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On several problems about automorphisms of the free group of rank two

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ABSTRACT

Let F_n be a free group of rank n generated by $x_1, \dots, x_n$. In this paper we discuss three algorithmic problems related to automorphisms of F_2.

A word $u = u(x_1, \dots, x_n)$ of F_n is called positive if no negative exponents of x_i occur in u. A word u in F_n is called potentially positive if $\phi(u)$ is positive for some automorphism ϕ of F_n. We prove that there is an algorithm to decide whether or not a given word in F_2 is potentially positive, which gives an affirmative solution to problem F34a in [G. Baumslag, A.G. Myasnikov, V. Shpilrain, Open problems in combinatorial group theory, second ed., in: Contemp. Math., vol. 296, 2002, pp. 1–38, online version: http://www.grouptheory.info] for the case of F_2.

Two elements u and v in F_n are said to be boundedly translation equivalent if the ratio of the cyclic lengths of $\phi(u)$ and $\phi(v)$ is bounded away from 0 and from ∞ for every automorphism ϕ of F_n. We provide an algorithm to determine whether or not two given elements of F_2 are boundedly translation equivalent, thus answering question F38c in the online version of [G. Baumslag, A.G. Myasnikov, V. Shpilrain, Open problems in combinatorial group theory, second ed., in: Contemp. Math., vol. 296, 2002, pp. 1–38, online version: http://www.grouptheory.info] for the case of F_2.

We also provide an algorithm to decide whether or not a given finitely generated subgroup of F_2 is the fixed point group of some automorphism of F_2, which settles problem F1b in [G. Baumslag, A.G. Myasnikov, V. Shpilrain, Open problems in combinatorial group theory, second ed., in: Contemp. Math., vol. 296, 2002, pp. 1–38, online version: http://www.grouptheory.info] in the affirmative for the case of F_2.

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1. Introduction

Let F_n be the free group of rank $n \geqslant 2$ with basis Σ. In particular, if $n = 2$, we let $\Sigma = \{a, b\}$, namely, F_2 is the free group with basis $\{a, b\}$. A word v in F_n is called *cyclically reduced* if all its cyclic permutations are reduced. A *cyclic word* is defined to be the set of all cyclic permutations of a cyclically reduced word. By $[v]$ we denote the cyclic word associated with a word v. Then by $\|v\|$ we denote the length of the cyclic word $[v]$ associated with v, that is, the number of cyclic permutations of a cyclically reduced word which is conjugate to v. The length $\|v\|$ is called the *cyclic length* of v. For two automorphisms ϕ and ψ of F_n, by writing $\phi \equiv \psi$ we mean the equality of ϕ and ψ over all cyclic words in F_n, that is, $\phi(w) = \psi(w)$ for every cyclic word w in F_n.

Recall that a *Whitehead automorphism* α of F_n is defined to be an automorphism of one of the following two types (cf. [8]):

(W1) α permutes elements in $\Sigma^{\pm 1}$.
(W2) α is defined by a letter $x \in \Sigma^{\pm 1}$ and a set $S \subset \Sigma^{\pm 1} \setminus \{x, x^{-1}\}$ in such a way that if $c \in \Sigma^{\pm 1}$ then (a) $\alpha(c) = cx$ provided $c \in S$ and $c^{-1} \notin S$; (b) $\alpha(c) = x^{-1}cx$ provided both $c, c^{-1} \in S$; (c) $\alpha(c) = c$ provided both $c, c^{-1} \notin S$.

If α is of type (W2), we write $\alpha = (S, x)$. Note that in the expression of $\alpha = (S, x)$ it is conventional to include the defining letter x in the defining set S, but for the sake of brevity of notation we will omit a from S as defined above.

Throughout the present paper, we let

$$\sigma = \big(\{a\}, b\big) \quad \text{and} \quad \tau = \big(\{b\}, a\big)$$

be Whitehead automorphisms of type (W2) of F_2. Then σ and τ are automorphisms of F_2 defined by

$$\sigma : \begin{cases} a \mapsto ab, \\ b \mapsto b \end{cases} \quad \text{and} \quad \tau : \begin{cases} a \mapsto a, \\ b \mapsto ba. \end{cases}$$

Recently the author [8] proved that every automorphism of F_2 can represented in one of two particular types over all cyclic words of F_2 as follows:

Lemma 1.1. *(Lemma 2.3 in [7].) For every automorphism ϕ of F_2, ϕ can be represented as $\phi \equiv \beta\phi'$, where β is a Whitehead automorphism of F_2 of type* (W1) *and ϕ' is a chain of one of the forms*

$$\text{(C1)} \quad \phi' \equiv \tau^{m_k}\sigma^{l_k} \cdots \tau^{m_1}\sigma^{l_1},$$
$$\text{(C2)} \quad \phi' \equiv \tau^{-m_k}\sigma^{-l_k} \cdots \tau^{-m_1}\sigma^{-l_1}$$

with $k \in \mathbb{N}$ and both $l_i, m_i \geqslant 0$ for every $i = 1, \ldots, k$.

With the notation of Lemma 1.1, we define the *length of an automorphism* ϕ of F_2 as $\sum_{i=1}^{k}(m_i + l_i)$, which is denoted by $|\phi|$. Then obviously $|\phi| = |\phi'|$.

In the present paper, with the help of Lemma 1.1, we resolve three algorithmic problems related to automorphisms of F_2. Indeed, the description of automorphisms ϕ of F_2 in the statement of Lemma 1.1 provides us with a very useful computational tool that facilitates inductive arguments on $|\phi|$ in the proofs of various statements.

The first problem we deal with is about potential positivity of elements in a free group the notion of which was first introduced by Khan [6].

Definition 1.2. Let F_n be generated by $x_1, \ldots, x_n$. A word $u = u(x_1, \ldots, x_n)$ of F_n is called positive if no negative exponents of x_i occur in u. A word u in F_n is called potentially positive if $\phi(u)$ is positive for some automorphism ϕ of F_n.

It was shown by Khan [6] and independently by Meakin and Weil [9] that the Hanna Neumann conjecture is satisfied if one of the subgroups is generated by positive elements.

In Section 2, we shall describe an algorithm to decide whether or not a given word in F_2 is potentially positive, which gives an affirmative solution to problem F34a in [1] for the case of F_2. This problem of detecting potential positive words of F_2 was previously settled by Goldstein [4] using different methods.

The second problem we discuss here is related to the notion of bounded translation equivalence which is one of generalizations of the notion of translation equivalence, due to Kapovich, Levitt, Schupp and Shpilrain [5].

Definition 1.3. Two elements u and v in F_n are called translation equivalent in F_n if $\|\phi(u)\| = \|\phi(v)\|$ for every automorphism ϕ of F_n.

Several different sources of translation equivalence in free groups were provided by Kapovich, Levitt, Schupp and Shpilrain [5] and the author [7]. In another paper of the author [8], it was proved that there is an algorithm to decide whether or not two given elements u and v of F_2 are translation equivalent. Generalizing the notion of translation equivalence, bounded translation equivalence is defined as follows:

Definition 1.4. Two elements u and v in F_n are said to be boundedly translation equivalent in F_n if there is $C > 0$ such that

$$\frac{1}{C} \leqslant \frac{\|\phi(u)\|}{\|\phi(v)\|} \leqslant C$$

for every automorphism ϕ of F_2.

Clearly every pair of translation equivalent elements in F_n are boundedly translation equivalent in F_n, but not vice versa. As one of specific examples of bounded translation equivalence, we mention that two elements a and $a[a,b]$ are boundedly translation equivalent in F_2. Indeed, if $u = a$ and $v = a[a,b]$, then we have, in view of Lemma 1.1, that

$$\big\|\phi(v)\big\| = \big\|\phi(u)\big\| + 4$$

for every automorphism ϕ of F_2, because $[a,b]$ is invariant under the action of $\sigma^{\pm 1}$ or $\tau^{\pm 1}$ and there can be no cancellation between $\phi'(a)$ and $[a,b]$ nor between $[a,b]$ and $\phi'(a)$ for every chain ϕ' of the form (C1) or (C2). Hence

$$\frac{1}{5} \leqslant \frac{\|\phi(u)\|}{\|\phi(v)\|} \leqslant 1$$

for every automorphism ϕ of F_2.

In Section 3, developing further the technique used in [8], we shall demonstrate that there is an algorithm to determine whether or not two given elements of F_2 are boundedly translation equivalent, thus affirmatively answering question F38c in the online version of [1] for the case of F_2.

Our last problem is concerned with the notion of fixed point groups of automorphisms of free groups.

Definition 1.5. A subgroup H of F_n is called the fixed point group of an automorphism ϕ of F_n if H is precisely the set of the elements of F_n which are fixed by ϕ.

Due to Bestvina and Handel [2], a subgroup of rank bigger than n cannot possibly be the fixed point group of an automorphism of F_n. Recently Martino and Ventura [10] provided an explicit description for the fixed point groups of automorphisms of F_n, generalizing the maximal rank case studied by Collins and Turner [3]. However, this description is not a complete characterization of all fixed point groups of automorphisms of F_n. On the other hand, Maslakova [11] proved that, given an automorphism ϕ of F_n, it is possible to effectively find a finite set of generators of the fixed point group of ϕ.

In Section 4, we shall present an algorithm to decide whether or not a given finitely generated subgroup of F_2 is the fixed point group of some automorphism of F_2, which settles problem F1b in [1] in the affirmative for the case of F_2.

2. Potential positivity in F_2

Recall that F_2 denotes the free group with basis $\Sigma = \{a, b\}$, and that σ and τ denote Whitehead automorphisms

$$\sigma = (\{a\}, b), \qquad \tau = (\{b\}, a)$$

of F_2 of type (W2). We also recall from [8] the definition of trivial or proper cancellation. For a cyclic word w in F_2 and a Whitehead automorphism, say σ, of F_2, a subword of the form ab^ra^{-1} ($r \neq 0$), if any, in w is invariant in passing from w to $\sigma(w)$, although there occurs cancellation in $\sigma(ab^ra^{-1})$ (note that $\sigma(ab^ra^{-1}) = ab \cdot b^r \cdot b^{-1}a^{-1} = ab^ra^{-1}$). Such cancellation is called *trivial cancellation*. And cancellation which is not trivial cancellation is called *proper cancellation*. For example, a subword $ab^{-r}a$ ($r \geqslant 1$), if any, in w is transformed to $ab^{-r+1}ab$ by applying σ, and thus the cancellation occurring in $\sigma(ab^{-r}a)$ is proper cancellation.

The following lemma from [8] will play a fundamental role throughout the present paper.

Lemma 2.1. *(Lemma 2.4 in [8].) Let u be a cyclic word in F_2, and let ψ be a chain of type* (C1) *(or* (C2)*). If ψ contains at least $\|u\|$ factors of σ (or σ^{-1}), then there cannot occur proper cancellation in passing from $\psi(u)$ to $\sigma\psi(u)$ (or $\psi(u)$ to $\sigma^{-1}\psi(u)$). Also if ψ contains at least $\|u\|$ factors of τ (or τ^{-1}), then there cannot occur proper cancellation in passing from $\psi(u)$ to $\tau\psi(u)$ (or $\psi(u)$ to $\tau^{-1}\psi(u)$).*

The main result of this section is

Theorem 2.2. *Let u be an element in F_2, and let Ω be the set of all chains of type* (C1) *or* (C2) *of length less than or equal to $2\|u\| + 3$. Suppose that the cyclic word $[\phi(u)]$ is positive for some automorphism ϕ of F_2. Then there exists $\psi \in \Omega$ and a Whitehead automorphism β of F_2 of type* (W1) *such that the cyclic word $[\beta\psi(u)]$ is positive (which is obviously equivalent to saying that there exists $c \in F_2$ such that $\pi_c\beta\psi(u)$ is positive, where π_c is the inner automorphism of F_2 induced by c).*

Once this theorem is proved, an algorithm to decide whether or not a given word in F_2 is potentially positive is naturally derived as follows.

Algorithm 2.3. Let u be an element in F_2, and let Ω be defined as in the statement of Theorem 2.2. Clearly Ω is a finite set. Check if there is $\psi \in \Omega$ and a Whitehead automorphism β of F_2 of type (W1) for which the cyclic word $[\beta\psi(u)]$ is positive. If so, conclude that u is potentially positive; otherwise conclude that u is not potentially positive.

Proof of Theorem 2.2. By Lemma 1.1, ϕ can be expressed as

$$\phi \equiv \beta\phi',$$

where β is a Whitehead automorphism of F_2 of type (W1) and ϕ' is a chain of type (C1) or (C2). By the hypothesis of the theorem,

$$\big[\phi(u)\big] = \big[\beta\phi'(u)\big] \quad \text{is positive.} \tag{1}$$

If $|\phi'| \leqslant 2\|u\| + 3$, then there is nothing to prove. So suppose that $|\phi'| > 2\|u\| + 3$. We proceed with the proof by induction on $|\phi'|$. Assume that ϕ' is a chain of type (C1) which ends in τ (the other cases are analogous). Write

$$\phi' = \tau\phi_1,$$

where ϕ_1 is a chain of type (C1). Since $|\phi_1| \geqslant 2\|u\| + 3$, ϕ_1 must contain at least $\|u\| + 2$ factors of σ or τ. We consider two cases separately.

Case 1. σ occurs at least $\|u\| + 2$ times in ϕ_1.

Write

$$\phi_1 = \tau^{m_t}\sigma^{\ell_t} \cdots \tau^{m_1}\sigma^{\ell_1},$$

where all $m_i, \ell_i > 0$ but ℓ_1 and m_t may be zero.

Case 1.1. $m_t \geqslant 1$.

In this case, put

$$\phi_1 = \tau^{m_t}\phi_2,$$

where ϕ_2 is a chain of type (C1). By Lemma 2.1, no proper cancellation can occur in passing from $[\sigma^{\ell_t - 1} \cdots \tau^{m_1}\sigma^{\ell_1}(u)]$ to $[\phi_2(u)]$, and hence the cyclic word $[\phi_2(u)]$ does not contain a subword of the form a^2 or a^{-2}. From this fact and the assumption $m_t \geqslant 1$, we can observe that no proper cancellation occurs in passing from $[\phi_1(u)]$ to $[\tau\phi_1(u)] = [\phi'(u)]$. This implies from (1) that the cyclic word $[\beta\phi_1(u)]$ is positive, and thus induction completes the case.

Case 1.2. $m_t = 0$.

In this case, we may put

$$\phi_1 = \sigma\phi_3,$$

where ϕ_3 is a chain of type (C1). Again by Lemma 2.1, no proper cancellation can occur in passing from $[\phi_3(u)]$ to $[\sigma\phi_3(u)] = [\phi_1(u)]$. Additionally, the proof of Theorem 1.2 of [8] shows that proper cancellation occurs in passing from $[\phi_3(u)]$ to $[\tau\phi_3(u)]$ exactly in the same place where proper cancellation occurs in passing from $[\phi_1(u)]$ to $[\tau\phi_1(u)] = [\phi'(u)]$. Therefore, by (1), the cyclic word $[\beta\tau\phi_3(u)]$ is positive. Since $|\tau\phi_3| = |\phi'| - 1$, we are done by induction.

Case 2. τ occurs at least $\|u\| + 2$ times in ϕ_1.

In this case, also by Lemma 2.1, no proper cancellation can occur in passing from $[\phi_1(u)]$ to $[\tau\phi_1(u)] = [\phi'(u)]$. It then follows from (1) that the cyclic word $[\beta\phi_1(u)]$ is positive; hence the required result follows by induction. □

3. Bounded translation equivalence in F_2

We begin this section by fixing notation. Following [5], if w is a cyclic word in F_2 and $x, y \in \{a, b\}^{\pm 1}$, we use $n(w; x, y)$ to denote the total number of occurrences of the subwords xy and $y^{-1}x^{-1}$ in w. Then clearly $n(w; x, y) = n(w; y^{-1}, x^{-1})$. Similarly we denote by $n(w; x)$ the total number of occurrences of x and x^{-1} in w. Again clearly $n(w; x) = n(w; x^{-1})$.

In this section, we shall prove that there is an algorithm to determine bounded translation equivalence in F_2. Let $u \in F_2$. We first establish four preliminary lemmas which demonstrate the difference between $\|\sigma\psi(u)\|$ or $\|\tau\psi(u)\|$ and $\|\psi(u)\|$, and which describe the situation when this difference becomes zero, in the case where ψ is a chain of type (C1) that contains a number of factors of σ. We remark that similar statements to the lemmas also hold if σ and τ are interchanged with each other, or (C1) is replaced by (C2) and σ and τ are replaced by σ^{-1} and τ^{-1}, respectively.

Lemma 3.1. *Let $u \in F_2$. Suppose that ψ is a chain of type* (C1) *which contains at least $\|u\| + 2$ factors of σ. We may write $\psi = \tau^m\sigma\psi_1$, where $m \geqslant 0$ and ψ_1 is a chain of type* (C1). *Then*

$$\text{(i)} \quad \|\sigma\psi(u)\| - \|\psi(u)\| = \|\sigma\tau^m\psi_1(u)\| - \|\tau^m\psi_1(u)\| + m\big(\|\sigma\psi_1(u)\| - \|\psi_1(u)\|\big);$$

$$\text{(ii)} \quad \|\tau\psi(u)\| - \|\psi(u)\| = \|\tau\tau^m\psi_1(u)\| - \|\tau^m\psi_1(u)\| + \|\sigma\psi_1(u)\| - \|\psi_1(u)\|.$$

Proof. By the proof of Case 1 of Theorem 1.2 in [8], we see that

$$n\big(\big[\tau^i\sigma\psi_1(u)\big]; b, a^{-1}\big) = n\big(\big[\tau^i\psi_1(u)\big]; b, a^{-1}\big) \tag{2}$$

for every $i \geqslant 0$, because ψ_1 contains at least $\|u\| + 1$ factors of σ. In particular,

$$n\big(\big[\psi(u)\big]; b, a^{-1}\big) = n\big(\big[\tau^m\psi_1(u)\big]; b, a^{-1}\big), \tag{3}$$

for $\psi = \tau^m\sigma\psi_1$. Since only a or a^{-1} can possibly cancel or newly occur in the process of applying τ, the number of b and b^{-1} remains unchanged if τ is applied. Thus

$$\begin{aligned} n\big(\big[\tau^i\sigma\psi_1(u)\big]; b\big) &= n\big(\big[\sigma\psi_1(u)\big]; b\big); \\ n\big(\big[\tau^i\psi_1(u)\big]; b\big) &= n\big(\big[\psi_1(u)\big]; b\big) \end{aligned} \tag{4}$$

for every $i \geqslant 0$. Also since only b or b^{-1} can possibly cancel or newly occur in the process of applying σ, we get

$$n\big(\big[\sigma\psi_1(u)\big]; b\big) = n\big(\big[\psi_1(u)\big]; b\big) + \|\sigma\psi_1(u)\| - \|\psi_1(u)\|.$$

By (4), this equality can be rewritten as

$$n\big(\big[\tau^i\sigma\psi_1(u)\big]; b\big) = n\big(\big[\tau^i\psi_1(u)\big]; b\big) + \|\sigma\psi_1(u)\| - \|\psi_1(u)\| \tag{5}$$

for every $i \geqslant 0$. In particular,

$$n\big(\big[\psi(u)\big]; b\big) = n\big(\big[\tau^m\psi_1(u)\big]; b\big) + \|\sigma\psi_1(u)\| - \|\psi_1(u)\|, \tag{6}$$

for $\psi = \tau^m\sigma\psi_1$.

Equality (5) together with (2) yields that

$$\begin{aligned} &n\big(\big[\tau^i\sigma\psi_1(u)\big]; b\big) - n\big(\big[\tau^i\sigma\psi_1(u)\big]; b, a^{-1}\big) \\ &\quad = n\big(\big[\tau^i\psi_1(u)\big]; b\big) - n\big(\big[\tau^i\psi_1(u)\big]; b, a^{-1}\big) + \|\sigma\psi_1(u)\| - \|\psi_1(u)\| \end{aligned} \tag{7}$$

for every $i \geqslant 0$. Here, since

$$\|\tau^{i+1}\sigma\psi_1(u)\| - \|\tau^i\sigma\psi_1(u)\| = n([\tau^i\sigma\psi_1(u)]; b) - 2n([\tau^i\sigma\psi_1(u)]; b, a^{-1});$$
$$\|\tau^{i+1}\psi_1(u)\| - \|\tau^i\psi_1(u)\| = n([\tau^i\psi_1(u)]; b) - 2n([\tau^i\psi_1(u)]; b, a^{-1}),$$

equality (7) can be rephrased as

$$\|\tau^{i+1}\sigma\psi_1(u)\| - \|\tau^i\sigma\psi_1(u)\| = \|\tau^{i+1}\psi_1(u)\| - \|\tau^i\psi_1(u)\| + \|\sigma\psi_1(u)\| - \|\psi_1(u)\|$$

for every $i \geqslant 0$. By summing up both sides of these equalities changing i from 0 to $m-1$, we have

$$\|\tau^m\sigma\psi_1(u)\| - \|\sigma\psi_1(u)\| = \|\tau^m\psi_1(u)\| - \|\psi_1(u)\| + m(\|\sigma\psi_1(u)\| - \|\psi_1(u)\|),$$

so that

$$\|\tau^m\sigma\psi_1(u)\| - \|\tau^m\psi_1(u)\| = (m+1)(\|\sigma\psi_1(u)\| - \|\psi_1(u)\|). \tag{8}$$

Since $\psi = \tau^m\sigma\psi_1$, equality (8) can be rephrased as

$$\|\psi(u)\| - \|\tau^m\psi_1(u)\| = (m+1)(\|\sigma\psi_1(u)\| - \|\psi_1(u)\|). \tag{9}$$

Clearly

$$n([\psi(u)]; a) = \|\psi(u)\| - n([\psi(u)]; b);$$
$$n([\tau^m\psi_1(u)]; a) = \|\tau^m\psi_1(u)\| - n([\tau^m\psi_1(u)]; b).$$

These equalities together with (6) and (9) yield that

$$n([\psi(u)]; a) = n([\tau^m\psi_1(u)]; a) + m(\|\sigma\psi_1(u)\| - \|\psi_1(u)\|). \tag{10}$$

It then follows from

$$\|\sigma\psi(u)\| - \|\psi(u)\| = n([\psi(u)]; a) - 2n([\psi(u)]; a, b^{-1});$$
$$\|\sigma\tau^m\psi_1(u)\| - \|\tau^m\psi_1(u)\| = n([\tau^m\psi_1(u)]; a) - 2n([\tau^m\psi_1(u)]; a, b^{-1})$$

together with (3) and (10) that

$$\|\sigma\psi(u)\| - \|\psi(u)\| = \|\sigma\tau^m\psi_1(u)\| - \|\tau^m\psi_1(u)\| + m(\|\sigma\psi_1(u)\| - \|\psi_1(u)\|),$$

thus proving the first assertion of the lemma.

On the other hand, we deduce from

$$\|\tau\psi(u)\| - \|\psi(u)\| = n([\psi(u)]; b) - 2n([\psi(u)]; b, a^{-1});$$
$$\|\tau\tau^m\psi_1(u)\| - \|\tau^m\psi_1(u)\| = n([\tau^m\psi_1(u)]; b) - 2n([\tau^m\psi_1(u)]; b, a^{-1})$$

together with (3) and (6) that

$$\|\tau\psi(u)\| - \|\psi(u)\| = \|\tau\tau^m\psi_1(u)\| - \|\tau^m\psi_1(u)\| + \|\sigma\psi_1(u)\| - \|\psi_1(u)\|,$$

which proves the second assertion of the lemma. □

Lemma 3.2. *Let $u \in F_2$. Suppose that ψ is a chain of type* (C1) *which contains at least $\|u\|$ factors of σ. Then*

$$\text{(i)}\quad \|\sigma\psi(u)\| - \|\psi(u)\| \geqslant 0;$$
$$\text{(ii)}\quad \|\tau\psi(u)\| - \|\psi(u)\| \geqslant 0.$$

Proof. Clearly

$$\begin{aligned}\|\sigma\psi(u)\| - \|\psi(u)\| &= n\big([\psi(u)]; a\big) - 2n\big([\psi(u)]; a, b^{-1}\big)\\ &= n\big([\psi(u)]; a, a\big) + n\big([\psi(u)]; a, b\big) - n\big([\psi(u)]; a, b^{-1}\big). \qquad (11)\end{aligned}$$

Since ψ contains at least $\|u\|$ factors of σ, by Lemma 2.1, there cannot occur proper cancellation in passing from $[\psi(u)]$ to $[\sigma\psi(u)]$. Hence every subword of $[\psi(u)]$ of the form ab^{-1} or ba^{-1} is necessarily part of a subword of the form $ab^{-r}a^{-1}$ or $ab^{r}a^{-1}$ $(r > 0)$, respectively. This implies that

$$n\big([\psi(u)]; a, b\big) \geqslant n\big([\psi(u)]; a, b^{-1}\big),$$

so that, from (11),

$$\|\sigma\psi(u)\| - \|\psi(u)\| \geqslant n\big([\psi(u)]; a, a\big) \geqslant 0,$$

thus proving (i).

On the other hand, clearly

$$\begin{aligned}\|\tau\psi(u)\| - \|\psi(u)\| &= n\big([\psi(u)]; b\big) - 2n\big([\psi(u)]; b, a^{-1}\big)\\ &= n\big([\psi(u)]; b, b\big) + n\big([\psi(u)]; b, a\big) - n\big([\psi(u)]; b, a^{-1}\big). \qquad (12)\end{aligned}$$

As above, every subword of $[\psi(u)]$ of the form ab^{-1} or ba^{-1} is necessarily part of a subword of the form $ab^{-r}a^{-1}$ or $ab^{r}a^{-1}$ $(r > 0)$, respectively. Observe that a subword of $[\psi(u)]$ of the form $ab^{\pm r}a^{-1}$ is actually part of either a subword of the form $ba^{s}b^{\pm r}a^{-1}$ or a subword of the form $a^{-1}b^{-t}a^{s}b^{\pm r}a^{-1}$ $(s, t > 0)$. This implies that

$$n\big([\psi(u)]; b, a\big) \geqslant n\big([\psi(u)]; b, a^{-1}\big), \qquad (13)$$

so that, from (12),

$$\|\tau\psi(u)\| - \|\psi(u)\| \geqslant n\big([\psi(u)]; b, b\big) \geqslant 0, \qquad (14)$$

thus proving (ii). □

Lemma 3.3. *Let $u \in F_2$. Suppose that ψ is a chain of type* (C1) *which contains at least $\|u\| + 1$ factors of σ. Then*

(i) *if $\|\sigma\psi(u)\| = \|\psi(u)\|$, then $\|\sigma^{i+1}\psi(u)\| = \|\sigma^{i}\psi(u)\|$ for every $i \geqslant 0$;*
(ii) *if $\|\sigma^{j+1}\psi(u)\| = \|\sigma^{j}\psi(u)\|$ for some $j \geqslant 0$, then $\|\sigma\psi(u)\| = \|\psi(u)\|$.*

Proof. For (i), assume that $\|\sigma\psi(u)\| = \|\psi(u)\|$. We shall prove $\|\sigma^{i+1}\psi(u)\| = \|\sigma^{i}\psi(u)\|$ by induction on $i \geqslant 0$. The case where $i = 0$ is clear. So let $i \geqslant 1$. By Lemma 3.1(i) with $m = 0$, we have

$$\|\sigma^{i+1}\psi(u)\| - \|\sigma^{i}\psi(u)\| = \|\sigma^{i}\psi(u)\| - \|\sigma^{i-1}\psi(u)\|.$$

It follows from the induction hypothesis that

$$\left\|\sigma^{i+1}\psi(u)\right\| = \left\|\sigma^{i}\psi(u)\right\|,$$

so proving (i).

For (ii), assume that $\|\sigma^{j+1}\psi(u)\| = \|\sigma^{j}\psi(u)\|$ for some $j \geqslant 0$. We use induction on $j \geqslant 0$. If $j = 0$, then there is nothing to prove. So let $j \geqslant 1$. It follows from Lemma 3.1(i) with $m = 0$ that

$$0 = \left\|\sigma^{j+1}\psi(u)\right\| - \left\|\sigma^{j}\psi(u)\right\| = \left\|\sigma^{j}\psi(u)\right\| - \left\|\sigma^{j-1}\psi(u)\right\|,$$

so that

$$\left\|\sigma^{j}\psi(u)\right\| = \left\|\sigma^{j-1}\psi(u)\right\|.$$

Then by the induction hypothesis, we get the required result. □

Lemma 3.4. *Let $u \in F_2$, and let $\psi = \sigma\psi_1$, where ψ_1 is a chain of type* (C1) *which contains at least $\|u\| + 1$ factors of σ. Suppose that $\|\tau\psi(u)\| = \|\psi(u)\|$. Then $\|\sigma^{i+1}\psi_1(u)\| = \|\sigma^{i}\psi_1(u)\|$ for every $i \geqslant 0$.*

Proof. By Lemma 3.1(ii) with $m = 0$, we have

$$0 = \left\|\tau\psi(u)\right\| - \left\|\psi(u)\right\| = \left\|\tau\psi_1(u)\right\| - \left\|\psi_1(u)\right\| + \left\|\sigma\psi_1(u)\right\| - \left\|\psi_1(u)\right\|.$$

Here, by Lemma 3.2(ii), $\|\tau\psi_1(u)\| - \|\psi_1(u)\| \geqslant 0$. Also by Lemma 3.2(i), $\|\sigma\psi_1(u)\| - \|\psi_1(u)\| \geqslant 0$. Hence we must have

$$\left\|\tau\psi_1(u)\right\| = \left\|\psi_1(u)\right\| \quad \text{and} \quad \left\|\sigma\psi_1(u)\right\| = \left\|\psi_1(u)\right\|.$$

The second equality $\|\sigma\psi_1(u)\| = \|\psi_1(u)\|$ yields from Lemma 3.3(i) that

$$\left\|\sigma^{i+1}\psi_1(u)\right\| = \left\|\sigma^{i}\psi_1(u)\right\|$$

for every $i \geqslant 0$, thus proving the assertion. □

For the proof of the main result of the present section, we need the following two technical corollaries of Lemmas 3.1–3.4. We remark that similar statements to the corollaries also hold if σ and τ are interchanged with each other, or (C1) is replaced by (C2) and σ and τ are replaced by σ^{-1} and τ^{-1}, respectively.

Corollary 3.5. *Let $u, v \in F_2$ with $\|u\| \geqslant \|v\|$, and let ψ be a chain of type* (C1) *with $|\psi| \geqslant 2\|u\| + 3$. Put $k = \|u\| + 1$. Suppose that u and v have the property that*

$$\left\|\sigma^{k+1}\psi'(u)\right\| = \left\|\sigma^{k}\psi'(u)\right\| \quad \textit{if and only if} \quad \left\|\sigma^{k+1}\psi'(v)\right\| = \left\|\sigma^{k}\psi'(v)\right\|;$$
$$\left\|\tau^{k+1}\psi'(u)\right\| = \left\|\tau^{k}\psi'(u)\right\| \quad \textit{if and only if} \quad \left\|\tau^{k+1}\psi'(v)\right\| = \left\|\tau^{k}\psi'(v)\right\|,$$

for every chain ψ' of type (C1) *with $|\psi'| < |\psi|$. Then we have*

$$\text{(i)} \quad \left\|\sigma^{k+1}\psi(u)\right\| = \left\|\sigma^{k}\psi(u)\right\| \quad \textit{if and only if} \quad \left\|\sigma^{k+1}\psi(v)\right\| = \left\|\sigma^{k}\psi(v)\right\|;$$
$$\text{(ii)} \quad \left\|\tau^{k+1}\psi(u)\right\| = \left\|\tau^{k}\psi(u)\right\| \quad \textit{if and only if} \quad \left\|\tau^{k+1}\psi(v)\right\| = \left\|\tau^{k}\psi(v)\right\|.$$

Proof. Suppose that ψ ends in τ (the case where ψ ends in σ is analogous). Since $|\psi| \geqslant 2\|u\| + 3$, either σ or τ occurs at least $\|u\| + 2$ times in ψ. We consider two cases separately.

Case 1. σ occurs at least $\|u\| + 2$ times in ψ.

First we shall prove (i). Suppose that $\|\sigma^{k+1}\psi(u)\| = \|\sigma^k\psi(u)\|$. By Lemma 3.3(ii), we have

$$\|\sigma\psi(u)\| = \|\psi(u)\|.$$

Write

$$\psi = \tau^\ell\sigma\psi_1,$$

where $\ell \geqslant 1$ and ψ_1 is a chain of type (C1). Clearly ψ_1 contains at least $\|u\| + 1$ factors of σ. By Lemma 3.1(i), we have

$$0 = \|\sigma\psi(u)\| - \|\psi(u)\| = \|\sigma\tau^\ell\psi_1(u)\| - \|\tau^\ell\psi_1(u)\| + \ell\big(\|\sigma\psi_1(u)\| - \|\psi_1(u)\|\big).$$

Here, since $\|\sigma\tau^\ell\psi_1(u)\| - \|\tau^\ell\psi_1(u)\| \geqslant 0$ and $\|\sigma\psi_1(u)\| - \|\psi_1(u)\| \geqslant 0$ by Lemma 3.2(i), the only possibility is that

$$\|\sigma\tau^\ell\psi_1(u)\| = \|\tau^\ell\psi_1(u)\| \quad \text{and} \quad \|\sigma\psi_1(u)\| = \|\psi_1(u)\|.$$

These equalities together with Lemma 3.3(i) yield that

$$\|\sigma^{k+1}\tau^\ell\psi_1(u)\| = \|\sigma^k\tau^\ell\psi_1(u)\| \quad \text{and} \quad \|\sigma^{k+1}\psi_1(u)\| = \|\sigma^k\psi_1(u)\|.$$

Since $|\tau^\ell\psi_1| < |\psi|$ and $|\psi_1| < |\psi|$, by the hypothesis of the corollary, we get

$$\|\sigma^{k+1}\tau^\ell\psi_1(v)\| = \|\sigma^k\tau^\ell\psi_1(v)\| \quad \text{and} \quad \|\sigma^{k+1}\psi_1(v)\| = \|\sigma^k\psi_1(v)\|.$$

Again by Lemma 3.3(ii), we have

$$\|\sigma\tau^\ell\psi_1(v)\| = \|\tau^\ell\psi_1(v)\| \quad \text{and} \quad \|\sigma\psi_1(v)\| = \|\psi_1(v)\|.$$

Therefore, by Lemma 3.1(i),

$$\begin{aligned}\|\sigma\psi(v)\| - \|\psi(v)\| &= \|\sigma\tau^\ell\psi_1(v)\| - \|\tau^\ell\psi_1(v)\| + \ell\big(\|\sigma\psi_1(v)\| - \|\psi_1(v)\|\big)\\ &= 0,\end{aligned}$$

namely, $\|\sigma\psi(v)\| = \|\psi(v)\|$. Then the desired equality $\|\sigma^{k+1}\psi(v)\| = \|\sigma^k\psi(v)\|$ follows from Lemma 3.3(i).

Conversely, if $\|\sigma^{k+1}\psi(v)\| = \|\sigma^k\psi(v)\|$, we can deduce, in the same way as above, that $\|\sigma^{k+1}\psi(u)\| = \|\sigma^k\psi(u)\|$.

Next we shall prove (ii). Assume that $\|\tau^{k+1}\psi(u)\| = \|\tau^k\psi(u)\|$. Apply Lemma 3.1(ii) to get

$$0 = \|\tau^{k+1}\psi(u)\| - \|\tau^k\psi(u)\| = \|\tau^{k+1}\tau^\ell\psi_1(u)\| - \|\tau^k\tau^\ell\psi_1(u)\| + \|\sigma\psi_1(u)\| - \|\psi_1(u)\|. \tag{15}$$

Here, since $\|\tau^{k+1}\tau^\ell\psi_1(u)\| - \|\tau^k\tau^\ell\psi_1(u)\| \geqslant 0$ by Lemma 3.2(ii), and since $\|\sigma\psi_1(u)\| - \|\psi_1(u)\| \geqslant 0$ by Lemma 3.2(i), we must have

$$\|\tau^{k+1}\tau^\ell\psi_1(u)\| = \|\tau^k\tau^\ell\psi_1(u)\| \quad \text{and} \quad \|\sigma\psi_1(u)\| = \|\psi_1(u)\|. \tag{16}$$

Since $|\tau^\ell \psi_1| < |\psi|$, by the hypothesis of the corollary, the first equality of (16) implies that

$$\|\tau^{k+1}\tau^\ell \psi_1(v)\| = \|\tau^k \tau^\ell \psi_1(v)\|.$$

Also, from the second equality of (16), arguing as above, we deduce that

$$\|\sigma \psi_1(v)\| = \|\psi_1(v)\|.$$

Therefore, by Lemma 3.1(ii),

$$\begin{aligned}\|\tau^{k+1}\psi(v)\| - \|\tau^k \psi(v)\| &= \|\tau^{k+1}\tau^\ell \psi_1(v)\| - \|\tau^k \tau^\ell \psi_1(v)\| + \|\sigma \psi_1(v)\| - \|\psi_1(v)\| \\ &= 0,\end{aligned}$$

that is, $\|\tau^{k+1}\psi(v)\| = \|\tau^k \psi(v)\|$, as required.

It is clear that the converse is also true.

Case 2. τ occurs at least $\|u\| + 2$ times in ψ.

Since ψ is assumed to end in τ, we may write

$$\psi = \tau \psi_2,$$

where ψ_2 is a chain of type (C1) that contains at least $\|u\| + 1$ factors of τ.

First we shall prove (i). Suppose that $\|\sigma^{k+1}\psi(u)\| = \|\sigma^k \psi(u)\|$. By Lemma 3.1(ii) with σ, τ interchanged, we have

$$0 = \|\sigma^{k+1}\psi(u)\| - \|\sigma^k \psi(u)\| = \|\sigma^{k+1}\psi_2(u)\| - \|\sigma^k \psi_2(u)\| + \|\tau \psi_2(u)\| - \|\psi_2(u)\|.$$

This is a similar situation to (15) with σ, τ interchanged. So arguing as in Case 1, we get the desired equality $\|\sigma^{k+1}\psi(v)\| = \|\sigma^k \psi(v)\|$. Clearly the converse also holds.

Next we shall prove (ii). Suppose that $\|\tau^{k+1}\psi(u)\| = \|\tau^k \psi(u)\|$. By Lemma 3.1(i) with σ, τ interchanged and $m = 0$, we have

$$0 = \|\tau^{k+1}\psi(u)\| - \|\tau^k \psi(u)\| = \|\tau^k \psi(u)\| - \|\tau^{k-1}\psi(u)\|.$$

So

$$\|\tau^k \psi(u)\| = \|\tau^{k-1}\psi(u)\|.$$

This equality can be rephrased as

$$\|\tau^{k+1}\psi_2(u)\| = \|\tau^k \psi_2(u)\|,$$

because $\psi = \tau \psi_2$. Since $|\psi_2| < |\psi|$, by the hypothesis of the corollary,

$$\|\tau^{k+1}\psi_2(v)\| = \|\tau^k \psi_2(v)\|,$$

that is,

$$\|\tau^k \psi(v)\| = \|\tau^{k-1}\psi(v)\|.$$

Thus, by Lemma 3.1(i) with σ, τ interchanged and $m = 0$, we obtain

$$\|\tau^{k+1}\psi(v)\| - \|\tau^k\psi(v)\| = \|\tau^k\psi(v)\| - \|\tau^{k-1}\psi(v)\| = 0,$$

namely, $\|\tau^{k+1}\psi(v)\| = \|\tau^k\psi(v)\|$, as required. Obviously the converse is also true. □

Corollary 3.6. *Let $u, v \in F_2$ with $\|u\| \geqslant \|v\|$, and let ψ be a chain of type* (C1)*. Put $k = \|u\| + 1$. Suppose that u and v have the property that*

$$\|\sigma^{k+1}\psi'(u)\| = \|\sigma^k\psi'(u)\| \quad \textit{if and only if} \quad \|\sigma^{k+1}\psi'(v)\| = \|\sigma^k\psi'(v)\|;$$
$$\|\tau^{k+1}\psi'(u)\| = \|\tau^k\psi'(u)\| \quad \textit{if and only if} \quad \|\tau^{k+1}\psi'(v)\| = \|\tau^k\psi'(v)\|,$$

for every chain ψ' of type (C1) *with $|\psi'| \leqslant |\psi|$. Then we have*

(i) *if ψ contains at least $\|u\| + 1$ factors of σ, then*

$$\|\sigma\psi(u)\| = \|\psi(u)\| \quad \textit{if and only if} \quad \|\sigma\psi(v)\| = \|\psi(v)\|;$$

(ii) *if $\|\tau\psi(u)\| = \|\psi(u)\|$ or $\|\tau\psi(v)\| = \|\psi(v)\|$, and $\psi = \sigma\psi_1$, where ψ_1 is a chain of type* (C1) *which contains at least $\|u\| + 1$ factors of σ, then*

$$\|\sigma\psi_1(u)\| = \|\psi_1(u)\| \quad \textit{and} \quad \|\sigma\psi_1(v)\| = \|\psi_1(v)\|;$$

(iii) *if ψ contains at least $\|u\| + 2$ factors of σ and ends in τ, then*

$$\|\tau\psi(u)\| = \|\psi(u)\| \quad \textit{if and only if} \quad \|\tau\psi(v)\| = \|\psi(v)\|.$$

Proof. For (i), let ψ contain at least $\|u\| + 1$ factors of σ, and suppose that $\|\sigma\psi(u)\| = \|\psi(u)\|$. By Lemma 3.3(i), we have $\|\sigma^{k+1}\psi(u)\| = \|\sigma^k\psi(u)\|$. Then by the hypothesis of the corollary, $\|\sigma^{k+1}\psi(v)\| = \|\sigma^k\psi(v)\|$. Finally by Lemma 3.3(ii), we get $\|\sigma\psi(v)\| = \|\psi(v)\|$. The converse also holds.

For (ii), let $\psi = \sigma\psi_1$, where ψ_1 is a chain of type (C1) containing at least $\|u\| + 1$ factors of σ, and suppose that $\|\tau\psi(u)\| = \|\psi(u)\|$. By Lemma 3.4, we have $\|\sigma\psi_1(u)\| = \|\psi_1(u)\|$. Then, by (i) of the corollary, $\|\sigma\psi_1(v)\| = \|\psi_1(v)\|$. The converse is proved similarly.

For (iii), let ψ contain at least $\|u\| + 2$ factors of σ, and let ψ end in τ. Assume that $\|\tau\psi(u)\| = \|\psi(u)\|$. Write

$$\psi = \tau^\ell\sigma\psi_2,$$

where $\ell \geqslant 1$ and ψ_2 is a chain of type (C1). By Lemma 3.1(ii), we have

$$0 = \|\tau\psi(u)\| - \|\psi(u)\| = \|\tau\tau^\ell\psi_2(u)\| - \|\tau^\ell\psi_2(u)\| + \|\sigma\psi_2(u)\| - \|\psi_2(u)\|.$$

Here, since $\|\tau\tau^\ell\psi_2(u)\| - \|\tau^\ell\psi_2(u)\| \geqslant 0$ by Lemma 3.2(ii) and $\|\sigma\psi_2(u)\| - \|\psi_2(u)\| \geqslant 0$ by Lemma 3.2(i), we must have

$$\|\tau\tau^\ell\psi_2(u)\| = \|\tau^\ell\psi_2(u)\| \quad \text{and} \quad \|\sigma\psi_2(u)\| = \|\psi_2(u)\|.$$

Since $\|\sigma\psi_2(u)\| = \|\psi_2(u)\|$, by (i) of the corollary,

$$\|\sigma\psi_2(v)\| = \|\psi_2(v)\|.$$

Also, the following claim shows that $\|\tau\tau^{\ell}\psi_2(v)\| = \|\tau^{\ell}\psi_2(v)\|$. Then by Lemma 3.1(ii), we have $\|\tau\psi(v)\| = \|\psi(v)\|$, as required.

Claim. $\|\tau\tau^{\ell}\psi_2(v)\| = \|\tau^{\ell}\psi_2(v)\|$.

Proof. Since $\|\tau\psi(u)\| = \|\psi(u)\|$, in view of (12), (13) and (14) in the proof of Lemma 3.2, we must have

$$n\big(\big[\psi(u)\big]; b, a\big) = n\big(\big[\psi(u)\big]; b, a^{-1}\big) \quad \text{and} \quad n\big(\big[\psi(u)\big]; b, b\big) = 0. \tag{17}$$

Since the chain ψ_2 contains at least $\|u\| + 1$ factors of σ, by Lemma 2.1, no proper cancellation occurs in passing from $[\psi_2(u)]$ to $[\sigma\psi_2(u)]$. This yields that

$$a^2 \text{ or } a^{-2} \text{ cannot occur in } \big[\sigma\psi_2(u)\big] \text{ as a subword.} \tag{18}$$

From this, we see that, since $\ell \geqslant 1$,

$$\text{no proper cancellation can occur in passing from } \big[\psi(u)\big] \text{ to } \big[\tau\psi(u)\big]. \tag{19}$$

In view of (17), (18) and (19), the cyclic word $[\psi(u)]$ must have the form

$$\big[\psi(u)\big] = \big[a^{\epsilon}ba^{-\epsilon}b^{-1}\cdots a^{\epsilon}ba^{-\epsilon}b^{-1}\big],$$

where either $\epsilon = 1$ or $\epsilon = -1$. Then, by applying $\sigma^{-1}\tau^{-\ell}$ to $[\psi(u)]$, we deduce that

$$\big[\psi_2(u)\big] = \big[\psi(u)\big] = \big[a^{\epsilon}ba^{-\epsilon}b^{-1}\cdots a^{\epsilon}ba^{-\epsilon}b^{-1}\big].$$

It then follows that

$$\big[\tau^{i}\psi_2(u)\big] = \big[\psi_2(u)\big]$$

for every $i \geqslant 0$, so that

$$\big\|\tau^{i+1}\psi_2(u)\big\| = \big\|\tau^{i}\psi_2(u)\big\| \tag{20}$$

for every $i \geqslant 0$. In particular,

$$\big\|\tau^{k+1}\psi_2(u)\big\| = \big\|\tau^{k}\psi_2(u)\big\|.$$

So by the hypothesis of the corollary,

$$\big\|\tau^{k+1}\psi_2(v)\big\| = \big\|\tau^{k}\psi_2(v)\big\|. \tag{21}$$

Then in the same way as obtaining (17), we get

$$n\big(\big[\tau^{k}\psi_2(v)\big]; b, a\big) = n\big(\big[\tau^{k}\psi_2(v)\big]; b, a^{-1}\big) \quad \text{and} \quad n\big(\big[\tau^{k}\psi_2(v)\big]; b, b\big) = 0. \tag{22}$$

Since the chain $\tau^{k}\psi_2$ contains at least $\|v\| + 1$ factors of τ, by Lemma 2.1, no proper cancellation may occur in passing from $[\tau^{k}\psi_2(v)]$ to $[\tau^{k+1}\psi_2(v)]$. This together with (22) yields that

$$\big[\tau^{k}\psi_2(v)\big] = \big[a^{s_1}ba^{t_1}b^{-1}\cdots a^{s_r}ba^{t_r}b^{-1}\big],$$

where every s_j, t_j is a nonzero integer. Then, by applying τ^{-k} to $[\tau^k \psi_2(v)]$, we deduce that

$$\bigl[\psi_2(v)\bigr] = \bigl[a^{s_1} b a^{t_1} b^{-1} \cdots a^{s_r} b a^{t_r} b^{-1}\bigr].$$

Thus it follows that

$$\bigl[\tau^i \psi_2(v)\bigr] = \bigl[\psi_2(v)\bigr]$$

for every $i \geqslant 0$, so that

$$\bigl\|\tau^{i+1}\psi_2(v)\bigr\| = \bigl\|\tau^i \psi_2(v)\bigr\|$$

for every $i \geqslant 0$. In particular, $\|\tau\tau^\ell \psi_2(v)\| = \|\tau^\ell \psi_2(v)\|$, as required. □

The proof of the corollary is now completed. □

For a Whitehead automorphism β of F_2, a chain ψ of Whitehead automorphisms of F_2 and an element w in F_2, we let $\|\beta : \psi : w\|$ denote the maximum of 1 and $\|\beta\psi(w)\| - \|\psi(w)\|$, that is,

$$\|\beta : \psi : w\| := \max\bigl\{1, \bigl\|\beta\psi(w)\bigr\| - \bigl\|\psi(w)\bigr\|\bigr\}.$$

Now we are ready to establish the main result of the present section as follows.

Theorem 3.7. *Let $u, v \in F_2$ with $\|u\| \geqslant \|v\|$, and let Ω be the set of all chains of type* (C1) *or* (C2) *of length less than or equal to $2\|u\| + 5$. Let Ω_1 be the subset of Ω consisting of all chains of type* (C1)*, and let Ω_2 be the subset of Ω consisting of all chains of type* (C2)*. Put $k = \|u\| + 1$. Suppose that u and v have the property that*

$$\bigl\|\sigma^{k+1}\psi_1(u)\bigr\| = \bigl\|\sigma^k \psi_1(u)\bigr\| \quad \text{if and only if} \quad \bigl\|\sigma^{k+1}\psi_1(v)\bigr\| = \bigl\|\sigma^k \psi_1(v)\bigr\|;$$
$$\bigl\|\tau^{k+1}\psi_1(u)\bigr\| = \bigl\|\tau^k \psi_1(u)\bigr\| \quad \text{if and only if} \quad \bigl\|\tau^{k+1}\psi_1(v)\bigr\| = \bigl\|\tau^k \psi_1(v)\bigr\|,$$

for every $\psi_1 \in \Omega_1$, and that

$$\bigl\|\sigma^{-k-1}\psi_2(u)\bigr\| = \bigl\|\sigma^{-k} \psi_2(u)\bigr\| \quad \text{if and only if} \quad \bigl\|\sigma^{-k-1}\psi_2(v)\bigr\| = \bigl\|\sigma^{-k} \psi_2(v)\bigr\|;$$
$$\bigl\|\tau^{-k-1}\psi_2(u)\bigr\| = \bigl\|\tau^{-k} \psi_2(u)\bigr\| \quad \text{if and only if} \quad \bigl\|\tau^{-k-1}\psi_2(v)\bigr\| = \bigl\|\tau^{-k} \psi_2(v)\bigr\|,$$

for every $\psi_2 \in \Omega_2$. Then u and v are boundedly translation equivalent in F_2.

More specifically,

$$\min \Delta \leqslant \frac{\|\phi(u)\|}{\|\phi(v)\|} \leqslant \max \Delta$$

for every automorphism ϕ of F_2, where

$$\Delta := \left\{ \frac{\|\psi(u)\|}{\|\psi(v)\|}, \frac{\|\alpha : \psi_1 : u\|}{\|\alpha : \psi_1 : v\|}, \frac{\|\alpha^{-1} : \psi_2 : u\|}{\|\alpha^{-1} : \psi_2 : v\|} \;\middle|\; \psi \in \Omega, \psi_i \in \Omega_i, \alpha = \sigma \text{ or } \tau \right\}.$$

(*Obviously, Δ is a finite set consisting of positive real numbers.*)

Proof. Let ϕ be an automorphism of F_2. By Lemma 1.1, ϕ can be represented as

$$\phi \equiv \beta\phi',$$

where β is a Whitehead automorphism of F_2 of type (W1) and ϕ' is of type either (C1) or (C2). We proceed with the proof of the theorem by induction on $|\phi'|$. Letting ϕ' be a chain of type (C1) with $|\phi'| > 2\|u\| + 5$ (the case for (C2) is similar), assume that

$$\begin{aligned} \left\|\sigma^{k+1}\psi(u)\right\| = \left\|\sigma^{k}\psi(u)\right\| \quad &\text{if and only if} \quad \left\|\sigma^{k+1}\psi(v)\right\| = \left\|\sigma^{k}\psi(v)\right\|; \\ \left\|\tau^{k+1}\psi(u)\right\| = \left\|\tau^{k}\psi(u)\right\| \quad &\text{if and only if} \quad \left\|\tau^{k+1}\psi(v)\right\| = \left\|\tau^{k}\psi(v)\right\|, \end{aligned}$$

and that

$$\min\Delta \leqslant \frac{\|\psi(u)\|}{\|\psi(v)\|}, \frac{\|\sigma:\psi:u\|}{\|\sigma:\psi:v\|}, \frac{\|\tau:\psi:u\|}{\|\tau:\psi:v\|} \leqslant \max\Delta,$$

for every chain ψ of type (C1) with $|\psi| < |\phi'|$.

By Corollary 3.5, it is easy to get

$$\begin{aligned} \left\|\sigma^{k+1}\phi'(u)\right\| = \left\|\sigma^{k}\phi'(u)\right\| \quad &\text{if and only if} \quad \left\|\sigma^{k+1}\phi'(v)\right\| = \left\|\sigma^{k}\phi'(v)\right\|; \\ \left\|\tau^{k+1}\phi'(u)\right\| = \left\|\tau^{k}\phi'(u)\right\| \quad &\text{if and only if} \quad \left\|\tau^{k+1}\phi'(v)\right\| = \left\|\tau^{k}\phi'(v)\right\|. \end{aligned}$$

In the following Claims A, B and C, we shall prove that

$$\min\Delta \leqslant \frac{\|\phi'(u)\|}{\|\phi'(v)\|}, \frac{\|\sigma:\phi':u\|}{\|\sigma:\phi':v\|}, \frac{\|\tau:\phi':u\|}{\|\tau:\phi':v\|} \leqslant \max\Delta,$$

which is clearly equivalent to showing that

$$\min\Delta \leqslant \frac{\|\phi(u)\|}{\|\phi(v)\|}, \frac{\|\sigma:\phi:u\|}{\|\sigma:\phi:v\|}, \frac{\|\tau:\phi:u\|}{\|\tau:\phi:v\|} \leqslant \max\Delta.$$

Suppose that ϕ' ends in τ (the case where ϕ' ends in σ is analogous).

Claim A.

$$\min\Delta \leqslant \frac{\|\phi'(u)\|}{\|\phi'(v)\|} \leqslant \max\Delta.$$

Proof. Since ϕ' ends in τ, we may write

$$\phi' = \tau\phi_1,$$

where ϕ_1 is a chain of type (C1). Then obviously

$$\begin{aligned} \left\|\phi'(u)\right\| &= \left\|\tau\phi_1(u)\right\| - \left\|\phi_1(u)\right\| + \left\|\phi_1(u)\right\|; \\ \left\|\phi'(v)\right\| &= \left\|\tau\phi_1(v)\right\| - \left\|\phi_1(v)\right\| + \left\|\phi_1(v)\right\|. \end{aligned} \tag{23}$$

If both $\|\tau\phi_1(u)\| \neq \|\phi_1(u)\|$ and $\|\tau\phi_1(v)\| \neq \|\phi_1(v)\|$, then equalities (23) can be rephrased as

$$\|\phi'(u)\| = \|\tau : \phi_1 : u\| + \|\phi_1(u)\|;$$
$$\|\phi'(v)\| = \|\tau : \phi_1 : v\| + \|\phi_1(v)\|. \tag{24}$$

Since

$$\min \Delta \leqslant \frac{\|\phi_1(u)\|}{\|\phi_1(v)\|}, \frac{\|\tau : \phi_1 : u\|}{\|\tau : \phi_1 : v\|} \leqslant \max \Delta$$

by the induction hypothesis, we obtain

$$\min \Delta \leqslant \frac{\|\phi'(u)\|}{\|\phi'(v)\|} \leqslant \max \Delta,$$

as required.

So assume that

$$\|\tau\phi_1(u)\| = \|\phi_1(u)\| \quad \text{or} \quad \|\tau\phi_1(v)\| = \|\phi_1(v)\|. \tag{25}$$

Clearly the chain ϕ_1 has length $|\phi_1| = |\phi'| - 1 \geqslant 2\|u\| + 5$. Hence either σ or τ occurs at least $\|u\| + 3$ times in ϕ_1. We consider two cases accordingly.

Case A.1. σ occurs at least $\|u\| + 3$ times in ϕ_1.

Since ϕ_1 is a chain of type (C1), ϕ_1 ends in either σ or τ.

Case A.1.1. ϕ_1 ends in σ.

Write

$$\phi_1 = \sigma\phi_2,$$

where ϕ_2 is a chain of type (C1). In view of Corollary 3.6(ii), our assumption (25) yields that

$$\|\sigma\phi_2(u)\| = \|\phi_2(u)\| \quad \text{and} \quad \|\sigma\phi_2(v)\| = \|\phi_2(v)\|. \tag{26}$$

This together with Lemma 3.1(ii) implies that

$$\|\tau\phi_1(u)\| - \|\phi_1(u)\| = \|\tau\phi_2(u)\| - \|\phi_2(u)\|;$$
$$\|\tau\phi_1(v)\| - \|\phi_1(v)\| = \|\tau\phi_2(v)\| - \|\phi_2(v)\|. \tag{27}$$

Since $\phi_1 = \sigma\phi_2$, we obtain from (26) that $\|\phi_1(u)\| = \|\phi_2(u)\|$ and $\|\phi_1(v)\| = \|\phi_2(v)\|$, so that, from (27),

$$\|\tau\phi_1(u)\| = \|\tau\phi_2(u)\|;$$
$$\|\tau\phi_1(v)\| = \|\tau\phi_2(v)\|. \tag{28}$$

Since $\phi' = \tau\phi_1$, (28) implies that

$$\frac{\|\phi'(u)\|}{\|\phi'(v)\|} = \frac{\|\tau\phi_2(u)\|}{\|\tau\phi_2(v)\|},$$

and thus, by the induction hypothesis,

$$\min \Delta \leqslant \frac{\|\phi'(u)\|}{\|\phi'(v)\|} \leqslant \max \Delta,$$

as desired.

Case A.1.2. ϕ_1 ends in τ.

In view of Corollary 3.6(iii), our assumption (25) yields that both $\|\tau\phi_1(u)\| = \|\phi_1(u)\|$ and $\|\tau\phi_1(v)\| = \|\phi_1(v)\|$. We then have from (23) that

$$\frac{\|\phi'(u)\|}{\|\phi'(v)\|} = \frac{\|\phi_1(u)\|}{\|\phi_1(v)\|},$$

so that, by the induction hypothesis,

$$\min \Delta \leqslant \frac{\|\phi'(u)\|}{\|\phi'(v)\|} \leqslant \max \Delta,$$

as required.

Case A.2. τ occurs at least $\|u\| + 3$ times in ϕ_1.

In view of Corollary 3.6(i) with τ in place of σ, we have from (25) both $\|\tau\phi_1(u)\| = \|\phi_1(u)\|$ and $\|\tau\phi_1(v)\| = \|\phi_1(v)\|$. It then follows from (23) that

$$\frac{\|\phi'(u)\|}{\|\phi'(v)\|} = \frac{\|\phi_1(u)\|}{\|\phi_1(v)\|},$$

so that, by the induction hypothesis,

$$\min \Delta \leqslant \frac{\|\phi'(u)\|}{\|\phi'(v)\|} \leqslant \max \Delta,$$

as desired. □

Claim B.

$$\min \Delta \leqslant \frac{\|\sigma : \phi' : u\|}{\|\sigma : \phi' : v\|} \leqslant \max \Delta.$$

Proof. As in the proof of Claim A, writing

$$\phi' = \tau\phi_1,$$

where ϕ_1 is a chain of type (C1), we consider two cases separately.

Case B.1. σ occurs at least $\|u\| + 3$ times in ϕ_1.

In this case, write

$$\phi_1 = \tau^{m-1}\sigma\phi_2,$$

where $m \geqslant 1$ and ϕ_2 is a chain of type (C1). Since $\phi' = \tau\phi_1$,

$$\phi' = \tau^m \sigma \phi_2.$$

Then by Lemma 3.1(i), we have

$$\begin{aligned}
\|\sigma\phi'(u)\| - \|\phi'(u)\| &= \|\sigma\tau^m\phi_2(u)\| - \|\tau^m\phi_2(u)\| + m\big(\|\sigma\phi_2(u)\| - \|\phi_2(u)\|\big); \\
\|\sigma\phi'(v)\| - \|\phi'(v)\| &= \|\sigma\tau^m\phi_2(v)\| - \|\tau^m\phi_2(v)\| + m\big(\|\sigma\phi_2(v)\| - \|\phi_2(v)\|\big).
\end{aligned} \tag{29}$$

Here, since ϕ_2 is a chain of type (C1) which contains at least $\|u\| + 2$ factors of σ, Corollary 3.6(i) yields that $\|\sigma\tau^m\phi_2(u)\| = \|\tau^m\phi_2(u)\|$ if and only if $\|\sigma\tau^m\phi_2(v)\| = \|\tau^m\phi_2(v)\|$. So if $\|\sigma\tau^m\phi_2(u)\| = \|\tau^m\phi_2(u)\|$ or $\|\sigma\tau^m\phi_2(v)\| = \|\tau^m\phi_2(v)\|$, then we get from (29) that

$$\begin{aligned}
\|\sigma\phi'(u)\| - \|\phi'(u)\| &= m\big(\|\sigma\phi_2(u)\| - \|\phi_2(u)\|\big); \\
\|\sigma\phi'(v)\| - \|\phi'(v)\| &= m\big(\|\sigma\phi_2(v)\| - \|\phi_2(v)\|\big).
\end{aligned}$$

This gives us

$$\frac{\|\sigma : \phi' : u\|}{\|\sigma : \phi' : v\|} = \frac{\|\sigma : \phi_2 : u\|}{\|\sigma : \phi_2 : v\|},$$

and hence the desired inequalities

$$\min \Delta \leqslant \frac{\|\sigma : \phi' : u\|}{\|\sigma : \phi' : v\|} \leqslant \max \Delta$$

follow by the induction hypothesis.

Now let us assume that

$$\|\sigma\tau^m\phi_2(u)\| \neq \|\tau^m\phi_2(u)\| \quad \text{and} \quad \|\sigma\tau^m\phi_2(v)\| \neq \|\tau^m\phi_2(v)\|.$$

Again by Corollary 3.6(i), we have $\|\sigma\phi_2(u)\| = \|\phi_2(u)\|$ if and only if $\|\sigma\phi_2(v)\| = \|\phi_2(v)\|$. Hence if $\|\sigma\phi_2(u)\| = \|\phi_2(u)\|$ or $\|\sigma\phi_2(v)\| = \|\phi_2(v)\|$, then, from (29),

$$\begin{aligned}
\|\sigma\phi'(u)\| - \|\phi'(u)\| &= \|\sigma\tau^m\phi_2(u)\| - \|\tau^m\phi_2(u)\|; \\
\|\sigma\phi'(v)\| - \|\phi'(v)\| &= \|\sigma\tau^m\phi_2(v)\| - \|\tau^m\phi_2(v)\|.
\end{aligned}$$

This yields

$$\frac{\|\sigma : \phi' : u\|}{\|\sigma : \phi' : v\|} = \frac{\|\sigma : \tau^m\phi_2 : u\|}{\|\sigma : \tau^m\phi_2 : v\|},$$

which gives us

$$\min \Delta \leqslant \frac{\|\sigma : \phi' : u\|}{\|\sigma : \phi' : v\|} \leqslant \max \Delta$$

by the induction hypothesis.

So let us further assume that

$$\|\sigma\phi_2(u)\| \neq \|\phi_2(u)\| \quad \text{and} \quad \|\sigma\phi_2(v)\| \neq \|\phi_2(v)\|.$$

It then follows from (29) that

$$\begin{aligned}\|\sigma\phi'(u)\| - \|\phi'(u)\| &= \|\sigma : \tau^m\phi_2 : u\| + m\|\sigma : \phi_2 : u\|;\\ \|\sigma\phi'(v)\| - \|\phi'(v)\| &= \|\sigma : \tau^m\phi_2 : v\| + m\|\sigma : \phi_2 : v\|.\end{aligned} \tag{30}$$

Since

$$\min\Delta \leqslant \frac{\|\sigma : \tau^m\phi_2 : u\|}{\|\sigma : \tau^m\phi_2 : v\|}, \frac{\|\sigma : \phi_2 : u\|}{\|\sigma : \phi_2 : v\|} \leqslant \max\Delta$$

by the induction hypothesis, we have from (30) that

$$\min\Delta \leqslant \frac{\|\sigma : \phi' : u\|}{\|\sigma : \phi' : v\|} \leqslant \max\Delta,$$

as required.

Case B.2. τ occurs at least $\|u\| + 3$ times in ϕ_1.

In this case, it follows from Lemma 3.1(ii) with σ, τ interchanged and $m = 0$ that

$$\begin{aligned}\|\sigma\phi'(u)\| - \|\phi'(u)\| &= \|\sigma\phi_1(u)\| - \|\phi_1(u)\| + \|\tau\phi_1(u)\| - \|\phi_1(u)\|;\\ \|\sigma\phi'(v)\| - \|\phi'(v)\| &= \|\sigma\phi_1(v)\| - \|\phi_1(v)\| + \|\tau\phi_1(v)\| - \|\phi_1(v)\|.\end{aligned} \tag{31}$$

Here, by Corollary 3.6(i) with τ in place of σ, we have $\|\tau\phi_1(u)\| = \|\phi_1(u)\|$ if and only if $\|\tau\phi_1(v)\| = \|\phi_1(v)\|$. Hence if $\|\tau\phi_1(u)\| = \|\phi_1(u)\|$ or $\|\tau\phi_1(v)\| = \|\phi_1(v)\|$, then, by (31),

$$\begin{aligned}\|\sigma\phi'(u)\| - \|\phi'(u)\| &= \|\sigma\phi_1(u)\| - \|\phi_1(u)\|;\\ \|\sigma\phi'(v)\| - \|\phi'(v)\| &= \|\sigma\phi_1(v)\| - \|\phi_1(v)\|,\end{aligned}$$

and thus

$$\begin{aligned}\|\sigma : \phi' : u\| &= \|\sigma : \phi_1 : u\|;\\ \|\sigma : \phi' : v\| &= \|\sigma : \phi_1 : v\|.\end{aligned}$$

Then by the induction hypothesis,

$$\min\Delta \leqslant \frac{\|\sigma : \phi' : u\|}{\|\sigma : \phi' : v\|} \leqslant \max\Delta,$$

as desired.

Now assume that

$$\|\tau\phi_1(u)\| \neq \|\phi_1(u)\| \quad \text{and} \quad \|\tau\phi_1(v)\| \neq \|\phi_1(v)\|.$$

We shall show that $\|\sigma\phi_1(u)\| = \|\phi_1(u)\|$ if and only if $\|\sigma\phi_1(v)\| = \|\phi_1(v)\|$. Let $\|\sigma\phi_1(u)\| = \|\phi_1(u)\|$. If ϕ_1 ends in σ, then, by Corollary 3.6(iii) with σ, τ interchanged, we have $\|\sigma\phi_1(v)\| = \|\phi_1(v)\|$. On the other hand, if ϕ_1 ends in τ, then, by Corollary 3.6(ii) with σ, τ interchanged, we get $\|\tau\phi_2(u)\| = \|\phi_2(u)\|$, where $\phi_1 = \tau\phi_2$. But then from Lemma 3.3(i) with σ, τ interchanged, it follows that $\|\tau^2\phi_2(u)\| = \|\tau\phi_2(u)\|$, namely, $\|\tau\phi_1(u)\| = \|\phi_1(u)\|$, which contradicts our assumption $\|\tau\phi_1(u)\| \neq \|\phi_1(u)\|$. Therefore, we must have $\|\sigma\phi_1(v)\| = \|\phi_1(v)\|$. Conversely, if $\|\sigma\phi_1(v)\| = \|\phi_1(v)\|$, then, for a similar reason, it must follow that $\|\sigma\phi_1(u)\| = \|\phi_1(u)\|$.

Thus if $\|\sigma\phi_1(u)\| = \|\phi_1(u)\|$ or $\|\sigma\phi_1(v)\| = \|\phi_1(v)\|$, then, from (31),

$$\left\|\sigma\phi'(u)\right\| - \left\|\phi'(u)\right\| = \left\|\tau\phi_1(u)\right\| - \left\|\phi_1(u)\right\|;$$
$$\left\|\sigma\phi'(v)\right\| - \left\|\phi'(v)\right\| = \left\|\tau\phi_1(v)\right\| - \left\|\phi_1(v)\right\|,$$

and so

$$\|\sigma : \phi' : u\| = \|\tau : \phi_1 : u\|;$$
$$\|\sigma : \phi' : v\| = \|\tau : \phi_1 : v\|.$$

Then by the induction hypothesis,

$$\min \Delta \leqslant \frac{\|\sigma : \phi' : u\|}{\|\sigma : \phi' : v\|} \leqslant \max \Delta,$$

as required.

So assume further that

$$\left\|\sigma\phi_1(u)\right\| \neq \left\|\phi_1(u)\right\| \quad \text{and} \quad \left\|\sigma\phi_1(v)\right\| \neq \left\|\phi_1(v)\right\|.$$

It follows from (31) that

$$\left\|\sigma\phi'(u)\right\| - \left\|\phi'(u)\right\| = \|\sigma : \phi_1 : u\| + \|\tau : \phi_1 : u\|;$$
$$\left\|\sigma\phi'(v)\right\| - \left\|\phi'(v)\right\| = \|\sigma : \phi_1 : v\| + \|\tau : \phi_1 : v\|. \tag{32}$$

Since

$$\min \Delta \leqslant \frac{\|\tau : \phi_1 : u\|}{\|\tau : \phi_1 : v\|}, \frac{\|\sigma : \phi_1 : u\|}{\|\sigma : \phi_1 : v\|} \leqslant \max \Delta$$

by the induction hypothesis, we obtain from (32) that

$$\min \Delta \leqslant \frac{\|\sigma : \phi' : u\|}{\|\sigma : \phi' : v\|} \leqslant \max \Delta,$$

as desired. □

Claim C.

$$\min \Delta \leqslant \frac{\|\tau : \phi' : u\|}{\|\tau : \phi' : v\|} \leqslant \max \Delta.$$

Proof. As in the proofs of Claims A and B, writing

$$\phi' = \tau\phi_1,$$

where ϕ_1 is a chain of type (C1), we consider two cases separately.

Case C.1. σ occurs at least $\|u\| + 3$ times in ϕ_1.

As in Case B.1, write

$$\phi_1 = \tau^{m-1}\sigma\phi_2,$$

where $m \geqslant 1$ and ϕ_2 is a chain of type (C1). Since $\phi' = \tau\phi_1$,

$$\phi' = \tau^m\sigma\phi_2.$$

It then follows from Lemma 3.1(ii) that

$$\begin{aligned} \|\tau\phi'(u)\| - \|\phi'(u)\| &= \|\tau\tau^m\phi_2(u)\| - \|\tau^m\phi_2(u)\| + \|\sigma\phi_2(u)\| - \|\phi_2(u)\|; \\ \|\tau\phi'(v)\| - \|\phi'(v)\| &= \|\tau\tau^m\phi_2(v)\| - \|\tau^m\phi_2(v)\| + \|\sigma\phi_2(v)\| - \|\phi_2(v)\|. \end{aligned} \tag{33}$$

By Corollary 3.6(i), we have $\|\sigma\phi_2(u)\| = \|\phi_2(u)\|$ if and only if $\|\sigma\phi_2(v)\| = \|\phi_2(v)\|$. Also by Corollary 3.6(iii), we get $\|\tau\tau^m\phi_2(u)\| = \|\tau^m\phi_2(u)\|$ if and only if $\|\tau\tau^m\phi_2(v)\| = \|\tau^m\phi_2(v)\|$. Hence we can apply a similar argument as in Cases B.1 and B.2 to obtain the desired inequalities

$$\min\Delta \leqslant \frac{\|\tau : \phi' : u\|}{\|\tau : \phi' : v\|} \leqslant \max\Delta.$$

Case C.2. τ occurs at least $\|u\| + 3$ times in ϕ_1.

By Lemma 3.1(i) with σ, τ interchanged and $m = 0$, we have

$$\begin{aligned} \|\tau\phi'(u)\| - \|\phi'(u)\| &= \|\tau\phi_1(u)\| - \|\phi_1(u)\|; \\ \|\tau\phi'(v)\| - \|\phi'(v)\| &= \|\tau\phi_1(v)\| - \|\phi_1(v)\|. \end{aligned}$$

It then follows that

$$\begin{aligned} \|\tau : \phi' : u\| &= \|\tau : \phi_1 : u\|; \\ \|\tau : \phi' : v\| &= \|\tau : \phi_1 : v\|, \end{aligned}$$

so that

$$\min\Delta \leqslant \frac{\|\tau : \phi' : u\|}{\|\tau : \phi' : v\|} \leqslant \max\Delta$$

by the induction hypothesis. This completes the proof of Claim C. □

Now the theorem is completely proved. □

The following theorem is the converse of Theorem 3.7.

Theorem 3.8. *Let $u, v \in F_2$ with $\|u\| \geqslant \|v\|$, and Ω, Ω_1 and Ω_2 be defined as in the statement of Theorem* 3.7. *Put $k = \|u\| + 1$. Suppose that u and v are boundedly translation equivalent in F_2. Then*

$$\|\sigma^{k+1}\psi_1(u)\| = \|\sigma^k\psi_1(u)\| \quad \text{if and only if} \quad \|\sigma^{k+1}\psi_1(v)\| = \|\sigma^k\psi_1(v)\|;$$
$$\|\tau^{k+1}\psi_1(u)\| = \|\tau^k\psi_1(u)\| \quad \text{if and only if} \quad \|\tau^{k+1}\psi_1(v)\| = \|\tau^k\psi_1(v)\|,$$

for every $\psi_1 \in \Omega_1$, and

$$\|\sigma^{-k-1}\psi_2(u)\| = \|\sigma^{-k}\psi_2(u)\| \quad \text{if and only if} \quad \|\sigma^{-k-1}\psi_2(v)\| = \|\sigma^{-k}\psi_2(v)\|;$$
$$\|\tau^{-k-1}\psi_2(u)\| = \|\tau^{-k}\psi_2(u)\| \quad \text{if and only if} \quad \|\tau^{-k-1}\psi_2(v)\| = \|\tau^{-k}\psi_2(v)\|,$$

for every $\psi_2 \in \Omega_2$.

Proof. Suppose on the contrary that

$$\|\sigma^{k+1}\psi_1(u)\| = \|\sigma^k\psi_1(u)\| \quad \text{but} \quad \|\sigma^{k+1}\psi_1(v)\| \neq \|\sigma^k\psi_1(v)\| \tag{34}$$

for some $\psi_1 \in \Omega_1$. (The treatment of the other cases is similar.) Put

$$K = \|\sigma^{k+1}\psi_1(v)\| - \|\sigma^k\psi_1(v)\|.$$

By Lemma 3.2(i) and the second inequality of (34), we have $K \geqslant 1$. By repeatedly applying Lemma 3.1(i), we deduce that

$$\|\sigma^{i+1}\psi_1(u)\| = \|\sigma^k\psi_1(u)\| \quad \text{for every } i \geqslant k;$$
$$\|\sigma^{i+1}\psi_1(v)\| = \|\sigma^k\psi_1(v)\| + K(i+1-k) \quad \text{for every } i \geqslant k.$$

Hence

$$\frac{\|\sigma^{i+1}\psi_1(u)\|}{\|\sigma^{i+1}\psi_1(v)\|} = \frac{\|\sigma^k\psi_1(u)\|}{\|\sigma^k\psi_1(v)\| + K(i+1-k)}$$

for every $i \geqslant k$, and thus

$$\lim_{i\to\infty} \frac{\|\sigma^{i+1}\psi_1(u)\|}{\|\sigma^{i+1}\psi_1(v)\|} = 0.$$

This contradiction to the hypothesis that u and v are boundedly translation equivalent in F_2 completes the proof. □

Consequently, in view of Theorems 3.7 and 3.8, we obtain the following algorithm to determine bounded translation equivalence in F_2.

Algorithm 3.9. Let $u, v \in F_2$ with $\|u\| \geqslant \|v\|$, and let Ω, Ω_1 and Ω_2 be defined as in the statement of Theorem 3.7. Put $k = \|u\| + 1$. Check if it is true that

$$\|\sigma^{k+1}\psi_1(u)\| = \|\sigma^k\psi_1(u)\| \quad \text{if and only if} \quad \|\sigma^{k+1}\psi_1(v)\| = \|\sigma^k\psi_1(v)\|;$$
$$\|\tau^{k+1}\psi_1(u)\| = \|\tau^k\psi_1(u)\| \quad \text{if and only if} \quad \|\tau^{k+1}\psi_1(v)\| = \|\tau^k\psi_1(v)\|,$$

for each $\psi_1 \in \Omega_1$, and if it is true that

$$\left\|\sigma^{-k-1}\psi_2(u)\right\| = \left\|\sigma^{-k}\psi_2(u)\right\| \quad \text{if and only if} \quad \left\|\sigma^{-k-1}\psi_2(v)\right\| = \left\|\sigma^{-k}\psi_2(v)\right\|;$$
$$\left\|\tau^{-k-1}\psi_2(u)\right\| = \left\|\tau^{-k}\psi_2(u)\right\| \quad \text{if and only if} \quad \left\|\tau^{-k-1}\psi_2(v)\right\| = \left\|\tau^{-k}\psi_2(v)\right\|,$$

for each $\psi_2 \in \Omega_2$. If so, conclude that u and v are boundedly translation equivalent in F_2; otherwise conclude that u and v are not boundedly translation equivalent in F_2.

4. Fixed point groups of automorphisms of F_2

In this section, we shall demonstrate that there is an algorithm to decide whether or not a given finitely generated subgroup of F_2 is the fixed point group of some automorphism of F_2. If $H = \langle u_1, \dots, u_k\rangle$ is a finitely generated subgroup of F_2, then we define

$$|H| := \max_{1 \leqslant i \leqslant k} |u_i|.$$

Clearly $\|u_i\| \leqslant |u_i| \leqslant |H|$ for every $i = 1, \dots, k$.

Theorem 4.1. *Let $H = \langle u_1, \dots, u_k\rangle$ be a finitely generated subgroup of F_2. Suppose that ϕ is a chain of type* (C1) *with $|\phi| \geqslant 4|H| + 5$ such that $\|\phi(u_i)\| = \|u_i\|$ for every $i = 1, \dots, k$. Then there exists a chain ψ of type* (C1) *with $|\psi| < |\phi|$ such that $[\psi(u_i)] = [\phi(u_i)]$ for every $i = 1, \dots, k$.*

Proof. Since ϕ is a chain of type (C1) with $|\phi| \geqslant 4|H| + 5$, ϕ contains at least $2|H| + 3$ factors of σ or τ. Suppose that ϕ contains at least $2|H| + 3$ factors of σ (the other case is similar). We may write

$$\phi = \tau^{m_t}\sigma^{\ell_t}\cdots\tau^{m_1}\sigma^{\ell_1}\phi', \tag{35}$$

where all $\ell_i, m_i > 0$ but ℓ_1 and m_t may be zero, and ϕ' is a chain of type (C1) which contains exactly $|H| + 2$ factors of σ.

Suppose that there exists u_j ($1 \leqslant j \leqslant k$) such that $\|\sigma\phi'(u_j)\| \neq \|\phi'(u_j)\|$. Put

$$K = \left\|\sigma\phi'(u_j)\right\| - \left\|\phi'(u_j)\right\|.$$

Since ϕ' contains at least $\|u_j\| + 2$ factors of σ, by Lemma 3.2(i), $K \geqslant 1$. Furthermore, since ϕ contains at least $2|H| + 3$ factors of σ and ϕ' contains exactly $|H| + 2$ factors of σ,

$$\sum_{i=1}^{t}\ell_i \geqslant |H| + 1 \geqslant \|u_j\| + 1. \tag{36}$$

From the following claim, we shall obtain a contradiction.

Claim. $\|\phi(u_j)\| - \|\phi'(u_j)\| \geqslant \|u_j\| + 1$.

Proof. First assume that $m_1 = 0$ in (35). Then $\phi = \sigma^{\ell_1}\phi'$, and so, from (36), $\ell_1 \geqslant \|u_j\| + 1$. By repeatedly applying Lemma 3.1(i), we have

$$\left\|\phi(u_j)\right\| - \left\|\phi'(u_j)\right\| = \ell_1 K.$$

Since $K \geqslant 1$, it follows that

$$\left\|\phi(u_j)\right\| - \left\|\phi'(u_j)\right\| \geqslant \ell_1 \geqslant \|u_j\| + 1,$$

as desired.

Next assume that $m_1 > 0$ in (35). In view of Lemmas 3.1 and 3.2, we can observe that

$$\begin{aligned}
\left\|\sigma^{\ell_1}\phi'(u_j)\right\| - \left\|\phi'(u_j)\right\| &= \ell_1 K;\\
\left\|\tau^{m_1}\sigma^{\ell_1}\phi'(u_j)\right\| - \left\|\sigma^{\ell_1}\phi'(u_j)\right\| &\geqslant m_1 K;\\
&\cdots\\
\left\|\sigma^{\ell_t}\cdots\tau^{m_1}\sigma^{\ell_1}\phi'(u_j)\right\| - \left\|\tau^{m_{t-1}}\cdots\tau^{m_1}\sigma^{\ell_1}\phi'(u_j)\right\| &\geqslant \ell_t K;\\
\left\|\tau^{m_t}\sigma^{\ell_t}\cdots\tau^{m_1}\sigma^{\ell_1}\phi'(u_j)\right\| - \left\|\sigma^{\ell_t}\cdots\tau^{m_1}\sigma^{\ell_1}\phi'(u_j)\right\| &\geqslant m_t K.
\end{aligned}$$

Summing up all of these inequalities together with (36) yields

$$\begin{aligned}
\left\|\phi(u_j)\right\| - \left\|\phi'(u_j)\right\| &\geqslant \sum_{i=1}^{t}(\ell_i + m_i)K\\
&\geqslant \left(\sum_{i=1}^{t}\ell_i\right)K\\
&\geqslant \sum_{i=1}^{t}\ell_i\\
&\geqslant \|u_j\| + 1,
\end{aligned}$$

as required. This completes the proof of the claim. □

It then follows from the claim that

$$\left\|\phi(u_j)\right\| \geqslant \left\|\phi'(u_j)\right\| + \|u_j\| + 1 \geqslant \|u_j\| + 1.$$

But this yields a contradiction to the hypothesis that $\|\phi(u_j)\| = \|u_j\|$. Therefore, we must have $\|\sigma\phi'(u_i)\| = \|\phi'(u_i)\|$ for every $i = 1, \ldots, k$. Then for each $i = 1, \ldots, k$,

$$\begin{aligned}
0 = \left\|\sigma\phi'(u_i)\right\| - \left\|\phi'(u_i)\right\| &= n\big(\big[\phi'(u_i)\big]; a\big) - 2n\big(\big[\phi'(u_i)\big]; a, b^{-1}\big)\\
&= n\big(\big[\phi'(u_i)\big]; a, a\big) + n\big(\big[\phi'(u_i)\big]; a, b\big) - n\big(\big[\phi'(u_i)\big]; a, b^{-1}\big).
\end{aligned} \tag{37}$$

Here, since ϕ' contains at least $\|u_i\| + 2$ factors of σ, by Lemma 2.1, there cannot occur proper cancellation in passing from $[\phi'(u_i)]$ to $[\sigma\phi'(u_i)]$, and so every subword of $[\phi'(u_i)]$ of the form ab^{-1} or ba^{-1} is necessarily part of a subword of the form $ab^{-r}a^{-1}$ or $ab^{r}a^{-1}$ $(r > 0)$, respectively. This implies that

$$n\big(\big[\phi'(u_i)\big]; a, b\big) \geqslant n\big(\big[\phi'(u_i)\big]; a, b^{-1}\big),$$

so that, from (37),

$$n\big(\big[\phi'(u_i)\big]; a, b\big) = n\big(\big[\phi'(u_i)\big]; a, b^{-1}\big) \quad \text{and} \quad n\big(\big[\phi'(u_i)\big]; a, a\big) = 0. \tag{38}$$

From the fact that no proper cancellation can occur in passing from $[\phi'(u_i)]$ to $[\sigma\phi'(u_i)]$ together with (38), each cyclic word $[\phi'(u_i)]$ must have the form

$$\big[\phi'(u_i)\big] = \big[b^{s_{i1}}ab^{t_{i1}}a^{-1}\cdots b^{s_{ir}}ab^{t_{ir}}a^{-1}\big],$$

where every s_{ij}, t_{ij} is a nonzero integer, and hence

$$[\sigma\phi'(u_i)] = [\phi'(u_i)]$$

for every $i = 1, \ldots, t$.

Thus letting

$$\psi = \tau^{m_t}\sigma^{\ell_t}\cdots\tau^{m_1}\sigma^{\ell_1-1}\phi',$$

we finally have

$$[\psi(u_i)] = [\phi(u_i)]$$

for every $i = 1, \ldots, t$. Obviously $|\psi| < |\phi|$, and so the proof of the theorem is completed. □

We remark that Theorem 4.1 also holds if (C1) is replaced by (C2). From now on, let

$$\delta_1 = (\{a^{\pm 1}\}, b), \qquad \delta_2 = (\{a^{\pm 1}\}, b^{-1}), \qquad \delta_3 = (\{b^{\pm 1}\}, a), \qquad \delta_4 = (\{b^{\pm 1}\}, a^{-1})$$

be Whitehead automorphisms of F_2 of type (W2).

Lemma 4.2. *Let α be a Whitehead automorphism of F_2 of type* (W2)*. Then α can be expressed as a composition of $\sigma^{\pm 1}$, $\tau^{\pm 1}$ and δ_i's.*

Proof. If α is not one of $\sigma^{\pm 1}$, $\tau^{\pm 1}$ and δ_i's, then α must be one of $(\{a^{-1}\}, b)$, $(\{a^{-1}\}, b^{-1})$, $(\{b^{-1}\}, a)$ and $(\{b^{-1}\}, a^{-1})$. Then the following easy identities

$$(\{a^{-1}\}, b) = \delta_1\sigma^{-1}; \qquad (\{a^{-1}\}, b^{-1}) = \delta_2\sigma;$$
$$(\{b^{-1}\}, a) = \delta_3\tau^{-1}; \qquad (\{b^{-1}\}, a^{-1}) = \delta_4\tau$$

imply the required result. □

The following two technical lemmas can be easily proved by direct calculations.

Lemma 4.3. *The following identities hold*:

$$\sigma\delta_1 = \delta_1\sigma; \qquad \sigma\delta_2 = \delta_2\sigma; \qquad \sigma\delta_3 = \delta_1\delta_3\sigma; \qquad \sigma\delta_4 = \delta_4\delta_2\sigma;$$
$$\tau\delta_1 = \delta_3\delta_1\tau; \qquad \tau\delta_2 = \delta_2\delta_4\tau; \qquad \tau\delta_3 = \delta_3\tau; \qquad \tau\delta_4 = \delta_4\tau;$$
$$\sigma^{-1}\delta_1 = \delta_1\sigma^{-1}; \qquad \sigma^{-1}\delta_2 = \delta_2\sigma^{-1}; \qquad \sigma^{-1}\delta_3 = \delta_2\delta_3\sigma^{-1}; \qquad \sigma^{-1}\delta_4 = \delta_4\delta_1\sigma^{-1};$$
$$\tau^{-1}\delta_1 = \delta_4\delta_1\tau^{-1}; \qquad \tau^{-1}\delta_2 = \delta_2\delta_3\tau^{-1}; \qquad \tau^{-1}\delta_3 = \delta_3\tau^{-1}; \qquad \tau^{-1}\delta_4 = \delta_4\tau^{-1}.$$

Lemma 4.4. *The following identities hold*:

$$\sigma\tau^{-1} = \pi\delta_1\sigma^{-1}; \qquad \sigma^{-1}\tau = \pi^{-1}\delta_3\sigma; \qquad \tau\sigma^{-1} = \pi^{-1}\delta_3\tau^{-1}; \qquad \tau^{-1}\sigma = \pi\delta_1\tau;$$
$$\sigma\pi = \pi\delta_3\tau^{-1}; \qquad \sigma\pi^{-1} = \pi^{-1}\tau^{-1}; \qquad \sigma^{-1}\pi = \pi\delta_4\tau; \qquad \sigma^{-1}\pi^{-1} = \pi^{-1}\tau;$$
$$\tau\pi = \pi\sigma^{-1}; \qquad \tau\pi^{-1} = \pi^{-1}\delta_1\sigma^{-1}; \qquad \tau^{-1}\pi = \pi\sigma; \qquad \tau^{-1}\pi^{-1} = \pi^{-1}\delta_2\sigma,$$

where π is a Whitehead automorphism of F_2 of type (W1) *that sends a to b and b to a^{-1}.*

The following corollary gives a nice description of automorphisms of F_2.

Corollary 4.5. *Every automorphism ϕ of F_2 can be represented as*

$$\phi = \beta\delta\phi',$$

where β is a Whitehead automorphism of F_2 of type (W1)*, δ is a composition of δ_i's, and ϕ' is a chain of type* (C1) *or* (C2)*.*

Proof. By Whitehead's Theorem (cf. [12]) together with Lemmas 4.2 and 4.3, an automorphism ϕ of F_2 can be expressed as

$$\phi = \beta'\delta'\tau^{q_t}\sigma^{p_t}\cdots\tau^{q_1}\sigma^{p_1}, \tag{39}$$

where β' is a Whitehead automorphism of F_2 of type (W1), δ' is a composition of δ_i's, and both p_j, q_j are (not necessarily positive) integers for every $j = 1, \ldots, t$. If not every p_j and q_j has the same sign (including 0), apply repeatedly Lemma 4.4 to the chain on the right-hand side of (39) to obtain that either $\phi = \beta'\pi^r\delta\tau^{m_k}\sigma^{l_k}\cdots\tau^{m_1}\sigma^{l_1}$ or $\phi = \beta'\pi^r\delta\tau^{-m_k}\sigma^{-l_k}\cdots\tau^{-m_1}\sigma^{-l_1}$, where π is as in Lemma 4.4, $r \in \mathbb{Z}$, δ is a composition of δ_i's, and both $l_j, m_j \geqslant 0$ for every $j = 1, \ldots, k$. Putting $\beta = \beta'\pi^r$, we obtain the required result. □

The following is the main result of this section.

Theorem 4.6. *Let $H = \langle u_1, \ldots, u_k\rangle$ be a finitely generated subgroup of F_2. Suppose that H is the fixed point group of an automorphism ϕ of F_2. Let Ω_1 be the set of all chains of type* (C1) *or* (C2) *of length less than or equal to $4|H| + 4$, and let Ω_2 be the set of all compositions of δ_i's of length less than or equal to $(2^{4|H|+4} + 1)|H|$. Put*

$$\Omega = \bigl\{\beta\delta'\psi' \bigm| \psi' \in \Omega_1, \delta' \in \Omega_2,\ \text{and } \beta \text{ is a Whitehead auto of } F_2 \text{ of type (W1)}\bigr\}.$$

Then there exists $\psi \in \Omega$ of which H is the fixed point group.

Proof. By Corollary 4.5, ϕ can be written as

$$\phi = \beta\delta\phi',$$

where β, δ and ϕ' are indicated as in the statement of Corollary 4.5.

Since $\phi(u_i) = u_i$ for every $i = 1, \ldots, k$, it is easy to see that

$$\bigl\|\phi'(u_i)\bigr\| = \|u_i\|$$

for every $i = 1, \ldots, k$. Then apply Theorem 4.1 continuously to obtain $\psi' \in \Omega_1$ such that

$$\bigl[\psi'(u_i)\bigr] = \bigl[\phi'(u_i)\bigr]$$

for every $i = 1, \ldots, k$. Since $|\delta\phi'(u_i)| = |\phi(u_i)| = |u_i| \leqslant |H|$ and $|\psi'(u_i)| \leqslant 2^{4|H|+4}|u_i| \leqslant 2^{4|H|+4}|H|$ for every $i = 1, \ldots, k$, we must have $\delta' \in \Omega_2$ such that

$$\delta'\psi'(u_i) = \delta\phi'(u_i)$$

for every $i = 1, \ldots, k$, and hence

$$\beta\delta'\psi'(u_i) = \beta\delta\phi'(u_i) = u_i$$

for every $i = 1, \ldots, k$. Therefore, letting

$$\psi = \beta\delta'\psi',$$

we finally have $\psi \in \Omega$ and that H is the fixed point subgroup of ψ. This completes the proof of the theorem. □

In conclusion, we naturally derive from Theorem 4.6 the following algorithm to decide whether or not a given finitely generated subgroup of F_2 is the fixed point group of some automorphism of F_2.

Algorithm 4.7. Let $H = \langle u_1, \ldots, u_k \rangle$ be a finitely generated subgroup of F_2. Let Ω_1, Ω_2 and Ω be defined as in the statement of Theorem 4.6. Clearly Ω is a finite set. Check if there is $\psi \in \Omega$ for which $\psi(u_i) = u_i$ holds for every $i = 1, \ldots, k$. If so, conclude that H is the fixed point group of some automorphism of F_2; otherwise conclude that H is not the fixed point group of any automorphism of F_2.

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