0 < i < n \). This implies that \( n \) divides \( t \). In particular, \( t \) is even if \( n \geq 4 \). We assume for the moment that \( t \) is even when \( n = 3 \) too.

Let \((x_0, \ldots, x_{2n-1}) \) be a good \((2n-1)\)-path and let \( a \) be an involution in \( U(x_0, \ldots, x_n) \). By Lemma 3 above, \( W = \{ x_{n-1}, x_{2n-2}, \ldots, x_n, a(x_{n+1}), \ldots, a(x_{2n-1}) \} \) is a good \((2n-1)\)-path. Since \( 2(n-1) \equiv n + 1 \), \( W \) determines a good \( t \)-circuit \( C \). Since \( a \) reverses \( W \), it reflects \( C \). In particular, \( a \) fixes the vertex \( y \), say, opposite \( x_n \) on \( C \). By condition (iii), \( d(x_n, y) \leq n \). If
t > 2n, there must be a path \((y_0, y_1, \ldots, y_k)\) for some \(k \leq n\) with \(y_0\) and \(y_k\) on \(C\) but not \(y_1\). The group \(G(W)\) contains an element \(b\) not fixing \(y_1\). Since \(G(W)\) fixes \(y_0\) and \(y_k\), \((y_k, y_{k-1}, \ldots, y_0, b(y_1), \ldots, b(y_k))\) is a closed 2k-path. It follows from condition (i) and the existence of a closed path of length at most 2n that \(I'\) is a generalised \(n\)-gon (so that \(t = 2n\) after all) and thus \(I' = \Delta\).

It remains only to show that \(t\) is even when \(n = 3\). Suppose, therefore, that \(n = 3\) but that \(t\) is odd. The involution \(a\) must then exchange the vertices \(u\) and \(v\), say, of the edge \(\{u, v\}\) opposite \(X_3\) on \(C\). Let \(X = U(u, v)\). By Theorem 2, \(|C_X(a)| = |X|^{1/2}\) or \(|X|^{1/3}\). By condition (iii), we have \(C_X(a) \subseteq G(x_1, x_2)\). Thus \(C_X(a) \cap G(x_1, x_2)\) contains an involution since, by condition (a), \(|X| > 8\). Again by condition (iii), it follows that \(x_1\) or \(x_2 \in I'(u) \cup I'(v)\) since every involution in \(X\) is conjugate in \(G\) to \(a\). But then \(a\) cannot exchange \(u\) and \(v\) after all. With this contradiction, the proof of Theorem 3 is complete.

REFERENCES

3. J. Faulkner, Groups with Steinberg relations and coordinatization of polygonal geometries, Mem. Amer. Math. Soc. 10 (1977), n. 185.

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