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A special case of Hadwiger's conjecture

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Abstract

We investigate Hadwiger's conjecture for graphs with no stable set of size 3. Such a graph on at least $2t-1$ vertices is not $t-1$ colorable, so is conjectured to have a K_t minor. There is a strengthening of Hadwiger's conjecture in this case, which states that there is a K_t minor in which the preimage of each vertex of K_t is a single vertex or an edge. We prove this strengthened version for graphs with an even number of vertices and fractional clique covering number less than 3. We investigate several possible generalizations and obtain counterexamples for some and improved results from others. We also show that for sufficiently large n, a graph on n vertices with no stable set of size 3 has a $K_{\frac{1}{9}n^{4/5}}$ minor using only vertices and single edges as preimages of vertices.

Keywords: Hadwiger's conjecture; Complete minor; Fractional clique covering number; Matching

1. Introduction

A graph H is a minor of a graph G if H can be obtained from a subgraph of G by contracting edges. If E is the set of contracted edges, we call a connected component of the graph $(V(G), E)$ a *prevertex*. Upon contraction of E, each prevertex becomes a vertex of H. One of the most difficult and beautiful problems in graph theory is Hadwiger's conjecture (see e.g. [4]):

Conjecture 1.1. *For each $t \geqslant 0$, every loopless graph with no K_{t+1} minor is t-colorable.*

At present, Hadwiger's conjecture has been proved for $t \leqslant 5$ and is open for all $t \geqslant 6$ [4]. We investigate Hadwiger's conjecture for t asymptotically comparable to the number of vertices

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in the graph. It is thought that if Hadwiger's conjecture is false, this is the most likely place for a counterexample. We restrict our attention to the case where $G = (V, E)$ has no stable set of size 3. This implies that there are at least $|V|/2$ color classes in a proper coloring of G, so Hadwiger's conjecture implies that G has a complete minor of size at least $|V|/2$. This special case is still open. A strengthening conjectured by Seymour is

Conjecture 1.2. *If $G = (V, E)$ has no stable set of size 3, then G has a complete minor of size at least $|V|/2$ using only edges or single vertices as prevertices.*

We call this the SSH conjecture; SS stands for Seymour's strengthening and stable set and H stands for Hadwiger. Hadwiger's conjecture for graphs with no stable set of size 3 as well as SSH were investigated by Plummer, Stiebitz, and Toft in [3]. An *inflation* of a graph is obtained by replacing vertices with complete graphs and edges with complete bipartite graphs. One of their main results states that Hadwiger's conjecture holds for any inflation of a graph with no stable set of size 3 and at most 11 vertices. In fact, their proof shows SSH holds as well, although they do not state this explicitly. In Section 4 we show that if we only consider inflations with an even number of vertices, the above result is a consequence of our main results.

The *clique covering number* of a graph G is the chromatic number of the complement of G. SSH is obviously true for graphs with clique covering number 2, so a natural question is whether it is true for graphs with clique covering number 3. Our main result states that

Theorem 1.3. *SSH is true for graphs with an even number of vertices and fractional clique covering number less than* 3 *and graphs with an even number of vertices and clique covering number* 3.

Incidentally, the question of whether Hadwiger's conjecture holds for graphs with no stable set of size 3 and clique covering number 3 was posed by Andreĭ Kotlov in [2], although his paper was not known to us until after we completed this work. Section 3 describes the proof of our main result and some attempts at generalization. Section 4 shows a neat way of using 2-satisfiability to determine limitations of our method.

In Section 2 we prove that SSH is true for any graph on n vertices with a cutset of size at most $\frac{n}{2}$. This result can also be found in [3]. It is easy, but important, and is used throughout the paper. Also in Section 2, we discuss some weakenings of SSH and related conjectures. We use an elementary counting argument to prove that for sufficiently large n, every graph with no stable set of size 3 has a complete minor of size $\frac{1}{9}n^{4/5}$ using prevertices of size one or two.

We strongly believe SSH is true because many attempts at constructing counterexamples have failed. However, our intuition for graphs with no stable set of size 3 is severely limited. We have much difficulty constructing graphs that are not proven to satisfy SSH by our results—graphs with no stable set of size 3, large connectivity, no dominating edges, and large clique covering number. The only random graphs we can construct with these properties are extremely dense and have large complete minors. Standard constructions of triangle free graphs with large chromatic number do not have complements that fit these requirements either. For example, the triangle free graphs of Mycielski's construction (see e.g. [7, 5.2]) have complements with small cutsets.

Before going any further, we need some notation. If A and B are sets, A *intersects* B means $A \cap B \neq \emptyset$. $[n]$ will denote the set $\{1, 2, \dots, n\}$.

All graphs in this paper are finite. Let G be a graph. We will sometimes write $G = (V, E)$, which means G has vertex set V and edge set E; we will also use $V(G)$ and $E(G)$ for the vertex and edge sets of G. When there is no ambiguity, we use n instead of $|V|$ without saying so explicitly. If $S \subseteq V(G)$, $G[S]$ is the induced subgraph $G\backslash(V(G)-S)$. $\overline{G}$ is the complement of G. $d_G(v)$ is the degree of v in G, and the subscript G will be omitted when there is no ambiguity. We will write (u,v) for an edge with ends u and v, and $u \sim v$ ($u \nsim v$) means edge (u,v) is (is not) present. If U and V are disjoint vertex sets, a (U,V) edge is some edge with one end in U and one end in V; the (U,V) edges is the set of all such edges.

We will say that the vertex sets U and V *touch* if they intersect or there is some edge with an end in each set. We will also speak of two edges touching or an edge and a vertex touching; we just identify the edge (u,v) with the set $\{u,v\}$ and use the notion of touching just mentioned. We say U is *complete* (*anticomplete*) to V if every (no) edge (u,v) $u \in U, v \in V$ is present. If v is a vertex, $N(v)$ will denote its set of neighbors (and will not include v); if V is a vertex set, $N(V) = \bigcup_{v \in V} N(v)$. A *dominating edge* of G is an edge that touches every vertex of G.

Vertices u, v are said to be *twins* if they are nonadjacent and $N(u) = N(v)$. Vertex *duplication* is the action of replacing a vertex by two nonadjacent vertices with the same neighbors as the original. Unfortunately, these are the standard definitions of twins and duplication, but we want a "complementary" definition. We say vertices u, v are *c-twins* if they are adjacent and $N(u) - v = N(v) - u$; c stands for complement and clique. Define *c-duplication* similarly.

An *antitriangle* is a stable set of size 3. Let $\mathfrak{A}$ be the set of graphs with no antitriangle.

2. First observations

A simple but important observation is that a minimal counterexample to SSH has no dominating edges. In fact, we can win in two ways. If G has a dominating edge, $e = (u,v)$, then we can use e as a prevertex together with a minor on $G\backslash\{u,v\}$ found inductively. Or we observe that $G\backslash e \in \mathfrak{A}$, and we can find a complete minor on it by induction on the number of edges.

Another preliminary result, also observed by Plummer, Stiebitz, and Toft in [3], gives a lower bound on the connectivity of a counterexample to SSH.

Lemma 2.1. *If $G = (V,E) \in \mathfrak{A}$ and has a cut set, M, of size at most $\frac{n}{2}$, then SSH holds.*

Proof. Choose M as small as possible. Let L, R be a partition of $V - M$ such that L and R do not touch. G has no antitriangle implies that L and R are cliques and that every vertex in M is either complete to L or complete to R. Let M_L, M_R partition M so that every vertex in M_L (M_R) is complete to L (R). Any $A \subseteq M_L$ of size at most $|R|$ is matchable into R. If not, by Hall's matching condition, there is an $S \subseteq A$ such that $|S| > |N(S) \cap R|$. But then $(M - S) \cup |N(S) \cap R|$ is a cutset because it separates $L \cup S$ and $R - N(S)$; it is smaller than M, contradiction. Now let Y be a matching from M_L to R of size $\min(|M_L|, |R|)$. The vertices of L, together with the edges of Y are the prevertices of a complete minor (any pair of edges in Y is adjacent because they both have an end in the clique R). We can, of course, do the same thing with vertices from R and a matching from M_R to L. So without loss of generality $|L| + |M_L| \geqslant |R| + |M_R|$. The size of the complete minor is $|L| + \min(|M_L|, |R|) = \min(|L| + |M_L|, |L| + |R|) \geqslant \frac{n}{2}$ by the assumption that $|M| \leqslant \frac{n}{2}$. $\square$

At first this result may seem not too helpful, because many of the graphs for which the SSH conjecture is most mysterious have vertex degrees $n - o(n)$ and connectivity $n - o(n)$. Nevertheless, it appears this lemma does away with some pathological cases that would otherwise present problems for a nice proof of the general result. In fact, we conjecture that if G has no cutset

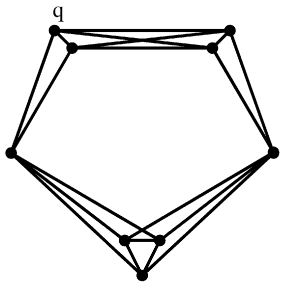

Fig. 1.

of size $\frac{n}{2}$ or smaller and no dominating edge, then G has a minor with any vertex $q \in V$ as a prevertex and all the other prevertices as edges. We can easily find graphs with no such minor, but all of them we found have dominating edges or a small cut set. For example, in the graph in Fig. 1, there is no K_5 minor that uses q as a prevertex.

2.1. *Constant factor weakenings are unsolved*

One approach to Hadwiger's conjecture for graphs with no antitriangle is to try to show there is a complete minor of size cn for some constant $c > 0$, rather than demanding $c = 1/2$. Even this weakening is unsolved for SSH. We present the progress made in this direction, and begin with an instructive result that seems to be part of the folklore of the field. Duchet and Meyniel prove more generally in [1] that a graph with independence number α on n vertices always has a $K_{\lceil n/(2\alpha-1)\rceil}$ minor.

Claim 2.2. *If $G = (V, E) \in \mathfrak{A}$, then G has a $K_{\lceil n/3\rceil}$ minor.*

Proof. We can obtain such a minor using induced paths of length 2 and single vertices as prevertices. If u, v, w are the vertices of an induced path of length 2, then $N(\{u, w\}) = V - \{u, w\}$ because G has no antitriangle. Choose a maximum number of vertex disjoint induced paths of length 2. Let their vertex sets be $Q_1, Q_2, \dots, Q_r$ and let $Q = \bigcup_i Q_i$. In $G \backslash Q$, there are no induced paths of length 2, so being connected by an edge is an equivalence relation. Thus $G \backslash Q$ is the disjoint union of at most two cliques; let C be the largest clique of $G \backslash Q$. $Q_1, Q_2, \dots, Q_r$ and the vertices of C are the prevertices of a $K_{r+|V(C)|}$ minor. $3r + 2|V(C)| \geqslant n$ implies $r + |V(C)| \geqslant n/3$. □

Induced paths of length 2 are a bit of a cheat because they let us ignore the complex structure of these graphs. For this reason the following problems are of interest.

Problem. Show that there is a constant

(i) $c > 1/3$ such that for every $G \in \mathfrak{A}$, G has a $K_{\lceil cn\rceil}$ minor.
(ii) $c > 0$ such that for every $G \in \mathfrak{A}$, G has a $K_{\lceil cn\rceil}$ minor using only cliques as prevertices.
(iii) $c > 0$ such that for every $G \in \mathfrak{A}$, G has a $K_{\lceil cn\rceil}$ minor using prevertices of size one or two.

Using an elementary counting argument, we show problem (iii) holds if cn is replaced by $cn^{4/5}$. The idea of the proof is that if $G \in \mathfrak{A}$ has large minimum degree, then lots of pairs of edges touch, making it easy to find a large complete minor, while if G has small minimum degree, this forces a large clique.

Theorem 2.3. *Let $G \in \mathfrak{A}$ have minimum degree $\delta(G) = n - c_1 n^\alpha$. Assume that $0 \leqslant \alpha < 1$ so that $|E(G)| = \frac{1}{2}n^2 + o(n^2)$. Then G has a complete minor of size $c_3 n^\beta + o(n^\beta)$ using prevertices of size one or two, where $\beta = \min(4 - 4\alpha, 1)$ and c_3 is a constant depending only on c_1.*

Proof. Let H be the graph with vertex set $E(G)$; edges e_1 and e_2 are adjacent in H if they share an end or do not touch. A stable set in H gives the prevertices of a complete minor in G. We will bound the degree of H to show that it has a large stable set.

If e is an edge, let $\overline{N(e)}$ be the set of vertices that do not touch e. A *vedge* is the simple graph with three vertices and one edge. We count the number of induced vedges in G in two different ways:

$$\sum_{v \in V(G)} \binom{n - d(v)}{2} = \begin{array}{c}\text{number of}\\ \text{induced vedges}\end{array} = \sum_{e \in E(G)} \left|\overline{N(e)}\right|. \tag{1}$$

$\binom{n-d(v)}{2}$ is the number of vedges with isolated vertex v, and $|\overline{N(e)}|$ is the number of vedges with edge e. Using the degree bound, we obtain

$$n\frac{c_1^2}{2}n^{2\alpha} \geqslant \sum_{v \in V(G)} \binom{n - d(v)}{2}. \tag{2}$$

Then the average value of $|\overline{N(e)}|$ is about $c_1^2 n^{2\alpha - 1}$. Let E' be the edges e for which $|\overline{N(e)}| \geqslant 2c_1^2 n^{2\alpha-1}$ (twice the average is arbitrary; other constant factors would do). We may now bound $|E'|$. Define c_2 so that $|E'| = c_2|E(G)|$. Then by (1) and (2)

$$\frac{c_1^2}{2}n^{2\alpha+1} \geqslant \sum_{e \in E(G)} \left|\overline{N(e)}\right| \geqslant c_2\left|E(G)\right| 2c_1^2 n^{2\alpha-1}$$

implies

$$c_2 \leqslant \frac{n^2}{4|E(G)|} = \frac{1}{2} + o(1). \tag{3}$$

Then for $|E(G) - E'| \geqslant (1 - (\frac{1}{2} + o(1)))|E(G)|$ edges e,

$$d_H(e) \leqslant 2n + \binom{|\overline{N(e)}|}{2} \leqslant 2n + 2c_1^4 n^{4\alpha-2}.$$

The bound on $d_H(e)$ comes from the trivial upper bound of $2n$ for the number of edges sharing an end with e, and $\binom{|\overline{N(e)}|}{2}$ is from the fact that $\overline{N(e)}$ is a clique containing all edges not touching e. Then $H \backslash E'$ has max degree $\Delta := 2n + 2c_1^4 n^{4\alpha-2}$ and a greedy coloring shows the chromatic number $\chi(H \backslash E') \leqslant \Delta + 1$. This together with (3) implies there is a stable set in $H \backslash E'$ of size at least

$$\frac{|E(G)| - |E'|}{\Delta + 1} \geqslant \frac{(1/2 + o(1))|E(G)|}{\Delta + 1} = \frac{n^2/4}{\Delta}\big(1 + o(1)\big).$$

Put

$$c_3 = \begin{cases} \frac{1}{8c_1^4} & \text{if } 4\alpha - 2 > 1, \\ \frac{1}{4(2+2c_1^4)} & \text{if } 4\alpha - 2 = 1, \\ \frac{1}{8} & \text{if } 4\alpha - 2 < 1. \end{cases}$$

Put $\beta = \min(4 - 4\alpha, 1)$. Then G has a complete minor of size

$$\frac{n^2/4}{\Delta}\bigl(1 + o(1)\bigr) = c_3 n^\beta + o\bigl(n^\beta\bigr). \qquad \square$$

The constants obtained in the proof are not the optimal obtainable by this method, but they will do. The corollary below follows easily.

Corollary 2.4. *For sufficiently large n, every $G \in \mathfrak{A}$ has a complete minor of size $\frac{n^{4/5}}{9}$ using prevertices of size one or two.*

Proof. In a graph with no antitriangle the nonneighbors of each vertex are a clique. Then $\delta(G) = n - c_1 n^\alpha$ implies G has a complete minor of size $c_1 n^\alpha$. Note that

$$\max\bigl(\min(4 - 4\alpha, 1), \alpha\bigr) \geqslant \frac{4}{5}.$$

Also observe that

$$\max\left(\frac{1}{4(2 + 2c_1^4)}, c_1\right) > 1/9$$

and the corollary follows. $\square$

Also of note is that this method shows that for sufficiently large n, G has a complete minor of size $\frac{n}{9}$ if $\delta(G) \geqslant n - n^{3/4}$. As mentioned in [3], the asymptotics of the Ramsey number $r(3, n)$ show that there exist graphs with no antitriangle and clique number of order $\sqrt{n \ln n}$. However, this result tells us that problem (iii) holds in this extreme case because the minimum degree is so large.

3. Good and bad edges

Let $c_1, c_2, \ldots, c_r$ be the cliques of G and let w be a function from $\{c_i\}$ to the nonnegative rationals. The *fractional clique covering number* of G is the minimum of $\sum_i w(c_i)$ over all maps w such that

$$\forall v \in V(G) \qquad \sum_{i \text{ s.t. } v \in c_i} w(c_i) \geqslant 1.$$

If G has fractional clique covering number less than 3, multiplying w by a common denominator shows that there is a list of k cliques (not necessarily distinct) such that every vertex is in more than $\frac{k}{3}$ of them. In particular this implies that G has no antitriangle. It is interesting to study the SSH conjecture for such graphs.

We observe that there is a natural way to partition the edges in a graph with fractional clique covering number less than 3. An edge (u, v) is *good* if there are more than $\frac{k}{2}$ cliques containing u

or v (or both). If (u, v) and (x, y) are good, there is a clique containing at least one of u, v and at least one of x, y, so every pair of good edges touch. An edge (u, v) is *bad* if there are $\frac{k}{2}$ or fewer cliques containing u or v (or both). If (u, v) and (v, w) are bad, then there are $< \frac{k}{6}$ cliques containing v and not u and $< \frac{k}{6}$ cliques containing v and not w. Since v is in more than $\frac{k}{3}$ cliques, there is a clique containing v and u that also contains v and w, i.e. it contains $\{u, v, w\}$. This shows there are no induced paths of length 2 that use only bad edges. This setup can be generalized slightly: if every vertex is in at least $\frac{k}{3}$ of the cliques (G has fractional clique covering number $\leqslant 3$) and if k is odd, then the same arguments show that $E(G)$ can be partitioned into good and bad edges satisfying the same rules. This is because there cannot be exactly $\frac{k}{2}$ cliques containing an edge.

It turns out that under some not too restrictive conditions, G has a perfect matching of good edges, and these edges are the prevertices of a complete minor. All that is needed to prove this are the conditions on pairs of good and bad edges. We therefore drop the fractional clique covering number condition and retain the conditions on edge pairs.

3.1. Perfect matching of good edges

Let $G = (V, E)$ be a graph with no antitriangle. Suppose E can be partitioned into good edges and bad edges, $E = \mathfrak{G} \cup \mathfrak{B}$, so that for every pair g_1, g_2 of good edges, g_1 and g_2 touch, and for every pair of bad edges that share an end, $b_1 = \{u, v\}$, $b_2 = \{v, w\}$, $\{u, w\}$ is an edge. We will call these conditions the good edge and bad edge axioms.

Theorem 3.1. *If $G = (V, E) \in \mathfrak{A}$, n is even, and $E = \mathfrak{G} \cup \mathfrak{B}$ as above, then G has a $K_{n/2}$ minor with prevertices of size at most* 2.

This subsection is devoted to proving this theorem. This theorem together with the discussion above yields Theorem 1.3 because if G has clique covering number 3, the generalization with fractional clique covering number $\leqslant 3$ and k odd applies.

Rather than prove Theorem 3.1 directly, we will prove a more technical theorem that implies it. Suppose G satisfies the hypotheses of Theorem 3.1. It is quite a bit easier to find a perfect matching of good edges if we make a special choice for the partition. Note that given any partition of E into good and bad edges with the axioms satisfied, an edge whose ends are c-twins can be made bad without violating the axioms. We can therefore choose the partition $E = \mathfrak{G} \cup \mathfrak{B}$ so that

(I) all edges between c-twins are bad, and
(II) given (I), $\mathfrak{G}$ is as large as possible.

To obtain a complete matching of good edges, it helps to do away with some exceptional cases first. If G has a dominating edge (u, v), then $E(G \backslash \{u, v\})$ can also be partitioned into good edges and bad edges. By induction on n, we obtain a complete minor of $G \backslash \{u, v\}$; by adding the prevertex (u, v), we obtain a complete minor of G. We know G has a $K_{n/2}$ minor if there is a small cutset (Lemma 2.1). Obviously, the minor exists if G has a clique of size $\geqslant n/2$. If none of these arguments works, we prove that G has a perfect matching of good edges. This gives the prevertices of a $K_{n/2}$ minor.

This theorem is almost true without the constraints on how good and bad edges are assigned. The proof is considerably more involved, but nicely illustrates the structure forced by the good and bad edge axioms. We therefore include this proof in Appendix A.

Theorem 3.2. *If $G = (V, E) \in \mathfrak{A}$, n is even, and $E = \mathfrak{G} \cup \mathfrak{B}$ satisfying* (I) *and* (II)*, then either G*

(a) *has a dominating edge, or*
(b) *has a cut set of size $\leqslant n/2$, or*
(c) *has a clique of size $\geqslant n/2$, or*
(d) *has a perfect matching of good edges.*

Proof. We will assume all the above are false and arrive at a contradiction. We apply Tutte's theorem to $G' := (V, \mathfrak{G})$: (d) is false implies that there is an $S \subseteq V$ such that $G' \backslash S$ has at least $|S| + 2$ odd components (components with an odd number of vertices). We will mostly be working with the graph G and will often think of it as the complete graph on $|V|$ vertices with three edge types: good ($\mathfrak{G}$), bad ($\mathfrak{B}$), and nonedges ($\mathfrak{N} := E(\overline{G})$). We will occasionally refer to G' so be sure not to confuse the two.

We begin by illustrating some of the structure that is forced by the good edge and bad edge axioms. This will be useful for the rest of the proof, Section 3.2, and Appendix A. A set of vertices P is *X-coupled*, $X \subseteq V$, if $\forall p_1, p_2 \in P$, $N(p_1) \cap X = N(p_2) \cap X$. If P and Q are vertex sets that are both X-coupled, we say P, Q is *X-anticoupled*, if $N(P) \cap X = X - N(Q)$. Also, we say an edge (u, v) is X-coupled if $\{u, v\}$ is X-coupled and X-anticoupled if u, v is X-anticoupled. Let L and R be nonempty subsets of V that do not touch in G' (the main application of this will be to the case when L and R are components in G').

Let M_1 be a component of bad and nonedges in L (a component in the subgraph of $(V, \mathfrak{B} \cup \mathfrak{N})$ induced by L). Only bad edges and nonedges cross between L and R; by the bad edge axiom and since G has no antitriangle, every bad edge in L is R-coupled and every nonedge in L is R-anticoupled. A pair of vertices cannot be both R-coupled and R-anticoupled because R is nonempty. Thus any cycle of nonedges in L must be even because any pair of vertices distance 2 apart in the cycle are necessarily R-coupled and being R-coupled is an equivalence relation. Now the graph of nonedges on M_1 must be bipartite, so M_1 can be partitioned into two sets, M_{1T} and M_{1B}, so that M_{1T}, M_{1B} is R-anticoupled. This depends on the fact that M_1 is a component of bad and nonedges and that being R-coupled is an equivalence relation while being R-anticoupled is an "antiequivalence" relation. Note that M_{1T} and M_{1B} are cliques (in G). We will call M_1 a *dipole* and call M_{1T} and M_{1B} *poles*. Given two poles of a dipole, we say one is the *antipole* of the other. If both (exactly one) poles of a dipole are nonempty, we will say the dipole is *proper* (*improper*).

Let l (r) be the number of dipoles in L (R). We have $L = M_1 \cup M_2 \cup \cdots \cup M_l$ and $R = N_1 \cup N_2 \cup \cdots \cup N_r$. By definition of the M_i, any edge $\{u, v\}$ with $u \in M_i, v \in M_j$ $(i \neq j)$ is good. An edge between M_{iT} and M_{iB} is dominating because it touches all of $L - M_i$ by the good edges just mentioned and touches all of R because it is R-anticoupled. Thus a pole does not touch its antipole (in G) because we are assuming (a) is false. If $V = L \cup R$, then every pair of vertices in a pole are c-twins. For every $i \in [l]$, $j \in [r]$, M_{iT} touches exactly one of N_{jT}, N_{jB}, because N_{jT}, N_{jB} are L-anticoupled. So either (M_{iT} is complete to N_{jT} and M_{iB} is complete to N_{jB}) or (M_{iT} is complete to N_{jB} and M_{iB} is complete to N_{jT})—if the former, we say the dipoles are *matched straight* and if the latter they are *matched twisted*.

Let $C_1, C_2, \ldots, C_m$ be the components of $G' \backslash S$; $m \geqslant |S| + 2$. We first prove a useful result that tells us how an odd antihole can lie in the C_i (step (1)). We then address the case $m = 2$ in step (2) and the $m > 2$ case in step (3).

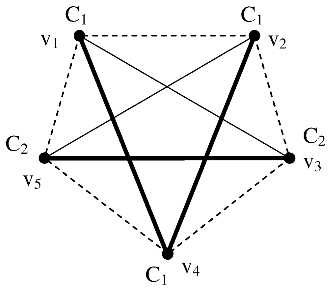

Fig. 2. The antihole of (1)(iii). The thicker (thinner) edges are good (bad), and the dotted edges are nonedges.

(**1**) Either
(i) $\overline{G}\backslash S$ is bipartite, or
(ii) $G\backslash S$ contains an odd antihole with vertex set contained in C_i for some $i \in [m]$, or
(iii) G contains an antihole of length 5 that belongs to two components and is isomorphic to the 5-antihole in Fig. 2.

If $\overline{G}\backslash S$ is not bipartite, it contains an odd cycle of length 5 or greater (a cycle of length 3 is an antitriangle in G). A shortest such cycle is an odd antihole in $G\backslash S$, which we label $Y = v_1, v_2, \ldots, v_l$. Throughout the proof of (1), we treat all subscripts mod l. Let $C(v_i)$ be the component containing v_i.

We will call $v_i, v_j, v_{j+1} \in Y$, a *forcing triple* if $d(i, j)$, $d(i, j+1) > 1$, where $d(i, j) = \min(|j-i|, l-|j-i|)$ is the “circular distance” between i and j. These conditions force at least one of (v_i, v_j) and (v_i, v_{j+1}) to be good and therefore either $C(v_i) = C(v_j)$ or $C(v_i) = C(v_{j+1})$. For example, v_4, v_1, v_2 in Fig. 2 is a forcing triple.

The forcing triples v_2, v_j, v_{j+1}, $j \in \{4, 5, \ldots, l\}$, show that among the set of vertices $\{v_4, v_5, \ldots, v_l\}$, at least half are in the component $C(v_2)$. Of course, v_2 was not special, and in general we have that any component intersecting Y intersects it in size at least $1 + \frac{l-3}{2}$. Then at most two components intersect Y because $3(1 + \frac{l-3}{2}) \geqslant l+1$ as $l \geqslant 5$. Either $l \geqslant 7$ (case A), or $l = 5$ (case B).

(A) Since Y intersects at most two components there is a j such that $C(v_j) = C(v_{j+1})$, and therefore $C(v_i) = C(v_j)$ for all $i \in [l] - \{j-1, j+2\}$. The forcing triples $v_{j+3}, v_{j+4}, v_{j-1}$ and $v_{j-2}, v_{j-3}, v_{j+2}$ show that $C(v_{j-1}) = C(v_j) = C(v_{j+2})$. Thus Y is contained in one component ((ii) holds).

(B) Begin as in (A) by supposing Y intersects at most two components, and choose j as in (A). The forcing triple v_{j-1}, v_j, v_{j+2} shows that $C(v_j) = C(v_{j+2})$ or $C(v_{j-1}) = C(v_{j+2})$. If the former, the forcing triple $v_{j+1}, v_{j+2}, v_{j-1}$ shows (ii) holds. If the latter holds, but the former does not, then we can relabel so that v_j, v_{j+1} becomes the pair v_1, v_2, and $C(v_1) = C(v_2) = C(v_4) = C_1$, and $C(v_3) = C(v_5) = C_2$ as in Fig. 2. Edges (v_1, v_3) and (v_2, v_5) must be bad because of the component assignments and then the bad edge axiom forces the other edges to be good. Therefore (iii) holds.

(**2**) The case $m = 2$ leads to a contradiction.

Since $m = 2$, $S = \emptyset$ and therefore 1(i) implies there is a large clique ((c) is true), so we can assume 1(i) does not hold. Set $L = V(C_1)$ and $R = V(C_2)$ and apply the discussion above to decompose L and R into dipoles. This shows that $\overline{G}\backslash L$ and $\overline{G}\backslash R$ are bipartite so 1(ii) does not hold. We may assume 1(iii) holds.

We will show that it is possible to make edge (v_1, v_3) of Fig. 2 good, which contradicts the constraint (II) on our choice of good and bad edges. From Fig. 2, we see that v_1 and v_3 are not c-twins so making (v_1, v_3) good does not violate (I). Let M_{1T} be the pole containing v_1 and let N_{1T} be the pole containing v_3. Because $V = L \cup R$, vertices in a pole are c-twins (as mentioned above), and then (I) implies all edges with both ends in the same pole are bad. Making edge (v_1, v_3) good does not violate the good edge axiom because all edges with both ends in $V - N(\{v_1, v_3\}) = M_{1B} \cup N_{1B}$ are bad. This completes step (2).

We may assume $m > 2$. It is here we reap the main rewards of (1). If (1)(ii) or (1)(iii) holds, then an antihole, Y, does not intersect all components of $G'\backslash S$. Let L (R) be the union of all components Y intersects (does not intersect). Non-edges in L are R -anticoupled and therefore, as seen earlier, an odd-cycle of nonedges in L is impossible. So we may assume (1)(i).

(1)(i) implies we can partition $V - S$ into A and B such that $(G\backslash S)[A]$ and $(G\backslash S)[B]$ are cliques. Let $C_i^A = C_i \cap A$ and $C_i^B = C_i \cap B$. Since $|C_i|$ is odd, $|C_i^A| \neq |C_i^B|$. Let $X = \bigcup_i$(smaller of C_i^A, C_i^B). Remembering that $m \geqslant |S| + 2$, we observe $|X \cup S| \leqslant n/2$. Therefore $X \cup S$ is not a cutset and $G\backslash X\backslash S$ is connected.

Without loss of generality $G\backslash X\backslash S$ is the union of $C_1^A, C_2^A, \ldots, C_k^A, C_{k+1}^B, C_{k+2}^B, \ldots, C_m^B$, $0 \leqslant k \leqslant m$. By symmetry we may assume $k \leqslant m - k$.

(**3**) The cases $k = 0$ (A), $k \geqslant 2$ (B), and $k = 1$ (C) each lead to a contradiction, which shows that $m > 2$ is impossible.

(A) $G\backslash X\backslash S$ is a clique and it is large enough to contradict the assumption that (c) is false.

(B) We can apply the arguments from the beginning of the proof to the pair $A - X$, $B - X$ in place of L, R because no good edges have one end in $A - X$ and one end in $B - X$ (remember, the C_i are components of good edges). Since $k \geqslant 2$, all of $A - X$ is a component of bad edges ($A - X$ is a pole). Since $m - k \geqslant 2$, $B - X$ is a pole. $G\backslash X\backslash S$ is connected, so there is an edge between $A - X$ and $B - X$, and therefore $A - X$ and $B - X$ are joined completely. This proves (c), which we assumed false.

(C) $m > 2$ implies $m - k \geqslant 2$. For this case, we will apply dipole structure to the partition $B - X, C_1$ (each $(B - X, C_1)$ edge is bad). As in (B), $B - X$ is a pole. Let $M_B = C_1^A \cap N(B - X)$. C_1^B is complete to M_B (in G) by the bad edge axiom. Since $|C_1|$ is odd, $|C_1^A - M_B|$ and $|C_1^B \cup M_B|$ are not equal. If $|C_1^A - M_B|$ is larger, then $G\backslash(X \cup M_B \cup S)$ is disconnected and this contradicts the assumption that (b) is false; if $|C_1^B \cup M_B|$ is larger, then $B \cup M_B$ is a clique and this contradicts the assumption that (c) is false. This proves (3). □

3.2. Extensions

Although the good edge axiom is what allows us to say anything about SSH, it seems too difficult to find a large number of edges that satisfy the good edge axiom (except for an obvious choice that is not that helpful for SSH—all the edges with at least one end in a clique). And finding a partition of good and bad edges is impossible for some graphs, as we will see in Section 4. We have therefore made many attempts to relax the good and bad edge axioms in such a way that we can still prove theorems about SSH. We were unsuccessful for the most part, but here we show a small attempt in this direction.

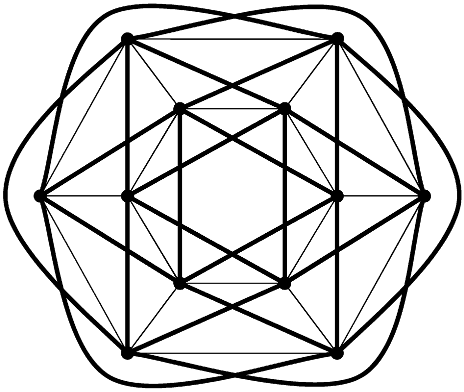

Fig. 3. The thicker (thinner) edges are medium (bad) edges.

Let $G = (V, E)$ be a graph in $\mathfrak{A}$ with an even number of vertices and suppose E is partitioned, $E = \mathfrak{M} \cup \mathfrak{B}$, so that the bad edge axiom holds for $\mathfrak{B}$, but the good edge axiom does not (necessarily) hold for $\mathfrak{M}$ (call them medium edges). Is it true that there is a perfect matching of medium edges? While a matching of medium edges would not necessarily give the prevertices of a complete minor, it would be necessary for there to be a complete minor that does not use bad edges as prevertices. It might be useful to know when we can ignore some edges (edges that do not touch many edges, perhaps) and still find a perfect matching in the remaining edges. In Appendix A we give a proof of a result similar to Theorem 3.2 without making the special choices (I) and (II) for the partition of good and bad edges. A matching of good edges is impossible in only one case—when G is an inflation of the complement of the Petersen graph. So an inflation of the complement of the Petersen graph is a counterexample to the question above, but are there others?

There are other counterexamples to this question—we will see that an inflation of Fig. 3 is one. However, we have quite a bit of control on the counterexamples. Note that the proof of Theorem 3.2 mentions the good edge axiom and uses the special choices (I) and (II) for the partition only in step (2). By an easy argument from the proof of Theorem A.1, any counterexample must have exactly two components of medium edges, L and R, with at least two dipoles in each.

Let T be the complete bipartite graph with vertex set the set of dipoles. If two dipoles are matched straight (twisted), label the corresponding edge in T straight (twisted). By exchanging labels of a pole and antipole, we may swap the edge type of all edges incident to a vertex of T. Note that T, together with the number of vertices in every pole, is enough information to reconstruct G, and graph(s) T that yield a fixed G are not necessarily unique. The graph in Fig. 3 corresponds to the bipartite graph $T = K_{3,3}$ in which 3 vertex disjoint edges are twisted. All poles are of size 1. The smallest cutset in this graph has size 7. This graph does have a perfect matching of medium edges, however, the graph with one pole in L of size $k + 1$ and one pole in R of size $k + 1$, and all other poles of size k is a counterexample for large k. This is because L and R are odd components in $(V, \mathfrak{M})$ so there is no matching of medium edges; the smallest cutset has size $7k$ which is $> n/2 = (12k + 2)/2$ for $k \geqslant 2$.

If we are willing to choose medium edges and bad edges with some additional properties, we can obtain a perfect matching of medium edges. We want to mimic (I) and (II) from Section 3.1, but if one of our restrictions is to maximize the number of medium edges, this may just end up making all edges medium and then we are proving a rather weak statement. A simple modification, however, gives a generalization of Theorem 3.2.

Corollary 3.3. *If $G = (V, E) \in \mathfrak{A}$, n is even, and $E = \mathfrak{M} \cup \mathfrak{B}$ such that*

(I′) *all edges between c-twins are bad,*
(II′) *given* (I′)*, the number of pairs of edges in $\mathfrak{M}$ that do not touch is as small as possible, and*
(III′) *given* (I′) *and* (II′)*, $\mathfrak{M}$ is as large as possible,*

then at least one of Theorem 3.2(a)–(d) *holds* (*replace good with medium in* (d)).

Proof. The proof is almost exactly the same as that of Theorem 3.2. In step (2) we still have that all edges with both ends in a pole are bad. Making edge (v_1, v_3) good does not increase the number of pairs in $\mathfrak{M}$ that do not touch because all edges with both ends in $V - N(\{v_1, v_3\}) = M_{1B} \cup N_{1B}$ are bad. Then $|\mathfrak{M}|$ was not maximum, contradicting (III′). □

This corollary generalizes Theorem 3.2 because if there is a way to partition the edges into good edges and bad edges, then the $\mathfrak{M}$ that satisfies (I′)–(III′), will satisfy the good edge axiom. As our attempts at a generalization of Theorem 3.2 show, we have little idea how to identify what edges should be used for the prevertices of a complete minor, but we have some idea of how to identify edges that should not be used. If an edge, e, is between two vertices that are c-twins, then our investigations strongly suggest we should be able to obtain a complete minor without using e as a prevertex. In fact, it is reasonable to believe that it is always possible to find the prevertices of a $K_{n/2}$ minor among medium edges satisfying (I′)–(III′).

Inflations of graphs come up naturally in investigating SSH. An interesting problem is to show that if G satisfies SSH, then any inflation of G satisfies SSH. We were unable to prove this, but an easy corollary to Theorem 3.1 is

Corollary 3.4. *If $G = (V, E) \in \mathfrak{A}$ and $E = \mathfrak{G} \cup \mathfrak{B}$ with the axioms satisfied, then any inflation of G with an even number of vertices satisfies SSH.*

Proof. Let H be an inflation of G. Label all edges in the complete bipartite graph of H corresponding to an edge of G good (bad) if that edge is in $\mathfrak{G}$ ($\mathfrak{B}$). Label all edges in the complete graph of H corresponding to a vertex of G bad. This gives a partition of $E(H)$ into good and bad edges with the axioms satisfied. □

4. Nonexistence of a good and bad edge partition

It turns out that there is a nice necessary and sufficient condition for whether a graph's edges can be partitioned into good and bad edges with the axioms satisfied. Finding an assignment of good and bad to a graph's edges is a 2-satisfiability problem: every pair of edges that do not touch corresponds to a clause requiring that at least one edge of the pair is bad, and every pair of edges (u, v), (v, w) such that $u \nsim w$ corresponds to a clause requiring that at least one edge of the pair is good. Equivalently, we may consider the graph $H = (E(G), N \cup B)$, where a pair of edges is in N if they do not touch and a pair of edges is in B if they induce a path of length 2. We seek a partition $V(H) = \mathfrak{G} \cup \mathfrak{B}$ such that $\mathfrak{G}$ is a stable set in $(V(H), N)$ and $\mathfrak{B}$ is a stable set in $(V(H), B)$. Such a partition exists if and only if there is a certain kind of alternating walk.

Let $H_N = (V(H), N)$ and $H_B = (V(H), B)$. For the theorem that follows we do not need the fact that H came from G; all we need is that H_N and H_B are graphs on the same vertex set. In this setting we define a *walk* of length l to be a sequence of vertices $v_1, v_2, \ldots, v_{l+1}$ (not necessarily distinct) such that $(v_i, v_{i+1}) \in N \cup B$, $i \in [l]$. A walk is *closed* if $v_1 = v_{l+1}$. A walk is *alternating* if edges of the form (v_{2i-1}, v_{2i}) are in N, and edges of the form (v_{2i}, v_{2i+1}) are in B (or the same with N and B switched). Closed alternating walks of odd length are possible with this definition, but if the vertex labels are cyclicly permuted, it is no longer alternating. We will call such a walk an *AACW* (almost alternating closed walk). The following result is due to Alexander Schrijver [5].

Theorem 4.1. *Exactly one of the following holds*:

(a) *There is a partition $V = \mathfrak{G} \cup \mathfrak{B}$ such that $\mathfrak{G}$ is a stable set in H_N and $\mathfrak{B}$ is a stable set in H_B.*
(b) *There is an even closed alternating walk such that two vertices an odd distance apart in the walk are identical.*

Figure 4 is a representation of a walk as described in (b), except this is a drawing of G, not H. If the only nonedges are those drawn, then this graph is in $\mathfrak{A}$. In H, this is two AACW's of length 7 connected by an edge in N.

The smallest value of $|V(G)|$ such that $H := (E(G), N \cup B)$ contains a walk as described in (b) is quite probably 18, although it seems like it would require a lot of case by case analysis to prove this. If the far left nonedge and the far right nonedge in Fig. 4 are identified, this gives a graph in $\mathfrak{A}$ on 18 vertices with no partition of edges into good and bad. We can prove that a walk as in (b) in H must correspond to a G with at least 12 vertices. We omit the proof, which requires a few cases and the easy fact that $G \in \mathfrak{A}$ implies any walk in H with edge type order N, B, N, B, N corresponds to a subgraph in G with 10 vertices (the largest number of vertices possible for such a walk, given that a B-edge in H corresponds to a 3 vertex subgraph in G). The corollary below then follows from Theorem 4.1 and Corollary 3.4.

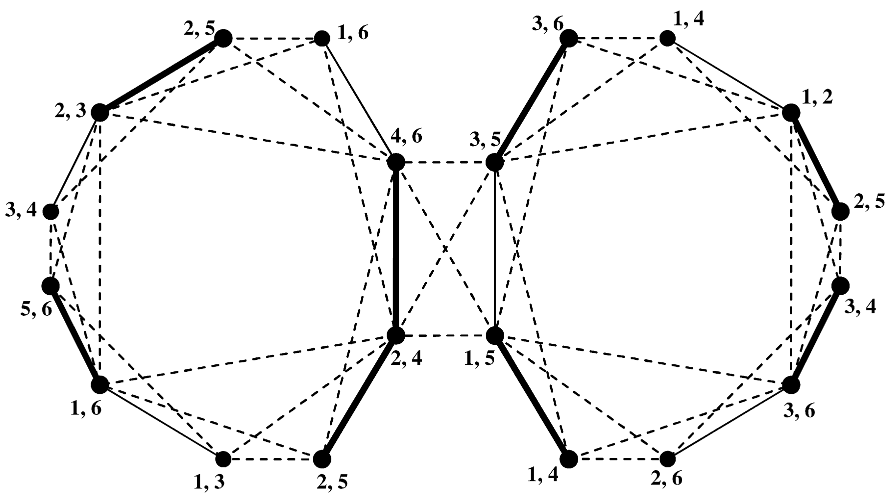

Fig. 4. A subgraph that makes it impossible to partition $E(G)$ into good and bad edges. The good and bad edge assignments shown fail to satisfy the axioms because the two good edges on the far right do not touch.

Corollary 4.2. *If $G \in \mathfrak{A}$ is an inflation of a graph on at most 11 vertices and $|V(G)|$ is even, then G satisfies SSH.*

Plummer, Stiebitz, and Toft prove this same result in [3] except without the restriction to graphs with an even number of vertices, however their proof requires quite a lot of case by case analysis. It is interesting that in this proof they also came across inflations of the complement of the Petersen graph as an important case to check SSH on (we come across this graph in Appendix A) even though their methods are very different.

At this point we can show that the application of Theorem 3.1 to graphs with fractional clique covering number less than 3 is in some sense best possible. The labels in Fig. 4 represent 6 different cliques and every vertex is in exactly 2 of them so this graph has fractional clique covering number at most 3. Evidently, its edges cannot be partitioned into good edges and bad edges satisfying the axioms so Theorem 3.1 cannot be applied. Also, it seems that even after deleting dominating edges, we obtain a graph with no large clique or small cutset, although we have not checked this carefully.

5. Conclusions, conjectures, and future work

Some questions one might have at this point are "will the good and bad edge axioms help us say anything about *all* graphs with no antitriangle?," "what do graphs with no antitriangle and minimum degree $n - O(n^{4/5})$ look like and why is SSH difficult in this range?," "does Theorem 3.1 extend to n odd?," and "why did not we try induction?" We will attempt some answers.

We can only construct random-like graphs with no antitriangle when the average degree is $n - O(n^{1/2})$ or larger [6]. In this degree range, Theorem 2.3 tells us a lot. Graphs with smaller minimum degree than $n - O(n^{1/2})$ have cliques too large for a typical random graph because the nonneighbors of every vertex are a clique. This suggests that in this density range, graphs with no antitriangle tend to have structure like that of an inflation of a smaller graph. It is here that we seek to apply results like Theorem 3.1 and Corollary 3.3. So far we are only successful for graphs with fractional clique covering number less than 3, in which case degrees are around $n - \frac{1}{3}n$. So, for example, how might we extend the results using the good and bad edge axioms to graphs with minimum degree $n - O(n^{4/5})$? Corollary 3.3 gives us one prescription, but what do the resulting medium and bad edges look like? Perhaps we could compute bounds on the number of pairs of medium edges that do not touch, which might lead to bounds on the size of a complete minor. It is strange that the graphs that give us the most trouble are denser than the graphs that our results apply to; large complete minors should be easier to find in denser graphs. In a sense, the problem is not that it is difficult to find a complete minor in these graphs (ones with minimum degree $n - cn^{4/5}$, say), but rather that we cannot say anything about them.

Another idea for extending the good and bad edge axioms is to assign weights to the edges. We may think of the weights as distances. Edges between vertices with "similar" neighbor sets (those that are close to being c-twins) will receive small weights and will be like bad edges of varying degrees. Edges that touch many other edges will receive large weights and will be like good edges of varying degrees. These two ways of choosing edge weights are similar, but do not agree exactly, and it is not clear what the right weighting function is.

Let G be as in Theorem 3.2. It would be nice if we could modify the proof of Theorem 3.2 to work for n odd. We think that if G has no dominating edge, small cutset, or large clique (as in (a), (b), and (c) of Theorem 3.2), then we can choose any vertex, q, to be a prevertex and use edges for the other prevertices.

Let Z_q be the clique of nonneighbors of q. An obvious idea is to apply Tutte's theorem to the graph $G' := (V - q, \mathfrak{G} - E(Z_q))$. A perfect matching in G' together with q are the prevertices of a complete minor of G. A similar proof to that of Theorem 3.2 works in quite a few cases, but not all. We have found a graph with no dominating edge, no small cutset, and no large clique, and no matching of edges in $\mathfrak{G} - E(Z_q)$ saturates $V - q$. It is possible that if a special set of good and bad edges is chosen, then G' does have a perfect matching. However, our investigations suggest that several natural choices for special sets of good and bad edges do not work. It seems best to try another approach for n odd.

Tutte's theorem is a fantastic structure theorem for graphs with no perfect matching, but it is not exactly what we need for this problem. Perhaps we can find an appropriate strengthening of SSH that we can apply inductively to get a perfect matching of medium edges. We want it to give us a special perfect matching of medium edges, not just any, as Tutte's theorem gives us.

It will require much cleverness to get induction to work on this problem. Suppose we have an edge, $e = (u, v)$, that seems like a good candidate to be a prevertex, and then we inductively obtain the prevertices of a minor on $G \backslash \{u, v\}$. It is not at all clear that the prevertices of the minor in $G \backslash \{u, v\}$ will touch e, so we have to find special prevertices, not just any. Labeling some edges bad provides a way for us to exclude *some* sets of prevertices, however, we need something more powerful to tackle the general case.

Acknowledgments

I thank Paul Seymour for his many awesome ideas, and for teaching me how to do good graph theory. I thank Ervin Győri for his help in fall semester 2003 and Benny Sudakov for his helpful discussions about random graphs.

Appendix A

Here we prove an analogue of Theorem 3.2 that does not require constraints (I) and (II) on the partition of good and bad edges. In addition to showing some of the neat structure that is forced by the good and bad edge axioms, the proof could also be useful if constraints (I) and (II) were modified (this might be useful, for instance, in tackling the odd vertex case). The precise statement is

Theorem A.1. *If $G = (V, E) \in \mathfrak{A}$, n is even, and $E = \mathfrak{G} \cup \mathfrak{B}$ with the axioms satisfied, then either G satisfies at least one of* (a)–(d) *of Theorem* 3.2*, or*

(e) *G is an inflation of the complement of the Petersen graph.*

Proof. We will use steps (1) and (3) of the proof of Theorem 3.2, which do not depend on constraints (I) and (II), and replace step (2) with what follows.

Let $L = M_1 \cup M_2 \cup \cdots \cup M_l$ and $R = N_1 \cup N_2 \cup \cdots \cup N_r$ be the partitions of $L := V(C_1)$ and $R := V(C_2)$ into dipoles as described in the proof of Theorem 3.2. We may assume 1(iii) holds by the same argument given in the proof of step (2). From Fig. 2, we see that v_3, v_5 are neither L-coupled nor L-anticoupled, so $r \geqslant 2$. v_1, v_2 are R-anticoupled, and v_1, v_4 are neither R-coupled nor R -anticoupled. So $l \geqslant 2$ and at least one dipole in L is proper.

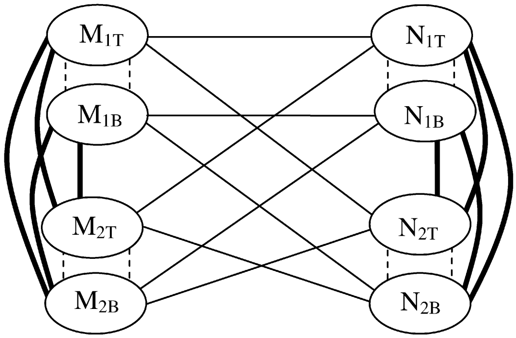

Fig. 5. An inflation of $\overline{V_8}$. All good and bad edges are drawn. Non-edges between poles and antipoles are drawn, but nonedges between L and R are not.

(2′) If two dipoles of L are proper, then either
  (i) G is an inflation of $\overline{V_8}$ as shown in Fig. 5, or
  (ii) G is an inflation of the complement of the Petersen graph as shown in Fig. 7.

Without loss of generality, M_1 and M_2 are proper. Consider $N(M_{1T}) \cap R$, $N(M_{1B}) \cap R$, $N(M_{2T}) \cap R$, $N(M_{2B}) \cap R$, and call them T_1, B_1, T_2, B_2 for brevity. Every pole in R is in exactly two of these sets. An edge, e, between M_{1T} and M_{2T} is good and therefore touches every good edge, so at most one dipole intersects $R - (T_1 \cup T_2) = B_1 \cap B_2$. If no dipole intersects $B_1 \cap B_2$, then e is dominating, so we may assume exactly one dipole intersects $B_1 \cap B_2$. Applying the same argument to edges between M_{1T} and M_{2B}, M_{1B} and M_{2T}, and M_{1B} and M_{2B} shows that exactly one dipole intersects $B_1 \cap T_2$, $T_1 \cap B_2$, and $T_1 \cap T_2$.

If R contains a proper dipole, N_1, say, then it must have nonempty intersection with each of T_1, B_1, T_2, and B_2 (because N_{1T}, N_{1B} is L-anticoupled). Then (up to symmetry between N_{1T} and N_{1B}) either ($N_{1T} = T_1 \cap T_2$ and $N_{1B} = B_1 \cap B_2$) or ($N_{1T} = T_1 \cap B_2$ and $N_{1B} = T_2 \cap B_1$). Clearly, since every vertex of a pole in R has the same neighbors in L, poles are contained in the sets $B_1 \cap B_2$, etc. As just seen, both poles of a dipole cannot be contained in $B_1 \cap B_2$, etc., so, in fact, $B_1 \cap B_2$, $B_1 \cap T_2$, $T_1 \cap B_1$, and $T_1 \cap T_2$ are poles. Up to symmetry, there are three possibilities, (A), (B) and (C), for the structure of R.

(A) R is the union of two proper dipoles: $N_{1T} = T_1 \cap T_2$, $N_{1B} = B_1 \cap B_2$, $N_{2T} = T_1 \cap B_2$, and $N_{2B} = T_2 \cap B_1$. Applying the argument above with L and R reversed, shows that L is the union of four poles, which implies $l = 2$. We now know the structure of G up to vertex c-duplication—G is an inflation of $\overline{V_8}$ ((i) holds).

(B) R is the union of a proper dipole and two improper dipoles: $N_{1T} = T_1 \cap T_2$, $N_{1B} = B_1 \cap B_2$, $N_2 = T_1 \cap B_2$, and $N_3 = T_2 \cap B_1$. An (N_2, N_3) edge is not dominating, so $l \geqslant 3$. Furthermore, there is a pole, M_{3T}, say, that does not touch N_2 or N_3. Without loss of generality, M_{3T} touches N_{1T} and not N_{1B}. But then an (N_2, N_{1B}) edge does not touch an (M_{2T}, M_{3T}) edge, contradicting the good edge axiom.

(C) R is the union of four improper dipoles: $N_1 = T_1 \cap T_2$, $N_2 = B_1 \cap B_2$, $N_3 = T_1 \cap B_2$, and $N_4 = T_2 \cap B_1$ (see Fig. 6). An (N_1, N_2) edge is not dominating so $l \geqslant 3$. Any dipole in L is proper, because suppose M_3 were improper. Then an edge from M_3 to R is dominating; if M_3 does not touch R, the good edge axiom is violated. Now apply the same argument to $\{M_1, M_j\}$

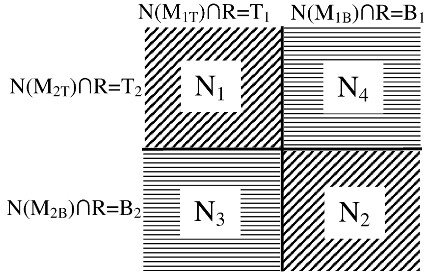

Fig. 6. Schematic drawing of the partitions $T_1 \uplus B_1$ and $T_2 \uplus B_2$. The different shadings represent the third partition $N(M_{3T}) \cap R \uplus N(M_{3B}) \cap R$.

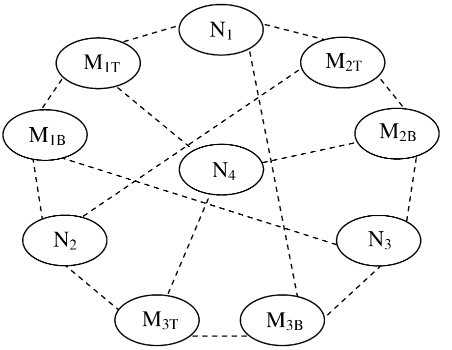

Fig. 7. An inflation of the complement of the Petersen graph. All nonedges are drawn.

and $\{M_2, M_j\}$ as was applied to $\{M_1, M_2\}$, $j \geqslant 3$. Each dipole in L partitions R into two sets each containing two poles. Moreover, every two such partitions must be isomorphic to the partitions defined by M_1 and M_2 ($T_1 \uplus B_1$ and $T_2 \uplus B_2$). That leaves room for only one more partition: $(N_1 \cup N_2) \uplus (N_3 \cup N_4)$. So $l = 3$ and without loss of generality, $N(M_{3T}) \cap R = N_1 \cup N_2$ and $N(M_{3B}) \cap R = N_3 \cup N_4$. We now know the structure of G up to vertex c-duplication—G is an inflation of the complement of the Petersen graph ((ii) holds). This proves (2′).

If (2′)(i) holds, then $G\backslash(M_1 \cup N_1)$ and $G\backslash(M_2 \cup N_2)$ are disconnected. At least one of $|M_1 \cup N_1|$ and $|M_2 \cup N_2|$ is $\leqslant n/2$ so (b) is true. We can also apply (2′) with L and R reversed, so we may assume that L and R have at most one proper dipole. We already know L has at least one proper dipole, so L has exactly one proper dipole. We now know a lot about the structure of G, and can finish up the remaining cases in (2″).

(2″) Given the conclusions of (1) and (2′), we may assume
- (i) L contains exactly one proper dipole,
- (ii) R contains at most one proper dipole,
- (iii) $l, r \geqslant 2$,
- (iv) L is the union of a proper dipole and an improper dipole, and
- (v) G is isomorphic to a graph represented by Fig. 5 with $M_{2B} = N_{2B} = \emptyset$ and all other sets nonempty except possibly N_{1T}.

As just discussed, (i) and (ii) hold. (iii) we have already seen. Without loss of generality, M_1 is proper. We proceed as in the proof of $(2')$. There is less symmetry so the arguments are a bit messier. Consider $N(M_{1T}) \cap R$, $N(M_{1B}) \cap R$, $N(M_2) \cap R$, $R - N(M_2)$, and call them T_1, B_1, T_2, B_2 for brevity. An edge, e, between M_{1T} and M_2 is good and therefore touches every good edge, so at most one dipole intersects $R - (T_1 \cup T_2) = B_1 \cap B_2$. If no dipole intersects $B_1 \cap B_2$, then e is dominating, so exactly one dipole intersects $B_1 \cap B_2$. Applying the same argument to edges between M_{1B} and M_2 shows that exactly one dipole intersects $T_1 \cap B_2$. Either R contains two improper dipoles (case A), or it does not (case B).

(A) R contains at least two improper dipoles, N_1 and N_2. N_1 is not contained in T_2 because then an (M_2, N_1) edge is dominating. Similarly for N_2. From the discussion above, we must have (up to symmetry of labeling) $N_1 = T_1 \cap B_2$, and $N_2 = B_1 \cap B_2$. Suppose for a contradiction that $l \geqslant 3$. M_3 is improper so any edge from $\{M_2 \cup M_3\}$ to $\{N_1 \cup N_2\}$ is dominating. If $\{M_2 \cup M_3\}$ does not touch $\{N_1 \cup N_2\}$ this violates the good edge axiom. So (iv) holds. If R contains another dipole, N_3, it is contained in T_2, but then either an (N_1, N_3) or an (N_2, N_3) edge is dominating, contradiction. We have determined the structure of G up to vertex c-duplication—G is isomorphic to Fig. 5 with $M_{2B} = N_{2B} = N_{1T} = \emptyset$ and all other sets nonempty ((v) holds).

(B) By (ii) and (iii), R is the union of a proper dipole and an improper dipole. Apply $(2'')$ with L and R reversed. If L contains more than one improper dipole, this is dealt with by (A). So we may assume L has exactly one proper dipole; this together with (i) implies (iv). The proper dipole of R, N_1, say, must have nonempty intersection with each of T_1, B_1 and T_2 (because N_{1T}, N_{1B} is L-anticoupled). We know from discussion above that $T_1 \cap B_2$ and $B_1 \cap B_2$ are poles. This determines the structure of G up to vertex c-duplication—G is isomorphic to Fig. 5 with $M_{2B} = N_{2B} = \emptyset$ and all other sets nonempty ((v) holds). This proves $(2'')$).

Now $(2'')$(v) implies $G \backslash (M_1 \cup N_1)$ and $G \backslash (M_2 \cup N_2)$ are disconnected (as in the $(2')$(i) case) so (b) is true. $\square$

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