Normal subgroups of the group of volume preserving diffeomorphisms of an open manifold*

Vicente Cervera

Departamento de Matemáticas e Informática, Universitat Jaume I, Castellón, Spain

Communicated by P. Michor
Received 11 December 1991


Abstract: Let \( \Omega \) be a volume element on an open manifold, \( M \), which is the interior of a compact manifold \( \bar{M} \). We study the lattice of the normal subgroups of the group of all \( \Omega \)-preserving diffeomorphisms isotopic to the identity by an \( \Omega \)-isotopy, \( \text{Diff}^0_\Omega(M) \).

Keywords: Normal subgroup, volume element, diffeomorphism.

MS classification: 57R50, 58A10.

0. Introduction

Let \( M \) be a connected open smooth \( n \)-manifold that is the interior of a compact one \( \bar{M} \), \( n \geq 3 \). We denote by \( (\text{Diff}_0(M))^r \) the group of all \( C^r \)-diffeomorphisms of \( M \) \( C^r \)-isotopic to the identity (i.e. the path-component of the identity of \( (\text{Diff}(M))^r \) with respect to the compact-open \( C^r \)-topology), \( r = 0, \ldots, \infty \).

If \( \partial \bar{M} = \emptyset \) then \( (\text{Diff}_0(M))^r \) is a simple group, except if \( r = n + 1 \) (See [3, 6, 12]). But if the boundary of \( \bar{M} \) is non-empty, then \( (\text{Diff}_n(M))^r \) contains non-trivial normal subgroups. So, if we denote by \( \partial_1 \bar{M}, \ldots, \partial_k \bar{M} \) the connected components of \( \partial \bar{M} \), McDuff in [7] proved that for each non-trivial normal subgroup \( N \) of \( (\text{Diff}_0(M))^r \), there is a unique subset \( J \subset K = \{1, \ldots, k\} \) such that \( [G_J, G_J] \subset N \subset G_J \), where \( G_J \) is the normal subgroup of all elements of \( (\text{Diff}_n(M))^r \) which are the identity near \( \partial J \bar{M} = \bigcup_{i \in J} \partial_i \bar{M} \).

The purpose of this paper is to get similar results for the group of all diffeomorphisms of \( M \) which preserve a given volume element \( \Omega \).

Correspondence to: Vicente Cervera, Departamento de Matemáticas e Informática, Escuela Sup. Tecnología y Ciencias Experimentales, Universitat Jaume I, 12071 Castellón, Spain.

*The author was partially supported by the CICYT grant n. PB90-0014-C03-01.
We denote by $\text{Diff}^\Omega(M)$ the group of all $C^\infty$-diffeomorphisms of $M$ preserving $\Omega$, and by $\text{Diff}^\Omega_0(M)$ the path component of the identity with respect to the compact-open $C^\infty$-topology. In general, $\text{Diff}^\Omega_0(M)$ is not a perfect group, and the flux homomorphism gives an obstruction to perfectness since the derived subgroup is contained in the kernel of flux $[9]$.

For each $i = 1, \ldots, k$ it is clear that the component, $\partial_i \tilde{M}$, of the boundary of $\tilde{M}$, determines only one end of the manifold $M$, that we will denote by $\partial_i$.

We will say that the end $\partial_i$ has finite $\Omega$-volume if for some compact subset $C \subset M$, the component of $M - C$ corresponding to $\partial_i$ has finite $\Omega$-volume. The end $\partial_i$ has infinite $\Omega$-volume if it does not have finite $\Omega$-volume.

By [4] we know that if a volume element $\Omega'$ has the same total volume as $\Omega$, and every end of $M$ has finite $\Omega$-volume if and only if it has finite $\Omega'$-volume also, then there is a diffeomorphism $f : M \to M$, such that $f^* \Omega = \Omega'$. Therefore, the groups $\text{Diff}^\Omega(M)$ and $\text{Diff}^{\Omega'}(M)$ are isomorphic. So, the group $\text{Diff}^\Omega(M)$ could be characterized by the total $\Omega$-volume of $M$, and by the $\Omega$-volume of its ends.

We begin the study of the normal subgroups of $\text{Diff}^\Omega_0(M)$ with the study of the group of volume preserving diffeomorphisms of $X \times \mathbb{R}^+$, which are the identity near $X \times \{0\}$, $\text{Diff}^\Omega(X \times \mathbb{R}^+, \text{rel} X \times \{0\})$; where $X$ is a closed connected manifold of dimension greater or equal than two, and $\mathbb{R}^+ = [0, \infty)$.

Our main interest for this group follows from the fact that any element $f \in \text{Diff}^\Omega_0(M)$, such that $f_\ell$ is in the kernel of the flux homomorphism

$$\phi : \text{Diff}^\Omega_0(M) \to H^{n-1}(M; \mathbb{R})$$

(see [2] or [9] for its definition), can be decomposed as $f = f_1 \circ \ldots \circ f_{k+1}$ where for each $i \leq k$, $f_i \in \text{Diff}^\Omega_0(M)$ and $\text{supp}(f_i)$ is contained in the interior of a subset diffeomorphic to $\partial_i M \times \mathbb{R}^+$; and $f_{k+1} \in \text{Diff}^\Omega_0(M)$, the subgroup of $\text{Diff}^\Omega(M)$ of all elements isotopic to the identity by an $\Omega$-isotopy with compact support. Thus, in Section 1 we will give the technical results needed to study the above group.

In the study of normal subgroups of $\text{Diff}^\Omega_0(M)$ we will separate out three cases, namely i) all the ends have finite volume (Theorem 2.4), ii) all the ends have infinite volume (Theorem 2.8), and iii) a mixture of ends having finite or infinite volume (Theorem 2.12).

In any case, $\text{Diff}^\Omega_0(M)$ contains non-trivial normal subgroups. For instance, for each $J \subset K$ we consider $G^\Omega_J$ the normal subgroup of all elements of $\text{Diff}^\Omega_0(M)$ which are the identity near $\partial_J \tilde{M}$. Note that $G^\Omega_\emptyset$ is the full group $\text{Diff}^\Omega_0(M)$, and $G^\Omega_K$ is the subgroup of all elements with compact support.

If $M$ has infinite $\Omega$-volume we have a new different type of normal subgroup, that is the subgroup of all elements of $\text{Diff}^\Omega_0(M)$ whose set of non-fixed points near $\partial_J \tilde{M}$ has finite $\Omega$-volume, we denote it by $W^\Omega_J$.

Some properties related with these subgroups and its intersections with the kernel of the flux homomorphism can be found in [1] and [2].

These subgroups will play an important role in the classification of all the normal subgroups of $\text{Diff}^\Omega_0(M)$. One of the main results of this paper is the following:
Corollary 2.9. For each non-trivial normal subgroup of \( \text{Diff}^0_0(M) \), \( N \), there are unique disjoint subsets \( J, J' \subset K \) such that \( N \subset G^0_0 \cap W^0_0 \). Moreover, \( N \) contains all the elements of \( \text{Diff}^0_0(M) \) with support in any locally finite disjoint union of cells included in some neighbourhood of the connected component \( \partial_i M \), for any \( i \in K - (J \cup J') \).

The final result of this paper is a refinement of Corollary 2.9.

Finally, in Section 3 we discuss some examples. In the case \( M = \mathbb{R}^n \), we get the same results that in [10]. We study also the case \( M = X \times \mathbb{R}^+ \), where \( X \) is a closed connected manifold, although \( M \) is not diffeomorphic to the interior of any compact manifold, we can apply the same techniques that above with slight modifications.

1. Study of the group \( \text{Diff}^0_0(X \times \mathbb{R}^+, \text{rel } X \times \{0\}) \)

In order to prove that any element \( f \in \text{Diff}^0_0(M) \), such that \( f_t \) is in the kernel of the flux homomorphism, could be decomposed as a finite product \( f = f_1 \circ \ldots \circ f_{k+1} \) where \( f_{k+1} \in \text{Diff}^0_0(M) \); and \( f_i \in \text{Diff}^0_0(M) \) has support in the interior of a subset diffeomorphic to \( \partial_i M \times \mathbb{R}^+ \), for each \( i \leq k \); we must consider a collar neighbourhood of \( \partial M \), this is an embedding \( \partial_i M \times [0,1] \subset \tilde{M} \), with \( \partial_i M \times \{1\} \) identified with \( \partial M \). Let \( M_0 \) be the compact submanifold \( \tilde{M} - \partial i \tilde{M} \times (0,1) \). (See Figure 1).

If \( J \) is any subset of \( K \), and \( J' = K - J \), for each \( \lambda \in [0,1) \), we denote by

\[
M_\lambda(J) = M_0 \cup \partial J \tilde{M} \times [0,\lambda] \cup \partial J' \tilde{M} \times [0,1) = M - \partial J \tilde{M} \times (\lambda,1).
\]

Let us consider the family of embeddings \( f_t| : M_0 \to M \), since \( M_0 \) is compact, there is
some $0 < \mu < 1$ such that the image of $M_0$ by the isotopy is included in

$$M' = M_0 \cup \bigcup_{i=1}^{k} \partial_i M \times [0, \mu).$$

Since $f_t \in \ker \phi$, there is no obstruction to extend $f_t : M_0 \to M'$ to a volume preserving isotopy that is the identity near $\partial_i M \times \{\mu\} ([1, 2])$. Then, there is a compactly supported $\Omega$-isotopy $g_t : M \to M$ such that for any $t \in [0, 1]$, $g_t$ agrees with $f_t$ on $M_0$, and we define $f_{k+1} = g_1$.

The element $f \circ (f_{k+1})^{-1}$ has support in $\partial M \times (0, 1) = \bigcup_{i \in K} \partial_i M \times (0, 1)$ and it is isotopic to the identity. So, because $f \circ (f_{k+1})^{-1}$ does not permute the components $\partial_i M$ we have that $f \circ (f_{k+1})^{-1} = f_1 \circ \ldots \circ f_k$ where $f_i \in \text{Diff}_0^\Omega(M)$ and $\text{supp}(f) \subseteq \partial_i M \times (0, 1)$, for each $i \leq k$.

Therefore, we start the study of $\text{Diff}_0^\Omega(X \times \mathbb{R}^+, \text{rel} X \times \{0\})$ giving the technical results needed to prove the main theorems on normal subgroups of the group of volume preserving diffeomorphisms of an open manifold. Many of those results are generalizations of similar ones of [10] for the group of volume preserving diffeomorphisms of $\mathbb{R}^n$, and they can be proved using the same techniques.

Such techniques are based on the decomposition of the $\Omega$-preserving diffeomorphisms in a finite product of $\Omega$-preserving diffeomorphisms with support in strips. Though we will be interested in the study of normal subgroups generated by diffeomorphisms with support in locally finite union of disjoint cells, the main reason for consider strips sometimes in our work is that two strips with the same $\Omega$-volume are diffeomorphic by an element of $\text{Diff}_0^\Omega(X \times \mathbb{R}^+, \text{rel} X \times \{0\})$ while the same result is not true for locally finite union of disjoint cells.

First of all, we will give some definitions and properties about strips.

**Definition 1.1.** Let $D$ be a cell in $X$. A **strip** is the image of $D \times \mathbb{R}^+$ under some element $g \in \text{Diff}_0^\Omega(X \times \mathbb{R}^+, \text{rel} X \times \{0\})$ such that for some point $z \in \text{int} D$, the projection $X \times \mathbb{R}^+ \to \mathbb{R}^+$ maps $g(\{z\} \times \mathbb{R}^+)$ diffeomorphically onto $\mathbb{R}^+$.

Notice that every strip does not have infinite $\Omega$-volume necessarily, even in the case that $\text{vol}_{\Omega}(X \times \mathbb{R}^+) = \infty$. From [8, Lemma A.2], and [8, Lemma 1.41], we get the following Lemma.

**Lemma 1.2.** Let $V_1$ and $V_2$ be two strips with the same $\Omega$-volume, and such that

$$\text{vol}_{\Omega}(X \times \mathbb{R}^+ - V_1) = \text{vol}_{\Omega}(X \times \mathbb{R}^+ - V_2) \quad \text{if} \quad \text{vol}_{\Omega} V_1 = \text{vol}_{\Omega} V_2 = \infty.$$

Then there is a diffeomorphism $h \in \text{Diff}_0^\Omega(X \times \mathbb{R}^+, \text{rel} X \times \{0\})$ such that $h(V_1) = V_2$.

For any subset $S \subseteq X \times \mathbb{R}^+$, we denote by $G_S$ the subgroup of all elements of $\text{Diff}_0^\Omega(X \times \mathbb{R}^+, \text{rel} X \times \{0\})$ with support in the interior of $S$.

Since it is not always a normal subgroup, we denote by $N(G_S)$ the normal subgroup of $\text{Diff}_0^\Omega(X \times \mathbb{R}^+, \text{rel} X \times \{0\})$ generated by $G_S$. 
Corollary 1.3. Let $V$ be a strip, and $f$ be an element of $\text{Diff}^\Omega(X \times \mathbb{R}^+ \times \mathbb{R}, \text{rel} X \times \{0\})$ with support in some strip $V'$ such that $\text{vol}_\Omega V' \leq \text{vol}_\Omega V$, and if $\text{vol}_\Omega V' = \text{vol}_\Omega V = \infty$ then $\text{vol}_\Omega((X \times \mathbb{R}^+) - V) = \text{vol}_\Omega((X \times \mathbb{R}^+) - V')$. Then, $f$ is in $N(G_V)$.

**Proof.** By Lemma 1.2, there is an element $h$ of $\text{Diff}^\Omega(X \times \mathbb{R}^+ \times \mathbb{R}, \text{rel} X \times \{0\})$ such that $h(V') \subset V$. So $h \circ f \circ h^{-1}$ belongs to $G_V$, and then $f \in N(G_V)$. □

The following three lemmas are generalizations of Lemmas 2.4, 2.5 and 2.6 from [10], respectively, and their proofs follow them closely.

**Lemma 1.4.** Let $V$ be a strip in $X \times \mathbb{R}^+$. Then $G_V$ is perfect.

**Lemma 1.5.** Let $f$ be an element of $\text{Diff}^\Omega(X \times \mathbb{R}^+ \times \mathbb{R}, \text{rel} X \times \{0\})$, and let $S$ be a subset of $X \times \mathbb{R}^+$, such that:

a) $f(S) \cap S = \emptyset$.

b) There exists an element $h \in \text{Diff}^\Omega(X \times \mathbb{R}^+ \times \mathbb{R}, \text{rel} X \times \{0\})$ such that $h(S) \cap S = \emptyset$ and $h(S) \cap f(S) = \emptyset$.

Then, the derived subgroup $[G_S, G_S]$ of $G_S$ is contained in the normal subgroup of $\text{Diff}^\Omega(X \times \mathbb{R}^+ \times \mathbb{R}, \text{rel} X \times \{0\})$ generated by $f$, $N(f)$.

**Lemma 1.6.** Let $f$ be an element of $\text{Diff}^\Omega(X \times \mathbb{R}^+ \times \mathbb{R}, \text{rel} X \times \{0\})$ such that there is a locally finite union of disjoint cells, $\{C_n\}_{n \geq 1}$, satisfying:

a) $\left( \bigcup_{n \geq 1} C_n \right) \cap \left( \bigcup_{n \geq 1} f(C_n) \right) = \emptyset$.

b) $\text{vol}_\Omega((X \times \mathbb{R}^+) - \bigcup_{n \geq 1} C_n) = \infty$ if $\text{vol}_\Omega(X \times \mathbb{R}^+) = \infty$,

and

$\text{vol}_\Omega\left( \bigcup_{n \geq 1} C_n \right) < \frac{1}{4} \text{vol}_\Omega((X \times \mathbb{R}^+) \text{ if } \text{vol}_\Omega((X \times \mathbb{R}^+) < \infty}$.

Then there is a strip, $V$, such that $\bigcup_{n \geq 1} C_n \subset \text{int} V$, and a diffeomorphism $f' \in N(f)$ such that $f'(V) \cap V = \emptyset$.

Let us suppose that a volume preserving diffeomorphism, $f$, has support of finite volume included in some strip of infinite volume, then we will prove that $f$ could be decomposed in a finite product of volume preserving diffeomorphisms with support in strips of finite volume. Moreover, we could get the volume of such strips be as small as we like.

**Lemma 1.7.** Let $f$ be an element of $\text{Diff}^\Omega(X \times \mathbb{R}^+ \times \mathbb{R}, \text{rel} X \times \{0\})$ with support of finite volume, included in the interior of some strip $V$, such that $\text{vol}_\Omega V = \infty$. Then, we can decompose $f$ as $f = f_1 \circ \ldots \circ f_4$, where each $f_i$ is a volume preserving diffeomorphism, and it has support in a strip $V_i$ of finite volume, for $i = 1, \ldots, 4$. 
Proof. Let us suppose that $V = \epsilon(T)$, for some embedding $\epsilon: T \rightarrow X \times \mathbb{R}^+$ of the standard tube $T$ of $\mathbb{R}^{n+1}$. Since $\text{supp}(f) \subseteq \text{int}V$, we define $f' = \epsilon^{-1} \circ f \circ \epsilon$.

Let us consider the volume element on $T$ given by $\Omega' = \epsilon^* \Omega$. It is easy to see that $f'$ preserves $\Omega'$. Since $\text{vol}_\Omega T = \text{vol}_\Omega \epsilon(T) = \infty$, by [10, Lemma 4.41], we get the following decomposition

$$f' = f_1' \circ \ldots \circ f_4' \quad \text{with} \quad \text{supp}(f_i') \subseteq V_i' \subset \text{int}T$$

where each $f_i'$ preserves $\Omega'$, and $V_i'$ is a strip in $\mathbb{R}^{n+1}$ with finite $\Omega'$-volume.

For each $i = 1, \ldots, 4$ we consider the diffeomorphism $f_i$ defined by $\epsilon \circ f_i' \circ \epsilon^{-1}$ on $V_i$, and equal to the identity elsewhere. So, all of them preserve $\Omega$ and the support of $f_i$ is included in the interior of $\epsilon(V_i')$, for any $i$. Moreover,

$$\text{vol}_\Omega(\epsilon(V_i')) = \text{vol}_{\Omega'} V_i' < \infty.$$ 

And, $f = \epsilon \circ f' \circ \epsilon^{-1} = f_1 \circ \ldots \circ f_4$. □

Lemma 1.8. Let $f$ be an element of $\text{Diff}^\Omega(X \times \mathbb{R}^+, \text{rel} X \times \{0\})$ $\Omega$-isotopic to the identity by an $\Omega$-isotopy $(f_t)$ with support in some strip of finite volume, $V$. Then, for any $\epsilon > 0$, $f$ can be decomposed as $f = f_1 \circ \ldots \circ f_m$, where, for any $i = 1, \ldots, m$, $f_i \in \text{Diff}^\Omega(X \times \mathbb{R}^+, \text{rel} X \times \{0\})$, and it has support in a strip $V_i$ of volume less than $\epsilon$.

Proof. Choose an arbitrary $\lambda > 0$, such that

$$\text{vol}_\Omega(V - (\text{supp}(f) \cap X \times [0, \lambda])) < \epsilon/2.$$ 

Then, there is some $\mu > 0$ such that $f_t(X \times [0, \lambda]) \subseteq (X \times [0, \mu]) \cap V$ for any $t \in [0, 1]$.

Thus, by [5], we can get an element $f_1 \in \text{Diff}^\Omega(X \times \mathbb{R}^+, \text{rel} X \times \{0\})$ equals to $f$ on $X \times [0, \lambda]$ and with support in $X \times [0, \mu] \cap V$.

Since the set $X \times [0, \mu] \cap V$ is a cell, applying [8, Lemma 2], we can decompose $f_1$ as a finite product of volume preserving diffeomorphisms with support in a cell of $\Omega$-volume less than $\epsilon$.

For each cell we can get a strip which contains the cell and with volume as near to the volume of the cell as we like, thus we get the decomposition of $f_1$.

Finally, since $f_2 = (f_1)^{-1} \circ f$ has support in the strip $V - X \times [0, \lambda]$ of volume less than $\epsilon$, then the proof is finished. □

2. The lattice of normal subgroups of $\text{Diff}_0^\Omega(M)$

In this section, one of the main tools is the decomposition of elements of $\text{Diff}_0^\Omega(M)$ in a finite product of volume preserving diffeomorphisms with support in a locally finite union of disjoint cells. We know (See [13, 1]) that such decompositions are not always possible for any element of $\text{Diff}_0^\Omega(M)$. So, it will be very useful to consider the following normal subgroups.
Definition 2.1. For each $i \in K$, let $H'_i$ be the subgroup of $\text{Diff}^\Omega_0(M)$ generated by all the elements with support in some locally finite family of disjoint cells contained in a neighbourhood of $\partial_i \bar{M}$.

For each subset $J \subset K$, we will denote by $H_J$ the product of the subgroups $H'_i$ for $i \in K - J$.

Since the derived subgroup $[\text{Diff}^\Omega_0(M), \text{Diff}^\Omega_0(M)]$ is generated by the elements of $\text{Diff}^\Omega_0(M)$ with support in cells $[1]$, then it is contained in $H_\emptyset$.

Lemma 2.2. Let $f$ be an element of $\text{Diff}^\Omega_0(M)$ such that it is not the identity near $\partial_i \bar{M}$ for some $i \in K$. Then, there is a locally finite union of disjoint cells, $\{C_n\}_{n \geq 1}$, satisfying:

a) $\bigcup_{n \geq 1} C_n$ and $\bigcup_{n \geq 1} f(C_n)$ are contained in $\partial_i \bar{M} \times (0,1)$

b) $(\bigcup_{n \geq 1} C_n) \cap (\bigcup_{n \geq 1} f(C_n)) = \emptyset$.

Proof. Since $f$ does not permute the boundary components of $\bar{M}$ we can choose some $\lambda \in (0,1)$ such that $f^{-1}(\partial_i \bar{M} \times (0,1)) \supset \partial_i \bar{M} \times (\lambda,1)$. And we choose a locally finite set of points of $\partial_i \bar{M} \times (\lambda,1)$, $\{z_n\}_{n \geq 1}$, such that $f(z_n) \neq z_m$, for any $n, m$. Then, we construct cells $C_n \subset \partial_i \bar{M} \times (\lambda,1)$ satisfying $C_n \cap f(C_m) = \emptyset$ for any $n, m$. \[\square\]

Given a volume element $\Omega$ on $M$, we denote by $K_1$ the subset of all indices $i \in K$ such that the end $\partial_i$ of $M$ has finite $\Omega$-volume. And let $K_2$ be the subset $K - K_1$.

As a consequence of the following lemma, we get that for each end of finite $\Omega$-volume, the normal subgroups of Definition 2.1 are the least normal subgroups of $\text{Diff}^\Omega_0(M)$ with non-compactly supported diffeomorphisms.

Lemma 2.3. Let $N$ be a non-trivial normal subgroup of $\text{Diff}^\Omega_0(M)$ which is contained in $H'_i$, for some $i \in K_1$. Then, $N$ must be the whole $H'_i$ or $N \subset G^\Omega_{K_1} \cap H'_i$.

Proof. Let us suppose that $N$ is not contained in $G^\Omega_{K_1}$. Then there is an element $f \in N - G^\Omega_{K_1}$. Since $N$ is included in $H'_i$, then $f$ it is not the identity near $\partial_i \bar{M}$.

It is easy to see, by Lemma 2.2, that there is a locally finite family of disjoint cells $\{C_n\}_{n \geq 1}$ in $\partial_i \bar{M} \times (0,1)$, such that

$$\left(\bigcup_{n \geq 1} C_n\right) \cap \left(\bigcup_{n \geq 1} f(C_n)\right) = \emptyset.$$

Then, by Lemma 1.6, we could get a strip $V$ in $\partial_i \bar{M} \times (0,1)$, containing $\bigcup_{n \geq 1} C_n$ in its interior and an element $f' \in N(f)$ such that $f'(V) \cap V = \emptyset$, and

$$\text{vol}_\Omega V < \frac{1}{4} \text{vol}_\Omega(\partial_i \bar{M} \times (0,1)).$$

Let $g$ be any element of $H'_i$. Then, we can decompose $g$ in a finite product $g = g_1 \circ \ldots \circ g_m$ where $\text{supp}(g_j) \subset \bigcup_{n} C_{n j} \subset V_j \subset \partial_i \bar{M} \times (0,1)$ for any $j$. And we could assume without loss of generality that the strip $V_j$ satisfies that

$$\text{vol}_\Omega V_j < \text{vol}_\Omega V.$$
Then, by Corollary 1.3, each element \( g_j \) belongs to \( N(G_V) \). So, if we prove that \( N(G_V) \subset N(f) \), then we will get the result.

Since \( \text{vol}_h V < \frac{1}{4} \text{vol}_h (\partial_t \tilde{M} \times (0,1)) \), we have enough room to construct a new strip \( V' \subset \partial_t \tilde{M} \times (0,1) \) such that

\[
\text{vol}_h V' = \text{vol}_h V, \quad V' \cap V = \emptyset \quad \text{and} \quad V' \cap f(V') = \emptyset.
\]

Thus, by Lemma 1.2, there is an element \( h \in \text{Diff}_0^\infty(M) \) with support in \( \partial_t \tilde{M} \times (0,1) \), and such that \( h(V) = V' \).

Finally, since \( G_V \) is a perfect subgroup (Lemma 1.4) and using Lemma 1.5, we get

\[
G_V = [G_V, G_V] \subset N(f') \subset N(f) \subset N.
\]

The following theorem is a consequence of Lemma 2.3 and gives us a first classification of the normal subgroups respect to the finite ends of \( M \).

**Theorem 2.4.** Let \( N \) be a non-trivial normal subgroup of \( \text{Diff}_0^\infty(M) \). Then, there is an unique subset \( J_1 \subset K_1 \) such that

\[
\Pi\{H'_i : i \in K_1 - J_1\} \subset N \subset G_{J_1}^\infty.
\]

**Proof.** Let \( J_1 \) be the subset \( \{i \in K_1 : N \cap H'_i \subset G_{J_1}^\infty\} \), it may be empty. Then, \( N \) is contained in \( G_{J_1}^\infty \). If we suppose that there is an element \( f \in N - G_{J_1}^\infty \). Then, for some \( j \in J_1 \), we can construct a locally finite family of disjoint cells \( \{C_n\}_{n \geq 1} \) in \( \partial_j \tilde{M} \times (0,1) \), such that

\[
\left( \bigcup_{n \geq 1} C_n \right) \cap \left( \bigcup_{n \geq 1} f(C_n) \right) = \emptyset.
\]

Let \( g \) be an element of \( \text{Diff}_0^\infty(M) \) with non-compact support contained in \( \bigcup_{n \geq 1} C_n \). Then, \([f, g]\) is an element of \( N \), equals to

\[
[f, g] = \begin{cases} 
  g^{-1} & \text{in } \bigcup_n C_n \\
  f \circ g \circ f^{-1} & \text{in } \bigcup_n f(C_n) \\
  \text{Identity} & \text{elsewhere}.
\end{cases}
\]

Therefore, \([f, g] \in N \cap H'_j \) and it has non-compact support, which contradicts the definition of \( J_1 \). So, \( N \subset G_{J_1}^\infty \).

On the other hand, \( N \cap H'_j \not\subset G_{K_1}^\infty \), for each \( j \in K_1 - J_1 \). Therefore, by the previous lemma, \( N \cap H'_j = H'_j \), and so that \( H'_j \subset N \), for each \( j \in K_1 - J_1 \).

Finally, we will prove that there is only one subset \( J \) of \( K_1 \) such that

\[
\Pi\{H'_i : i \in K_1 - J\} \subset N \subset G_{J_1}^\infty.
\]

Note that the first inclusion implies that \( H'_i \subset N \), for each \( i \in K_1 - J \). While the second one implies that \( H'_i \not\subset N \) for each \( i \in J \). So

\[
J = \{i \in K_1 : H'_i \not\subset N\} = \{i \in K_1 : N \cap H'_i \subset G_{K_1}^\infty\} = J_1.
\]

\( \square \)
Now, we will study the normal subgroups of $\text{Diff}^0(M)$ if there exist some end of $M$ with infinite $\Omega$-volume.

First of all, we get a generalization of Lemma 2.2 when the set of non-fixed points of a diffeomorphism with support in $\partial_i \bar{M} \times (0,1)$ has infinite $\Omega$-volume.

Let $f$ be an element of $\text{Diff}^0(M)$, we denote by $W_f$ the following subset of $M$: $W_f = \{x \in M : f(x) \neq x\}$. By a slight modification of [10, Lemma 4.1] we have the following.

**Lemma 2.5.** Let $f$ be an element of $\text{Diff}^0(M)$ and let $A$ be any open subset of $W_f \cap \partial_i \bar{M} \times (0,1)$ with compact closure. Then, there is a finite number of disjoint cells $C_1, \ldots, C_m$ included in $A$, satisfying:

a) $(\bigcup_{j=1}^m C_j) \cap (\bigcup_{j=1}^m f(C_j)) = \emptyset$.

b) $\sum_{j=1}^m \text{vol}_\Omega C_j > \frac{1}{16} \text{vol}_\Omega A$.

**Lemma 2.6.** Let $f$ be an element of $\text{Diff}^0(M)$ such that $\text{vol}_\Omega(W_f \cap \partial_i \bar{M} \times (0,1)) = \infty$ for some $i \in K$. Then, there is a locally finite family of disjoint cells, $\{D_n\}_{n \geq 1}$, such that

a) $\bigcup_{n \geq 1} D_n$ and $\bigcup_{n \geq 1} f(D_n)$ are contained in $\partial_i \bar{M} \times (0,1)$

b) $\sum_{n \geq 1} \text{vol}_\Omega D_n = \infty$ and $(\bigcup_{n \geq 1} D_n) \cap (\bigcup_{n \geq 1} f(D_n)) = \emptyset$.

**Proof.** Inductively we get a family of open subsets of $W_f \cap \partial_i \bar{M} \times (0,1)$, $\{A_j\}_{j \geq 1}$, with compact closure and satisfying

\[
\text{vol}_\Omega A_i - i, \quad A_i \cap A_j = \emptyset, \quad A_i \cap f(A_j) = \emptyset, \quad A_i \cap f^{-1}(A_j) = \emptyset \quad \text{if } i \neq j.
\]

Now, applying Lemma 2.5 to each $A_j$, we get a locally finite family of disjoint cells, $\{D_n\}_{n \geq 1}$, satisfying

\[
\left(\bigcup_{n \geq 1} D_n\right) \cap \left(\bigcup_{n \geq 1} f(D_n)\right) = \emptyset.
\]

and

\[
\sum_{n \geq 1} \text{vol}_\Omega D_n \geq \frac{1}{16} \sum_{j \geq 1} \text{vol}_\Omega A_j = \infty.
\]

Now, we prove that for each end $\partial_i$ of infinite $\Omega$-volume the subgroups $H_i^J$ are the least normal subgroups with diffeomorphisms whose set of non-fixed points has infinite volume.

For each $J \subset K_2$, let $W_J^\Omega$ be the subgroup of all elements of $\text{Diff}^\Omega(M)$ whose set of non-fixed points near $\partial J \bar{M}$ has finite $\Omega$-volume.

**Lemma 2.7.** Let $N$ be a non-trivial normal subgroup of $\text{Diff}^\Omega(M)$ which is contained in $H_i^J$, for some $i \in K_2$. Then $N$ must be the whole $H_i^J$, or $N$ is contained in $W_J^\Omega \cap H_i^J$.

**Proof.** Let us suppose that there is an element $f \in N - W_J^\Omega$. So, we have that the
support of \( f \) is in \( \partial_i \overline{M} \times (0,1) \), and \( f \) moves an infinite \( \Omega \)-volume set of points of \( \partial_i \overline{M} \times (0,1) \).

Then, by Lemma 2.6 we can find, in \( \partial_i \overline{M} \times (0,1) \), a locally finite family of disjoint cells, \( \{D_n\}_{n \geq 1} \), such that

\[
\left( \bigcup_{n \geq 1} D_n \right) \cap \left( \bigcup_{n \geq 1} f(D_n) \right) = \emptyset \quad \text{and} \quad \sum_{n \geq 1} \text{vol}_\Omega D_n = \infty.
\]

Applying Lemma 1.6, we get a strip \( V \) with infinite \( \Omega \)-volume, such that it contains \( \{D_n\}_{n \geq 1} \) in its interior, and a diffeomorphism \( f' \in N(f) \) such that \( f'(V) \cap V = \emptyset \) and \( \text{vol}_\Omega(\partial_i \overline{M} \times (0,1) - [V \cup f'(V)]) = \infty \).

Let \( g \) be any element of \( H'_2 \), then \( g \) is decomposed in a finite product \( g_1 \circ \ldots \circ g_m \) with \( \text{supp}(g_k) \subset \bigcup j C_{kj} \subset V_k \). Without loss of generality we could assume that the strip \( V_k \) satisfies that

\[
\text{vol}_\Omega(\partial_i \overline{M} \times (0,1) - V_k) = \infty.
\]

So, by Corollary 1.3, we know that \( g_k \in N(G_V) \) for each \( k \).

Finally, we prove that \( N(G_V) \subset N(f) \). Since \( \text{vol}_\Omega(\partial_i \overline{M} \times (0,1) - [V \cup f'(V)]) = \infty \), we can find a strip \( V' \) in \( \partial_i \overline{M} \times (0,1) - [V \cup f'(V)] \) of infinite \( \Omega \)-volume, and by Lemma 1.2 we could get an element \( h \in \text{Diff}_G^\Omega(M) \) such that \( h(V) = V' \). So, using a similar method to the one used in Lemma 2.3, we get

\[
[G_V, G_V] \subset N(f') \subset N(f).
\]

Since \( G_V \) is a perfect subgroup, we have proved the result. \( \square \)

The last lemma gives us the following classification of the normal subgroups respect to the ends of infinite volume.

**Theorem 2.8.** Let \( N \) be a non-trivial normal subgroup of \( \text{Diff}_G^\Omega(M) \). Then, there is a unique subset \( J_2 \subset K_2 \) such that

\[
\Pi \{H'_i : i \in K_2 - J_2 \} \subset N \subset W_{J_2}^\Omega.
\]

**Proof.** Let \( J_2 \) be the subset \( \{i \in K_2 : N \cap H'_i \subset W_{J_2}^\Omega \} \). Let us suppose that there is an element \( f \in N - W_{J_2}^\Omega \). Thus, we have that \( \text{vol}_\Omega(W_f \cap \partial_j \overline{M} \times (0,1)) = \infty \), for some \( j \in J_2 \).

We can choose a locally finite family of disjoint cells such that

\[
\bigcup_{n \geq 1} D_n \subset \partial_j \overline{M} \times (0,1), \quad \sum_{n \geq 1} \text{vol}_\Omega D_n = \infty,
\]

and

\[
\left( \bigcup_{n \geq 1} D_n \right) \cap \left( \bigcup_{n \geq 1} f(D_n) \right) = \emptyset.
\]

Let \( g \) be an element of \( \text{Diff}_G^\Omega(M) \), which support is in \( \bigcup_{n \geq 1} D_n \). Then, as in Theorem 2.4 we get that \([f, g] \in N \cap H'_i \). Therefore, if we choose a suitable \( g \) such that
vol_\Omega(W_{\{f,g\}}) = \infty$, then $[f,g]$ is an element of $N \cap H_i'$ that not belongs to $W_\Omega^\Omega_{K_2}$, which contradicts the definition of $J_2$. So $N \subset W_\Omega^\Omega_{J_2}$.

On the other hand, if $i \in K_2 - J_2$, by Lemma 2.7, we have that $N \cap H_i' = H_i'$, so that $H_i' \subset N$.

Finally, as in Theorem 2.4, we prove that there is only one subset $J$ of $K_2$ such that

$$\Pi\{H_i' : i \in K_2 - J\} \subset N \subset W_\Omega^\Omega_{J_2}.$$ 

From the first inclusion, it follows that $H_i' \subset N$ for each $i \in K_2 - J$. While the second one implies that $H_i' \not\subset N$ for each $i \in J$. So

$$J = \{i \in K_2 : H_i' \not\subset N\} = \{i \in K_2 : N \cap H_i' \subset W_\Omega^\Omega_{K_2}\}. \quad \square$$

From Theorems 2.4 and 2.8, we get the following result in the case that $\Omega$ is a volume element on $M$ of infinite total volume.

**Corollary 2.9.** For each non-trivial normal subgroup $N$ of $\text{Diff}_0^\Omega(M)$, there are unique subsets $J_1 \subset K_1$ and $J_2 \subset K_2$ such that

$$H_{J_1 \cup J_2} = \Pi\{H_i' : i \in K - (J_1 \cup J_2)\} \subset N \subset G_\Omega^\Omega_{J_1} \cap W_\Omega^\Omega_{J_2}.$$ 

Now, we are going to improve the last result. First of all, we will introduce a new type of normal subgroup.

For each $i \in K_2$, we will denote by $F_i'$ the normal subgroup generated by the following subset

$$\left\{ h \in \text{Diff}_0^\Omega(M) : \text{supp}(h) \subset \bigcup_n C_n \subset \partial_i \bar{M} \times (0,1), \text{ and } \text{vol}_\Omega(\text{supp}(h)) < \infty \right\}.$$

Obviously, this new subgroup is contained in $H_i'$, for each $i \in K_2$. And we will denote by $F_J'$ the product $\Pi\{F_i' : i \in J\}$.

As a consequence of the following Lemma we will get that the subgroups $F_i'$ are the least normal subgroups which contains diffeomorphisms with non-compact support of finite volume.

For each non-trivial normal subgroup $N$ of $\text{Diff}_0^\Omega(M)$, let $\Lambda_N$ denote the subset $\{i \in J_2 : N \cap H_i' \subset F_i'\}$. Then, we have the following result.

**Lemma 2.10.** For each $i \in \Lambda_N$, then, $N \cap H_i' = F_i'$ or $N \cap H_i' \subset G_\Omega^\Omega_K$.

**Proof.** It follows closely the proof of Lemma 2.3.

Suppose that for some $i \in \Lambda_N$ we have that $N \cap H_i' \not\subset G_\Omega^\Omega_K$. Then, there must be an element $f \in N \cap H_i'$ such that $f$ is not equal to the identity near $\partial_i \bar{M}$. Since the support of $f$ has finite $\Omega$-volume, there is a locally finite family of disjoint cells $\{C_n\}_{n \geq 1}$ such that

$$\bigcup_{n \geq 1} C_n \subset \partial_i \bar{M} \times (0,1), \quad \left( \bigcup_{n \geq 1} C_n \right) \cap \left( \bigcup_{n \geq 1} f(C_n) \right) = \emptyset.$$
and 
\[ \sum_{n \geq 1} \text{vol}_\Omega C_n < \infty. \]

Then, as in Lemma 2.3, we could find a strip $V$ of finite $\Omega$-volume and a diffeomorphism $f' \in N(f)$ such that $f'(V) \cap V = \emptyset$.

Every element $g \in F_1'$, can be decomposed in a finite product of volume preserving diffeomorphisms $g_1 \circ \ldots \circ g_m$ with $\text{supp}(g_k) \subset \bigcup_j C_{kj} \subset V_k \subset \partial_i \tilde{M} \times (0,1)$ and $\text{vol}_\Omega(\text{supp}(g_k)) < \infty$.

By Lemmas 1.7 and 1.8, we could assume without loss of generality that for each $k$, the $\Omega$-volume of the strips $V_k$ is less or equal than $\text{vol}_\Omega V$. So, each diffeomorphism $g_k$ lies in $N(G_V)$; therefore $g \in N(G_V)$.

The inclusion $N(G_V) \subset N(f)$ can be proved as in Lemma 2.3. □

Applying the above lemma, we get the following result.

**Theorem 2.11.** Let $N$ be a non-trivial normal subgroup of $\text{Diff}_0^\Omega(M)$. Then, there is a unique subset $J_3$ of $\Lambda_N$, such that

\[ F_{\Lambda_N-J_3}' = \Pi \{ F_i' : i \in \Lambda_N - J_3 \} \subset N \subset G_{J_3}. \]

**Proof.** The result follows by taking $J_3$ as the set \{ $i \in \Lambda_N : N \cap H_i' \subset G_{J_3}$ \}, and repeating the same argument as in Theorem 2.4. □

Thurston in [13] proved that the derived subgroup $[\text{Diff}_0^\Omega(M), \text{Diff}_0^\Omega(M)]$ is the smallest non-trivial normal subgroup of $\text{Diff}_0^\Omega(M)$. So, from this result, Corollary 2.9 and Theorem 2.11, we get the following theorem, which resumes the main results given up to now.

**Theorem 2.12.** For each non-trivial normal subgroup $N$ of $\text{Diff}_0^\Omega(M)$, there are unique subsets

\[ J_1 \subset K_1 \text{ and } J_3 \subset \Lambda_N \subset J_2 \subset K_2 \]

such that:

\[ [\text{Diff}_\text{co}^\Omega(M), \text{Diff}_\text{co}^\Omega(M)] \cdot H_{J_1 \cup J_2} \cdot F_{\Lambda_N-J_3}' \subset N \subset G_{J_1 \cup J_3} \cap W_{J_3}^\Omega. \]

**Proof.** For each normal subgroup $N$, we can define the following subsets:

\[ J_1 = \{ i \in K_1 : N \cap H_i' \subset G_{J_1}^\Omega \}, \quad J_2 = \{ i \in K_2 : N \cap H_i' \subset W_{K_2}^\Omega \}, \]

\[ \Lambda_N = \{ i \in J_2 : N \cap H_i' \subset F_i' \}, \quad J_3 = \{ i \in \Lambda_N : N \cap H_i' \subset G_{J_3}^\Omega \}. \]

From Corollary 2.9 and the result from Thurston [13], we get the following result

\[ [\text{Diff}_\text{co}^\Omega(M), \text{Diff}_\text{co}^\Omega(M)] \cdot H_{J_1 \cup J_2} \subset N \subset G_{J_1}^\Omega \cap W_{J_2}^\Omega; \]

and from Theorem 2.11, it follows

\[ [\text{Diff}_\text{co}^\Omega(M), \text{Diff}_\text{co}^\Omega(M)] \cdot H_{J_1 \cup J_2} \cdot F_{\Lambda_N-J_3}' \subset N \subset G_{J_1 \cup J_3} \cap W_{J_3}^\Omega. \quad \Box \]
3. Examples

First we will make some considerations for the case $M = \mathbb{R}^n, n \geq 3$. Since $M$ has only one end, there are only two different cases to discuss, namely $\text{vol}_\Omega(\mathbb{R}^n) < \infty$ and $\text{vol}_\Omega(\mathbb{R}^n) = \infty$.

In both cases, using the results on decomposition of $\Omega$-preserving diffeomorphisms in a finite product of $\Omega$-preserving diffeomorphisms with support in strips [10], and applying Theorem 2.12, we get, using the same notation of [10], the following chain of normal subgroups if $\text{vol}_\Omega(\mathbb{R}^n) < \infty$,

$$\{\text{Id}\} \rightarrow \text{Diff}_0^\Omega(\mathbb{R}^n) \subset \text{Diff}_c^\Omega(\mathbb{R}^n) \rightarrow \text{Diff}^\Omega(\mathbb{R}^n).$$

And in the case that $\text{vol}_\Omega(\mathbb{R}^n) = \infty$ we get that

$$\{\text{Id}\} \rightarrow \text{Diff}_0^\Omega(\mathbb{R}^n) \subset \text{Diff}_c^\Omega(\mathbb{R}^n) \rightarrow \text{Diff}^\Omega(\mathbb{R}^n) \subset \text{Diff}^\Omega_0(\mathbb{R}^n) \rightarrow \text{Diff}^\Omega(\mathbb{R}^n),$$

where "—" means that there is no normal subgroup in between. Those chains are the same that the ones obtained in [10].

Now, we will illustrate the results of Section 2 for $M = X \times \mathbb{R}^+$, where $X$ is a closed connected manifold. Although $M$ is not diffeomorphic to the interior of any compact manifold, we can apply all the preceding results with slight modifications only.

Recall that the group $\text{Diff}^\Omega(X \times \mathbb{R}^+, \text{rel} X \times \{0\})$ is contractible, so any $\Omega$-preserving diffeomorphism is also $\Omega$-isotopic to the identity. On the other hand, because $M$ has only one end, as above there are only two different cases to discuss.

Let $N$ be a non-trivial normal subgroup of $\text{Diff}^\Omega(X \times \mathbb{R}^+, \text{rel} X \times \{0\})$.

Case 1. Case of finite total $\Omega$ volume

1a) Let us suppose that there is some element $f \in N$ with non-compact support. Then, as in Lemma 2.2, we can get a locally finite family of disjoint cells satisfying

$$(\bigcup_{n \geq 1} C_n) \cap (\bigcup_{n \geq 1} f(C_n)) = \emptyset.$$

Let $g$ be a given element of $\text{Diff}^\Omega(X \times \mathbb{R}^+, \text{rel} X \times \{0\})$ with non-compact support in $\bigcup_{n \geq 1} C_n$. So, the element $[f, g] \in N \cap H'_1$ and it has non-compact support.

Following the same arguments as in Lemma 2.3, we get that $H'_1 \subset N$, and then we obtain the following result

$$[\text{Diff}_0^\Omega(M), \text{Diff}_c^\Omega(M)] \cdot H'_1 \subset N \subset \text{Diff}^\Omega(X \times \mathbb{R}^+, \text{rel} X \times \{0\}).$$

1b) If we assume that all the elements of $N$ have compact support, then it follows immediately that

$$[\text{Diff}_0^\Omega(M), \text{Diff}_c^\Omega(M)] \subset N \subset \text{Diff}^\Omega(X \times \mathbb{R}^+, \text{rel} X \times \{0\}).$$
Case 2. Case of infinite total $\Omega$-volume

2a) Let us suppose that there is some element $f \in N$ such that $\text{vol}_{\Omega} W_f = \infty$. Then, as in Lemma 2.6, we construct a locally finite family of disjoint cells such that

$$\left( \bigcup_{n \geq 1} C_n \right) \cap \left( \bigcup_{n \geq 1} f(C_n) \right) = \emptyset \quad \text{and} \quad \sum_{n \geq 1} \text{vol}_{\Omega} C_n = \infty.$$ 

If we consider an element $g \in \text{Diff}^\Omega(X \times \mathbb{R}^+, \text{rel} X \times \{0\})$ with support in $\bigcup_{n \geq 1} C_n$ and such that $\text{vol}_{\Omega} W_g = \infty$, then as above the element $[f, g] \in N \cap H'_1$ and $\text{vol}_{\Omega} W_{[f, g]} = \infty$. So, we have that $N \cap H'_1 = H'_1$, and it follows that

$$[\text{Diff}^\Omega_{co}(M), \text{Diff}^\Omega_{co}(M)] \cdot H'_1 \subset N \subset \text{Diff}^\Omega(X \times \mathbb{R}^+, \text{rel} X \times \{0\}).$$

2b) Let us suppose that for any element $f \in N$ we have that $\text{vol}_{\Omega} W_f < \infty$ and $\text{vol}_{\Omega} \text{supp}(f) < \infty$.

2b1) If there is some element $f \in N$ with non-compact support, then we can construct a locally finite family of disjoint cells such that $\left( \bigcup_{n \geq 1} C_n \right) \cap \left( \bigcup_{n \geq 1} f(C_n) \right) = \emptyset$ and $\sum_{n \geq 1} \text{vol}_{\Omega} C_n < \infty$. Therefore, as in Lemma 2.10, we can prove that $F'_1 \subset N$ and it follows that

$$[\text{Diff}^\Omega_{co}(M), \text{Diff}^\Omega_{co}(M)] \cdot F'_1 \subset N \subset \text{Diff}^\Omega(X \times \mathbb{R}^+, \text{rel} X \times \{0\}).$$

2b2) Any element of $N$ has compact support, then it follows immediately that

$$[\text{Diff}^\Omega_{co}(M), \text{Diff}^\Omega_{co}(M)] \subset N \subset \text{Diff}^\Omega(X \times \mathbb{R}^+, \text{rel} X \times \{0\}).$$

Notice that in Case 2 remains to study when there is some element $f \in N$ such that $\text{vol}_{\Omega} W_f < \infty$ but $\text{vol}_{\Omega} \text{supp}(f) = \infty$ (See [10]). Unfortunately the arguments used in this paper do not work in such case.

References


