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Type and Cotype with Respect to Arbitrary Orthonormal Systems

STEFAN GEISS*, †

Mathematisches Institut der Friedrich Schiller Universität Jena, UHH 17. OG, D 07743 Jena, Germany

AND

MARIUS JUNGE*, ‡

Mathematisches Seminar der Universität Kiel, Ludewig Meyn Strasse 4, D 24098 Kiel, Germany

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Let $\Phi = (\phi_k)_{k \in \mathbb{N}}$ be an orthonormal system on some σ-finite measure space (Ω, p). We study the notion of cotype with respect to Φ for an operator T between two Banach spaces X and Y, defined by $c_\Phi(T) := \inf c$ such that

$$\left\| \sum_k \phi_k \, T x_k \right\|_{L_2(Y)} \leqslant c \left\| \sum_k g_k x_k \right\|_{L_2(X)} \qquad \text{for all} \quad (x_k) \subset X,$$

where $(g_k)_{k \in \mathbb{N}}$ is the sequence of independent and normalized gaussian variables. It is shown that this Φ-cotype coincides with the usual notion of cotype 2 iff $c_\Phi(I_{l_2^n}) \sim \sqrt{n/(\log(n+1))}$ uniformly in n iff there is a positive $\eta > 0$ such that for all $n \in \mathbb{N}$ one can find an orthonormal $\Psi = (\psi_l)_1^n \subset \operatorname{span}\{\phi_k \mid k \in \mathbb{N}\}$ and a sequence of disjoint measurable sets $(A_l)_1^n \subset \Omega$ with

$$\int_{A_l} |\psi_l|^2 \, dp \geqslant \eta \qquad \text{for all} \quad l = 1, ..., n.$$

A similar result holds for the type situation. The study of type and cotype with respect to orthonormal systems of a given length provides the appropriate approach to this result. We intend to give a quite complete picture for orthonormal systems in measure space with few atoms.

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† E-mail address: geiss@minet.uni-jena.de.

‡ E-mail address: nms06@rz.uni-kiel.d400.de.

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Introduction and Notation

The theory of type and cotype in Banach spaces is closely connected to the probability theory and provides a good frame work to distinguish relevant local properties of Banach spaces. In this paper we develop a connection between this theory and geometric properties of orthonormal systems. We do this by means of a certain approximation of orthonormal systems by systems of functions having disjoint supports. The main technical tool is the use of operator ideal techniques.

Throughout this paper the standard notation of the Banach space theory is used. All Banach spaces are real or complex, in particular $\mathbb{K}$ stands for the real or complex scalars. Given a Banach space X then B_X denotes the closed unit ball, X^* the dual and I_X the identity. If $x \in X$ and $a \in X^*$ then $\langle x, a \rangle := a(x)$, whereas for the scalar product in a Hilbert space we use $(\cdot, \cdot)$. If (Ω, p) is a σ-finite measure space (we will always assume σ-finiteness) and if $1 \leqslant r < \infty$ then $L_r(X)$ is the Banach space of all X-valued strongly measurable functions $f\colon \Omega \to X$ such that $\|f\|_{L_r(X)} := (\int_\Omega \|f\|^r \, dp)^{1/r} < \infty$. For two Banach spaces X and Y as usual $\mathscr{L}(X, Y)$ is the Banach space of all linear and continuous operators from X into Y equipped with the operator norm $\|Tx\| = \sup\{\|Tx\| \mid x \in B_X\}$. To shorten some statements let us denote the formal identity $l_p^n \to l_q^n$ by ι_{pq}^n. In the whole paper $(g_k)_{k \in \mathbb{N}}$ is a sequence of independent, normalized gaussian variables, where we use complex variables whenever the underlying Banach spaces are complex. Finally, let us fix the orthonormal systems $G_n := (g_k)_{k=1}^n$ and $U_n := (e_k)_1^n$, where (e_k) is the unit vector basis of l_2^n.

The starting point for our investigations is the following well-known observation.

THEOREM 1. *Let $T \in \mathscr{L}(X, Y)$ and let $\Phi = (\phi_k)_{k \in \mathbb{N}}$ be a orthonormal system.*

(i) *If T is of cotype 2, then one has for all finite sequences $(x_k) \subset X$*

$$\Big\| \sum_k \phi_k \, T x_k \Big\|_{L_2(Y)} \leqslant c_2(T) \Big\| \sum_k g_k x_k \Big\|_{L_2(X)}.$$

(ii) *If T is of type 2, then one has for all finite sequences $(x_k) \subset X$*

$$\Big\| \sum_k g_k \, T x_k \Big\|_{L_2(Y)} \leqslant t_2(T) \Big\| \sum_k \phi_k x_k \Big\|_{L_2(X)}.$$

Let us recall that an operator $T \in \mathscr{L}(X, Y)$ is of cotype 2 or type 2 if there are constants $c>0$ or $t>0$ such that for all finite sequences $(x_k) \subset X$

$$\left(\sum_k \|Tx_k\|^2\right)^{1/2} \leqslant c \left\|\sum_k g_k x_k\right\|_{L_2(X)} \quad \text{or} \quad \left\|\sum_k g_k Tx_k\right\|_{L_2(Y)} \leqslant t\left(\sum_k \|x_k\|^2\right)^{1/2}.$$

As usual $c_2(T) := \inf c$ and $t_2(T) := \inf t$. Restricting the above inequalities to n vectors $(x_k)_1^n$ we obtain $c_2^n(T)$ and $t_2^n(T)$ which can be defined for all $T \in \mathscr{L}(X, Y)$.

For a further discussion of Theorem 1 as well as for the first occurrence we refer to [9] (Theorems 9.24, 9.25). Clearly, Theorem 1 expresses the extreme position of gaussian variables among arbitrary orthonormal systems. An easy approximation argument yields a converse of Theorem 1 in the case of complete orthonormal systems.

THEOREM 2. *Let $\Phi = (\phi_k)_{k \in \mathbb{N}}$ be a* complete *orthonormal system in $L_2(0, 1)$ and let $T \in \mathscr{L}(X, Y)$. If there is a $c>0$ such that for all finite sequences $(x_k) \subset X$*

$$\left\|\sum_k \phi_k Tx_k\right\|_{L_2(Y)} \leqslant c \left\|\sum_k g_k x_k\right\|_{L_2(X)},$$

then T is of cotype 2. If there is a $c>0$ such that for all finite sequences $(x_k) \subset X$

$$\left\|\sum_k g_k Tx_k\right\|_{L_2(Y)} \leqslant c \left\|\sum_k \phi_k x_k\right\|_{L_2(X)},$$

then T is of type 2.

Proof. For example, one can use $\sum_{l=1}^N \|y_l\|^2 = \int_0^1 \|\sum_{k=1}^\infty (\sum_{l=1}^N (f_l, \phi_k) y_l) \phi_k(t)\|^2 \, dt$ for $f_l(t) := \sqrt{l(l+1)}\, \chi_{(1/(l+1)), 1/l)}(t)$ and the fact that $g_l' = \sum_{k=1}^\infty (f_l, \phi_k) g_k$ are again standard gaussian variables which are independent. ∎

In this paper we mainly discuss the following two problems.

(P1) Characterize those (not necessarily complete) orthonormal systems $\Phi = (\phi_k)_{k \in \mathbb{N}}$ such that the conclusions of Theorem 2 hold true.

(P2) Find a local version of Theorem 2 in the sense that we consider systems $\Phi = (\phi_k)_1^n$ of a given length and ask for the usual cotype and type constants restricted to n vectors.

To give a systematic treatment let $\Phi = (\phi_k)_{k \in I} \subset L_2(\Omega, p)$, ($I$ is a countable index set) be an orthonormal system. An operator $T \in \mathscr{L}(X, Y)$ is said to be of Φ-cotype and Φ-type, respectively, if there is a constant $c \geqslant 0$ such that

$$\left\| \sum_{k \in I} \phi_k \, Tx_k \right\|_{L_2(Y)} \leqslant c \left\| \sum_{k \in I} g_k \, x_k \right\|_{L_2(X)}$$

and

$$\left\| \sum_{k \in I} g_k \, Tx_k \right\|_{L_2(Y)} \leqslant c \left\| \sum_{k \in I} \phi_k \, x_k \right\|_{L_2(X)},$$

respectively. The best possible constants will be denoted by $c_\Phi(T)$ and $t_\Phi(T)$. Furthermore, $T \in \mathscr{L}(X, Y)$ is said to be of the modified Φ-type, if there is a constant $c \geqslant 0$ such that for all $f \in L_2(X)$

$$\left\| \sum_{k \in I} g_k \left(\int Tf\bar{\phi}_k \, dp \right) \right\|_{L_2(Y)} \leqslant c \, \|f\|_{L_2(X)}.$$

The best possible constant is denoted by $\hat{t}_\Phi(T)$. In the same way one could define the remaining case $\hat{c}_\Phi(T)$. Since this case follows by duality from the modified $\bar{\Phi}$-type ($\bar{\Phi} := (\bar{\phi}_k)$ is the conjugate system) we will omit his case. To consider the above quantities we use ideal norms as one of the main tools. Given an orthonormal system $\Phi = (\phi_k)_1^n$ we first introduce for $u \in \mathscr{L}(l_2^n, X)$

$$\Phi(u) := \left\| \sum_{k=1}^{n} \phi_k \, ue_k \right\|_{L_2(X)}.$$

The dual norm Φ^* on $\mathscr{L}(X, l_2^n)$ is given by trace duality, that is

$$\Phi^*(v) := \sup\{ |\mathrm{tr}(vu)| \mid \Phi(u) = 1 \}.$$

Note that in general the definition of $\Phi(u)$ and $\Phi^*(v)$ depends on the special choice of the orthonormal system $\{e_1, ..., e_n\} \subset l_2^n$ (we can consider Φ as a norm on X^n via $\Phi((x_1, ..., x_n)) := \|\sum \phi_k x_k\|_{L_2(X)}$ and Φ^* as the dual norm on $[X^n, \Phi]^*$). There are two standard procedures to generate ideal norms starting from Φ. For $T \in \mathscr{L}(X, Y)$ we define

$$\pi_\Phi(T) := \sup\{ \Phi(Tu) \mid \|u : l_2^n \to X\| \leqslant 1 \},$$

and, if T is a finite rank operator,

$$\nu_\Phi(T) := \inf\left\{ \sum_{j=1}^{N} \Phi(u_j) \, \|v_j\| \;\middle|\; T = \sum_{j=1}^{N} u_j v_j, \; u_j \in \mathscr{L}(l_2^n, Y), \; v_j \in \mathscr{L}(X, l_2^n) \right\}.$$

It is an easy exercise to check that π_Φ is an ideal norm on the class of all bounded operators and that ν_Φ is an ideal norm on the class of finite rank operators. That is, we have for $\alpha \in \{\pi_\Phi, \nu_\Phi\}$ the norm properties and the relations

$$\|BTA\| \leqslant \alpha(BTA) \leqslant \|B\| \, \alpha(T) \, \|A\|$$

and $\alpha(a \otimes y) = \|a\|_{X^*} \, \|y\|_Y$ for $a \otimes y \in \mathscr{L}(X, Y)$. To shorten the notation we will write in the sequel $\alpha(X)$ instead of $\alpha(I_X)$. The connection to the approximation theory is given by the geometric interpretation of the inequalities $\pi_\Phi(l^n_\infty) \geqslant \delta' \sqrt{n}$ and $\pi_\Phi(l^n_{2,\infty}) \geqslant \delta'' \sqrt{n}$. They correspond to the conditions (G'_n) and (G''_n), respectively, in Theorem 5 below. To compare the usual type and cotype with the Φ-type and cotype we will compare the π_Φ-norm with the π^n_2-norm directly.

Let us remember that an operator $T \in \mathscr{L}(X, Y)$ is absolutely q-summing $(1 \leqslant q < \infty)$ provided there is a constant $c > 0$ such that for all finite sequences $(x_k) \subset X$ one has

$$\left(\sum_k \|Tx_k\|^q\right)^{1/q} \leqslant c \sup\left\{\left(\sum_k |\langle x_k, a\rangle|^q\right)^{1/q} \,\middle|\, a \in B_{X^*}\right\}.$$

The best possible constant is denoted by $\pi_q(T)$. Considering the above inequality for n vectors $(x_k)_1^n$ only we get $\pi^n_q(T)$ which is again defined for all $T \in \mathscr{L}(X, Y)$.

The concept of the π_Φ-norms connects in a natural way the usual l-norm with the π^n_2-norm. Namely, if $G_n = (g_k)_1^n$ and $U_n = (e_k)_1^n$ are defined as below then we recover the l-norm and the π^n_2-norm for $u \in \mathscr{L}(l^n_2, X)$ by

$$l(u) = \pi_{G_n}(u) = \left\|\sum_1^n g_k u e_k\right\|_{L_2(X)} \qquad \text{and} \qquad \pi^n_2(u) = \pi_{U_n}(u).$$

These are the extreme situations since in any case

$$l(u) \leqslant \pi_\Phi(u) \leqslant \pi_2(u) \leqslant \sqrt{2}\pi^n_2(u) \qquad \text{whenever} \quad u \in \mathscr{L}(l^n_2, X)$$

(see Remark 3.10, Lemma 2.1 and [17] for the latter inequality). The following example has served us a pro type for the whole investigation and also as a motivation for the introduction of π_Φ and ν_Φ norms in connection with the problems concerning type and cotype.

EXAMPLE 3. *Let* $\Phi = E_n := (e^{ikt})_{k=1}^n \subset L_2(\Pi)$ *be the trigonometric system. Then*

$$c^n_2(T) \leqslant c c_{E_n}(T) \qquad \textit{and} \qquad t^n_2(T) \leqslant c t_{E_n}(T) \qquad \textit{for all} \quad T \in \mathscr{L}(X, Y),$$

whereas $c > 0$ *is an absolute constant independent from* n.

Proof. Using the Marcinkiewicz–Zygmund-inequality (see [18](II, p. 30), [12])

$$\left(\frac{1}{n}\sum_{l=1}^{n}\left\|\sum_{k=1}^{n} e^{2\pi i(kl/n)}x_k\right\|_X^2\right)^{1/2} \leqslant c\left(\int_0^{2\pi}\left\|\sum_{k=1}^{n} e^{ikt}x_k\right\|_X^2 \frac{dt}{2\pi}\right)^{1/2}$$

implies, by a simple rotation argument, for all $u \in \mathscr{L}(l_2^n, X)$

$$\begin{aligned}\pi_2^n(u) &= \sup_{\|w: l_2^n \to l_2^n\|=1}\left\{\left(\sum_{k=1}^{n}\|uwe_k\|^2\right)^{1/2}\right\}\\ &= \sup_{\|w: l_2^n \to l_2^n\|=1}\left\{\left(\sum_{l=1}^{n}\left\|\sum_{k=1}^{n}\frac{e^{2\pi i(kl/n)}}{\sqrt{n}}uwe_k\right\|^2\right)^{1/2}\right\}\\ &\leqslant c \sup_{\|w: l_2^n \to l_2^n\|=1}\left\{\left(\int_0^{2\pi}\left\|\sum_{k=1}^{n} e^{ikt}uwe_k\right\|^2\frac{dt}{2\pi}\right)^{1/2}\right\} = c\pi_{E_n}(u).\end{aligned}$$

Consequently,

$$\pi_2^n(u) \leqslant c\pi_{E_n}(u). \qquad (*)$$

Hence $\pi_2^n(Tu) \leqslant c\pi_{E_n}(Tu) \leqslant cc_{E_n}(T)\, l(u)$ and $c_2^n(T) \leqslant cc_{E_n}(T)$ for all $T \in \mathscr{L}(X, Y)$. Let us turn to the type situation. To deduce the type equivalence from $(*)$ we cannot use the Riesz-projections since this would require UMD-properties (for example) for X (see e.g. [12]). Instead of this we use the de la Vallée Poussin kernel and find a sequence $\lambda_1, ..., \lambda_n \geqslant 0$ with $\lambda_i = 1$ for $n_0 \leqslant i \leqslant n_1$, whereas $n_1 - n_0 \geqslant n/3$, such that

$$\left\|\sum_{k=1}^{n}\lambda_k \hat{f}(k)\, e^{ikt}\right\|_{L_2(X)} \leqslant c_1 \|f\|_{L_2(X)}$$

for all $f \in L_2(X)$ and some absolute constant $c_1 > 0$ independent from n $(\hat{f}(k) = \int_0^{2\pi} f(t)\, e^{-ikt}\, dt/2\pi)$. Now let $1 \leqslant r \leqslant n/3$ and $u \in \mathscr{L}(l_2^r, X)$ be such that $\hat{f}(k+N_0) = ue_k$ for $k = 1, ..., r$. Defining $J: l_2^r \to l_2^n$ by $Je_k := e_{k+n_0-1}$ and $\tilde{u}: l_2^n \to X$ by $\tilde{u} := \sum_{k=1}^{n}\lambda_k e_k \otimes \hat{f}(-n_0+1+N_0+k)$ we obtain $u = \tilde{u}J$ and

$$v_{E_n}(u) \leqslant E_n(\tilde{u}) \leqslant c_1 \|fe^{-i(-n_0+1+N_0)t}\|_2 = c_1 \|f\|_2.$$

Using a simple blocking argument and the definition of v_{E_n} this means for $w \in \mathscr{L}(l_2^n, X)$

$$v_{E_n}(w) \leqslant c_2 \inf\{\|f\|_{L_2(X)} \mid \hat{f}(k) = we_k,\ k = 1, ..., n\}. \qquad (**)$$

This allows us to consider the bigger norm $\hat{t}_{E_n}$. Indeed, we get

$$\left\| \sum_{k=1}^{n} T\hat{f}(k)\, g_k \right\|_{L_2(Y)} \leqslant t_{E_n}(T)\, v_{E_n}\left(\sum_{k=1}^{n} e_k \otimes \hat{f}(k) \right) \leqslant c_2\, t_{E_n}(T)\, \|f\|_{L_2(X)}$$

such that $\hat{t}_{E_n}(T) \leqslant c_2\, t_{E_n}(T)$. Now we are in the position to use a duality argument. The Marcinkiewicz–Zygmund-inequality gives for $g'_l := 1/\sqrt{n}\ \sum_{k=1}^{n} e^{2\pi i(kl/n)} \bar{g}_k$ and $h \in L_2(Y^*)$

$$\left(\sum_{l=1}^{n} \left\| \int T^* h \bar{g}'_l \, dp \right\|_{X^*}^2 \right)^{1/2} = \left(\sum_{l=1}^{n} \left\| \sum_{k=1}^{n} \frac{e^{-2\pi i(kl/n)}}{\sqrt{n}} \int T^* h g_k \, dp \right\|_{X^*}^2 \right)^{1/2}$$
$$\leqslant c \left(\int_0^{2\pi} \left\| \sum_{k=1}^{n} e^{-ikt} \int T^* h g_k \, dp \right\|_{X^*}^2 \frac{dt}{2\pi} \right)^{1/2}.$$

Using the convenient duality properties of the modified type we can continue with

$$\left(\int_0^{2\pi} \left\| \sum_{k=1}^{n} e^{-ikt} \left(\int T^* h g_k \, dp \right) \right\|_{X^*}^2 \frac{dt}{2\pi} \right)^{1/2} \leqslant \hat{t}_{E_n}(T)\, \|h\|_{L_2(Y^*)}.$$

Consequently,

$$\left(\sum_{l=1}^{n} \left\| \int T^* h \bar{g}'_l \, dp \right\|_{X^*}^2 \right)^{1/2} \leqslant c c_2\, t_{E_n}(T)\, \|h\|_{L_2(Y^*)}.$$

Using again duality we arrive at $t_2^n(T) \leqslant c c_2\, t_{E_n}(T)$. ∎

The paper is organized in the following way. First we consider the problem (P2) mentioned above and derive as a simple consequence the answer of (P1). Concerning the problem (P2) our main theorem states that it is sufficient to test cotype and type conditions on rather extreme operators. More precisely we prove

THEOREM 4. *Let $\Phi = (\phi_k)_1^n$ be an orthonormal system and let $\delta > 0$. For some absolute constant $c > 0$ not depending on δ, n, and Φ, the following holds true.*

1. *If $c_\Phi(l_\infty^n) \geqslant \delta \sqrt{n/(\log(n+1))}$ or $\hat{t}_\Phi(l_1^n) \geqslant \delta \sqrt{n}$ then $\delta^3 \pi_2(w) \leqslant c \pi_\Phi(w)$ for all $w \in \mathscr{L}(l_2^n, X)$. Consequently, for all $T \in \mathscr{L}(X, Y)$*

$$\frac{\delta^3}{c}\, c_2^n(T) \leqslant c_\Phi(T) \leqslant \sqrt{2}\, c_2^n(T) \qquad \textit{and} \qquad \frac{\delta^3}{c}\, t_2^n(T) \leqslant \hat{t}_\Phi(T) \leqslant \sqrt{2}\, t_2^n(T).$$

2. *If* $t_\Phi(l_1^n) \geqslant \delta\sqrt{n}$ *then* $\delta^3 v_\Phi(w) \leqslant c\pi_2(w^*)$ *for all* $w \in \mathscr{L}(l_2^n, X)$. *Consequently, for all* $T \in \mathscr{L}(X, Y)$

$$\frac{\delta^3}{c}\, t_2^n(T) \leqslant t_\Phi(T) \leqslant \hat{t}_\Phi(T) \leqslant \sqrt{2}\, t_2^n(T).$$

It turns out that the conditions $(*)$ and $(**)$ from Example 3 are necessary in general. Note, that $\pi_2(u^*) \leqslant \Phi^*(u^*) \leqslant \|f\|_2$ if $\hat{f}(k) = ue_k$ $(k = 1, ..., n)$ (see section 2) such that Theorem 4(2) implies $(**)$ with $c_2 = c/\delta^3$. In section 1 we establish an abstract version of Theorem 4 in the terms of operator ideals whereas the connection to the notion of cotype and type is given in section 2. The proof of Theorem 4(1) consists of several steps formulated in the next theorem which is verified in section 2. In the first step $(G_n) \to (G_n')$ one we get rid of the logarithmical factor but lose the orthogonality. In the second one $(G_n') \to (G_n'')$, which is an essential part of the proof of Theorem 4, we come back to an orthonormal system. The last condition in the abstract corresponds to (G_n'') via an observation of Bourgain.

THEOREM 5. *Let* $\Phi = (\phi_k)_1^n \subset L_2(\Omega, p)$ *be an orthonormal system and let* $\delta, \delta', \delta'' > 0$. *Let us define the following conditions.*

(G_n) $c_\Phi(l_\infty^n) \geqslant \delta\sqrt{n/(\log(n+1))}$.

(G_n') *There exists n functions* $h_j \in \operatorname{span}\{\phi_k \mid k = 1, ..., n\}$ *with* $\|h_j\|_2 \leqslant 1$ *and*

$$\left(\int_\Omega \sup_{j=1,...,n} |h_j|^2\, dp\right)^{1/2} \geqslant \delta' \sqrt{n}.$$

(G_n'') *There exists an orthonormal system* $\Psi = (\psi_j)_1^n \subset \operatorname{span}\{\phi_k \mid k = 1, ..., n\}$ *with*

$$\left(\int_\Omega \sup_{j=1,...,n} |\psi_j|^2\, dp\right)^{1/2} \geqslant \delta'' \sqrt{n}.$$

Then $(G_n) \to (G_n') \to (G_n'') \to (G_n)$ *with* $\delta' = \delta/(20c_0\sqrt{1+\log((c_0/\delta)+1)})$, $\delta'' = (1/c_1)\,\delta'^3$, *and* $\delta = (1/c_2)\,\delta''$ *for numerical constants* $c_0, c_1, c_2 > 0$.

In section 3 we consider orthonormal systems defined on measure spaces with few atoms and no continuous part. We prove the following Theorem 6 which uses the local theory of Banach spaces to clarify the relations between cotype and type conditions.

THEOREM 6. *Let $1 \leqslant n \leqslant N$. Then one has the following.*

(1) $c_2^n(T) \leqslant 12\sqrt{N/n}\, c_\Phi(T)$ *for all* $T \in \mathscr{L}(X, Y)$ *and all* $\Phi = (\phi_k)_1^n \subset l_2^N$.

(2) *For all* $2 < q < \infty$ *there is an orthonormal system* $\Phi = (\phi_k)_1^n \subset l_2^N$ *such that*

$$c_\Phi(l_\infty^n) \leqslant c_q \max\left\{\sqrt{q}\, n^{1/q - 1/2}, \left(\frac{n}{N}\right)^{1/q}\right\} c_2^n(l_\infty^n).$$

(3) *As long as* $0 < \varepsilon < 1$ *and* $1 \leqslant n \leqslant (1-\varepsilon)N$ *there is an orthonormal system* $\Phi = (\phi_k)_1^n \subset l_2^N$ *with*

$$t_\Phi(l_1) \leqslant c_0 \sqrt{\frac{1}{\varepsilon} \log\left(1 + \frac{1}{\varepsilon}\right)}.$$

$c_0 > 0$ *is an absolute constant whereas* $c_q > 0$ *depends on* q *only.*

Assertion (1) shows that as long as N is proportional to n the corresponding cotype constants are equivalent, whereas in (3) "pathological" orthonormal systems are found for the notion of type. The theory of Λ_p-sets is involved for the construction of the orthonormal systems in the second assertion. Choosing $n \sim N^\delta$ we obtain systems which fail the first conclusion of Theorem 6.

1. Abstract Theory

Throughout this section we will say that a norm α on $\mathscr{L}(l_2^n, \cdot)$ (which means the collection of all $\mathscr{L}(l_2^n, X)$, where n is fixed and X is an arbitrary Banach space) is an ideal norm if

$$\|TuA\| \leqslant \alpha(TuA) \leqslant \|T\|\, \alpha(u)\, \|A\| \quad \text{and} \quad \alpha(a \otimes x) = \|a\|_{l_2^n} \|x\|_X$$

for all $T \in \mathscr{L}(X, Y)$, $u \in \mathscr{L}(l_2^n, X)$, $A \in \mathscr{L}(l_2^n, l_2^n)$, $a \in l_2^n$ and $x \in X$. Similarly, β is an ideal norm on $\mathscr{L}(\cdot, l_2^n)$ if

$$\|AvT\| \leqslant \beta(AvT) \leqslant \|A\|\, \beta(v)\, \|T\| \quad \text{and} \quad \beta(b \otimes y) = \|b\|_{Y^*} \|y\|_{l_2^n}$$

for all $A \in \mathscr{L}(l_2^n, l_2^n)$, $v \in \mathscr{L}(Y, l_2^n)$, $T \in \mathscr{L}(X, Y)$, $b \in Y^*$ and $y \in l_2^n$. The adjoint ideal norms α^* on $\mathscr{L}(X, l_2^n)$ and β^* on $\mathscr{L}(l_2^n, X)$ are given by

$$\alpha^*(v) = \sup_{\alpha(u:\, l_2^n \to X) \leqslant 1} |\mathrm{tr}(vu)| \quad \text{and} \quad \beta^*(u) = \sup_{\beta(v:\, X \to l_2^n) \leqslant 1} |\mathrm{tr}(vu)|.$$

Furthermore, we define $\mathbb{1}-\pi_2^n(T) := \inf c$ such that

$$\left(\sum_{i=1}^{n} \|Tx_i\|^2\right)^{1/2} \leqslant c \sup\left\{\left(\sum_{i=1}^{n} |\langle x_i, a\rangle|^2\right)^{1/2} \,\middle|\, a \in B_{X^*}\right\}$$

for all $x_1, ..., x_n \in X$ with $\|Tx_1\| = \cdots = \|Tx_n\|$. The following lemma is the key for what follows. The proof is similar to the proof of [3] (Theorem 3.1). We thank Th. Kühn for his hints to improve the constant appearing in Lemma 1.1.

LEMMA 1.1. *Let $T \in \mathscr{L}(X, Y)$ and $n \in \mathbb{N}$. Then*

$$\pi_2^n(T) \leqslant \sqrt{6}\,\mathbb{1}-\pi_2^n(T).$$

Proof. Assume $x_1, ..., x_n \in X$ with $\sum_1^n \|Tx_i\|^2 = 1$ whereas $\|Tx_i\| > 0$ for all i. Setting

$$\sigma_m := \{i \in \{1, ..., n\} \mid 2^{-m} < \|Tx_i\|^2 \leqslant 2^{1-m}\}$$

we obtain $\sum_{m=1}^{\infty} |\sigma_m| = n$. Let $m_0 \in \mathbb{N}$ such that $2^{m_0-1} < 3n \leqslant 2^{m_0}$. Then we get

$$\sum_{m=m_0+1}^{\infty} \sum_{i \in \sigma_m} \|Tx_i\|^2 \leqslant \sum_{m=m_0+1}^{\infty} |\sigma_m|\, 2^{1-m} \leqslant 2^{-m_0} \sum_{m=m_0+1}^{\infty} |\sigma_m| \leqslant 2^{-m_0} n \leqslant \tfrac{1}{3}.$$

Now we define

$$I := \{(i, j) \mid j = 1, ..., 2^{m_0-m} \text{ if } i \in \sigma_m;\ m = 1, ..., m_0\}$$

and obtain

$$|I| = \sum_{m=1}^{m_0} |\sigma_m|\, 2^{m_0-m} \leqslant 2^{m_0} \sum_{m=1}^{\infty} \sum_{i \in \sigma_m} \|Tx_i\|^2 \leqslant 2^{m_0} < 6n$$

as well as

$$|I| = \sum_{m=1}^{m_0} |\sigma_m|\, 2^{m_0-m} \geqslant 2^{m_0-1} \sum_{m=1}^{m_0} \sum_{i \in \sigma_m} \|Tx_i\|^2$$

$$\geqslant 2^{m_0-1}\left(1 - \sum_{m=m_0+1}^{\infty} \sum_{i \in \sigma_m} \|Tx_i\|^2\right) \geqslant 2^{m_0-1}(1-\tfrac{1}{3}) \geqslant n.$$

Defining $y_{ij} := x_i/\|Tx_i\|$ for $(i, j) \in I$ and choosing a subset $J \subseteq I$ with $|J| = n$ we deduce

$$\begin{aligned} n=|J| &\leqslant 1\!\!1-\pi_2^n(T)^2 \sup_{a\in B_{X^*}} \Big(\sum_J |\langle y_{ij}, a\rangle|^2\Big) \\ &\leqslant 1\!\!1-\pi_2^n(T)^2 \sup_{a\in B_{X^*}} \Big(\sum_I |\langle y_{ij}, a\rangle|^2\Big) \\ &\leqslant 1\!\!1-\pi_2^n(T)^2 \sup_{a\in B_{X^*}} \Big(\sum_{m=1}^{m_0}\sum_{i\in\sigma_m}\sum_{j=1}^{2^{m_0-m}} \Big|\Big\langle \frac{x_i}{\|Tx_i\|}, a\Big\rangle\Big|^2\Big) \\ &\leqslant 2^{m_0} 1\!\!1-\pi_2^n(T)^2 \sup_{a\in B_{X^*}} \Big(\sum_{m=1}^{m_0}\sum_{i\in\sigma_m} |\langle x_i, a\rangle|^2 \frac{2^{-m}}{\|Tx_i\|^2}\Big) \\ &\leqslant 2^{m_0} 1\!\!1-\pi_2^n(T)^2 \sup_{a\in B_{X^*}} \Big(\sum_{m=1}^{m_0}\sum_{i\in\sigma_m} |\langle x_i, a\rangle|^2\Big) \\ &\leqslant 2^{m_0} 1\!\!1-\pi_2^n(T)^2 \sup_{a\in B_{X^*}} \Big(\sum_{i=1}^{n} |\langle x_i, a\rangle|^2\Big). \end{aligned}$$

Consequently,

$$\pi_2^n(T) \leqslant \sqrt{\frac{2^{m_0}}{n}}\, 1\!\!1-\pi_2^n(T) \leqslant \sqrt{6}\, 1\!\!1-\pi_2^n(T). \quad \blacksquare$$

Lemma 1.2. *Let α and β be ideal norms on $\mathscr{L}(l_2^n, \cdot)$ and $\mathscr{L}(\cdot, l_2^n)$, respectively. Then for all $u\in\mathscr{L}(l_2^n, X)$ and $v\in\mathscr{L}(X, l_2^n)$*

(1) $\alpha(\iota_{2,\infty}^n)\leqslant\alpha(u)$ *whenever* $\|ue_i\|=1$ *for* $i=1, ..., n$,

(2) $\beta(\iota_{1,2}^n)\leqslant\beta(v)$ *whenever* $\|v^*e_i\|=1$ *for* $i=1, ..., n$.

Proof. (1) Choosing $a_1, ..., a_n\in B_{X'}$ with $\langle ue_i, a_i\rangle=1$ and setting $w:=\sum_{i=1}^n a_i\otimes e_i\in\mathscr{L}(X, l_\infty^n)$ we obtain

$$n=\operatorname{tr}(\iota_{\infty,2}^n wu)\leqslant\alpha^*(\iota_{\infty,2}^n)\,\|w\|\,\alpha(u)\leqslant\alpha^*(\iota_{\infty,2}^n)\,\alpha(u).$$

Using $\alpha(\iota_{2,\infty}^n)\,\alpha^*(\iota_{\infty,2}^n)=n$ from [10] (9.1.8) we arrive at our assertion. (2) For $\varepsilon>0$ we choose $x_1, ..., x_n\in B_X$ with $\langle x_i, v^*e_i\rangle\geqslant 1-\varepsilon$ and set $w:=\sum e_i\otimes x_i\in\mathscr{L}(l_1^n, X)$. Hence

$$(1-\varepsilon)n\leqslant\operatorname{tr}(\iota_{2,1}^n vw)\leqslant\beta^*(\iota_{2,1}^n)\,\beta(v)\,\|w\|$$

and $n\leqslant\beta^*(\iota_{2,1}^n)\,\beta(v)$ such that we finish as in (1). $\blacksquare$

Lemma 1.3. *Let α and β be ideal norms on $\mathscr{L}(l_2^n, \cdot)$ and $\mathscr{L}(\cdot, l_2^n)$, respectively. Then*

(1) $\alpha(\iota^n_{2,\infty})\,\mathbb{1} - \pi^n_2(u) \leqslant \sqrt{n}\alpha(u)$ *for all* $u \in \mathscr{L}(l^n_2, X)$,

(2) $\beta(\iota^n_{1,2})\,\mathbb{1} - \pi^n_2(v^*) \leqslant \sqrt{n}\beta(v)$ *for all* $v \in \mathscr{L}(X, l^n_2)$.

Proof. (1) Let $w \in \mathscr{L}(l^n_2, l^n_2)$ be such that $\|uwe_i\| = 1$ for $i = 1, ..., n$. Then $\alpha(\iota^n_{2,\infty}) \leqslant \alpha(uw) \leqslant \alpha(u)\,\|w\|$ and

$$\alpha(\iota^n_{2,\infty}) \sup \left\{ \frac{\sqrt{n}}{\|w\|} \;\middle|\; \|uwe_i\| = 1 \text{ for } i = 1, ..., n \right\} \leqslant \sqrt{n}\alpha(u).$$

(2) Let $w \in \mathscr{L}(l^n_2, l^n_2)$ be such that $\|v^*w^*e_i\| = 1$ for $i = 1, ..., n$. Then $\beta(\iota^n_{1,2}) \leqslant \beta(wv) \leqslant \|w\|\,\beta(v)$ and

$$\beta(\iota^n_{1,2}) \sup \left\{ \frac{\sqrt{n}}{\|w^*\|} \;\middle|\; \|v^*w^*e_i\| = 1 \text{ for } i = 1, ..., n \right\} \leqslant \sqrt{n}\beta(v). \quad \blacksquare$$

Combining Lemmata 1.1, 1.3, and the fact that $\pi_2(T) \leqslant \sqrt{2}\pi^n_2(T)$ whenever $\text{rank}(T) \leqslant n$ (see [17]) we get

THEOREM 1.4. *Let* α *and* β *be ideal norms on* $\mathscr{L}(l^n_2, \cdot)$ *and* $\mathscr{L}(\cdot, l^n_2)$, *respectively. Then*

(1) $\alpha(\iota^n_{2,\infty})\,\pi_2(u) \leqslant \sqrt{12}\,\sqrt{n}\alpha(u)$ *for all* $u \in \mathscr{L}(l^n_2, X)$,

(2) $\beta(\iota^n_{1,2})\,\pi_2(v^*) \leqslant \sqrt{12}\,\sqrt{n}\beta(v)$ *for all* $v \in \mathscr{L}(X, l^n_2)$.

Let us recall that the nth approximation number [11] of an operator $T \in \mathscr{L}(X, Y)$ is defined by

$$a_n(T) := \inf\{\|T - L\| \mid L \in \mathscr{L}(X, Y), \text{rank}(L) < n\}.$$

To bring the above theorem in a form we need we will use

LEMMA 1.5. *Let* β *be a norm on* $\mathscr{L}(l^n_1, l^n_2)$ *such that for all* $v \in \mathscr{L}(l^n_1, l^n_2)$ *and all orthogonal matrices* $w \in \mathscr{L}(l^n_2, l^n_2)$ *one has* $\beta(wv) = \beta(v) \leqslant \nu(v)$. *If* $\sup\{\beta(v) \mid \|v : l^n_1 \to l^n_2\| \leqslant 1\} \geqslant \delta\sqrt{n}$ *for some* $\delta > 0$, *then* $\beta(\iota^n_{1,2}) \geqslant \delta^3/c\,\sqrt{n}$ *where* $c > 0$ *is an absolute constant.*

Proof. Using Grothendieck's inequality (see [14] (Theorem 5.10)) our assumption ensures the existence of some $v \in \mathscr{L}(l^n_1, l^n_2)$ with $\pi_2(v) \leqslant K_G$ and $\beta(v) \geqslant \delta\sqrt{n}$. Trace duality gives some $u \in \mathscr{L}(l^n_2, l^n_1)$ with $\pi_2(u) \geqslant \delta/K_G\sqrt{n}$ and $\beta^*(u) \leqslant 1$ (note that $\beta \leqslant \nu$ implies $\|\cdot\| \leqslant \beta^*$). Exploiting [11] (2.7.4) we deduce for $\theta > 0$

$$\frac{\delta}{K_G}\sqrt{n} \leqslant \pi_2(u) \leqslant c \sum_{k=1}^{n} \frac{a_k(u)}{\sqrt{k}} \leqslant 2c(\sqrt{[\theta n]} + \sqrt{n}a_{[\theta n]+1}(u))$$

and $\delta/(2cK_G) \leqslant \sqrt{\theta} + a_{[\theta n]+1}(u)$. Setting $\theta := (\delta/(4cK_G))^2$ we obtain (using [11] (2.11.6, 2.11.8)) for some orthogonal $w \in \mathscr{L}(l_2^n, l_2^n)$

$$\begin{aligned} \frac{\delta}{4cK_G} &\leqslant a_{[\theta n]+1}(u) \leqslant \sqrt{n}\, a_{[\theta n]+1}(\iota_{1,2}^n u) \\ &\leqslant \frac{\sqrt{n}}{[\theta n]+1} \sum_1^n a_k(\iota_{1,2}^n u) = \frac{\sqrt{n}}{[\theta n]+1} |\mathrm{tr}(w\iota_{1,2}^n u)| \\ &\leqslant \frac{1}{\theta\sqrt{n}} \beta(w\iota_{1,2}^n)\, \beta^*(u) = \frac{1}{\theta\sqrt{n}} \beta(\iota_{1,2}^n)\, \beta^*(u). \end{aligned}$$

Hence $\delta^3/(4cK_G)^3 \sqrt{n} \leqslant \beta(\iota_{1,2}^n)$. ∎

Now Theorem 1.4 and Lemma 1.5 imply

COROLLARY 1.6. *Let α and β be ideal norms on $\mathscr{L}(l_2^n, \cdot)$ and $\mathscr{L}(\cdot, l_2^n)$, respectively. Then*

(1) $\sup\{\alpha(w)/\sqrt{n} \mid \|w: l_2^n \to l_\infty^n\| = 1\}^3 \pi_2(u) \leqslant c\alpha(u)$ *for all* $u \in \mathscr{L}(l_2^n, X)$,

(2) $\sup\{\beta(w)/\sqrt{n} \mid \|w: l_1^n \to l_2^n\| = 1\}^3 \pi_2(v^*) \leqslant c\beta(v)$ *for all* $v \in \mathscr{L}(X, l_2^n)$,

where $c > 0$ is an absolute constant.

Proof. (1) Setting $\beta(v) := \alpha(v^*)$ for $v \in \mathscr{L}(X, l_2^n)$ and $\delta > 0$ such that

$$\delta\sqrt{n} = \sup\{\alpha(w) : \|w: l_2^n \to l_\infty^n\| = 1\} = \sup\{\beta(v) : \|v: l_1^n \to l_2^n\| = 1\}$$

we obtain from Theorem 1.4 and Lemma 1.5

$$c\sqrt{n}\alpha(u) \geqslant \alpha(\iota_{2,\infty}^n)\, \pi_2(u) \geqslant \beta(\iota_{1,2}^n)\, \pi_2(u) \geqslant \frac{\delta^3}{c'} \sqrt{n}\pi_2(u).$$

Consequently, $\delta^3\pi_2(u) \leqslant cc'\alpha(u)$. (2) follows directly. ∎

In the following Corollary 1.6 is made applicable to our problems concerning type and cotype with respect to arbitrary orthonormal systems. To do this we need the Weyl numbers and nuclear operators. The nth Weyl number [11] of an operator $T \in \mathscr{L}(X, Y)$ is given by

$$x_n(T) := \sup\{a_n(Tu) \mid u \in \mathscr{L}(l_2^n, X),\ \|u\| = 1\}.$$

An operator $T \in \mathscr{L}(X, Y)$ is nuclear [10] provided that T can be written as

$$T = \sum_1^\infty a_n \otimes y_n$$

with $a_n \in X^*$, $y_n \in Y$, and $\sum_1^\infty \|a_n\| \, \|y_n\| < \infty$. We set $\nu(T) := \inf \sum_1^\infty \|a_n\| \, \|y_n\|$ where the infimum is taken over all possible representations.

LEMMA 1.7. *Let α be an ideal norm on $\mathscr{L}(l_2^n, \cdot)$ and let $u \in \mathscr{L}(l_2^n, l_\infty^n)$ be such that*

$$l(u) \leqslant 1 \qquad \text{and} \qquad \alpha(u) \geqslant \delta \sqrt{\frac{n}{\log(n+1)}}$$

for some $\delta > 0$. Then there exists an operator $\tilde{u} \in \mathscr{L}(l_2^n, l_\infty^n)$ with

$$\|\tilde{u}\| = 1 \qquad \text{and} \qquad \alpha(\tilde{u}) \geqslant \frac{\delta}{c_0 \sqrt{A}} \sqrt{n}$$

for all $A > 1$ whenever $n \geqslant (c_0 \sqrt{A}/\delta)^{2A/(A-1)}$. Moreover

$$\alpha(\tilde{u}) \geqslant \frac{\delta}{20 c_0 \sqrt{1 + \log((c_0/\delta) + 1)}} \sqrt{n}$$

for $n = 1, 2, \ldots$. The constant $c_0 > 0$ is independent from n, δ, A and α.

Proof. First we observe that $a_r(u) \leqslant c(l(u)/\sqrt{\log(r+1)})$ for $u \in \mathscr{L}(l_2^n, l_\infty^n)$, which follows for example from the much deeper factorization $u = BDA$ due to Talagrand used in the proof of Lemma 3.3. This gives the existence of an orthogonal projection $P \in \mathscr{L}(l_2^n, l_2^n)$ with $\operatorname{rank}(P) > n - r$ and $\|uP\| = a_r(u) \leqslant c/\sqrt{1 + \log(r+1)}$. Hence via trace duality we find an operator $v \in \mathscr{L}(l_\infty^n, l_2^n)$ with $\alpha^*(v) = 1$ and

$$\begin{aligned}
\delta \sqrt{\frac{n}{\log(n+1)}} &\leqslant |\operatorname{tr}(vu)| \leqslant |\operatorname{tr}(vuP)| + |\operatorname{tr}(vu(I-P))| \\
&\leqslant \nu(u) \, \|uP\| + 2 \sum_{k=1}^{r-1} x_k(v) \, a_k(u(I-P)) \\
&\leqslant \nu(u) \, a_r(u) + 2 \sum_{k=1}^{r-1} x_k(v) \, a_k(u).
\end{aligned}$$

Since Grothendieck's inequality [14] implies

$$x_k(v) \leqslant k^{-1/2} \pi_2(v) \leqslant K_G k^{-1/2} \|v\| \leqslant K_G k^{-1/2} \alpha^*(v) \leqslant K_G k^{-1/2}$$

we can continue to

$$\frac{\delta}{c}\sqrt{\frac{n}{\log(n+1)}} \leqslant \frac{\nu(v)}{\sqrt{\log(r+1)}} + 2K_G \sum_{k=1}^{r-1} \frac{1}{\sqrt{k}\log(k+1)}$$
$$\leqslant \frac{\nu(v)}{\sqrt{\log(r+1)}} + 2K_G c' \sqrt{\frac{r}{\log(r+1)}}.$$

Hence, for $1 \leqslant r \leqslant n$,

$$\frac{\delta}{c}\sqrt{\frac{\log(r+1)}{\log(n+1)}}\sqrt{n} - 2c'K_G\sqrt{r} \leqslant \nu(v).$$

Now we pick for $A > 1$ an $r \in \mathbb{N}$ with $1 \leqslant r \leqslant n$ and $r \leqslant (n+1)^{1/A} \leqslant r+1$ and obtain

$$\frac{\delta}{c\sqrt{A}} n^{1/2} - 4c'K_G n^{1/(2A)} \leqslant \nu(v).$$

Consequently, $n \geqslant (8cc'K_G\sqrt{A}/\delta)^{2A/(A-1)}$ implies $\nu(v) \geqslant \delta/(2c\sqrt{A})\, n^{1/2}$. Finally, the desired operator $\tilde{u} \in \mathscr{L}(l_2^n, l_\infty^n)$ is chosen such that $\|\tilde{u}\| = 1$ and $|\mathrm{tr}(v\tilde{u})| = \nu(v)$. Setting $c_0 := \max(8cc'K_G, 2c)$ we arrive at the first part of our assertion. To prove the second assertion we put $A_0 := 2 + 2\log(c_0/\delta + 1) \geqslant 2$. It is clear that it remains to consider the situation $c_0\sqrt{A_0}/\delta \geqslant 1$ and $n < (c_0\sqrt{A_0}/\delta)^{2A_0/(A_0-1)}$. Here we use $A_0/(A_0-1) \leqslant 1 + 2/A_0$ and $(c_0\sqrt{A_0}/\delta)^{2/A_0} \leqslant e^2$ to conclude (for any $\tilde{u}$ with $\|\tilde{u}\| = 1$)

$$\frac{\delta}{e^2 c_0\sqrt{A_0}}\sqrt{n} \leqslant \left(\frac{\delta}{c_0\sqrt{A_0}}\right)^{1+(2/A_0)}\sqrt{n}$$
$$\leqslant \left(\frac{\delta}{c_0\sqrt{A_0}}\right)^{A_0/(A_0-1)}\sqrt{n} \leqslant 1 \leqslant \alpha(\tilde{u}). \quad \blacksquare$$

The main result of this section is

THEOREM 1.8. *Let α be an ideal norm on $\mathscr{L}(l_2^n, \cdot)$ and let $\delta > 0$.*

(1) *If there is an operator $u \in \mathscr{L}(l_2^n, l_\infty^n)$ such that $l(u) \leqslant 1$ and $\alpha(u) \geqslant \delta\sqrt{n/\log(n+1)}$ then*

$$\delta^3 \pi_2(w) \leqslant c\alpha(w) \qquad \textit{for all} \quad w \in \mathscr{L}(l_2^n, X).$$

(2) *If there is an operator $u \in \mathscr{L}(l_2^n, l_1^n)$ such that $\alpha(u) \leqslant 1$ and $l(u) \geqslant \delta\sqrt{n}$ then*

$$\delta^3 \alpha(w) \leqslant c\pi_2(w^*) \qquad \textit{for all} \quad w \in \mathscr{L}(l_2^n, X).$$

$c > 0$ is an absolute constant independent from n, δ and α.

Proof. (1) Corollary 1.6 and Lemma 1.7 imply (for some c, $c_0>0$) $(\delta/(c_0\sqrt{A}))^3\,\pi_2(w)\leqslant c\alpha(w)$ for all $w\in\mathscr{L}(l_2^n, X)$ whenever $n\geqslant (c_0\sqrt{A}/\delta)^{2A/(A-1)}$. Setting $A=3/2$ we obtain

$$\pi_2(w)\leqslant\sqrt{n}\,\|w\|\leqslant\sqrt{n}\alpha(w)\leqslant\left(\frac{c_0\sqrt{A}}{\delta}\right)^3\alpha(w)$$

in the case $n<(c_0\sqrt{A}/\delta)^{2A/(A-1)}$. (2) Using trace duality we find $v\in\mathscr{L}(l_1^n, l_2^n)$ with $l^*(v)\leqslant 1$ and $\alpha^*(v)\geqslant\delta\sqrt{n}$. Applying [13] and [17] (Theorem 12.7) we get $l(v^*)\leqslant K\sqrt{\log(n+1)}$ for some numerical constant $K>0$. Lemma 1.7 (applied to $\tilde{\alpha}(w)=\alpha^*(w^*)$) produces an operator $\tilde{u}\in\mathscr{L}(l_2^n, l_\infty^n)$ with $\|\tilde{u}\|\leqslant 1$ and $\alpha^*(\tilde{u}^*)\geqslant\delta/(Kc_0\sqrt{A})\sqrt{n}$ whenever $n\geqslant(Kc_0\sqrt{A}/\delta)^{2A/(A-1)}$. Corollary 1.6 (applied to $\beta(w)=\alpha^*(w)$) shows $(\delta/(Kc_0\sqrt{A}))^3\;\pi_2(w^*)\leqslant c\alpha^*(w)$ for all $w\in\mathscr{L}(X, l_2^n)$. Now trace duality gives for $\tilde{w}\in\mathscr{L}(l_2^n, X)$ the existence of some $w\in\mathscr{L}(X, l_2^n)$ with $\alpha^*(w)=1$ and

$$\alpha(\tilde{w})=\operatorname{tr}(w\tilde{w})\leqslant\pi_2(w^*)\,\pi_2(\tilde{w}^*)\leqslant\alpha^*(w)c\left(\frac{Kc_0\sqrt{A}}{\delta}\right)^3\pi_2(\tilde{w}^*)$$
$$\leqslant c\left(\frac{Kc_0\sqrt{A}}{\delta}\right)^3\pi_2(\tilde{w}^*).$$

In the case $n<(c_0\sqrt{A}/\delta)^{2A/(A-1)}$ we can continue as in (1) since $\alpha(\tilde{w})\leqslant v(\tilde{w})\leqslant\sqrt{n}\pi_2(\tilde{w}^*)$. ∎

2. Orthonormal Systems in Connection with Type and Cotype

To handle the Φ-type and cotype norms it is sometimes convenient to introduce for orthonormal systems $\Phi=(\phi_k)_1^n$ and $\Psi=(\psi_k)_1^n$ and an operator $T\in\mathscr{L}(X, Y)$ the quantity $\delta(T\mid\Phi,\Psi):=\inf c$, such that

$$\left\|\sum_{k=1}^n\phi_k\left(\int Tf\bar{\psi}_k\,dp\right)\right\|_{L_2(Y)}\leqslant c\,\|f\|_{L_2(X)}$$

for all $f\in L_2(X)$ (see [12]). It is clear that $\delta(T\mid\Phi,\Psi)=\delta(T^*\mid\bar{\Psi},\bar{\Phi})$ whereas $\bar{\Phi}:=(\bar{\phi}_k)_1^n$ is the conjugate system of $\Phi=(\phi_k)_1^n$. Using the same arguments as in [7] (Lemma 9.2) ([17] (Theorem 12.7)) it turns out that

$$\delta(T\mid\Phi,\Psi)=\sup\{\Phi(Tu)\mid\bar{\Psi}^*(u^*)\leqslant 1,\ u\in\mathscr{L}(l_2^n, X)\},$$

where $\bar{\Psi}^*$ stands for $(\bar{\Psi})^*$. In fact, for $u \in \mathscr{L}(l_2^n, X)$ one obtains

$$\begin{aligned}\bar{\Psi}^*(u^*) &= \sup\left\{\left|\sum_1^n \langle ue_k, a_k\rangle\right| \,\middle|\, \left\|\sum_1^n \bar{\psi}_k a_k\right\|_2 \leqslant 1\right\}\\ &= \sup\left\{\left|\int \left\langle f(\omega), \sum_1^n \bar{\psi}_k(\omega) a_k\right\rangle dp(\omega)\right| \,\middle|\, \left\|\sum_1^n \bar{\psi}_k a_k\right\|_2 \leqslant 1,\right.\\ &\qquad\left.\int f\bar{\psi}_k\, dp = ue_k\right\}\\ &= \inf\left\{\left\| f\colon \operatorname{span}\left\{\sum_1^n \bar{\psi}_k a_k\right\} \to \mathbb{K}\right\| \,\middle|\, f \in L_2(X),\ \int f\bar{\psi}_k\, dp = ue_k\right\}\end{aligned}$$

and

$$\delta(T \mid \Phi, \Psi) \leqslant \sup\{\Phi(Tu) \mid \bar{\Psi}^*(u^*) \leqslant 1,\ u \in \mathscr{L}(l_2^n, X)\}.$$

For the reverse inequality we take $g \in L_2(Y^*)$ with $\|g\|_2 \leqslant 1 + \varepsilon$ and $\Phi(Tu) = \langle \sum_1^n \phi_k Tue_k, g\rangle$ such that

$$\begin{aligned}\Phi(Tu) &= \int \left\langle \sum_{k=1}^n \psi_k ue_k, \sum_{l=1}^n \bar{\psi}_l \int \phi_l T^*g\, dp\right\rangle dp\\ &\leqslant \left\| \sum_{k=1}^n \psi_k ue_k : \operatorname{span}\left\{\sum_1^n \bar{\psi}_k a_k\right\} \to \mathbb{K}\right\| \delta(T^* \mid \bar{\Psi}, \bar{\Phi})\, \|g\|_2\\ &\leqslant (1+\varepsilon)\, \bar{\Psi}^*(u^*)\, \delta(T \mid \Phi, \Psi).\end{aligned}$$

To apply the results from section 1 we remark that for $T \in \mathscr{L}(X, Y)$

$$t_\Phi(T) = \sup_{v_\Phi(u:\, l_2^n \to X) = 1} l(Tu) \quad \text{and} \quad c_\Phi(T) = \sup_{l(u:\, l_2^n \to X) = 1} \pi_\Phi(Tu).$$

Using $\pi_2^n(u) = \pi_{U_n}(u)$ and $t_2^n(T) = \delta(T \mid G_n, U_n) = \delta(T^* \mid U_n, G_n)$ (U_n and G_n are defined in the introduction) it is clear that (cf. [17] (Theorem 25.5))

$$t_2^n(T) = \sup_{w \in \mathscr{L}(l_2^n, Y^*),\, l^*(w^*) = 1} \pi_2^n(T^*w) \quad \text{and} \quad c_2^n(T) = \sup_{l(u:\, l_2^n \to X) = 1} \pi_2^n(Tu).$$

Furthermore, via $t_\Phi(T) = \delta(T^* \mid \bar{\Phi}, G_n)$ we obtain

$$\hat{t}_\Phi(T) = \sup_{w \in \mathscr{L}(l_2^n, Y^*),\, l^*(w^*) = 1} \pi_{\bar{\Phi}}(T^*w).$$

Sometimes we will use

$$t_\Phi(l_1^n) \leqslant \hat{t}_\Phi(l_1^n) \leqslant K\sqrt{\log(n+1)}\, c_\Phi(l_\infty^n)$$

which is an easy consequence of $\hat{t}_\Phi(l_1^n) = \delta(l_\infty^n \mid \Phi, G_n)$ and of $l(u) \leqslant K\sqrt{\log(n+1)}\, l^*(u^*)$ in the case $u \in \mathscr{L}(l_2^n, l_\infty^n)$ ([13], [17] (Theorem 12.7)).

Let us start with the following standard lemma (cf. [11](6.2.7)).

Lemma 2.1. *Let $u \in \mathscr{L}(l_2^n, X)$ and $\Phi = (\phi_k)_1^n$ be an orthonormal system. Then*

$$\pi_2(u^*) \leqslant \bar{\Phi}^*(u^*) \leqslant \Phi(u) \leqslant u_2(u).$$

Proof. For some normalized Borel measure μ on $B_{l_2^n}$ (see [10](17.3.2)) we get

$$\begin{aligned}
\Phi(u) &= \left(\int_\Omega \left\| \sum_{k=1}^n \phi_k u e_k \right\|^2 dp\right)^{1/2} \\
&\leqslant \pi_2(u) \left(\int_\Omega \int_{B_{l_2^n}} \left| \left\langle \sum_1^n \phi_k e_k, a \right\rangle \right|^2 d\mu(a)\, dp\right)^{1/2} \\
&\leqslant \pi_2(u) \left(\int \sum_1^n |\langle e_k, a\rangle|^2 d\mu(a)\right)^{1/2} \leqslant \pi_2(u).
\end{aligned}$$

Using trace duality and $\pi_2^*(u^*) = \pi_2(u^*)$ from [10] (19.2.14) we obtain $\pi_2(u^*) \leqslant \Phi^*(u^*)$ and in the same way $\pi_2(u^*) \leqslant \bar{\Phi}^*(u^*)$. Finally, assuming $w \in \mathscr{L}(l_2^n, X^*)$ the inequality $\bar{\Phi}^*(u^*) \leqslant \Phi(u)$ follows from

$$\begin{aligned}
|\mathrm{tr}(u^* w)| &= \left| \sum_{k=1}^n \langle u e_k, w e_k \rangle \right| \\
&= \left| \int \left\langle \sum_k \phi_k u e_k, \sum_l \bar{\phi}_l w e_l \right\rangle dp \right| \leqslant \Phi(u)\, \bar{\Phi}(w). \quad \blacksquare
\end{aligned}$$

Lemma 2.1 together with $\pi_2(u) \leqslant \sqrt{2}\pi_2^n(u)$ for $u \in \mathscr{L}(l_2^n, X)$ ([17]) imply the easy part of Theorem 4.

Corollary 2.2. *For all $T \in \mathscr{L}(X, Y)$ one has*

$$t_\Phi(T) \leqslant \hat{t}_\Phi(T) \leqslant \sqrt{2}\, t_2^n(T) \quad \text{and} \quad c_\Phi(T) \leqslant \sqrt{2}\, c_2^n(T).$$

We come to the non-trivial part.

Proof of Theorem 4 *in the Introduction.* (1) Since $\hat{\imath}_\Phi(l_1^n) \geq \delta\sqrt{n}$ implies $c_\Phi(l_\infty^n) \geq \delta/K\sqrt{n/\log(n+1)}$ Theorem 1.8 (1) gives

$$(\delta/K)^3\,\pi_2(w) \leqslant c\pi_\Phi(w) \qquad \text{for all} \quad w \in \mathscr{L}(l_2^n, X)$$

such that the conclusion with respect to the cotype will be clear. In the type situation we have to observe that $c_\Phi(l_\infty^n) = c_{\bar\Phi}(l_\infty^n)$ and hence $(\delta/K)^3\,\pi_2(w) \leqslant c\pi_{\bar\Phi}(w)$. (2) One obtains $\delta^3 v_\Phi(w) \leqslant c\pi_2(w^*)$ from Theorem 1.8(2). Then we use for $T \in \mathscr{L}(X, Y)$ the equality $t_2^n(T) = \sup\{l(Tu) \mid \pi_2^n(u^*),\ u \in \mathscr{L}(l_2^n, X)\}$, see [17] (Theorem 25.5, 24.2) to conclude. ∎

Remark 2.3. Since $t_\Phi(l_1^n) \leqslant \hat{\imath}_\Phi(l_1^n) \leqslant K\sqrt{\log(n+1)}\,c_\Phi(l_\infty^n)$ the assumptions in (1) of Theorem 4 are weaker than the assumption made in (2) of Theorem 4. Theorem 6 shows that the assumptions of (1) are "strictly" weaker than the assumption of (2).

Proof of Theorem 5 *in the Introduction.* Let $w \in \mathscr{L}(l_2^n, l_\infty^n)$ and $h_j := \sum_{k=1}^n \langle we_k, e_j\rangle \phi_k$. Then $\Phi(u) = (\sup_{j=1,\ldots,n} |h_j(\omega)|^2\, dp(\omega))^{1/2}$ and $\|h_j\|_2 = \|w^* e_j\|$ such that

$$\pi_\Phi(l_\infty^n) = \sup\{\Phi(w) \mid w \in \mathscr{L}(l_2^n, l_\infty^n),\ \|w\| \leqslant 1\}$$

$$= \sup\left\{\left(\int_\Omega \sup_{j=1,\ldots,n} |h_j(\omega)|^2\, dp(\omega)\right)^{1/2} \,\middle|\, (h_j)_1^n \subseteq \operatorname{span}\{\phi_k\},\ \|h_j\|_2 \leqslant 1\right\}$$

and

$$\pi_\Phi(\iota_{2,\infty}^n) = \sup\{\Phi(\iota_{2,\infty}^n w) \mid w \in \mathscr{L}(l_2^n, l_2^n),\ \|w\| \leqslant 1\}$$

$$= \sup\left\{\left(\int_\Omega \sup_{j=1,\ldots,n} |\psi_j(\omega)|^2\, dp(\omega)\right)^{1/2} \,\middle|\, (\psi_j)_1^n \subseteq \operatorname{span}\{\phi_k\} \text{ orthonormal}\right\}.$$

For the latter equality we use the fact that it is sufficient to take the supremum over all orthogonal matrices $w \in \mathscr{L}(l_2^n, l_2^n)$. The implications $(G_n) \to (G_n') \to (G_n'')$ follow immediately from Lemma 1.7 and Corollary 1.6 $(\pi_2(\iota_{2,\infty}^n) = \sqrt{n})$. For $(G_n'') \to (G_n)$ we use Theorem 1.4 to deduce $\delta''\pi_2(u) \leqslant \sqrt{12}\pi_\Phi(u)$ such that $\delta'' c_2^n(l_\infty^n) \leqslant \sqrt{12}c_\Phi(l_\infty^n)$. ∎

Finally we prove the infinite versions of Theorem 4. Before doing this we need

LEMMA 2.4. *Let* $(f_l)_1^n \subset L_2(\Omega, p)$ *be a normalized sequence and* $H \subseteq L_2(\Omega, p)$ *be an n-dimensional subspace such that*

$$\sum_1^n |(f_l, h)|^2 \leqslant \|h\|^2$$

for all $h \in H$. *If* $\sup\{|(f_l, h)| \mid h \in B_H\} \geqslant \theta$ *for all* $l = 1, ..., n$ *then there exists an orthonormal basis* $(\psi_l)_1^n$ *of H and a subset* $I \subseteq \{1, ..., n\}$ *with* $|I| \geqslant (\theta^6/c)n$ *such that*

$$|(f_l, \psi_l)| \geqslant \frac{\theta^3}{c} \qquad \textit{for all} \quad k \in I,$$

where $c \geqslant 1$ *is an absolute constant.*

Proof. Our assumption ensures $(f_l, h_l) \geqslant \theta$ for some $h_1, ..., h_n \in B_H$. We fix an isometry $T: l_2^n \to H$ and define a norm β on $\mathscr{L}(l_1^n, l_2^n)$ by

$$\beta(u) := \sup_{\|w: l_2^n \to l_2^n\| \leqslant 1} \left(\sum_1^n |(f_l, Twu(e_l))|^2 \right)^{1/2}.$$

Let us note that the first inequality of our assumption gives $\beta(a \otimes x) \leqslant \|a\|_{l_\infty^n} \|x\|_{l_2^n}$, which implies $\beta \leqslant v$ on the component $\mathscr{L}(l_1^n, l_2^n)$. Furthermore, the operator $u := \sum_1^n e_l \otimes T^{-1}(h_l) \in \mathscr{L}(l_1^n, l_2^n)$ is of norm at most one and satisfies $\beta(u) \geqslant \theta \sqrt{n}$. In this situation we can apply Lemma 1.5 to deduce for some $c \geqslant 1$

$$\beta(\iota_{1,2}^n) \geqslant \frac{\theta^3}{c} \sqrt{n}.$$

By convexity we find an orthogonal matrix O such that $\sum_1^n |(f_l, TO(e_l))|^2 \geqslant (\theta^6/c^2)n$. Clearly,

$$\psi_l := TO(e_l) \frac{|TO(e_l), f_l)|}{(TO(e_l), f_l)},$$

where we assume $0/0 = 1$, defines an orthonormal basis in H. From $|(f_l, TO(e_l))| \leqslant 1$ we derive that the set

$$I := \left\{ l \,\middle|\, (\psi_l, f_l) \geqslant \frac{\theta^3}{c\sqrt{2}} \right\}$$

is of cardinality at least $(\theta^6/2c^2)n$. ∎

THEOREM 2.5. *Let* $\Phi = (\phi_k)_{k \in \mathbb{N}} \subset L_2(\Omega, p)$ *be an orthonormal system. Then the following assertions are equivalent.*

(1) *There exists a constant* $c > 0$ *such that* $c_2(T) \leqslant c c_\Phi(T)$ *for all operators* $T \in \mathscr{L}(X, Y)$.

(2) *There exists* $\delta > 0$ *such that* $c_\Phi(l_\infty^n) \geqslant \delta \sqrt{n/\log(n+1)}$ *for all* $n = 1, 2, \ldots$

(3) *There exists* $\eta > 0$ *such that for all* $n = 1, 2, \ldots$ *there is an orthonormal system* $\Psi = (\psi_l)_{l=1}^n \subset \operatorname{span}\{\phi_k\}$ *and disjoint measurable subsets* $A_1, \ldots, A_n$ *such that*

$$\int_{A_l} |\psi_l|^2 \, dp \geqslant \eta \qquad \text{for} \quad l = 1, \ldots, n.$$

(4) *There exists* $\theta > 0$ *such that for all* $n = 1, 2, \ldots$ *there is an n-dimensional subspace* $H \subseteq \operatorname{span}\{\phi_k\}$ *and an orthonormal system* $f = (f_l)_1^n \subset L_2(\Omega, p)$ *such that the* f_l *have disjoint support and*

$$\sup_{h \in B_H} |(f_l, h)| \geqslant \theta \qquad \text{for} \quad l = 1, \ldots, n.$$

Proof. (1) → (2) is trivial.

(2) → (3) For fix n there are $x_1, \ldots, x_N \in l_\infty^n$ such that

$$\left\| \sum_1^N \phi_k x_k \right\|_2 \geqslant \frac{\delta}{2} \sqrt{\frac{n}{\log(n+1)}} \qquad \text{and} \qquad \left\| \sum_1^N g_k x_k \right\|_2 \leqslant 1.$$

Assuming $\operatorname{span}\{x_1, \ldots, x_N\} = l_\infty^n$ it is easy to see that there are $y_1, \ldots, y_n \in l_\infty^n$ and a matrix $(p_{ij})_{i=1, j=1}^{n, N}$ such that $\sum_{j=1}^N p_{ij} \bar{p}_{kj} = \delta_{ik}$ and $x_j = \sum_i p_{ij} y_i$ for $j = 1, \ldots, N$. Consequently,

$$\left\| \sum_1^n \psi_i y_i \right\|_2 = \left\| \sum_1^N \phi_j x_j \right\|_2 \geqslant \frac{\delta}{2} \sqrt{\frac{n}{\log(n+1)}}$$

and

$$\left\| \sum_1^n g_i' y_i \right\|_2 = \left\| \sum_1^N g_j x_j \right\|_2 \leqslant 1$$

if $\Psi := (\psi_i)_1^n = (\sum_{j=1}^N p_{ij} \phi_j)_{i=1}^n$. Applying Theorem 1.8 yields $(\delta/2)^3 \pi_2(w) \leqslant c\pi_\Phi(w)$ for all $w \in \mathscr{L}(l_2^n, X)$. Especially, $\pi_\Psi(l_{2,\infty}^n) \geqslant 1/c\, (\delta/2)^3 \sqrt{n}$ such that there is an orthonormal system $(h_k)_1^n \subset \operatorname{span}\{\psi_i\}$ with

$$\int \sup_k |h_k|^2 \, dp \geqslant \left(\frac{1}{c}\right)^2 \left(\frac{\delta}{2}\right)^6 n.$$

Applying [17] (Lemma 31.3) we find an index set $J \subseteq \{1, ..., n\}$ with $|J| \geq ((\delta/2)^6/(2c^2))n$ and disjoint measurable sets A_k such that $\int_{A_k} |h_k|^2 \, dp \geq (\delta/2)^6/(2c^2)$ for $k \in J$.

$(3) \to (1)$ It is clear that we have

$$\int \sup_k |\psi_k|^2 \, dp \geq \eta n$$

such that $\pi_\Psi(\iota^n_{2,\infty}) \geq \sqrt{\eta}\sqrt{n}$. Applying Theorem 1.4 we obtain $\pi_2(u)\eta^{1/2} \leq c\pi_\Psi(u)$ for all $u \in \mathscr{L}(l^n_2, Y)$. Assuming $\psi_i = \sum_{j=1}^N p_{ij}\phi_j$ such that $\sum_{j=1}^N p_{ij}\bar{p}_{kj} = \delta_{ik}$ we obtain for $P := (p_{ij}) \in \mathscr{L}(l^N_2, l^n_2)$, $u \in \mathscr{L}(l^n_2, X)$, $T \in \mathscr{L}(X, Y)$ and some $A \in \mathscr{L}(l^n_2, l^n_2)$ with $\|A\| = 1$

$$\pi_2(Tu) \leq c\eta^{-1/2}\pi_\Psi(Tu) = c\eta^{-1/2}\Psi(TuA) = c\eta^{-1/2}\Phi(TuAP)$$

such that

$$\pi_2(Tu) \leq c\eta^{-1/2}c_\Phi(T)\, l(uAP) \leq c\eta^{-1/2}c_\Phi(T)\, l(u).$$

$(3) \to (4)$ We take $H := \operatorname{span}\{\psi_l \mid l = 1, ..., n\}$ and $f_l := \psi_l\chi_{A_l}/\|\psi_l\chi_{A_l}\|$ such that $|(f_l, \psi_l)| \geq \sqrt{\eta}$.

$(4) \to (3)$ We apply Lemma 2.4 and get an orthonormal basis $\Psi = (\psi_l)_1^n$ of H and a proportional subset $I \subseteq \{1, ..., n\}$ with $|I| \geq (\theta^6/c)n$ such that for $l \in I$ and $A_l := \operatorname{supp}(f_l)$ one obtains

$$\frac{\theta^3}{c} \leq (f_l, \psi_l) \leq \|f_l\| \, \|\psi_l\chi_{A_l}\|.$$

Hence $\int_{A_l} |\psi_l|^2 \, dp \geq \theta^6/c^2$ for $l \in I$. ∎

THEOREM 2.6. *Let $\Phi = (\phi_k)_{k \in \mathbb{N}} \subset L_2(\Omega, p)$ be an orthonormal system. Then the following assertions are equivalent.*

(1) *There exists a constant $c > 0$ such that $t_2(T) \leq ct_\Phi(T)$ for all operators $T \in \mathscr{L}(X, Y)$.*

(2) *There exists $\delta > 0$ such that $t_\Phi(l^n_1) \geq \delta\sqrt{n}$ for all $n = 1, 2, \ldots$.*

Proof. Clearly it remains to show $(2) \to (1)$. Using the argument as in $(2) \to (3)$ of the above theorem we find an orthonormal system $\Psi_n = (\psi_i)_1^n \subset \operatorname{span}\{\phi_j\}$ and $y_1, ..., y_n \in l^n_1$ such that

$$\left\|\sum_1^n \psi_i y_i\right\|_2 \leq 1 \qquad \text{and} \qquad \left\|\sum_1^n g_i' y_i\right\|_2 \geq \frac{\delta}{2}\sqrt{n}.$$

Theorem 1.8 yields

$$\left(\frac{\delta}{2}\right)^3 \nu_{\Psi_n}(w) \leqslant c\pi_2(w^*) \qquad \text{for all} \quad w \in \mathscr{L}(l_2^n, X).$$

Consequently, for all $T \in \mathscr{L}(X, Y)$ and all $n = 1, 2, \ldots$

$$\frac{\delta^3}{8c} t_2^n(T) \leqslant t_{\Psi_n}(T) \leqslant t_\Phi(T). \quad \blacksquare$$

Remark 2.7. In Proposition 3.11 we will see that there exists an orthonormal system $\Phi = (\phi_k)_1^\infty \subseteq (e^{ikt})_{k \in \mathbb{N}} \subseteq L_2(\Pi)$ such that the Φ-cotype and type does not coincide with the usual cotype 2 and type 2, but is non-trivial, i.e., there are Banach spaces without Φ-cotype and type.

3. Orthonormal Systems on Discrete Measure Spaces

In this section we compare (sometimes for simplicity in the real situation) ordinary type and cotype constants with the Φ-type and cotype constants in the case that the orthonormal system Φ lives on a discrete measure space $\Omega := \{\omega_1, \ldots, \omega_N\}$. We start with the positive part by showing that the π_Φ-norm and the π_2^n-norm are close to each other whenever $n \sim N$ (which clearly implies the same for the corresponding cotype constants). In order to apply Theorem 1.4 we need the following lemma which contains an argument discovered in a discussion with B. Kashin.

Lemma 3.1. *Let $1 \leqslant n \leqslant N$ and let $H \subset l_2^N$ be an n-dimensional subspace. Then there exists an orthonormal basis $(h_k)_1^n$ and pair wise different coordinates $j_k \in \{1, \ldots, N\}$ such that*

$$|(h_k, e_{j_k})|^2 \geqslant \frac{n}{3N} \qquad \textit{for all} \quad k \leqslant \frac{n}{3}.$$

Proof. Let P_H be the orthogonal projection onto H. It is well-known that

$$n = \pi_2(P_H)^2 = \sum_1^N \|P_H(e_j)\|^2.$$

Hence there exists $j_1 \in \{1, \ldots, N\}$ such that

$$\|P_H(e_{j_1})\|^2 \geqslant \frac{n}{N}.$$

We consider the normalized element $h_1 = P_H(e_{j_1})/\|P_H(e_{j_1})\|$ which satisfies (P_H is a projection)

$$(h_1, e_{j_1}) = \frac{(P_H(e_{j_1}), e_{j_1})}{\|P_H(e_{j_1})\|} = \|P_H(e_{j_1})\| \geqslant \sqrt{\frac{\dim(H)}{N}}.$$

Now we can proceed by induction setting $H^1 := H$ and $H^{k+1} := H^k \cap \operatorname{span}\{e_{j_k}, h_k\}^{\perp}$. Here j_k and $h_k \in H_j$ are chosen by the construction above and satisfy $\|h_k\| = 1$ and

$$|(h_k, e_{j_k})|^2 \geqslant \frac{\dim(H_k)}{N} \geqslant \frac{n-2k+2}{N}.$$

If we continue as far as $k \leqslant n/3$ we get an orthonormal sequence $(h_k)_{k \leqslant n/3}$ in H with the desired properties. Note that by construction the elements e_{j_k} are disjoint. Finally we complete the sequence (h_k) to an orthonormal basis of H. ∎

Now we compare the π_Φ-norm with the π_2^n-norm.

PROPOSITION 3.2. *Let $\Phi = (\phi_k)_1^n \subset l_2^N$ be an orthonormal system. Then*

$$\pi_2^n(u) \leqslant 12 \sqrt{\frac{N}{n}}\, \pi_\Phi(u) \qquad \textit{for all} \quad u \in \mathscr{L}(l_2^n, X).$$

Proof. First we show $\sqrt{n} \leqslant 2\sqrt{3}\sqrt{N/n}\, \pi_\phi(\iota_{2,\infty}^n)$. Since $\pi_\phi(\iota_{2,\infty}^n) \geqslant 1$ we can assume $n \geqslant 12$. Setting $H := \operatorname{span}\{\phi_k\}$ we choose an orthonormal system $(h_k)_1^n$ and pair wise different k_j according to Lemma 3.1. Now the lower estimate of $\pi_\phi(\iota_{2,\infty}^n)$ follows from

$$\pi_\phi(\iota_{2,\infty}^n)^2 = \sup\left\{ \sum_{j=1}^N \sup_{k=1,\dots,n} |\psi_k(j)|^2 \;\middle|\; (\psi_k)_1^n \subseteq \operatorname{span}\{\phi_k\}_1^n \text{ orthonormal} \right\}$$

$$\geqslant \sum_{j=1}^N \sup_{k=1,\dots,n} |(h_k, e_j)|^2 \geqslant \sum_{k \leqslant n/3} |(h_k, e_{j_k})|^2 \geqslant \frac{n}{4}\,\frac{n}{3N}.$$

Finally, Theorem 1.4 yields the desired assertion. ∎

The above proposition gives

$$c_2^n(T) \leqslant 12 \sqrt{\frac{N}{n}}\, \pi_\Phi(T) \qquad \text{whenever} \quad \Phi = (\phi_k)_1^n \subset l_2^N$$

which was claimed in Theorem 6(1). To prove the remaining parts of Theorem 1.4 we have to construct orthonormal systems with small type or cotype constants. The notion of a Λ_p-system originally introduced for

Fourier series turns out to be a useful tool. For $1 \leqslant p < \infty$ an orthonormal system $\Phi = (\phi_i)_{i \in I}$ (I is a countable index set) on a probability space (Ω, p) is said to be a Λ_p-system if there exists a constant $c \geqslant 0$ such that for all finitely supported sequences $(\alpha_i)_I \subset \mathbb{K}$ one has

$$\left(\int_\Omega \left|\sum_i \alpha_i \phi_i\right|^p dp\right)^{1/p} \leqslant c \int_\Omega \left|\sum_i \alpha_i \phi_i\right| dp$$

By $\Lambda_p(\Phi)$ we denote the best constant in the inequality above. In order to construct orthonormal systems with small cotype constants we also need the notion of a K_q-system. For $2 \leqslant q < \infty$ an orthonormal system $\Phi = (\phi_i)_{i \in I}$ on a probability space (Ω, p) is said to be a K_q-system if there exists a constant $c \geqslant 0$ such that for all finite sequences $(\alpha_i)_I \subset \mathbb{K}$ one has

$$\left(\int_\Omega \left|\sum_i \alpha_i \phi_i\right|^q d\mu\right)^{1/q} \leqslant c \left(\sum_i |\alpha_i|^2\right)^{1/2}.$$

By $K_q(\Phi)$ we again denote the best constant in the inequality above. In fact for $q > 2$ every K_q-system is a Λ_q-system and vice versa. Note that any finite orthonormal system, that is the index set I is assumed to be finite, $\Phi \subset L_p$ is a Λ_p-system and any finite orthonormal system $\Phi \subset L_q$ is a K_q-system but the constants could be different and are important in the sequel. In the following it will be convenient to use $L_p^N := [\mathbb{K}^N, \|\ \|_{L_p^N}]$ with

$$\|(\xi_k)\|_{L_p^N} := \left(\frac{1}{N} \sum_1^N |\xi_k|^p\right)^{1/p}$$

instead of l_p^n. We will start with the construction of orthonormal systems Φ such that $c_\Phi(l_\infty^n)$ is small.

Lemma 3.3. *Let $2 < q < \infty$ and let $\Phi = (\phi_k)_1^N$ be an orthonormal system. Then*

$$c_\Phi(l_\infty^n) \leqslant c_q \frac{n^{1/q}}{\sqrt{\log(n+1)}} K_q(\Phi),$$

where $c_q > 0$ is an absolute constant depending on q only.

Proof. (1) First we show for $u \in \mathscr{L}(l_2^n, l_\infty^n)$

$$\pi_q(u) \leqslant c \frac{n^{1/q}}{\sqrt{\log(n+1)}} l(u).$$

Applying [9] (Theorem 12.10) in the situation

$$X_t := \left\langle \sum_{i=1}^{n} g_i u e_i, e_t \right\rangle \qquad \text{for} \quad t \in T := \{1, ..., n\},$$

where $(e_t)_1^n$ is the unit vector basis of l_1^n, yields a sequence $(Y_k)_{k \geqslant 1}$ of gaussian variables with $\| Y_k \|_2 \leqslant cl(u)/\sqrt{\log(k+1)}$ such that

$$X_t \overset{L_2}{=} \sum_{k \geqslant 1} \alpha_k(t) \, Y_k$$

for $\alpha_k(t) \geqslant 0$ with $\sum_k \alpha_k(t) \leqslant 1$ (in the complex case we consider the real and complex part separately and obtain complex $\alpha_k(t)$ with $\sum_k |\alpha_k(t)| \leqslant 1$). Setting $u_k := \sqrt{\log(k+1)}(\langle Y_k, g_i \rangle)_{i=1}^n \in l_2^n$, $v_k := (\alpha_k(t))_{i=1}^n \in l_\infty^n$, $A := \sum_{k \geqslant 1} u_k \otimes e_k \in \mathscr{L}(l_2^n, l_\infty)$, $D := \sum_{k \geqslant 1} 1/(\sqrt{\log(k+1)}\, e_k \otimes e_k \in \mathscr{L}(l_\infty, l_\infty)$, $B := \sum_{k \geqslant 1} e_k \otimes v_k \in \mathscr{L}(l_\infty, l_\infty^n)$ we deduce $\|u_k\| \leqslant \sqrt{\log(k+1)} \, \| Y_k \|_2 \leqslant cl(u)$, $\|A\| \leqslant cl(u)$, $\|B\| \leqslant 1$, and the factorization

$$u \colon l_2^n \xrightarrow{A} l_\infty \xrightarrow{D} l_\infty \xrightarrow{B} l_\infty^n.$$

Considering $D = D_1 + D_2$, whereas D_1 is the diagonal operator associated to the sequence $(1/\sqrt{\log 2}, ..., 1/\sqrt{\log(n+1)}, 0, ...)$ we obtain from [6] (Theorem 5) and $\pi_q(D_a) \leqslant \|a\|_q$, if the diagonal operator $D_a \in \mathscr{L}(l_\infty, l_\infty)$ is generated by the sequence a,

$$\begin{aligned}
\pi_q(u) &\leqslant \|B\| \; \pi_q(D_1) \; \|A\| + \pi_q^n(B) \frac{1}{\sqrt{\log(n+1)}} \|A\| \\
&\leqslant c_q \, \|B\| \frac{n^{1/q}}{\sqrt{\log(n+1)}} \|A\| + n^{1/q} \, \|B\| \frac{1}{\sqrt{\log(n+1)}} \|A\| \\
&\leqslant c'_q \frac{n^{1/q}}{\sqrt{\log(n+1)}} \, l(u).
\end{aligned}$$

(2) Assuming $v \in \mathscr{L}(l_2^N, l_\infty^n)$ it is easy to see that there is a factorization $v = uP$ where $P \in \mathscr{L}(l_2^N, l_2^n)$ and $u \in \mathscr{L}(l_2^n, l_\infty^n)$ such that $\|P\| = 1$ and $l(u) = l(v)$. Using the continuous version of the q-summing norm (which follows by an easy approximation argument, see [14] (Proposition 1.2)) we can deduce

$$\Phi(v) \leqslant \left(\int_\Omega \left\|\sum_1^N \phi_k v(e_k)\right\|^q dp\right)^{1/q} \leqslant \pi_q(v) \sup_{\|(\alpha_k)\|_2 \leqslant 1} \left(\int_\Omega \left|\sum_1^N \alpha_k \phi_k\right|^q dp\right)^{1/q}$$

$$\leqslant \pi_q(v)\, K_q(\Phi) \leqslant \pi_q(u)\, K_q(\Phi) \leqslant c'_q \frac{n^{1/q}}{\sqrt{\log(n+1)}} K_q(\Phi)\, l(u)$$

$$\leqslant c'_q \frac{n^{1/q}}{\sqrt{\log(n+1)}} K_q(\Phi)\, l(v). \quad \blacksquare$$

The following proposition provides small orthonormal systems with small cotype constants.

PROPOSITION 3.4. *Let $2 < q < \infty$ and $1 \leqslant n \leqslant N$. Then there exists a real orthonormal system $\Phi = (\phi_k)_1^n \subset L_2^N$ with*

(1) $\sqrt{n}/N^{1/q} \leqslant K_q(\Phi) \leqslant c \max\{\sqrt{q}, \sqrt{n}/N^{1/q}\}$,

(2) $c_\Phi(l_\infty^n)\sqrt{\log(n+1)/n} \leqslant c_q \max\{\sqrt{q}n^{1/q-1/2}, (n/N)^{1/q}\}$,

where $c > 0$ is an absolute constant and $c_q > 0$ depends on q only. In particular, for $0 < \delta < 1$ and $N = [n^{1/\delta}]$ one has $c_\Phi(l_\infty^n) \leqslant c_\delta n^{\delta/2}$.

Proof. (1) We will use a random argument. By the comparison principle for random orthonormal matrices and gaussian variables of Marcus and Pisier, see [1], and Chevet's inequality, see [5], we deduce

$$\sqrt{N}\mathbb{E}\left\|\sum_{k=1}^n \sum_{j=1}^N o_{k,j} e_k \otimes e_j : l_2^n \to L_q^N\right\|$$

$$\leqslant \frac{c_0}{N^{1/q}} \mathbb{E}\left\|\sum_{k=1}^n \sum_{j=1}^N g_{k,j} e_k \otimes e_j : l_2^n \to l_q^N\right\|$$

$$\leqslant \frac{c_0\sqrt{2}}{N^{1/q}}\left(\mathbb{E}\left\|\sum_{j=1}^N g_j e_j\right\|_{l_q^N} + \mathbb{E}\left\|\sum_{k=1}^n g_k e_k\right\|_{l_2^n}\right)$$

$$\leqslant c_0 c_1 \sqrt{2}(\sqrt{q} + \sqrt{n}N^{-1/q}).$$

Here the expectation is taken with respect to the Haar-measure on the group $\mathcal{O}(N)$ of orthonormal matrices and with respect to the standard gaussian density in $\mathbb{R}^{nN}$. For a random matrix o satisfying the above inequality we define the orthonormal system $\Phi = (\phi_k)_1^n \subset L_2^N$ by

$$\phi_k := \sqrt{N} \sum_{j=1}^N o_{k,j} e_j.$$

Therefore we have proved

$$\mathbb{E}K_q(\Phi) \leqslant 3c_0 c_1 \max\{\sqrt{q}, \sqrt{n}N^{-1/q}\}.$$

and can choose our system randomly (with an obvious change of constants the same estimate for the K_q-constant is valid if we compute this constant with complex coefficients). To obtain the lower estimate of the K_q-constant we first claim that $\sqrt{n} \leqslant \pi_\Phi(l_\infty^N)$. In order to prove this claim we find $(a_j)_1^N \subset B_{l_2^n}$ such that $\langle \sum_{k=1}^n \phi_k(j) e_k, a_j\rangle = \|\sum_{k=1}^n \phi_k(j) e_k\|_{l_2^n}$ and define the operator $R := \sum_1^N a_j \otimes e_j \in \mathscr{L}(l_2^n, l_\infty^N)$ with $\|R\| \leqslant 1$. Then it follows that

$$\sqrt{n} = \left(\sum_1^n \|\phi_k\|_{L_2^N}^2\right)^{1/2} = \left(\frac{1}{N}\sum_{j=1}^N \left\|\sum_{k=1}^n \phi_k(j) e_k\right\|_{l_2^n}^2\right)^{1/2} \leqslant \Phi(R) \leqslant \pi_\Phi(l_\infty^N).$$

Using the argument given in the end of the proof of Lemma 3.3 we continue with

$$\begin{aligned} \pi_\Phi(l_\infty^N) &= \sup\{\Phi(u) \mid \|u: l_2^n \to l_\infty^N\| \leqslant 1\} \\ &\leqslant K_q(\Phi) \sup\{\pi_q(u) \mid \|u: l_2^n \to l_\infty^N\| \leqslant 1\}. \end{aligned}$$

Finally, [6] (Theorem 5) implies $\pi_q(u) \leqslant N^{1/q} \|u\|$. (2) is a consequence of (1) and Lemma 3.3. The last assertion follows by $q = 2/\delta$. ∎

We continue by constructing orthonormal systems with small type constants in the proportional case.

LEMMA 3.5. *Let $\Phi = (\phi_k)_1^n$ be an orthonormal system. Then $t_\Phi(l_1) \leqslant cA_2(\Phi)$ for some absolute constant $c > 0$.*

Proof. Let $x_1, ..., x_n \subset l_1$. Using the Kahane inequality for gaussian averages due to Hoffmann-Jörgensen, see [9], we deduce

$$\begin{aligned} \left\|\sum_1^n g_k x_k\right\|_{L_2(l_1)} &\leqslant c \int \sum_{j\in\mathbb{N}} \left|\sum_{k=1}^n g_k \langle x_k, e_j\rangle\right| dp \leqslant c \sum_{j\in\mathbb{N}} \left(\sum_{k=1}^n |\langle x_k, e_j\rangle|^2\right)^{1/2} \\ &\leqslant cA_2(\Phi) \sum_{j\in\mathbb{N}} \int_\Omega \left|\sum_{k=1}^n \phi_k \langle x_k, e_j\rangle\right| dp \\ &= cA_2(\Phi) \left\|\sum_{k=1}^n \phi_k x_k\right\|_{L_1(l_1)}. \quad ∎ \end{aligned}$$

A more abstract version of this argument can be applied for Banach lattices with finite cotype, see [8]. In contrast to the previous results large orthonormal systems will now be constructed in L_2^N.

PROPOSITION 3.6. *For all* $1 \leqslant n \leqslant N$ *there exists a real orthonormal system* $\Phi = (\phi_k)_1^n \subset L_2^N$ *such that for some absolute constant* $c > 0$

(1) $\Lambda_2(\Phi) \leqslant c\sqrt{N/(N-n)\log(1+N/(N-n))}$,

(2) $t_\Phi(l_1) \leqslant c\sqrt{N/(N-n)\log(1+N/(N-n))}$.

In particular, for $0 < \varepsilon < 1$ *and* $1 \leqslant n \leqslant (1-\varepsilon)N$ *there is an orthonormal system satisfying*

$$t_\Phi(l_1) \leqslant c\sqrt{\frac{1}{\varepsilon}\log\left(1+\frac{1}{\varepsilon}\right)}.$$

Proof. By [4] (Theorem 2.2) there exists a subspace $E \subset l_1^N$ with $n = N - m = \dim(E)$ such that for all $x \in E$

$$\|x\|_2 \leqslant c\sqrt{\frac{\log(1+N/m)}{m}}\,\|x\|_1.$$

Now we consider E as a subspace of L_1^N, L_2^N, respectively. Then we have for all $x \in E$

$$\begin{aligned}\|x\|_{L_2^N} &= \frac{1}{\sqrt{N}}\|x\|_2 \leqslant \frac{c}{\sqrt{N}}\sqrt{\frac{\log(1+N/m)}{m}}\,\|x\|_1 \\ &= c\sqrt{\frac{N}{m}\log\left(1+\frac{N}{m}\right)}\,\|x\|_{L_1^N}.\end{aligned}$$

If we choose an orthonormal system $\Phi = (\phi_k)_1^n$ in E with respect to the scalar product of L_2^N we obtain for all sequences $(\alpha_k)_1^n \in \mathbb{R}^n$ (and with the constant $2c$ instead of c also for complex (α_k))

$$\left(\sum_1^n |\alpha_k|^2\right)^{1/2} = \left\|\sum_1^n \alpha_k\phi_k\right\|_{L_2^N} \leqslant c\sqrt{\frac{N}{m}\log\left(1+\frac{N}{m}\right)}\left\|\sum_1^n \alpha_k\phi_k\right\|_{L_1^N}.$$

Therefore assertion (1) is proved. Assertion (2) follows immediately from Lemma 3.5. ∎

For $n = \delta N$ we can again choose a random orthonormal system in L_2^N satisfying the assertion of the above proposition, because for random subspaces the corresponding norm estimate is valid, see [16] (cf. [15] (Theorem 6.1)). We are now in position to complete the

Proof of Theorem 6 *in the Introduction.* (1) follows from Proposition 3.2. (2) and (3) are consequences of Proposition 3.4 (2) and 3.6. ∎

Analyzing Theorem 4 one can ask whether it is possible or not to replace the space l_∞^n or l_1^n by an arbitrary space. We will show in Proposition 3.8 that this is not possible in some sense. Let us begin with the following construction which yields n-dimensional quotients of l_1^N with large Φ-type constants. Given an arbitrary orthonormal system $\Phi = (\phi_k)_1^n \subset L_2(\Omega, p)$ we consider the convex body

$$B_\Phi := \overline{\text{absconv}} \left\{ \frac{\sum_1^n \phi_k(\omega)\, e_k}{(\sum_1^n |\phi_k(\omega)|^2)^{1/2}} \,\middle|\, \sum_1^n |\phi_k(\omega)|^2 > 0 \right\} \subset l_2^n$$

and the associated Banach space $E_\Phi := [\mathbb{K}^n, B_\Phi]$ via the Minkowski functional of B_Φ (we will see that E_Φ is correctly defined). From the definition it is clear that, for example, B_Φ is the convex combination of at most N points if the measure space has N atoms and no continuous part. The next lemma summarizes some properties of the convex body B_Φ.

LEMMA 3.7. *Let $\Phi = (\phi_k)_1^n$ be an orthonormal system. Then the following holds true.*

(1) *There exists a normalized Borel measure μ supported by $S_{n-1} \cap B_\Phi$ satisfying $\int_{B_{l_2^n}} \langle x, e_i \rangle \overline{\langle x, e_j \rangle}\, d\mu(x) = (1/n)\delta_{ij}$ and, for all Banach spaces X and all $u \in \mathcal{L}(l_2^n, X)$,*

$$\Phi(u) = \sqrt{n} \left(\int_{B_{l_2^n}} \|ux\|^2 \, d\mu(x) \right)^{1/2}.$$

(2) *If $\iota_\Phi : l_2^n \to E_\Phi$ is the formal identity then $\|\iota_\Phi^{-1}\| \leqslant 1$ and $\pi_2(\iota_\Phi^*) \leqslant \sqrt{n}$.*

(3) *In the real situation one has*

$$\frac{l(\iota_\Phi)}{\sqrt{n}} \geqslant \left(\frac{\text{vol}(B_{l_2^n})}{\text{vol}(B_\Phi)} \right)^{1/n} \quad \textit{and} \quad \frac{r(\iota_\Phi)}{\sqrt{n}} \geqslant \frac{1}{c} \left(\frac{\text{vol}(B_{l_2^n})}{\text{vol}(B_\Phi)} \right)^{1/n}$$

where $c > 0$ is an absolute constant, $r(u) := (\mathbb{E}\, \|\sum_1^n \varepsilon_k u e_k\|^2)^{1/2}$, and $(\varepsilon_k)_1^n$ is the sequence of independent random variables with $p(\varepsilon_k = 1) = p(\varepsilon_k = -1) = \frac{1}{2}$.

Proof. (1) Considering $F: [\Omega, \mathcal{F}, p] \to [l_2^n, \mathcal{B}(l_2^n)]$ with $F(\omega) := (\phi_1(\omega), ..., \phi_n(\omega))$ we obtain an image measure p_F on $\mathcal{B}(l_2^n)$ such that

$$\int_{l_2^n} \frac{\|x\|_2^2}{n} \, dp_F(x) = \frac{1}{n} \int_\Omega \sum_1^n |\phi_k(\omega)|^2 \, dp(\omega) = 1.$$

Hence $d\nu(x) := (\|x\|_2^2/n)\, dp_F(x)$ defines a normalized Borel measure on l_2^n with $\nu(\{0\}) = 0$.

Finally, the image measure μ of ν with respect to

$$P: [l_2^n\backslash\{0\}, \mathscr{B}(l_2^n\backslash\{0\})] \to [B_{l_2^n}, \mathscr{B}(B_{l_2^n})] \qquad \text{whereas} \quad P(x) := \frac{x}{\|x\|}$$

is the desired measure. (2) Since we have a measure μ supported by B_Φ with a standard coveriance matrix it is easy to see that $\dim(\operatorname{span}\{B_\Phi\}) = n$. Consequently, the formal identity ι_Φ is correctly defined and we have $\|\iota_\Phi^{-1}\| \leqslant 1$ as well as $\pi_2(\iota_\Phi^*) \leqslant \Phi(\iota_\Phi) = \sqrt{n}$ (see Lemma 2.1). (3) For the first inequality we can use well-known volume estimates for the corresponding l-norm, namely

$$\frac{l(\iota_\Phi)}{\sqrt{n}} = \left(\int_{S_{n-1}} \|x\|_{E_\Phi}^2\, d\sigma_n(x)\right)^{1/2}$$
$$\geqslant \left(\int_{S_{n-1}} \|x\|_{E_\Phi}^{-n}\, d\sigma_n(x)\right)^{-1/n} = \left(\frac{\operatorname{vol}(B_{l_2^n})}{\operatorname{vol}(B_\Phi)}\right)^{1/n},$$

where σ_n is the normalized Haar measure on the sphere S_{n-1}. The second inequality of (3) follows from the results of Carl and Pajor, see [4] (Corollary 1.4(b)), since

$$\left(\frac{\operatorname{vol}(B_{l_2^n})}{\operatorname{vol}(B_\Phi)}\right)^{1/n} \leqslant 2e_n(\iota_\Phi),$$

where $e_n(\iota_\Phi)$ denotes the nth dyadic entropy number of ι_Φ. ∎

We deduce the lower estimates for the Φ-type and -cotype constants which emphasize the special role of the spaces l_1^n and l_∞^n in Theorem 4.

Proposition 3.8. *Let $\Phi = (\phi_k)_1^n \subset L_2^N$ be an orthonormal system. Then, for some constant $\theta > 0$ not depending on Φ, n, and N, the following holds true.*

(1) *There exists an n-dimensional subspace $E \subseteq l_\infty^N$ with*

$$c_\Phi(E) \geqslant \theta \sqrt{\frac{\log(n+1)}{\log(N+1)}} \sqrt{\frac{n}{\log(n+1)}}.$$

(2) *In the real situation there exists an n-dimensional quotient l_1^N/L with*

$$t_\Phi(l_1^N/L) \geqslant \theta \sqrt{\frac{1}{\log(N/n+1)}} \sqrt{n}.$$

Proof. (1) We define the operator $R \in \mathscr{L}(l_2^n, l_\infty^N)$ with $\|R\| \leqslant 1$ and $\sqrt{n} \leqslant \Phi(R)$ as in the proof of Proposition 3.4. Then we consider the space $E := R(l_2^n)$ of dimension at most n. From $l(R) \leqslant c_0 \sqrt{\log(N+1)}\, \|R\|$ and the above inequality it follows that

$$\sqrt{\frac{n}{\log(n+1)}} \leqslant c_0\, c_\Phi(E) \sqrt{\frac{\log(N+1)}{\log(n+1)}}.$$

(2) Since $L_2(\Omega, p) = L_2^N$ we get $B_\Phi = \overline{\operatorname{absconv}}\{x_1, ..., x_N\}$ for some $x_1, ..., x_N \in S_{n-1}$ and $E_\Phi = l_1^N/L$. Corollary 2.4(i) in [4] implies

$$\left(\frac{\operatorname{vol}(B_\Phi)}{\operatorname{vol}(B_{l_2^n})}\right)^{1/2} \leqslant c \sqrt{\frac{\log(N/n+1)}{n}}$$

for some absolute constant $c > 0$. Using Lemma 3.7 (3) we deduce

$$\sqrt{\frac{n}{\log(N/n+1)}} \leqslant c\, \frac{l(\iota_\Phi)}{\sqrt{n}} \leqslant c t_\Phi(E_\Phi)\, \frac{\Phi(\iota_\Phi)}{\sqrt{n}} = c t_\Phi(E_\Phi). \quad \blacksquare$$

Remark 3.9. Note that assertion (1) can formulated as follows in the cotype situation: As long as N does not grow faster than a polynomial in n there is an n dimensional subspace of l_∞^N with "worst possible" cotype-Φ constant, although this constant can be quite small on l_∞^n, see Proposition 3.4. On the other hand assertion (2) in the type situation means: As long as $N \sim n$ there is an n dimensional quotient of l_1^N with "worst possible" type-Φ constant, although this constant can be bounded on l_1, see Proposition 3.6.

Remark 3.10. 1. For any orthonormal system $\Phi = (\phi_k)_1^n$ and any $u \in \mathscr{L}(l_2^n, X)$ we have $l(u) \leqslant \pi_\Phi(u)$. This is known and follows (for example) from Lemma 3.7 (1).

In fact, we have

$$\begin{aligned}
\pi_\Phi(u) &= \sqrt{n} \sup_{w \in \mathcal{O}_n} \left(\int_{B_{l_2^n}} \|uwx\|^2 \, d\mu(x) \right)^{1/2} \\
&\geqslant \sqrt{n} \left(\int_{w \in \mathcal{O}_n} \int_{B_{l_2^n}} \|uwx\|^2 \, d\mu(x)\, dw \right)^{1/2} \\
&= \sqrt{n} \left(\int_{B_{l_2^n}} \|ux\|^2 \, d\sigma_n(x) \right)^{1/2},
\end{aligned}$$

where dw stands for the integration with respect to the Haar measure on the group $\mathcal{O}(n)$ of the n-dimensional orthogonal matrices.

2. Let $\Phi=(\phi_k)_1^n \subset l_2^N$ be an (real) orthonormal system such that

$$\left\|\sum_1^n \varepsilon_k x_k\right\|_2 \leqslant c_0 \left\|\sum_1^n \phi_k x_k\right\|_2$$

holds for all Banach spaces X and all $x_1, ..., x_n \in X$. Then $N \geqslant n(e^{n/(cc_0)^2}-1)$ for some numerical constant $c>0$. To see this we consider $E_\Phi = l_1^N/L$ as in the proof of Proposition 3.8 and observe

$$\frac{r(\iota_\Phi)}{\sqrt{n}} \geqslant \frac{1}{c_1}\left(\frac{\operatorname{vol}(B_{l_2^n})}{\operatorname{vol}(B_\Phi)}\right)^{1/n} \geqslant \frac{1}{c_1 c_2}\sqrt{\frac{n}{\log(N/n+1)}}$$

according to Lemma 3.7 (3) and [4] (Corollary 2.4). Since $\Phi(\iota_\Phi)=\sqrt{n}$ we get

$$\sqrt{n}=\Phi(\iota_\Phi) \geqslant \frac{r(\iota_\Phi)}{c_0} \geqslant \frac{1}{c_0 c_1 c_2}\sqrt{n}\sqrt{\frac{n}{\log(N/n+1)}}$$

such that $n/(c_0 c_1 c_2)^2 \leqslant \log(N/n+1)$.

Finally, we are in a position to complete Remark 2.7.

PROPOSITION 3.11. *There exists an orthonormal system* $\Phi=(\phi_k)_1^\infty \subseteq (e^{ikt})_{k\in\mathbb{N}} \subseteq L_2(\Pi)$ *such that*

$$\sup_n \frac{c_2(l_\infty^n)}{c_\Phi(l_\infty^n)} = \sup_n \frac{t_2(l_1^n)}{t_\Phi(l_1^n)} = \infty \qquad \textit{and} \qquad c_\Phi(l_\infty) = t_\Phi(l_\infty) = \infty.$$

Proof. We use the system constructed in [2] (Theorem 2). For $2<q<\infty$ and $k=1,2,...$ there are chosen subsets $S_k \subseteq \{n : 2^k \leqslant n < 2^{k+1}\}$ satisfying $|S_k| = [4^{k/q}]$ such that for $\Lambda = \bigcup_{k=1}^\infty S_k$ the system $\Phi=(e^{ikt})_{k\in\Lambda}$ is a K_q system. Using Lemma 3.3 we obtain

$$c_\Phi(l_\infty^n) \leqslant c_q \frac{n^{1/q}}{\sqrt{\log(n+1)}} K_q(\Phi) \qquad \text{and} \qquad \sup_n \frac{c_2(l_\infty^n)}{c_\Phi(l_\infty^n)} = \infty.$$

Applying $t_\Phi(l_1^n) \leqslant K\sqrt{\log(n+1)}\, c_\Phi(l_\infty^n)$ one also gets $\sup_n t_2(l_1^n)/t_\Phi(l_1^n) = \infty$. Now let us consider the system $\Psi_k := (e^{ilt})_{l\in S_k}$. It follows from the Marcinkiewicz–Zygmund-inequality and a shift argument that

$$\left(\frac{1}{2^k}\sum_{l=1}^{2^k}\left\|\sum_{s+2^k-1\in S_k} e^{2\pi i(sl/2^k)} x_s\right\|^2\right)^{1/2} \leqslant c\left(\int_0^{2\pi}\left\|\sum_{s\in S_k} e^{ist} x_s\right\|^2 \frac{dt}{2\pi}\right)^{1/2}$$

for all $x_1, \ldots, x_{|S_k|} \in X$ and all Banach spaces X. Setting $\Psi_k^0 := ((e^{2\pi i(sl/2^k)})_{l=1}^{2^k})_{s=1}^{|S_k|} \subset L_2^{2^k}$ we get

$$\pi_{\Psi_k^0}(T) \leqslant c\pi_{\Psi_k}(T) \qquad \text{for all} \quad T \in \mathscr{L}(X, Y).$$

Applying Proposition 3.2 we continue to

$$\frac{1}{12}\left(\frac{|S_k|}{2^k}\right)^{1/2} \pi_2^{|S_k|}(T) \leqslant \pi_{\Psi_k^0}(T) \leqslant c\pi_{\Psi_k}(T)$$

for all $T \in \mathscr{L}(X, Y)$. Setting $n = 2^k$ this reads as

$$\frac{1}{12c}\left(\frac{[n^{2/q}]}{n}\right)^{1/2} \pi_2^{[n^{2/q}]}(T) \leqslant \pi_{\Psi_k}(T).$$

If T is the embedding of $l_2^{[n^{2/q}]}$ into $l_\infty^{[n^{2/q}]}$ this gives

$$\frac{1}{12c}\, [n^{2/q}]^{1/2+1/2}\, n^{-1/2} \leqslant \pi_{\Psi_k}(\iota_{2,\infty}^{[n^{2/q}]}).$$

On the other hand,

$$l(\iota_{2,\infty}^{[n^{2/q}]}) \sim \sqrt{\log[n^{2/q}]}$$

which yields for $2/q - 1/2 > 0$ $(q < 4)$ the equality $\sup_n c_\Phi(l_\infty^n) = \infty$. Finally, we have for any subsystem $(\psi_k)_1^n \subset \Phi$ the estimates ((e_k) is the standard basis of l_∞^n)

$$\left\|\sum_1^n \psi_k e_k\right\|_{L_2(l_\infty^n)} = 1 \qquad \text{and} \qquad \left\|\sum_1^n g_k e_k\right\|_{L_2(l_\infty^n)} \sim \sqrt{\log(n+1)}$$

such that $\sup_n t_\Phi(l_\infty^n) = \infty$. ∎

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