ELSEVIER
Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT
Applied Mathematics Letters 17 (2004) 167–172
Applied Mathematics Letters
www.elsevier.com/locate/aml

The Diameter of Honeycomb Rhombic Tori

XIAOFAN YANG
College of Computer Science
and National Education Ministry's Key Laboratory
of Electro-Optical Technique and System
Chongqing University, Chongqing, 400044, P.R. China
yxf640126@sina.com

(Received October 2002; accepted November 2002)

Communicated by A. Tucker

Abstract—The effectiveness of an interconnection network is one of the dominating performance factors for the associated parallel computers. Honeycomb rhombic torus networks are attractive alternatives to torus networks due to the smaller node degree, leading to lower complexity and lower implementation cost. Diameter of an interconnection network is one of the most important evaluation indexes because it characterizes the maximum communication delay in the network. In this paper, the diameter of a honeycomb rhombic torus is determined, which is lower than that of a torus of the same order. Therefore, the superiority of honeycomb rhombic torus networks is greatly strengthened.

Keywords—Interconnection network, Honeycomb rhombic torus, Diameter.

1. INTRODUCTION

The effectiveness of an interconnection network is one of the dominating performance factors for the associated parallel computer. Due to this, the design and analysis of interconnection networks have been a main topic of research in the field of parallel computing [1]. An interconnection network is usually modeled by a graph with vertices representing processing nodes and edges corresponding to communication links [2,3].

Honeycomb-type torus networks are attractive alternatives to torus networks due to the smaller node degree [4,5], which leads to lower complexity and lower implementation cost. Stojmenovic [5] proposed three classes of honeycomb-type tori: *honeycomb tori*, *honeycomb rectangular tori*, and *honeycomb rhombic tori*. For honeycomb tori, Stojmenovic [5] addressed their topological properties and presented the associated routing and broadcasting algorithms, Megson *et al.* [6,7] proved that they contain Hamiltonian cycles, even in the presence of node failures, and Yang and Huang [8] proved that they are rotational Cayley graphs. For honeycomb rectangular tori, Parhami and Kwai [9] found that they can be seen as a pruned version of tori, and they are

This research was supported by the Visiting Scholar's Funds of National Education Ministry's Key Laboratory of Electro-Optical Technique and System, Chongqing University.

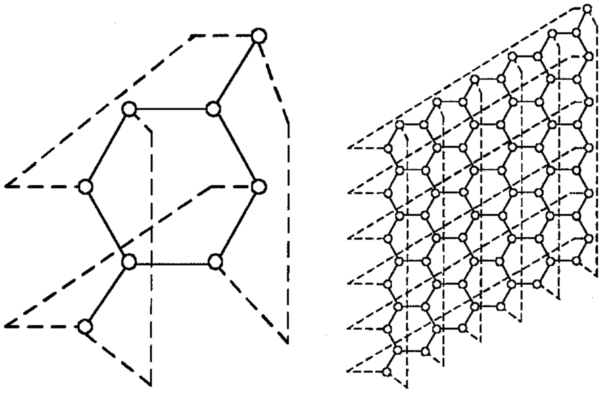
0893-9659/04/$ - see front matter Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
doi: 10.1016/S0893-9659(03)00235-0

Cayley graphs. As for honeycomb rhombic tori, Yang *et al.* [10] proved that they are also Cayley graphs.

The diameter of an interconnection network is one of the most important evaluation indexes because it characterizes the maximum communication delay in the network. Usually those graphs of smaller node degree and with smaller diameter are preferred as interconnection networks. Stojmenovic [3] claimed that a honeycomb rhombic torus of size t has $\lfloor 3t/2 \rfloor$ as diameter, but this is not the case. In this paper, we completely determine the diameter of a honeycomb rhombic torus. The result demonstrates that a honeycomb rhombic torus has a diameter smaller than that of a torus of the same order. Due to this, and therefore, the remarkably lower cost (defined as the product of degree and diameter), the superiority of honeycomb rhombic torus networks is greatly strengthened.

2. DEFINITION OF HONEYCOMB RHOMBIC TORUS

A honeycomb rhombic torus of size $t \geq 2$, denoted by HRoT(t), is built according to the following process: fill in a rhombic region with a set of t vertical zigzag paths of length $2t-1$ (see Figure 1). Sequentially, from top to bottom, label the vertices of the i^{th} path counted from left as $u_1^i v_1^i u_2^i v_2^i \cdots u_t^i v_t^i$. For any i and j such that $1 \leq i \leq t-1$ and $1 \leq j \leq t$, join u_j^i and v_j^{i+1} with an edge (see those horizontal edges in Figure 1). For any i such that $1 \leq i \leq t$, join u_1^i and v_t^i with an edge, and join v_i^1 and u_i^t with an edge (see those dashed lines in Figure 1).



(a) $t = 2$. (b) $t = 6$.

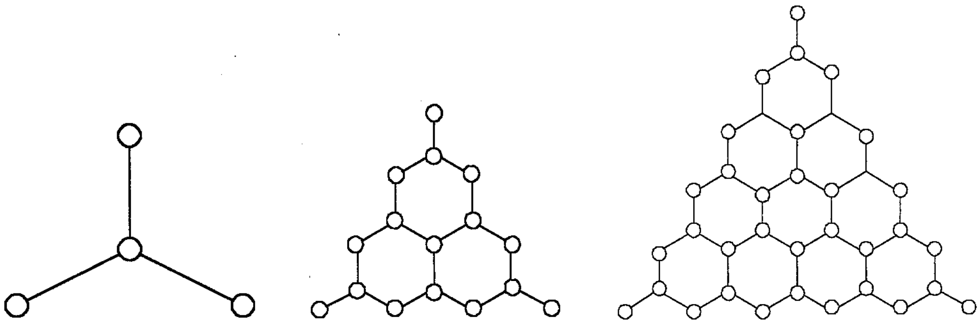
Figure 1. Illustrations of honeycomb rhombic torus HRoT(t) of size t.

Let diam(G) denote the diameter of the graph G, i.e., the maximum distance between two vertices of G. Let cost$(G) = \Delta(G) \times$ diam(G), where $\Delta(G)$ denotes the maximum degree of a vertex of G.

3. DIAMETER OF HONEYCOMB RHOMBIC TORUS

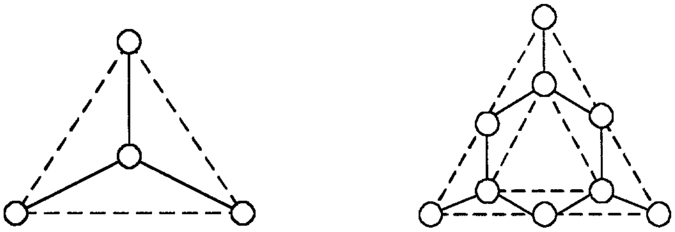
We begin by building a family of graphs $\{T_p\}_{p=0}^{\infty}$ in a recursive way. T_0 is a single-vertex graph. T_1 is the claw graph $K_{1,3}$ (see Figure 2a). In general, T_p is obtained from T_{p-1} and a horizontal zigzag path P of length $2p+1$ as follows: label the p vertices lying on the bottom line of T_{p-1} from left to right as $x_1 x_2 \cdots x_p$, label the $2p+1$ vertices of P from left to right as $z_0 y_1 z_1 y_2 z_2 \cdots y_p z_p$, and join x_i and y_i with an edge for each i $(1 \leq i \leq p+1)$. See Figure 2 for three examples of T_p.

Let S_p^0 denote the set of vertices in T_p which lie on the boundary triangle of T_p. In general, let S_p^k denote the set of vertices in T_p which are at distance k from S_p^0. It can be seen from Figure 3 that all the vertices on each S_p^k lie on a dashed equilateral triangle (where a single point

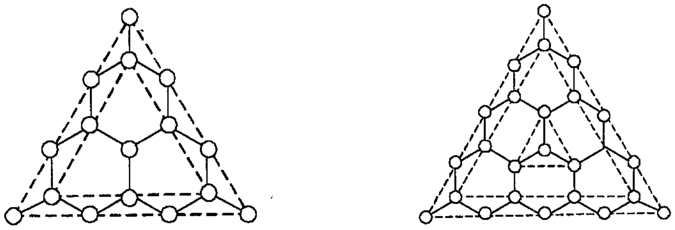


(a) $p = 1$. (b) $p = 3$. (c) $p = 5$.

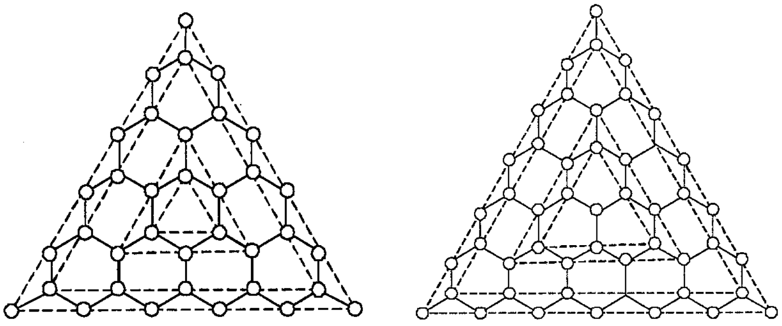
Figure 2. Illustrations of the graph T_p.



(a) $p = 1$. (b) $p = 2$.



(c) $p = 3$. (d) $p = 4$.



(e) $p = 5$. (f) $p = 6$.

Figure 3. Illustrations of families of equidistant triangles in T_p.

is considered as a degenerate triangle). Let $g(p) = \max\{k : S_p^k \neq \Phi\}$. That is, $g(p)$ denotes the maximum distance between a vertex in S_p^0 and a vertex of T_p. It is easily verified that

$$g(0) = 0, \quad g(1) = 1, \quad g(2) = 1, \quad g(3) = 2, \quad g(4) = 3, \quad g(5) = 3, \quad g(6) = 4,$$
$$g(7) = 5, \quad g(8) = 5, \quad g(9) = 6, \quad g(10) = 7, \quad g(11) = 7, \quad g(12) = 8, \ldots.$$

In general, we have the following.

LEMMA 1.

$$g(p) = \begin{cases} \dfrac{2p}{3}, & \textit{if } p \equiv 0 \bmod 3, \\ \dfrac{2p+1}{3}, & \textit{if } p \equiv 1 \bmod 3, \\ \dfrac{2p-1}{3}, & \textit{if } p \equiv 2 \bmod 3. \end{cases}$$

PROOF. By induction on p. ∎

The following properties of the function $g(p)$ are obvious.

LEMMA 2.

(a) $g(p+1) - 1 \leq g(p) \leq g(p+1)$ *for every nonnegative integer* p.
(b) $g(p) = g(p+1)$ *if and only if* $p \equiv 1 \bmod 3$.

THEOREM 3.

$$\operatorname{diam}(\mathrm{HRoT}(t)) = \begin{cases} \left\lfloor \dfrac{4t}{3} \right\rfloor, & \textit{if } t \equiv 1 \textit{ or } 4 \bmod 6, \\ \left\lceil \dfrac{4t}{3} \right\rceil, & \textit{otherwise.} \end{cases}$$

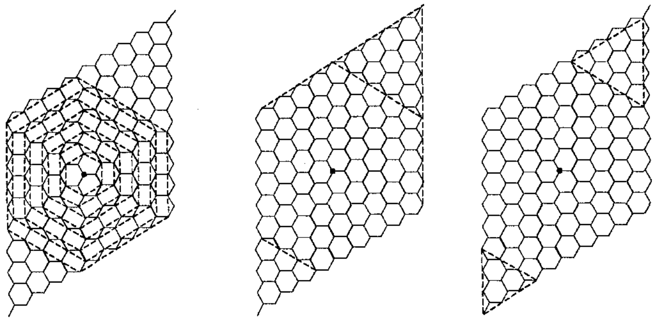
PROOF. It is easily verified that diam (HRoT(2)) $= 3 = \lceil (4 \times 2)/3 \rceil$. Now we assume $k \geq 3$. Due to the fact that HRoT(t) is Cayley graph [10], we may fix any vertex, say X, and calculate the maximum distance between X and any vertex of HRoT(t) to determine diam (HRoT(t)) [2,3]. We distinguish between two cases.

CASE 1. t is odd. Fix the vertex $X = v_{(t+1)/2}^{(t+1)/2}$. (See Section 2 and the black node in Figure 4. For clarity, here and in what follows those wrap-around edges of HRoT(t) are not drawn.) Let S_k denote the set of vertices at distance k from X. It is easily verified that

(i) for each k such that $2 \leq k \leq t-1$, S_k forms a dashed hexagon as shown in Figure 5a,
(ii) S_t forms those dashed lines in Figure 5b, and
(iii) S_{t+1} forms those dashed lines in Figure 5c.

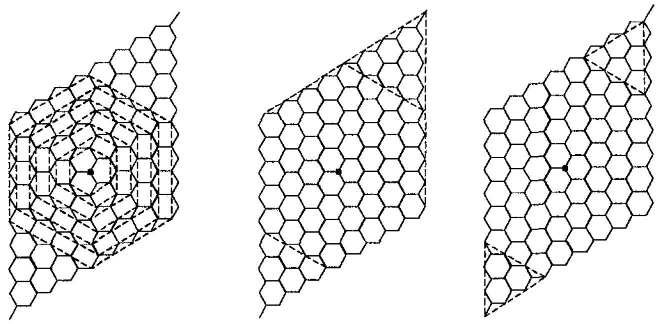
Furthermore, the graph in the dashed triangle on the upper-right corner of Figure 5b is isomorphic to $T_{(t-1)/2}$, and the graph in the dashed triangle on the bottom-left corner of Figure 5c is isomorphic to $T_{(t-3)/2}$. It follows from Lemmas 1 and 2 that

$$\begin{aligned} \operatorname{diam}(\mathrm{HRoT}(t)) &= \max\left\{ t + g\left(\frac{t-1}{2}\right),\ t+1+g\left(\frac{t-3}{2}\right) \right\} \\ &= \begin{cases} t + \dfrac{2}{3} \times \dfrac{t-1}{2}, & \text{if } \dfrac{t-1}{2} \equiv 0 \bmod 3, \\ t + \dfrac{1}{3}\left(2\dfrac{t-1}{2}+1\right), & \text{if } \dfrac{t-1}{2} \equiv 1 \bmod 3, \\ t+1+\dfrac{1}{3}\left(2\dfrac{t-3}{2}+1\right), & \text{if } \dfrac{t-1}{2} \equiv 2 \bmod 3, \text{ i.e., } \dfrac{t-3}{2} \equiv 1 \bmod 3; \end{cases} \\ &= \begin{cases} \dfrac{4t-1}{3}, & \text{if } t \equiv 1 \bmod 6, \\ \dfrac{4t}{3}, & \text{if } t \equiv 3 \bmod 6, \\ \dfrac{4t+1}{3}, & \text{if } t \equiv 5 \bmod 6. \end{cases} \end{aligned}$$



(a) S_k for $2 \le k \le t-1$. (b) S_t. (c) S_{t+1}.

Figure 4. Equidistance lines from the vertex X when $t = 10$.



(a) S_k for $2 \le k \le t-1$. (b) S_t. (c) S_{t+1}.

Figure 5. Equidistance lines from the vertex X when $t = 9$.

CASE 2. t is even. Fix the vertex $X = v^{t/2}_{(t+2)/2}$. (See Section 2 and the black node in Figure 4.) Let S_k denote the set of vertices at distance k from X. It is easily verified that

(i) for each k such that $2 \le k \le t-1$, S_k forms a dashed hexagon as shown in Figure 4a,
(ii) S_t forms those dashed lines in Figure 4b, and
(iii) S_{t+1} forms those dashed lines in Figure 4c.

Furthermore, the graph in the dashed triangle on the upper-right corner of Figure 4b is isomorphic to $T_{t/2}$, and the graph in the dashed triangle on the bottom-left corner of Figure 4c is isomorphic to $T_{(t-4)/2}$. It follows from Lemmas 1 and 2 that

$$\operatorname{diam}(\mathrm{HRoT}(t)) = \max\left\{t + g\left(\frac{t}{2}\right),\ t+1+g\left(\frac{t}{2}-2\right)\right\}$$

$$= \begin{cases} t + \dfrac{2}{3} \times \dfrac{t}{2}, & \text{if } \dfrac{t}{2} \equiv 0 \bmod 3, \\[2mm] t + \dfrac{1}{3}\left(2\dfrac{t}{2}+1\right), & \text{if } \dfrac{t}{2} \equiv 1 \bmod 3, \\[2mm] t + \dfrac{1}{3}\left(2\dfrac{t}{2}-1\right), & \text{if } \dfrac{t}{2} \equiv 2 \bmod 3; \end{cases}$$

$$= \begin{cases} \dfrac{4t}{3}, & \text{if } t \equiv 0 \bmod 6, \\[2mm] \dfrac{4t+1}{3}, & \text{if } t \equiv 2 \bmod 6, \\[2mm] \dfrac{4t-1}{3}, & \text{if } t \equiv 4 \bmod 6. \end{cases}$$

By carefully combining the previous results, we conclude the proposition. ∎

Finally, we make a comparison between HRoT(t) and a square torus of the same order, which is $\sqrt{2}\,t$ rows by $\sqrt{2}\,t$ columns and is denoted by Torus($\sqrt{2}\,t$) since the order of HRoT(t) is $2t^2$. It is obvious that diam(Torus($\sqrt{2}\,t$)) $= \sqrt{2}\,t \approx 1.414t$, and *cost* (Torus($\sqrt{2}\,t$)) $\approx 5.656t$. However, diam(HRoT(t)) $\approx 4t/3 \approx 1.333t$, and cost(HRoT($t$)) $\approx 4t$. Therefore, HRoT(t) is 5.73% and 29.28% better than Torus($\sqrt{2}\,t$) in terms of diameter and cost, respectively.

REFERENCES

1. B. Parhami, *An Introduction to Parallel Processing: Algorithms and Architectures*, Plenum Press, (1999).
2. S. Akers and B. Krishnamurthy, A group theoretic model for symmetric interconnection networks, *IEEE Trans. Computers* **38** (4), 555–566, (1989).
3. M.-C. Heydemann, Cayley graphs and interconnection networks, In *Graph Symmetry*, (Edited by G. Hahn and G. Sabidussi), pp. 167–224, Kluwer Academic, (1997).
4. D. Milutinovic, V. Milutinovic and B. Soucek, The honeycomb architecture, *Computer* **20** (4), 81–83, (1987).
5. I. Stojmenovic, Honeycomb networks: Topological properties and communication algorithms, *IEEE Trans. Parallel Distributed Systems* **8** (10), 1036–1042, (1997).
6. G.M. Megson, X. Yang and X. Liu, Honeycomb tori are Hamiltonian, *Inform. Process. Lett.* **72**, 99–103, (1999).
7. G.M. Megson, X. Liu and X. Yang, Fault-tolerant ring embedding in a honeycomb torus with node failures, *Parallel Process. Lett.* **9** (4), 551–562, (1999).
8. X. Yang and X. Huang, On the symmetry of honeycomb torus networks, (submitted).
9. B. Parhami and D.-M. Kwai, A unified formulation of honeycomb and diamond networks, *IEEE Trans. Parallel Distributed Systems* **12** (1), 74–80, (2001).
10. X. Yang, G.M. Megson and X. Huang, On the symmetry of honeycomb parallelogramic torus networks, *Parallel Algorithms and Applications* (to appear).