Discrete Applied Mathematics 6 (1983) 135–138
North-Holland

THE VERTICES OF THE KNAPSACK POLYTOPE

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Received 26 November 1981
Revised 8 February 1982 and 10 March 1982

The number of vertices of a polytope associated to the Knapsack integer programming problem is shown to be small. An algorithm for finding these vertices is discussed.

1. Introduction

We begin with a statement of the well-known Knapsack problem, namely:

$$\begin{aligned} &\text{maximise} && c_1x_1+c_2x_2+\cdots+c_nx_n,\\ &\text{subject to} && a_1x_1+a_2x_2+\cdots+a_nx_n\le b, \qquad (1)\\ & && a_j, c_j, x_j, b \text{ non-negative integers, } 1\le j\le n.\end{aligned}$$

We define the *Knapsack polytope, K*, to be the convex hull of the feasible solutions of the inequalities associated with (1). That is, we define:

$$K=\operatorname{conv}\{\boldsymbol{x}=(x_1,\ldots,x_n)\in\mathbb{Z}^n : a_1x_1+\cdots+a_nx_n\le b,\ x_j\ge 0, 1\le j\le n\}.$$

Then the principal result of this paper is to show that the set V of vertices of K has only a small number of elements. More precisely, letting $|V|$ denote the cardinality of V, we show that

$$|V|<(\log_2\sigma)^n \qquad (2)$$

where $\sigma=(4b)/\min\{a_1,\ldots,a_n\}$. In addition, the method of the proof of (2) reveals something about the distribution of the vertices on K. In the final section of the paper we use our geometric results to obtain an algorithm to find V explicitly.

2. The geometry

In order to prove (2), we shall partition the lattice points of K into 'boxes' in such

* The research of this author was carried out while he was in receipt of a Science Research Council Research Studentship.

a way that no box may contain more than one vertex of K. We begin by defining a sequence $\{X_j\}_{j=0}^{\infty}$ of integers by

$$X_0=0, \qquad X_j=2^{j-1}, \quad j\geq 1.$$

For each $i=1,2,\dots,n$ define an integer N_i by

$$X_{N_i-1}\leq (b/a_i)<X_{N_i}.$$

Then it is clear that $N_i<\log_2(4b/a_i)$.

Let I_j denote the closed-open interval $[X_{j-1},X_j)$, and let β' be the set of boxes

$$\beta'=\left\{\prod_{j=1}^{n} I_{k_j}: 1\leq k_j\leq N_j\right\}.$$

From the definition of K and of β' it follows that

$$K\subseteq \bigcup_{B\in\beta'} B.$$

We note that the number of elements of β' is

$$\prod_{j=1}^{n} N_j<(\log_2\sigma)^n$$

where $\sigma=(4b)/\min\{a_1,\dots,a_n\}$. It is the case that some members of β' will not meet K. Let $\beta\subseteq\beta'$ be those elements of β' which met K. We show:

Lemma. *No box in β contains more than one vertex of K.*

Proof. Let H denote the hyperplane $\{\boldsymbol{x}: a_1x_1+a_2x_2+\cdots a_nx_n=b\}$ and let $\boldsymbol{u}$ denote the normal of H, outward with respect to K. Let α be the real number such that $\langle \boldsymbol{x},\boldsymbol{u}\rangle=\alpha$ for all $\boldsymbol{x}\in H$, where $\langle\ ,\ \rangle$ is the standard Euclidean inner product.

Now suppose for the moment that we have proven the result for the cases of dimension $2,3,\dots,n-1$. Suppose $v=(v_1,v_2,\dots,v_n)$ and $\boldsymbol{w}=(w_1,w_2,\dots,w_n)$ are vertices of K lying in some common element B of β. Let $B=\prod_{j=1}^{n} I_{k_j}$, $1\leq k_j\leq N_j$ for some fixed choice of the k_j. Suppose $\langle \boldsymbol{u},\boldsymbol{w}\rangle\leq\langle \boldsymbol{u},v\rangle$. We show that $2\boldsymbol{w}-v\in K$ contradicting the fact that $\boldsymbol{w}$ is a vertex of K. The condition above ensures that $\langle \boldsymbol{u},2\boldsymbol{w}-v\rangle\leq\alpha$ and so it only remains to show that $2w_i-v_i\geq 0$, $i=1,\dots,n$. We may suppose that B has no $k_j=1$. For if some k_j is 1, we have an obvious 1-1 correspondence between the boxes

$$\left\{\prod_{i=1}^{j-1} I_{k_i}\times I_1\times\prod_{i=j+1}^{n} I_{k_i}\right\} \quad\text{and}\quad \left\{\prod_{\substack{i=1\\ i\neq j}}^{n} I_{k_i}\right\}$$

and hence we may inductively apply the $(n-1)$-dimensional result to the reduced problem in the hyperplane $\{\boldsymbol{x}: x_j=0\}$.

So, supposing $k_j\geq 2$ for each $j=1,\dots,n$, we have since $v,\boldsymbol{w}$ are in B that,

$$|w_j-v_j|\geq X_{k_j}-X_{k_j-1}=2^{k_j-2}=X_{k_j-1}$$

Hence

$$2w_j - v_j \geq w_j - |v_j - w_j| \geq X_{k_j-1} - X_{k_j-1} = 0.$$

Thus we have proven the lemma provided we establish the case $n=2$. Any difficulties arising in copying the above proof occur only if B has $k_j = 1$ for some $j \in \{1, 2\}$. But this restricts $\boldsymbol{u}$ and $\boldsymbol{w}$ to lie in either $\mathbb{R} \times [0,1)$ or $[0,1) \times \mathbb{R}$,and thus we are left considering the existence of two vertices in B both of which lie on the same axis, which is clearly impossible.

If we now note that $|\beta| \leq |\beta'| = (\log_2 \sigma)^n$, we have:

Theorem. $|V| < (\log_2 \sigma)^n$.

As a simple consequence of the method of proof of the lemma we obtain:

Corollary. *Any vertex of K in a box B is the unique point of K maximising $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ over integer points of K in B.*

Proof. If $\boldsymbol{u} \in V$ and $\boldsymbol{w} \in K$ and both are in some $B \in \beta$ with $\langle \boldsymbol{u}, \boldsymbol{w} \rangle \geq \langle \boldsymbol{u}, \boldsymbol{v} \rangle$ then exactly as in the proof above we show that $2\boldsymbol{w} - \boldsymbol{v} \in K$, contradicting $\boldsymbol{v} \in V$.

To conclude this section we remark that the theorem implies that the number of $(n-1)$ faces (facets) of K is surprisingly small. By the Upper Bound Theorem for convex polytopes [4], the maximum number of facets of a polytope in n dimensions with v vertices is at most $V^{n/2}$. It therefore follows that the number of facets of K is at most $(\log_2 \sigma)^{n^2/2}$.

3. The algorithm

Due to the interesting property that no $B \in \beta$ may contain more than one vertex of K, we can supply an algorithm to find V. Suppose we have an algorithm A to yield an optimal solution to the problem:

$$\begin{aligned} &\text{maximise} && a_1x_1 + a_2x_2 + \cdots + a_nx_n, \\ &\text{subject to} && a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b, \\ &&& L_j \leq x_j \leq U_j, \ 1 \leq j \leq n, \\ &&& L_j, U_j, a_j, x_j, b \text{ non-negative integers, } 1 \leq j \leq n. \end{aligned} \tag{3}$$

Certainly such an algorithm exists; for example, see Lenstra [3]. Given a box $B \in \beta$ it is clear that the maximisation of $a_1x_1 + \cdots + a_nx_n$ over integer points of K in B is a problem of the form of (3). We note here that we can easily select from β' those $B \in \beta$ since $B = \prod_{j=1}^{n} I_{k_j} \in \beta$ if, and only if, $a_1X_{k_1-1} + a_2X_{k_2-1} + \cdots + a_nX_{k_n-1} \leq b$. We now apply A to find an optimal solution $\boldsymbol{x}(B)$ for each box $B \in \beta$. Let

$S=\{x(B):B\in\beta\}$. Then it follows from the uniqueness assertion of the corollary that if a box B contains a vertex $v\in V$, then $x(B)=v$ and hence $V\subseteq S$. The set S is small since $|S|<(\log_2 \sigma)^n$. Of course, not every $B\in\beta$ will contain a vertex $v\in V$, and so in this case $x(B)\notin V$. In order to reduce S to V we have to discard some elements of S. To do this, we firstly note that if $v\in S$ then $v\in V$ if, and only if, v is not a convex combination in $S-\{v\}$ (since each vertex is an extreme point and each point of S is a convex combination in V).

Let $v_1,\dots,v_t$ be an enumeration of S and suppose $v_1=\mathbf{0}$. Then v_t is a convex combination in $S-\{v_t\}$ if, and only if, there are constants $\lambda_1,\dots,\lambda_{t-1}$ such that:

$$\begin{aligned} &v_t=\sum_{i=1}^{t-1}\lambda_i v_i,\\ &\sum_{i=1}^{t-1}\lambda_i=1, \qquad \lambda_i\geq 0, \quad 1\leq i\leq t-1. \end{aligned} \tag{4}$$

Since each $v_i\in\mathbb{Z}^n$, we can re-write (4) as a system of inequalities with integer coefficients in $\lambda=(\lambda_1,\dots,\lambda_{t-1})$. Call this system Ω. From (4) it follows that Ω has a solution if, and only if, $v_t\notin V$. Checking the solvability of Ω we conclude one of two possibilities:

(a) *Ω is unsolvable implying $v_t\in V$.* Relabelling S as $v_1^{(1)},\dots,v_t^{(1)}$ with $v_1^{(1)}=0$ and $v_2^{(1)}=v_t$, $v_i^{(1)}=v_{i-1}$, $3\leq i\leq t$ we may repeat the above procedure on $S^{(1)}=S$.

(b) *Ω is solvable implying $v_t\notin V$.* Then $V\subseteq S-\{v_t\}$ and we repeat the procedure to $S^{(1)}=S-\{v_t\}$.

If we now repeat this process inductively to obtain sets $S\supseteq S^{(1)}\supseteq S_2^{(2)}\supseteq\cdots$, then after $k\leq(t-1)$ steps we will have established that the only points remaining in $S^{(k)}$ are points of V. Thus we will have reduced S to V.

Finally, it may be of theoretical interest to note that the solvability of Ω can be determined in polynomial time (Khachiyan [1] and [2]). For *fixed n*, the algorithm of Lenstra [3] is also polynomial time.

Acknowledgement

The authors would like to take this opportunity to thank V.L. Klee for some interesting discussions.

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