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Analysis on the minimal representation of $\mathrm{O}(p, q)$ I. Realization via conformal geometry

Toshiyuki Kobayashi[a,*] and Bent Ørsted[b]

[a] *RIMS, Kyoto University, Sakyo-ku, Kyoto 606-8502, Japan*
[b] *Department of Mathematics and Computer Science, SDU - Odense University, Campusvej 55, Odense M, DK-5230, Denmark*

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Abstract

This is the first in a series of papers devoted to an analogue of the metaplectic representation, namely the minimal unitary representation of an indefinite orthogonal group; this representation corresponds to the minimal nilpotent coadjoint orbit in the philosophy of Kirillov–Kostant. We begin by applying methods from conformal geometry of pseudo-Riemannian manifolds to a general construction of an infinite-dimensional representation of the conformal group on the solution space of the Yamabe equation. By functoriality of the constructions, we obtain different models of the unitary representation, as well as giving new proofs of unitarity and irreducibility. The results in this paper play a basic role in the subsequent papers, where we give explicit branching formulae, and prove unitarization in the various models.

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Contents

*Corresponding author.
E-mail addresses: toshi@kurims.kyoto-u.ac.jp (T. Kobayashi), orsted@imada.sdu.dk (B. Ørsted).

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1. Introduction

1.0. This is the first in a series of papers devoted to a study of the so-called minimal representation of the semisimple Lie group $G = \mathrm{O}(p,q)$. We have taken the point of view that a rather complete treatment of this representation and its various realizations can be done in a self-contained way; also, such a study involves many different tools from other parts of mathematics, such as differential geometry (conformal geometry and pseudo-Riemannian geometry), analysis of solution spaces of ultrahyperbolic differential equations, Sobolev spaces, special functions such as hypergeometric functions of two variables, Bessel functions, analysis on semisimple symmetric spaces, and Dolbeault cohomology groups. Furthermore, the representation theory yields new results back to these areas, so we feel it is worthwhile to illustrate such an interaction in as elementary a way as possible. The sequel (Part II) to the present paper contains Sections 4–9, and we shall also refer to these here. Part III is of more independent nature.

Working on a single unitary representation we essentially want to analyze it by understanding its restrictions to natural subgroups, and to calculate intertwining operators between the various models—all done very explicitly. We are in a sense studying the symmetries of the representation space by breaking the large symmetry present originally with the group G by passing to a subgroup. Geometrically the restriction is from the conformal group G to the subgroup of isometries H, where different geometries (all locally conformally equivalent) correspond to different choices of H. Changing H will give rise to radically different models of the representation, and at the same time allow calculating the spectrum of H.

Thus, the overall aim is to elucidate as many aspects as possible of a distinguished unitary irreducible representation of $\mathrm{O}(p,q)$, including its explicit branching laws to natural subgroups and its explicit inner product on each geometric model. Our approach is also useful in understanding the relation between the representation and a certain coadjoint orbit, namely the minimal one, in the dual of the Lie algebra. In order to give a good view of the perspective in our papers, we are giving below a rather careful introduction to all these aspects.

For a semisimple Lie group G a particularly interesting unitary irreducible representation, sometimes called the minimal representation, is the one corresponding via "geometric quantization" to the minimal nilpotent coadjoint orbit. It is still a little mysterious in the present status of the classification problem of the unitary dual of semisimple Lie groups. In recent years several authors have considered the minimal representation, and provided many new results, in particular, Kostant, Torasso, Brylinski, Li, Binegar, Zierau, and Sahi, mostly by algebraic methods [2–5,11,12,29,32]. For the double cover of the symplectic group, this is the metaplectic representation, introduced many years ago by Segal, Shale, and Weil. The explicit treatment of the metaplectic representation requires various methods from analysis and geometry, in addition to the algebraic methods; and it is our aim in a series of papers to present for the case of $G = \mathrm{O}(p,q)$ the aspects pertaining to branching laws. From an algebraic view point of representation theory, our representations $\varpi^{p,q}$ are:

(i) minimal representations if $p+q\geqslant 8$ (i.e. the annihilator is the Joseph ideal);
(ii) *not* spherical if $p\neq q$ (i.e. no non-zero K-fixed vector);
(iii) *not* highest weight modules of $SO_0(p,q)$ if $p,q\geqslant 3$.

Apparently our case provides examples of new phenomena in representation theory, and we think that several aspects of our study can be applied to other cases as well. The metaplectic representation has had many applications in representation theory and in number theory. A particularly useful concept has been Howe's idea of dual pairs, where one considers a mutually centralizing pair of subgroups in the metaplectic group and the corresponding restriction of the metaplectic representation. In Part II of our papers, we shall initiate a similar study of explicit branching laws for other groups and representations analogous to the classical case of Howe. Several such new examples of dual pairs have been studied in recent years, mainly by algebraic techniques. Our case of the real orthogonal group presents a combination of abstract representation theory and concrete analysis using methods from conformal differential geometry. Thus, we can relate the branching law to a study of the Yamabe operator and its spectrum in locally conformally equivalent manifolds; furthermore, we can prove the existence of and construct explicitly an infinite discrete spectrum in the case where both factors in the dual pair are non-compact.

The methods we use are further motivated by the theory of spherical harmonics, extending analysis on the sphere to analysis on hyperboloids [13,30], and at the same time using elliptic methods in the sense of analysis on complex quadrics and the theory of Zuckerman–Vogan's derived functor modules and their Dolbeault cohomological realizations [31,34,36,38]. Also important are general results on discrete decomposability of representations and explicit knowledge of branching laws [14,17–19].

It is noteworthy, that as we have indicated, this representation and its theory of generalized Howe correspondence, illustrates several interesting aspects of modern representation theory. Thus, we have tried to be rather complete in our treatment of the various models of the representations occurring in the branching law. See for example Fact 5.4, where we give three realizations: derived functor modules or Dolbeault cohomology groups, eigenspaces on semisimple symmetric spaces, and quotients of generalized principal series, of the representations attached to minimal elliptic orbits.

Most of the results of Parts I and II were announced in [22], and the branching law in the discretely decomposable case (Theorem 7.1) was obtained in 1991, from which our study grew out. We have here given the proofs of the branching laws for the minimal unipotent representation and postpone the detailed treatment of the corresponding classical orbit picture as announced in [22] to another paper. Also, the branching laws for the representations associated to minimal elliptic orbits will appear in another paper by one of the authors [21].

It is possible that a part of our results could be obtained by using sophisticated results from the theory of dual pairs in the metaplectic group, for example the see–saw rule (for which one may let our representation correspond to the trivial representation of one $SL(2,\mathbb{R})$ member of the dual pair [11,12]. We emphasize, however, that our approach is quite explicit and has the following advantages:

(a) It is not only an abstract representation theory but also attempts new interaction of the minimal representation with analysis on manifolds. For example, in Part II we use in an elementary way conformal differential geometry and the functorial properties of the Yamabe operator to construct the minimal representation and the branching law in a way which seems promising for other cases as well; each irreducible constituent is explicitly constructed by using explicit intertwining operators via local conformal diffeomorphisms between spheres and hyperboloids.

(b) For the explicit intertwining operators we obtain Parseval–Plancherel-type theorems, i.e. explicit L^2 versions of the branching law and the generalized Howe correspondence. This also gives a good perspective on the continuous spectrum, in particular yielding a natural conjecture for the complete Plancherel formula.

A special case of our branching law illustrates the physical situation of the conformal group of space–time $O(2,q)$; here the minimal representation may be interpreted either as the mass-zero spin-zero wave equation, or as the bound states of the Hydrogen atom (in $q-1$ space dimensions). Studying the branching law means breaking the symmetry by for example restricting to the isometry group of De Sitter space $O(2,q-1)$ or anti De Sitter space $O(1,q)$. In this way the original system (particle) breaks up into constituents with less symmetry.

In Part III, we shall realize the same representation on a space of solutions of the ultrahyperbolic equation $\square_{\mathbb{R}^{p-1,q-1}} f = 0$ on $\mathbb{R}^{p-1,q-1}$, and give an intrinsic inner product as an integration over a non-characteristic hypersurface.

Completing our discussion of different models of the minimal representation, we find yet another explicit intertwining operator, this time to an L^2-space of functions on a hypersurface (a cone) in the nilradical of a maximal parabolic P in G. We find the K-finite functions in the case of $p+q$ even in terms of modified Bessel functions. Integration formulae involving various special functions naturally appear in our analysis on the minimal representation [6–8]. We remark that Vogan pointed out a long-time ago that there is no minimal representation of $O(p,q)$ if $p+q>8$ is odd [33]. On the other hand, we have found a new interesting phenomenon that in the case $p+q$ is odd there still exists a geometric model of a "minimal representation" of $\mathfrak{o}(p,q)$ with a natural inner product (see Part III). Of course, such a representation does not have non-zero K-finite vectors for $p+q$ odd, but have K'-finite vectors for smaller K'. What we construct in this case is an element of the category of $(\mathfrak{g},P)$ modules in the sense that it globalizes to P (but not K); we feel this concept perhaps plays a role for other cases of the orbit method as well.

In summary, we give a geometric and intrinsic model of the minimal representation $\varpi^{p,q}$ (not coming from the construction of $\varpi^{p,q}$ by the θ-correspondence) on $S^{p-1}\times S^{q-1}$ and on various pseudo-Riemannian manifolds which are conformally equivalent, using the functorial properties of the Yamabe operator, a key element in conformal differential geometry. The branching law for

$\varpi^{p,q}$ gives at the same time new perspectives on conformal geometry, and relates analysis on hyperboloids to that of minimal representations, with new phenomena in both areas. The main interest in this special case of a small unitary representation is not only to obtain the formulae, but also to investigate the geometric and analytic methods, which provide new ideas in representation theory.

Leaving the general remarks, let us now for the rest of this introduction be a little more specific about the contents of the present paper.

1.1. Let G be a reductive Lie group, and G' a reductive subgroup of G. We denote by $\hat{G}$ the unitary dual of G, the equivalence classes of irreducible unitary representations of G. Likewise $\widehat{G'}$ for G'. If $\pi \in \hat{G}$, then the restriction $\pi|_{G'}$ is not necessarily irreducible. By a branching law, we mean an explicit irreducible decomposition formula:

$$\pi|_{G'} \simeq \int_{\widehat{G'}}^{\oplus} m_\pi(\tau)\tau \, \mathrm{d}\mu(\tau) \quad \text{(direct integral)}, \tag{1.1.1}$$

where $m_\pi(\tau) \in \mathbb{N} \cup \{\infty\}$ and $d\mu$ is a Borel measure on $\widehat{G'}$.

1.2. We denote by $\mathfrak{g}_0$ the Lie algebra of G. The *orbit method* due to Kirillov–Kostant in the unitary representation theory of Lie groups indicates that the coadjoint representation $\mathrm{Ad}^* : G \to GL(\mathfrak{g}_0^*)$ often has a surprising intimate relation with the unitary dual $\hat{G}$. It works perfectly for simply connected nilpotent Lie groups. For real reductive Lie groups G, known examples suggest that the set of coadjoint orbits $\sqrt{-1}\mathfrak{g}_0^*/G$ (with certain integral conditions) still gives a fairly good approximation of the unitary dual $\hat{G}$.

1.3. Here is a rough sketch of a unitary representation π_λ of G, attached to an elliptic element $\lambda \in \sqrt{-1}\mathfrak{g}_0^*$: The elliptic coadjoint orbit $\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda$ carries a G-invariant complex structure, and one can define a G-equivariant holomorphic line bundle $\widetilde{\mathscr{L}_\lambda} := \mathscr{L}_\lambda \otimes (\wedge^{\mathrm{top}} T^*\mathcal{O}_\lambda)^{\frac{1}{2}}$ over $\mathcal{O}_\lambda$, if λ satisfies some integral condition. Then, we have a Fréchet representation of G on the Dolbeault cohomology group $H^S_{\bar\partial}(\mathcal{O}_\lambda, \widetilde{\mathscr{L}_\lambda})$, where $S := \dim_{\mathbb{C}} \mathrm{Ad}^*(K)\lambda$ (see [38] for details), and of which a unique dense subspace we can define a unitary representation π_λ of G [35] if λ satisfies certain positivity. The unitary representation π_λ is irreducible and non-zero if λ is sufficiently regular. The underlying $(\mathfrak{g}, K)$-module is the so-called "$A_\mathfrak{q}(\lambda)$" in the sense of Zuckerman–Vogan after certain ρ-shift [34,36,37].

In general, decomposition (1.1.1) contains both discrete and continuous spectrum. The condition for the discrete decomposition (without continuous spectrum) has been studied in [14,17–19], especially for π_λ attached to elliptic orbits $\mathcal{O}_\lambda$. It is likely that if $\pi \in \hat{G}$ is "attached to" a nilpotent orbit, which is contained in the limit set of $\mathcal{O}_\lambda$, then the discrete decomposability of $\pi|_{G'}$ should be inherited from that of the elliptic case $\pi_\lambda|_{G'}$. We shall see in Theorem 4.2 that this is the case in our situation.

1.4. There have been a number of attempts to construct representations attached to nilpotent orbits. Among all, the Segal–Shale–Weil representation (or the oscillator representation) of $\widetilde{Sp}(n,\mathbb{R})$, for which we write $\tilde{\varpi}$, has been best studied, which is supposed to be attached to the minimal nilpotent orbit of $\mathfrak{sp}(n,\mathbb{R})$. The restriction of $\tilde{\varpi}$ to a reductive dual pair $G' = G'_1G'_2$ gives Howe's correspondence [10].

The group $\widetilde{Sp}(n,\mathbb{R})$ is a split group of type C_n, and analogously to $\tilde{\varpi}$, Kostant constructed a minimal representation of $\mathrm{SO}(n,n)$, a split group of type D_n. Then Binegar–Zierau generalized it for $\mathrm{SO}(p,q)$ with $p+q\in 2\mathbb{N}$. This representation (precisely, of $\mathrm{O}(p,q)$, see Section 3) will be denoted by $\varpi^{p,q}$.

1.5. Let $G' := G'_1G'_2 = \mathrm{O}(p',q')\times \mathrm{O}(p'',q'')$, $(p'+p''=p, q'+q''=q)$, be a subgroup of $G=\mathrm{O}(p,q)$. Our object of study in Part II will be the branching law $\varpi^{p,q}|_{G'}$. We note that G'_1 and G'_2 form a mutually centralizing pair of subgroups in G.

It is interesting to compare the feature of the following two cases:

(i) the restriction $\tilde{\varpi}|_{G'_1G'_2}$ (the Segal–Shale–Weil representation for type C_n),
(ii) the restriction $\varpi^{p,q}|_{G'_1G'_2}$ (the Kostant–Binegar–Zierau representation for type D_n).

The reductive dual pair $(G,G') = (G, G'_1G'_2)$ is of the $\otimes$-type in (i), that is, induced from $\mathrm{GL}(V)\times \mathrm{GL}(W)\to \mathrm{GL}(V\otimes W)$; is of the $\oplus$-type in (ii), that is, induced from $\mathrm{GL}(V)\times \mathrm{GL}(W)\to \mathrm{GL}(V\oplus W)$. On the other hand, both of the restrictions in (i) and (ii) are discretely decomposable in the sense of [14,17–19] if one factor G'_2 is compact. Furthermore, the resulting branching laws are multiplicity free. (See [10,16,20] for general theory.) On the other hand, $\tilde{\varpi}$ is (essentially) a highest weight module in (i), while $\varpi^{p,q}$ is not if $p,q>2$ in (ii).

1.6. Let $p+q\in 2\mathbb{N}$, $p,q\geqslant 2$, and $(p,q)\neq(2,2)$. In this section we state the main results of the present paper and the sequels (mainly Part II; an introduction of Part III will be given separately in [24]). The first Theorem A (Theorem 2.5) says that there is a general way of constructing representations of a conformal group by twisted pull-backs (see Section 2 for notation). It is the main tool to give different models of our representation.

Theorem A. *Suppose that a group G acts conformally on a pseudo-Riemannian manifold M of dimension n.*

(1) *Then, the Yamabe operator* (see (2.2.1) *for the definition*)

$$\tilde{\Delta}_M : C^\infty(M)\to C^\infty(M)$$

is an intertwining operator from $\varpi_{\frac{n-2}{2}}$ *to* $\varpi_{\frac{n+2}{2}}$ (see (2.5.1) *for the definition of* ϖ_λ).

(2) *The kernel* Ker $\tilde{\Delta}_M$ *is a subrepresentation of G through* $\varpi_{\frac{n-2}{2}}$.

Theorem B. (1) *The minimal representation* $\varpi^{p,q}$ *of* $\mathrm{O}(p,q)$ *is realized as the kernel of the Yamabe operator on* $S^{p-1} \times S^{q-1}$.

(2) $\varpi^{p,q}$ *is also realized as a subspace* (*roughly, half*) *of the kernel of the Yamabe operator on the hyperboloid* $\{(x,y) \in \mathbb{R}^{p,q}\colon |x|^2 - |y|^2 = 1\}$.

(3) $\varpi^{p,q}$ *is also realized in a space of solutions to the Yamabe equation on* $\mathbb{R}^{p-1,q-1}$ *which is a standard ultrahyperbolic constant coefficient differential equation.*

(4) $\varpi^{p,q}$ *is also realized as the unique non-trivial subspace of the Dolbeault cohomology group* $H_{\bar{\partial}}^{p-2}(G/L, \mathscr{L}_{\frac{p+q-4}{2}})$.

In Theorem B (1) is contained in Part I, Theorem 3.6.1; (2) in Part II, Corollary 7.2.1; and (3) in Part III, Theorem 4.7. In each of these models, an explicit realization is given. In models (2) and (3), the situation is subtle because the "action" of $\mathrm{O}(p,q)$ is no more smooth but only meromorphic. Then Theorem A does not hold in its original form, and we need to carry out a careful analysis for it (see Parts II and III). The proof of statement (4) will appear in another paper. Here G/L is an elliptic coadjoint orbit as in Section 1.3, and $L = \mathrm{SO}(2) \times \mathrm{O}(p-2,q)$.

The branching laws in Theorems C and D are the main themes in Part II; for notation see Sections 7 and 9.

Theorem C. *If* $q'' \geqslant 1$ *and* $q' + q'' = q$, *then the twisted pull-back* $\widetilde{\Phi_1^*}$ *of the local conformal map* Φ_1 *between spheres and hyperboloids gives an explicit irreducible decomposition of the unitary representation* $\varpi^{p,q}$ *when restricted to* $\mathrm{O}(p,q') \times \mathrm{O}(q'')$:

$$\varpi^{p,q}|_{\mathrm{O}(p,q') \times \mathrm{O}(q'')} \simeq \sum_{l=0}^{\infty} \pi^{p,q'}_{+,l+\frac{q''}{2}-1} \boxtimes \mathscr{H}^l(\mathbb{R}^{q''}).$$

In addition, we give in Section 8, Theorem 8.6 the Parseval–Plancherel theorem for the situation in Theorem C on the "hyperbolic space model". This may be also regarded as the unitarization of the minimal representation $\varpi^{p,q}$.

The twisted pull-back for a locally conformal diffeomorphism is defined for an arbitrary pseudo-Riemannian manifold (see Definition 2.3).

Theorem D. *The twisted pull-back of the locally conformal diffeomorphism also constructs*

$$\sum_{\lambda \in A'(p',q') \cap A'(q'',p'')}^{\oplus} \pi^{p',q'}_{+,\lambda} \boxtimes \pi^{p'',q''}_{-,\lambda} \oplus \sum_{\lambda \in A'(q',p') \cap A'(p'',q'')}^{\oplus} \pi^{p',q'}_{-,\lambda} \boxtimes \pi^{p'',q''}_{+,\lambda}$$

as a discrete spectra in the branching law.

1.7. The papers (Parts I and II) are organized as follows: Section 2 provides a conformal construction of a representation on the kernel of a shifted Laplace–Beltrami operator. In Section 3, we construct an irreducible unitary representation,

$\varpi^{p,q}$ of $\mathrm{O}(p,q)$ $(p+q\in 2\mathbb{N}, p,q\geqslant 2)$ "attached to" the minimal nilpotent orbit applying Theorem 2.5. This representation coincides with the minimal representation studied by Kostant, Binegar–Zierau [2,25]. In Section 3 we give a new intrinsic characterization of the Hilbert space for the minimal representation in this model, namely as a certain Sobolev space of solutions, see Theorem 3.9.3 and Lemma 3.10. Such Sobolev estimate will be used in the construction of discrete spectrum of the branching law in Section 9. Section 4 contains some general results on discrete decomposable restrictions [17,18], specialized in detail to the present case. Theorem 4.2 characterizes which dual pairs in our situation provide discrete decomposable branching laws of the restriction of the minimal representation $\varpi^{p,q}$. In Section 5, we introduce unitary representations, $\pi^{p,q}_{\pm,\lambda}$ of $\mathrm{O}(p,q)$ "attached to" minimal elliptic coadjoint orbits. In Sections 7 and 9, we give a discrete spectrum of the branching law $\varpi^{p,q}|_{G'}$ in terms of $\pi^{p',q'}_{\pm,\lambda}\in\widehat{\mathrm{O}(p',q')}$ and $\pi^{p'',q''}_{\pm,\lambda}\in\widehat{\mathrm{O}(p'',q'')}$. In particular, if one factor $G'_2=\mathrm{O}(p'',q'')$ is compact (i.e. $p''=0$ or $q''=0$), the branching law is completely determined together with a Parseval–Plancherel theorem in Section 8.

Following the suggestion of the referee, we have included a full account of our proof of the unitarity of the minimal representation. This proof is independent of earlier proofs by Kostant, Binegar–Zierau, Howe–Tan, and others, and we feel it in itself deserves attention. Our argument is purely analytical, based on analysis on hyperboloids, and avoids combinatorial calculations of the actions of Lie algebras. The key statement is in Theorem 3.9.1 with the immediate application to the unitarity in Corollary 3.9.2. The proof of Theorem 3.9.1 will be given in Section 8.3, and it uses a factorization (see (8.3.8)) of the Knapp–Stein intertwining operator as the product of a Poisson transform into an affine symmetric space (a hyperboloid), and a boundary value map. This gives the explicit eigenvalues of the Knapp–Stein intertwining operators on generalized principal series representations, and not only on some subrepresentations. We think this method is promising with regard to some higher-rank situations. In particular, one is free to choose "intermediate" affine symmetric spaces.

Finally, we have included the proofs of the explicit formulas for the Jacobi functions used in Section 8, mainly Lemmas 8.1 and 8.2. These formulas lead to the Parseval–Plancherel formulas (see Theorem 8.6) for the branching laws of the minimal representation realized on hyperboloids. (Incidentally, this can be applied to give a proof of the unitarity of a certain Zuckerman–Vogan's derived functor module even outside the weakly fair range.)

Notation: $\mathbb{N}=\{0,1,2,\dots\}$.

2. Conformal geometry

2.1. The aim of this section is to associate a distinguished representation ϖ_M of the conformal group $\mathrm{Conf}(M)$ to a general pseudo-Riemannian manifold M (see Theorem 2.5).

2.2. Let M be an n-dimensional manifold with pseudo-Riemannian metric g_M $(n \geqslant 2)$. Let ∇ be the Levi–Civita connection for the pseudo-Riemannian metric g_M. The curvature tensor field R is defined by

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X, Y, Z \in \mathfrak{X}(M).$$

We take an orthonormal basis $\{X_1, \ldots, X_n\}$ of $T_x M$ for a fixed $x \in M$. Then the scalar curvature K_M is defined by

$$K_M(x) := \sum_{i=1}^{n} \sum_{j=1}^{n} g_M(R(X_i, X_j)X_i, X_j).$$

The right side is independent of the choice of the basis $\{X_i\}$ and so K_M is a well-defined function on M. We denote by Δ_M the Laplace–Beltrami operator on M. The *Yamabe operator* is defined to be

$$\tilde{\Delta}_M := \Delta_M - \frac{n-2}{4(n-1)} K_M. \tag{2.2.1}$$

See for example [26] for a good discussion of the geometric meaning and applications of this operator. Our choice of the signature of K_M and Δ_M is illustrated as follows:

Example 2.2. We equip $\mathbb{R}^n$ and S^n with standard Riemannian metric. Then

$$\text{For } \mathbb{R}^n: \quad K_{\mathbb{R}^n} \equiv 0, \qquad \tilde{\Delta}_{\mathbb{R}^n} = \Delta_{\mathbb{R}^n} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

$$\text{For } S^n: \quad K_{S^n} \equiv (n-1)n, \qquad \tilde{\Delta}_{S^n} = \Delta_{S^n} - \frac{1}{4} n(n-2).$$

2.3. Suppose (M, g_M) and (N, g_N) are pseudo-Riemannian manifolds of dimension n. A local diffeomorphism $\Phi : M \to N$ is called a *conformal map* if there exists a positive valued function Ω on M such that

$$\Phi^* g_N = \Omega^2 g_M.$$

Φ is isometry if and only if $\Omega \equiv 1$ by definition.

We denote the group of conformal transformations (respectively, isometries) of a pseudo-Riemannian manifold (M, g_M) by

$$\mathrm{Conf}(M) := \{\Phi \in \mathrm{Diffeo}(M) : \Phi : M \to M \text{ is conformal}\},$$

$$\mathrm{Isom}(M) := \{\Phi \in \mathrm{Diffeo}(M) : \Phi : M \to M \text{ is isometry}\}.$$

Clearly, $\mathrm{Isom}(M) \subset \mathrm{Conf}(M)$.

If Φ is conformal, then we have (e.g. [9, Chapter II, Excercise A.5] [27,28])

$$\Omega^{\frac{n+2}{2}}(\Phi^* \tilde{\Delta}_N f) = \tilde{\Delta}_M(\Omega^{\frac{n-2}{2}} \Phi^* f) \tag{2.3.1}$$

for any $f \in C^\infty(N)$. We define a twisted pull-back

$$\Phi_\lambda^* : C^\infty(N) \to C^\infty(M), \quad f \mapsto \Omega^\lambda(\Phi^* f), \tag{2.3.2}$$

for each fixed $\lambda \in \mathbb{C}$. Then formula (2.3.1) is rewritten as

$$\Phi^*_{\frac{n+2}{2}} \tilde{\Delta}_N f = \tilde{\Delta}_M \Phi^*_{\frac{n-2}{2}} f. \tag{2.3.1$'$}$$

The case when $\lambda = \frac{n-2}{2}$ is particularly important. Thus, we write the twisted pull-back for $\lambda = \frac{n-2}{2}$ as follows:

Definition 2.3. $\widetilde{\Phi^*} = \Phi^*_{\frac{n-2}{2}} : C^\infty(N) \to C^\infty(M),\ f \mapsto \Omega^{\frac{n-2}{2}}(\Phi^* f).$

Then formula (2.3.1) implies that

$$\tilde{\Delta}_N f = 0 \quad \text{on } \Phi(M) \quad \text{if and only if} \quad \tilde{\Delta}_M(\widetilde{\Phi^*} f) = 0 \quad \text{on } M \tag{2.3.3}$$

because Ω is nowhere vanishing.

If $n = 2$, then $\tilde{\Delta}_M = \Delta_M$, $\tilde{\Delta}_N = \Delta_N$, and $\widetilde{\Phi^*} = \Phi^*$. Hence, (2.3.3) implies a well-known fact in the two-dimensional case that *a conformal map Φ preserves harmonic functions*, namely,

$$f \text{ is harmonic } \Leftrightarrow \Phi^* f \text{ is harmonic.}$$

2.4. Let G be a Lie group acting conformally on a pseudo-Riemannian manifold (M, g_M). We write the action of $h \in G$ on M as $L_h : M \to M, x \mapsto L_h x$. By the definition of conformal transformations, there exists a positive valued function $\Omega(h, x)$ $(h \in G, x \in M)$ such that

$$L_h^* g_M = \Omega(h, \cdot)^2 g_M \quad (h \in G).$$

Then we have

Lemma 2.4. *For $h_1, h_2 \in G$ and $x \in M$, we have*

$$\Omega(h_1 h_2, x) = \Omega(h_1, L_{h_2} x)\ \Omega(h_2, x).$$

Proof. It follow from $L_{h_1 h_2} = L_{h_1} L_{h_2}$ that

$$L^*_{h_1 h_2} g_M = L^*_{h_2} L^*_{h_1} g_M.$$

Therefore we have $\Omega(h_1h_2,\cdot)^2\ g_M = L^*_{h_1h_2}g_M = L^*_{h_2}(L^*_{h_1}g_M) = L^*_{h_2}(\Omega(h_1,\cdot)^2\ g_M) = \Omega(h_1, L_{h_2}\cdot)^2\Omega(h_2,\cdot)^2 g_M$. Since Ω is a positive valued function, we conclude that $\Omega(h_1h_2,x) = \Omega(h_1, L_{h_2}x)\Omega(h_2,x)$. $\square$

2.5. For each $\lambda \in \mathbb{C}$, we form a representation $\varpi_\lambda \equiv \varpi_{M,\lambda}$ of the conformal group G on $C^\infty(M)$ as follows:

$$(\varpi_\lambda(h^{-1})f)(x) := \Omega(h,x)^\lambda f(L_h x), \quad (h \in G, f \in C^\infty(M), x \in M). \tag{2.5.1}$$

Then Lemma 2.4 assures that $\varpi_\lambda(h_1)\varpi_\lambda(h_2) = \varpi_\lambda(h_1h_2)$, namely, ϖ_λ is a representation of G.

Denote by dx the volume element on M defined by the pseudo-Riemannian structure g_M. Then we have

$$L^*_h(dx) = \Omega(h,x)^n dx \quad \text{for } h \in G.$$

Therefore, the map $f \mapsto f\,dx$ gives a G-intertwining operator from $(\varpi_n, C^\infty(M))$ into the space of distributions $\mathscr{D}'(M)$ on M.

Here is a construction of a representation of the group of conformal diffeomorphisms of M.

Theorem 2.5. *Suppose that a group G acts conformally on a pseudo-Riemannian manifold M of dimension n. Retain the notation before.*

(1) *Then, the Yamabe operator*

$$\tilde{\Delta}_M : C^\infty(M) \to C^\infty(M)$$

is an intertwining operator from $\varpi_{\frac{n-2}{2}}$ to $\varpi_{\frac{n+2}{2}}$.

(2) *The kernel* $\operatorname{Ker} \tilde{\Delta}_M$ *is a subrepresentation of G through $\varpi_{\frac{n-2}{2}}$.*

Proof. Statement (1) is a representation theoretic counterpart of formula (2.3.1). Statement (2) follows immediately from (1). $\square$

The representation of G on $\operatorname{Ker} \tilde{\Delta}_M$ given in Theorem 2.5 (2) will be denoted by $\varpi \equiv \varpi_M$.

2.6. Here is a naturality of the representation of the conformal group $\mathrm{Conf}(M)$ in Theorem 2.5:

Proposition 2.6. *Let M and N be pseudo-Riemannian manifolds of dimension n, and a local diffeomorphism $\Phi : M \to N$ be a conformal map. Suppose that Lie groups G' and G act conformally on M and N, respectively. The actions will be denoted by L_M and L_N, respectively. We assume that there is a homomorphism $i : G' \to G$ such that*

$$L_{N,i(h)} \circ \Phi = \Phi \circ L_{M,h} \quad (\text{for any } h \in G').$$

We write conformal factors Ω_M, Ω_N *and* Ω *as follows*:

$$L_{M,h}^* g_M = \Omega_M(h,\cdot)^2 g_M \quad (h \in G'),$$

$$L_{N,h}^* g_N = \Omega_N(h,\cdot)^2 g_N \quad (h \in G),$$

$$\Phi^* g_N = \Omega^2 g_M.$$

(1) *For* $x \in M$ *and* $h \in G'$, *we have*

$$\Omega(L_{M,h}x)\ \Omega_M(h,x) = \Omega(x)\Omega_N(i(h),\Phi(x)). \tag{2.6.1}$$

(2) *Let* $\lambda \in \mathbb{C}$ *and* $\Phi_\lambda^* : C^\infty(N) \to C^\infty(M)$ *be the twisted pull-back defined in* (2.3.2). *Then* Φ_λ^* *respects the G-representation* $(\varpi_{N,\lambda}, C^\infty(N))$ *and the* G'*-representation* $(\varpi_{M,\lambda}, C^\infty(M))$ *through* $i : G' \to G$.

(3) $\widetilde{\Phi^*} = \Phi^*_{\frac{n-2}{2}} : C^\infty(N) \to C^\infty(M)$ *sends* Ker $\tilde{\Delta}_N$ *into* Ker $\tilde{\Delta}_M$. *In particular, we have a commutative diagram*:

$$\begin{array}{ccc} \operatorname{Ker}\widetilde{\Delta}_N & \xrightarrow{\widetilde{\Phi^*}} & \operatorname{Ker}\widetilde{\Delta}_M \\ {\scriptstyle \varpi_N(i(h))}\downarrow & & \downarrow{\scriptstyle \varpi_M(h)} \\ \operatorname{Ker}\widetilde{\Delta}_N & \xrightarrow[\widetilde{\Phi^*}]{} & \operatorname{Ker}\widetilde{\Delta}_M \end{array} \tag{2.6.2}$$

for each $h \in G'$.

(4) *If* Φ *is a diffeomorphism onto* N, *then* $(\Phi^{-1})_\lambda^*$ *is the inverse of* Φ_λ^* *for each* $\lambda \in \mathbb{C}$. *In particular*, $\widetilde{\Phi^*}$ *is a bijection between* Ker $\tilde{\Delta}_N$ *and* Ker $\tilde{\Delta}_M$ *with inverse* $\widetilde{(\Phi^{-1})^*}$.

Proof. (1) Because $L_{N,i(h)} \circ \Phi = \Phi \circ L_{M,h}$ for $h \in G'$, we have

$$(\Phi^* L_{N,i(h)}^* g_N)(x) = (L_{M,h}^* \Phi^* g_N)(x) \quad \text{for } x \in M.$$

Hence,

$$\Omega_N(i(h),\Phi(x))^2 \Omega(x)^2 g_M(x) = \Omega(L_{M,h}x)^2 \Omega_M(h,x)^2 g_M(x).$$

Because all conformal factors are positive-valued functions, we have proved (2.6.1).

(2) We want to prove

$$(\varpi_{M,\lambda}(h^{-1})\Phi_\lambda^* f)(x) = (\Phi_\lambda^* \varpi_{N,\lambda}(i(h^{-1}))f)(x) \tag{2.6.3}$$

for any $x \in M$, $h \in G'$ and $\lambda \in \mathbb{C}$. In view of the definition, we have

$$\begin{aligned}\text{the left-hand side of } (2.6.3) &= (\varpi_{M,\lambda}(h^{-1})(\Omega^{\lambda}\Phi^* f))(x)\\ &= \Omega_M(h,x)^{\lambda}\Omega(L_{M,h}x)^{\lambda}(\Phi^* f)(L_{M,h}x)\\ &= \Omega(x)^{\lambda}\Omega_N(i(h),\Phi(x))^{\lambda} f(\Phi \circ L_{M,h}x).\end{aligned}$$

Here the last equality follows from (2.6.1).

$$\begin{aligned}\text{The right-hand side of } (2.6.3) &= (\Phi_{\lambda}^* \Omega_N(i(h),\cdot)^{\lambda} f(L_{N,h}\cdot))(x)\\ &= \Omega(x)^{\lambda}\Omega_N(i(h),\Phi(x))^{\lambda} f(L_{N,i(h)} \circ \Phi(x)).\end{aligned}$$

Therefore, we have (2.6.3), because $L_{N,i(h)} \circ \Phi = \Phi \circ L_{M,h}$.

(3) If $f \in C^{\infty}(N)$ satisfies $\tilde{\Delta}_N f = 0$, then $\tilde{\Delta}_M(\widetilde{\Phi^*} f) = \Omega^{\frac{n+2}{2}}(\Phi^* \tilde{\Delta}_N f) = 0$ by (2.3.1). Hence $\widetilde{\Phi^*}(\operatorname{Ker} \tilde{\Delta}_N) \subset \operatorname{Ker} \tilde{\Delta}_M$. The commutativity of the diagram (2.6.2) follows from (2) and Theorem 2.5 (2), if we put $\lambda = \frac{n-2}{2}$.

(4) Because $(\Phi^{-1})^* g_M = (\Omega \circ \Phi^{-1})^{-2} g_N$, the twisted pull-back $(\Phi^{-1})_{\lambda}^* F$ is given by the following formula from definition (2.3.2):

$$(\Phi^{-1})_{\lambda}^* : C^{\infty}(M) \to C^{\infty}(N), \quad F \mapsto (\Phi^{-1})_{\lambda}^* F = (\Omega \circ \Phi^{-1})^{-\lambda}(F \circ \Phi^{-1}).$$

Now statement (4) follows immediately. $\square$

3. Minimal unipotent representations of $\mathbf{O}(p,q)$

3.1. In this section, we apply Theorem 2.5 to the specific setting where $M = S^{p-1} \times S^{q-1}$ is equipped with an indefinite Riemannian metric, and where the indefinite orthogonal group $G = O(p,q)$ acts conformally on M. The resulting representation, denoted by $\varpi^{p,q}$, is non-zero, irreducible and unitary if $p+q \in 2\mathbb{N}, p,q \geqslant 2$ and if $(p,q) \neq (2,2)$. This representation coincides with the one constructed by Kostant [25] and Binegar–Zierau [2], which has the Gelfand–Kirillov dimension $p+q-3$ (see Part II, Lemma 4.4). This representation is supposed to be attached to the unique minimal nilpotent coadjoint orbit, in the sense that its annihilator in the enveloping algebra $U(\mathfrak{g})$ is the Joseph ideal if $p+q \geqslant 8$, which is the unique completely prime primitive ideal of minimum non-zero Gelfand–Kirillov dimension.

Our approach based on conformal geometry gives a geometric realization of the minimal representation $\varpi^{p,q}$ for $\mathrm{O}(p,q)$. One of the advantages using conformal geometry is the naturality of the construction (see Proposition 2.6), which allows us naturally different realizations of $\varpi^{p,q}$, not only on the K-picture (a compact picture in Section 3), but also on the N-picture (a flat picture) (see Part III), and on the hyperboloid picture (see Part II, Section 7, Corollary 7.2.1), together with the Yamabe operator in each realization. In later sections, we shall reduce the branching problems of $\varpi^{p,q}$ to the analysis on different models on which the minimal representation $\varpi^{p,q}$ is realized.

The case of $\mathrm{SO}(3,4)$ was treated by Sabourin [29]; his method was generalized in [32] to cover all simple groups with admissible minimal orbit, as well as the case of a local field of characteristic zero.

3.2. We write a standard coordinate of $\mathbb{R}^{p+q}$ as $(x,y)=(x_1,\dots,x_p,y_1,\dots,y_q)$. Let $\mathbb{R}^{p,q}$ be the pseudo-Riemannian manifold $\mathbb{R}^{p+q}$ equipped with the pseudo-Riemannian metric:

$$ds^2 = dx_1^2+\cdots+dx_p^2-dy_1^2-\cdots-dy_q^2. \tag{3.2.1}$$

We assume $p,q\geqslant 1$ and define submanifolds of $\mathbb{R}^{p,q}$ by

$$\Xi := \{(x,y)\in\mathbb{R}^{p,q}\colon\ |x|=|y|\}\setminus\{0\}, \tag{3.2.2}$$

$$M := \{(x,y)\in\mathbb{R}^{p,q}\colon\ |x|=|y|=1\}\simeq S^{p-1}\times S^{q-1}. \tag{3.2.3}$$

We define a diagonal matrix by $I_{p,q}:=\mathrm{diag}(1,\dots,1,-1,\dots,-1)$. The indefinite orthogonal group

$$G=\mathrm{O}(p,q):=\{g\in GL(p+q,\mathbb{R})\colon\ {}^tgI_{p,q}g=I_{p,q}\}$$

acts isometrically on $\mathbb{R}^{p,q}$ by the natural representation, denoted by $z\mapsto g\cdot z$ $(g\in G, z\in\mathbb{R}^{p,q})$. This action stabilizes the light cone Ξ. The multiplicative group $\mathbb{R}_+^\times :=\{r\in\mathbb{R}: r>0\}$ acts on Ξ as a dilation and the quotient space $\Xi/\mathbb{R}_+^\times$ is identified with M. Because the action of G commutes with that of $\mathbb{R}_+^\times$, we can define the action of G on the quotient space $\Xi/\mathbb{R}_+^\times$, and also on M through the diffeomorphism $M\simeq\Xi/\mathbb{R}_+^\times$. This action will be denoted by

$$L_h: M\to M, x\mapsto L_hx \quad (x\in M, h\in G).$$

In summary, we have a G-equivariant principal $\mathbb{R}_+^\times$-bundle:

$$\Phi:\Xi\to M,\quad (x,y)\mapsto\left(\frac{x}{|x|},\frac{y}{|y|}\right)=\frac{1}{v(x,y)}(x,y), \tag{3.2.4}$$

where $v:\Xi\to\mathbb{R}_+$ is defined by

$$v(x,y)=|x|=|y|. \tag{3.2.5}$$

3.3. Suppose N is a $(p+q-2)$-dimensional submanifold of Ξ. We say N is *transversal to rays* if $\Phi|_N: N\to M$ is locally diffeomorphic. Then, the standard pseudo-Riemannian metric on $\mathbb{R}^{p,q}$ induces a pseudo-Riemannian metric on N which has the codimension 2 in $\mathbb{R}^{p,q}$. The resulting pseudo-Riemannian metric is denoted by g_N, which has the signature $(p-1, q-1)$. In particular, $M \simeq S^{p-1} \times S^{q-1}$ itself is transversal to rays, and the induced metric $g_{S^{p-1} \times S^{q-1}} = g_{S^{p-1}} \oplus (-g_{S^{q-1}})$, where $g_{S^{n-1}}$ denotes the standard Riemannian metric on the unit sphere S^{n-1}.

Lemma 3.3. *Assume that N is transversal to rays. Then $\Phi|_N : N \to M$ is a conformal map. Precisely, we have*

$$(\Phi^* g_M)_z = v(z)^{-2} (g_N)_z \quad \textit{for } z = (x, y) \in N. \tag{3.3.1}$$

Proof. Write the coordinates as $(u_1, \ldots, u_p, v_1, \ldots, v_q) = \Phi(x, y) \in S^{p-1} \times S^{q-1}$. Then

$$\Phi^*(du_j) = \frac{dx_j}{|x|} - \frac{x_j}{|x|^3} \sum_{i=1}^{p} x_i \, dx_i.$$

Therefore, we have

$$\begin{aligned}\Phi^*\left(\sum_{j=1}^{p} (du_j)^2\right) &= |x|^{-2} \sum_{j=1}^{p} (dx_j)^2 - 2\,|x|^{-4}\left(\sum_{j=1}^{p} x_j \, dx_j\right)^2 + |x|^{-6}\left(\sum_{j=1}^{p} x_j^2\right)\left(\sum_{i=1}^{p} x_i \, dx_i\right)^2 \\ &= |x|^{-2} \sum_{j=1}^{p} (dx_j)^2 - |x|^{-4}\left(\sum_{j=1}^{p} x_j \, dx_j\right)^2.\end{aligned}$$

Similarly, we have

$$\Phi^*\left(\sum_{j=1}^{q} (dv_j)^2\right) = |y|^{-2} \sum_{j=1}^{q} (dy_j)^2 - |y|^{-4}\left(\sum_{j=1}^{q} y_j \, dy_j\right)^2.$$

Because $|x|^2 = |y|^2$ and $\sum_{j=1}^{p} x_j \, dx_j = \sum_{k=1}^{q} y_k \, dy_k$, we have

$$\Phi^*\left(\sum_{j=1}^{p} (du_j)^2 - \sum_{j=1}^{q} (dv_j)^2\right) = \frac{1}{|x|^2}\left(\sum_{j=1}^{p} (dx_j)^2 - \sum_{k=1}^{q} (dy_k)^2\right).$$

Hence, we have proved (3.3.1) from our definition of g_M and g_N. $\square$

3.4. If we apply Lemma 3.3 to the transformation on the pseudo-Riemannian manifold $M = S^{p-1} \times S^{q-1}$, we have:

Lemma 3.4.1. *G acts conformally on M. That is, for $h \in G, z \in M$, we have*

$$L_h^* g_M = \frac{1}{\nu(h \cdot z)^2} g_M \quad at\ T_z M.$$

Proof. The transformation $L_h : M \to M$ is the composition of the isometry $M \to h \cdot M$, $z \mapsto h \cdot z$, and the conformal map $\Phi|_{h \cdot M} : h \cdot M \to M$, $\xi \mapsto \frac{\xi}{\nu(\xi)}$. Hence Lemma 3.4.1 follows. $\square$

Several works in differential geometry treat the connection between the geometry of a manifold and the structure of its conformal group. For the identity

$$\mathrm{Conf}(S^{p-1} \times S^{q-1}) = \mathrm{O}(p,q), \quad (p>2, q>2),$$

see for example [15, Chapter IV].

As in Example 2.2, the Yamabe operator on $M = S^{p-1} \times S^{q-1}$ is given by the formula:

$$\begin{aligned}\tilde{\Delta}_M &= \Delta_{S^{p-1}} - \Delta_{S^{q-1}} - \frac{p+q-4}{4(p+q-3)}((p-1)(p-2) - (q-1)(q-2)) \\ &= (\Delta_{S^{p-1}} - \frac{1}{4}(p-2)^2) - (\Delta_{S^{q-1}} - \frac{1}{4}(q-2)^2) \\ &= (\tilde{\Delta}_{S^{p-1}} - \frac{1}{4}) - (\tilde{\Delta}_{S^{q-1}} - \frac{1}{4}).\end{aligned} \tag{3.4.1}$$

We define a subspace of $C^\infty(S^{p-1} \times S^{q-1})$ by

$$V^{p,q} := \{f \in C^\infty(S^{p-1} \times S^{q-1}) : \tilde{\Delta}_M f = 0\}. \tag{3.4.2}$$

By applying Theorem 2.5, we have

Theorem 3.4.2. *Let $p, q \geqslant 1$. For $h \in \mathrm{O}(p,q), z \in M = S^{p-1} \times S^{q-1}$, and $f \in V^{p,q}$, we define*

$$(\varpi^{p,q}(h^{-1})f)(z) := \nu(h \cdot z)^{-\frac{p+q-4}{2}} f(L_h z). \tag{3.4.3}$$

Then $(\varpi^{p,q}, V^{p,q})$ is a representation of $\mathrm{O}(p,q)$.

3.5. In order to describe the K-type formula of $\varpi^{p,q}$, we recall the basic fact of spherical harmonics. Let $p \geqslant 2$. The space of spherical harmonics of degree $k \in \mathbb{N}$ is defined to be

$$\mathscr{H}^k(\mathbb{R}^p) = \{f \in C^\infty(S^{p-1}) : \Delta_{S^{p-1}} f = -k(k+p-2)f\},$$

which is rewritten in terms of $\tilde{\Delta}_{S^{p-1}} = \Delta_{S^{p-1}} - \frac{1}{4}(p-1)(p-3)$ (see Example 2.2) as

$$= \left\{ f \in C^\infty(S^{p-1}) : \tilde{\Delta}_{S^{p-1}} f = \left(\frac{1}{4} - \left(k + \frac{p-2}{2} \right)^2 \right) f \right\}. \tag{3.5.1}$$

The orthogonal group $O(p)$ acts on $\mathscr{H}^k(\mathbb{R}^p)$ irreducibly and we have the dimension formula:

$$\dim_{\mathbb{C}} \mathscr{H}^k(\mathbb{R}^p) = \binom{p+k-2}{k} + \binom{p+k-3}{k-1}. \tag{3.5.2}$$

For $p=1$, it is convenient to define representations of $O(1)$ by

$$\mathscr{H}^k(\mathbb{R}^1) := \begin{cases} \mathbb{C} \text{ (trivial representation)} & (k=0), \\ \mathbb{C} \text{ (signature representation)} & (k=1), \\ 0 & (k \geqslant 2). \end{cases}$$

Then we have irreducible decompositions as $O(p)$-modules for $p \geqslant 1$:

$$L^2(S^{p-1}) \simeq \sum_{k=0}^{\infty}{}^{\oplus} \mathscr{H}^k(\mathbb{R}^p) \quad \text{(Hilbert direct sum)}.$$

3.6. Here is a basic property of the representation $(\varpi^{p,q}, V^{p,q})$.

Theorem 3.6.1. *Suppose that p, q are integers with $p \geqslant 2$ and $q \geqslant 2$.*

(1) *The underlying $(\mathfrak{g}, K)$-module $(\varpi^{p,q})_K$ of $\varpi^{p,q}$ has the following K-type formula*:

$$(\varpi^{p,q})_K \simeq \bigoplus_{\substack{a,b \in \mathbb{N} \\ a+\frac{p}{2}=b+\frac{q}{2}}} \mathscr{H}^a(\mathbb{R}^p) \boxtimes \mathscr{H}^b(\mathbb{R}^q). \tag{3.6.1}$$

(2) *In the Harish-Chandra parametrization, the $\mathscr{Z}(\mathfrak{g})$-infinitesimal character of $\varpi^{p,q}$ is given by $(1, \frac{p+q}{2}-2, \frac{p+q}{2}-3, \ldots, 1, 0)$.*

(3) *$V^{p,q}$ is non-zero if and only if $p+q \in 2\mathbb{N}$.*

(4) *If $p+q \in 2\mathbb{N}$ and if $(p,q) \neq (2,2)$, then $(\varpi^{p,q}, V^{p,q})$ is an irreducible representation of $G = O(p,q)$ and the underlying $(\mathfrak{g}, K)$-module $(\varpi^{p,q}_K, V^{p,q}_K)$ is unitarizable.*

Although Theorem 3.6.1 overlaps with the results of Kostant [25], Binegar–Zierau [2], Howe–Tan [11], Huang–Zhu [12] obtained by algebraic methods, we shall give a self-contained and new proof from our viewpoint: conformal geometry, discrete decomposability of the restriction with respect to non-compact subgroups, and analysis on affine symmetric spaces (hyperboloids). The method of finding K-types will be generalized to the branching law for non-compact subgroups (Sections 7, 9). The idea of proving irreducibility (see Theorem 7.6) is new and seems interesting by its simplicity, because we do not need rather complicated computations (cf. [2,11]) but just use the discretely decomposable branching law with respect to $O(p,q') \times O(q'')$. The point here is that we have flexibility in choosing (q',q'') such that $q' + q'' = q$. We shall give a new proof of the unitarizability of $\varpi^{p,q}$ because of the importance of "small" representations in the current status of unitary representation theory, see Theorem 3.9.1, Corollary 3.9.2 and Part II [23, Section 8.3].

Proof. Let $F \in V^{p,q} \subset C^\infty(M)$. Then F is developed as

$$F = \sum_{a,b\in\mathbb{N}} F_{a,b} \quad (F_{a,b} \in \mathscr{H}^a(\mathbb{R}^p) \boxtimes \mathscr{H}^b(\mathbb{R}^q)),$$

where the right side converges in the topology of $C^\infty(M)$. Applying the Yamabe operator, we have

$$\tilde{\Delta}_M F = \sum_{a,b\in\mathbb{N}} \left(-\left(a + \frac{p-2}{2}\right)^2 + \left(b + \frac{q-2}{2}\right)^2 \right) F_{a,b}.$$

Since $\tilde{\Delta}_M F = 0$, $F_{a,b}$ can be non-zero if and only if

$$\left|a + \frac{p-2}{2}\right| = \left|b + \frac{q-2}{2}\right|, \tag{3.6.2}$$

whence (1) and (3). Statement (2) follows from Lemma 3.7.2 and (3.7.4). An explicit (unitarizable) inner product for $\varpi^{p,q}$ will be given in Section 3.9 (see also Remark in Sections 3.9 and 8.3).

We shall give a simple proof of the irreducibility of $\varpi^{p,q}$ in Theorem 7.6 by using discretely decomposable branching laws to non-compact subgroups (Theorems 4.2 and 7.1). $\square$

Remark 3.6.2. (1) $\varpi^{2,2}$ contains the trivial one-dimensional representation as a subrepresentation. The quotient $\varpi^{2,2}/\mathbb{C}$ is irreducible as an $O(2,2)$-module and splits into a direct sum of four irreducible $SO_0(2,2)$-modules. The short exact sequence of $O(2,2)$-modules $0 \to \mathbb{C} \to \varpi^{2,2} \to \varpi^{2,2}/\mathbb{C} \to 0$ does not split, and $\varpi^{2,2}$ is not unitarizable as an $O(2,2)$-module.

This case is the only exception that $\varpi^{p,q}$ is not unitarizable as a $\mathrm{Conf}(S^{p-1} \times S^{q-1})$-module.

(2) The K-type formula for the case $p=1$ or $q=1$ is obtained by the same method as in Theorem 3.6.1. Then we have that

$$V^{p,q} \simeq \begin{cases} \mathbb{C}^4 & \text{if } (p,q)=(1,1), \\ \mathbb{C}^2 & \text{if } (p,q)=(1,3),(3,1), \\ \{0\} & \text{if } p=1 \text{ or } q=1 \text{ with } p+q>4 \text{ or if } p+q\notin 2\mathbb{N}. \end{cases}$$

$V^{p,q}$ consists of locally constant functions on $S^{p-1}\times S^{q-1}$ if $(p,q)=(1,1),(1,3)$ and (3,1).

(3) In the case of the Kepler problem, i.e. the case of $G=\mathrm{O}(4,2)$, the above K-type formula has a nice physical interpretation, namely: the connected component of G acts irreducibly on the space with positive Fourier components for the action of the circle $\mathrm{SO}(2)$, the so-called *positive energy subspace*; the Fourier parameter $n=1,2,3,\ldots$ corresponds to the energy level in the usual labeling of the bound states of the Hydrogen atom, and the dimension (also called the degeneracy of the energy level) for the spherical harmonics is n^2, as it is in the labeling using angular momentum and its third component of the wave functions ψ_{nlm}. Here n corresponds to our b.

3.7. Let us understand $\varpi^{p,q}$ as a subrepresentation of a degenerate principal series.

For $v\in\mathbb{C}$, we denote by the space

$$S^v(\Xi) := \{f\in C^\infty(\Xi) : f(t\xi)=t^v f(\xi),\ \xi\in\Xi, t>0\} \tag{3.7.1}$$

of smooth functions on Ξ of homogeneous degree v. Furthermore, for $\varepsilon=\pm 1$, we put

$$S^{v,\varepsilon}(\Xi) := \{f\in S^v(\Xi) : f(-\xi)=\varepsilon f(\xi),\ \xi\in\Xi\}.$$

Then we have a direct sum decomposition

$$S^v(\Xi)=S^{v,1}(\Xi)+S^{v,-1}(\Xi),$$

on which G acts by left translations, respectively.

Lemma 3.7.1. *The restriction $C^\infty(\Xi)\to C^\infty(M)$, $f\mapsto f|_M$ induces the isomorphism of G-modules between $S^{-\lambda}(\Xi)$ and $(\varpi_\lambda, C^\infty(M))$ (given in (2.5.1)) for any $\lambda\in\mathbb{C}$.*

Proof. If $f\in S^{-\lambda}(\Xi)$, $h\in G$ and $z\in M$, then

$$f(h\cdot z)=f\left(v(h\cdot z)\frac{h\cdot z}{v(h\cdot z)}\right)=v(h\cdot z)^{-\lambda}f(L_h z)=(\varpi_\lambda(h^{-1})f|_M)(z),$$

where the last formula follows from definition (2.5.1) and Lemma 3.4.1. $\square$

Let us also identify $S^{\nu,\varepsilon}(\Xi)$ with degenerate principal series representations in standard notation. The indefinite orthogonal group $G = \mathrm{O}(p,q)$ acts on the light cone Ξ transitively. We put

$$\xi^o := {}^t(1,0,\ldots,0,0,\ldots,0,1) \in \Xi. \tag{3.7.2}$$

Then the isotropy subgroup at ξ^o is of the form $M_+^{\max} N^{\max}$, where $M_+^{\max} \simeq \mathrm{O}(p-1, q-1)$ and $N^{\max} \simeq \mathbb{R}^{p+q-2}$ (abelian Lie group). We set

$$E := E_{1,p+q} + E_{p+q,1} \in \mathfrak{g}_0,$$

where E_{ij} denotes the matrix unit. We define an abelian Lie group by $A^{\max} := \exp \mathbb{R}E$ $(\subset G)$, and put

$$m_0 := -I_{p+q} \in G. \tag{3.7.3}$$

We define $M^{\max}$ to be the subgroup generated by $M_+^{\max}$ and m_0, then

$$P^{\max} := M^{\max} A^{\max} N^{\max}$$

is a Langlands decomposition of a maximal parabolic subgroup $P^{\max}$ of G. If $a = \exp(tE)$ $(t \in \mathbb{R})$, we put $a^\lambda := \exp(t\lambda E)$ for $\lambda \in \mathbb{C}$. We put

$$\rho := \frac{p+q-2}{2}.$$

For $\varepsilon = \pm 1$, we define a character χ_ε of $M^{\max}$ by the composition

$$\chi_\varepsilon : M^{\max} \to M^{\max}/M_+^{\max} \simeq \{1, m_0\} \to \mathbb{C}^\times,$$

such that $\chi_\varepsilon(m_0) := \varepsilon$. We also write sgn for χ_{-1} and $\mathbf{1}$ for χ_1. We define $\mathscr{F}$ to be the $\mathscr{A}$, $\mathscr{B}$, C^∞ or $\mathscr{D}'$ valued degenerate principal series by

$$\mathscr{F}\text{-}\mathrm{Ind}_{P^{\max}}^G(\varepsilon \otimes \mathbb{C}_\lambda) := \{f \in \mathscr{F}(G) : f(gman) = \chi_\varepsilon(m^{-1}) a^{-(\lambda+\rho)} f(g)\},$$

which has $\mathscr{Z}(\mathfrak{g})$-infinitesimal character

$$\left(\lambda, \frac{p+q}{2} - 2, \frac{p+q}{2} - 3, \ldots, \frac{p+q}{2} - \left[\frac{p+q}{2}\right]\right) \tag{3.7.4}$$

in the Harish-Chandra parametrization. The underlying $(\mathfrak{g}, K)$-module will be denoted by $\mathrm{Ind}_{P^{\max}}^G(\varepsilon \otimes \mathbb{C}_\lambda)$. We note that $\mathrm{Ind}_{P^{\max}}^G(\varepsilon \otimes \mathbb{C}_\lambda)$ is unitarizable if $\lambda \in \sqrt{-1}\mathbb{R}$.

In view of the commutative diagram of G-spaces:

$$\begin{array}{ccccc}
 & G/M_+^{\max}N^{\max} & \xrightarrow{\sim} \Xi, & & gM_+^{\max}N^{\max} \mapsto g \cdot \xi^o \\
 & \downarrow & \downarrow \Phi & & \\
G/P^{\max} \overset{\mathbb{Z}_2}{\leftarrow} & G/M_+^{\max}A^{\max}N^{\max} & \xrightarrow{\sim} M \simeq \Xi/\mathbb{R}_+^\times & &
\end{array}$$

we have an isomorphism of G-modules:

$$C^\infty\text{-}\mathrm{Ind}_{P^{\max}}^{G}(\varepsilon \otimes \mathbb{C}_\lambda) \simeq S^{-\lambda-\frac{p+q-2}{2},\varepsilon}(\Xi). \tag{3.7.5}$$

It follows from Theorem 2.5 and Lemma 3.7.1 that $(\varpi^{p,q}, V^{p,q})$ is a subrepresentation of $S^{-\frac{p+q-4}{2}}(\Xi)$. Furthermore, $\varpi^{p,q}(m_0)$ acts on each K-type $\mathscr{H}^a(\mathbb{R}^p) \boxtimes \mathscr{H}^b(\mathbb{R}^q)$ $(a + \frac{p}{2} = b + \frac{q}{2})$ as a scalar

$$(-1)^{a+b} = (-1)^{2a+\frac{p-q}{2}} = (-1)^{\frac{p-q}{2}}.$$

Hence, we have the following:

Lemma 3.7.2. *$\varpi^{p,q}$ is a subrepresentation of $S^{a,\varepsilon}(\Xi)$ with $a = -\frac{p+q-4}{2}$ and $\varepsilon = (-1)^{\frac{p-q}{2}}$, or equivalently, of $C^\infty\text{-}\mathrm{Ind}_{P^{\max}}^{G}((-1)^{\frac{p-q}{2}} \otimes \mathbb{C}_{-1})$.*

The quotient will be described in (5.5.5).

Remark 3.7.3. (1) $\varpi^{p,q}$ splits into two irreducible components as $SO(p,q)$-modules, say $\varpi^{p,q}_{\pm}$, if $p = 2$ and $q \geqslant 4$. Then, $\varpi^{p,q}$ (or $\varpi^{p,q}_{\pm}$ if $p = 2$ and $q \geqslant 4$) coincides with the "minimal representations" constructed in [2,25,32].

(2) In [2], it was claimed that the minimal representations of $SO(p,q)$ are realized in the subspace of $\{\psi \in C^\infty(S^{p-1} \times S^{q-1}) : \psi(-y) = (-1)^d \psi(y)\}$ for $d = 2 - \frac{p+q}{2}$. But this parity is not correct when both p and q are odd.

(3) Our parametrization of $S^{a,\varepsilon}(\Xi)$ is the same with $S^{a,\varepsilon}(X^0)$ in the notation of [11].

3.8. Let $p \geqslant 2$. The differential operator $-\Delta_{S^{p-1}} + \frac{(p-2)^2}{4}$ acts on the space $\mathscr{H}^a(\mathbb{R}^p)$ of spherical harmonics as a scalar $a(a+p-2) + \frac{1}{4}(p-2)^2 = (a + \frac{p-2}{2})^2$. Therefore, we can define a non-negative self-adjoint operator

$$D_p : L^2(S^{p-1}) \to L^2(S^{p-1}) \tag{3.8.1}$$

by

$$D_p := \left(-\Delta_{S^{p-1}} + \frac{(p-2)^2}{4}\right)^{\frac{1}{4}}$$

with the domain of definition given by

$$\mathrm{Dom}(D_p) := \left\{ F = \sum_{a=0}^{\infty} F_a \in L^2(S^{p-1}) : \sum_{a=0}^{\infty} \left(a + \frac{p-2}{2}\right) ||F_a||^2_{L^2(S^{p-1})} < \infty \right\}.$$

Here is a convenient criterion which assures a given function to be in $\mathrm{Dom}(D_p)$:

Lemma 3.8.1. *If $F \in L^2(S^{p-1})$ satisfies $YF \in L^{2-\frac{2}{p}}(S^{p-1})$ for any smooth vector field Y on S^{p-1} then $F \in \mathrm{Dom}(D_p)$. Namely, D_pF is well-defined and $D_pF \in L^2(S^{p-1})$.*

In order to prove Lemma 3.8.1, we recall an inequality due to Beckner:

Fact 3.8.2 (Beckner [1, Theorem 2]). *Let $1 \leqslant \delta \leqslant 2$ and $F \in L^\delta(S^n)$. Let $F = \sum_{k=0}^\infty F_k$ be the expansion in terms of spherical harmonics $F_k \in \mathscr{H}^k(\mathbb{R}^{n+1})$, which converges in the distribution sense. Then*

$$\sum_{k=0}^{\infty} \gamma_k ||F_k||^2_{L^2(S^n)} \leqslant ||F||^2_{L^\delta(S^n)}, \quad \gamma_k \coloneqq \frac{\Gamma(\frac{n}{\delta})\Gamma(k+n-\frac{n}{\delta})}{\Gamma(n-\frac{n}{\delta})\Gamma(k+\frac{n}{\delta})}. \tag{3.8.2}$$

For our purpose, we need to give a lower estimate of γ_k in Fact 3.8.2. By Stirling's formula for the Gamma function, we have

$$k^{b-a}\frac{\Gamma(k+a)}{\Gamma(k+b)} \sim 1 + \frac{(a-b)(a+b-1)}{2k} + \cdots$$

as $k \to \infty$. Hence, there exists a positive constant C depending only on n and δ so that

$$Ck^{n(1-\frac{2}{\delta})} \leqslant \gamma_k \tag{3.8.3}$$

for any $k \geqslant 1$. Combining (3.8.2) and (3.8.3), we have:

$$C\sum_{k=1}^{\infty} k^{n(1-\frac{2}{\delta})}||F_k||^2_{L^2(S^n)} \leqslant ||F||^2_{L^\delta(S^n)}. \tag{3.8.4}$$

Now we are ready to prove Lemma 3.8.1.

Proof of Lemma 3.8.1. Let $\{X_i\}$ be an orthonormal basis of $\mathfrak{o}(p)$ with respect to $(-1)\times$ the Killing form. The action of $O(p)$ on S^{p-1} induces a Lie algebra homomorphism $L : \mathfrak{o}(p) \to \mathfrak{X}(S^{p-1})$. Then we have $\Delta_{S^{p-1}} = \sum_i L(X_i)^2$. We write $F = \sum_{k=0}^\infty F_k$ where $F_k \in \mathscr{H}^k(\mathbb{R}^p)$. We note that $L(X)F_k \in \mathscr{H}^k(\mathbb{R}^p)$ for any k and for any $X \in \mathfrak{o}(p)$, because $\Delta_{S^{p-1}}$ commutes with $L(X)$. If we apply (3.8.4) with $\delta = 2 - \frac{2}{p}$ and $n = p-1$, then we have

$$C\sum_{k=1}^{\infty} k^{-1}||L(X_i)F_k||^2_{L^2(S^{p-1})} \leqslant ||L(X_i)F||^2_{L^{2-\frac{2}{p}}(S^{p-1})}.$$

Because $L(X_i)$ is a skew-symmetric operator, we have

$$\sum_i ||L(X_i)F_k||^2_{L^2(S^{p-1})} = -\sum_i (\Delta_{S^{p-1}}F_k, F_k)_{L^2(S^{p-1})} = k(k+p-2)||F_k||^2_{L^2(S^{p-1})},$$

and therefore

$$C\sum_{k=1}^{\infty}(k+p-2)||F_k||^2_{L^2(S^{p-1})}\leqslant\sum_i||L(X_i)F||^2_{L^{2-\frac{2}{p}}(S^{p-1})}<\infty.$$

Hence, we have proved that D_pF is well-defined and

$$||D_pF||^2_{L^2(S^{p-1})}=\sum_{k=0}^{\infty}\left(k+\frac{p-2}{2}\right)||F_k||^2_{L^2(S^{p-1})}<\infty.$$

This completes the proof of Lemma 3.8.1. $\square$

3.9. Let $p\geqslant 2$ and $q\geqslant 2$. We extend D_p to a self-adjoint operator (with the same notation) on $L^2(M)$. Then D_p is a pseudo-differential operator acting on $\mathscr{H}^a(\mathbb{R}^p)\boxtimes L^2(S^{q-1})$ as a scalar $\sqrt{a+\frac{p-2}{2}}$. Likewise, we define D_q as a self-adjoint operator on $L^2(S^{q-1})$ and extend it to that on $L^2(M)$. It follows from (3.6.2) that

$$D_p=D_q \quad \text{on } V_K^{p,q}. \tag{3.9.1}$$

Let us unitarize $(\varpi^{p,q},V^{p,q})$ by finding an explicit inner product by means of the operator D_p (or D_q).

First, we note that the meromorphic continuations of the distributions $|x|^\nu$ and $|x|^\nu\operatorname{sgn}x$ have simple poles at $\nu=-1,-3,-5,\ldots,$ and at $\nu=-2,-4,-6,\ldots,$ respectively. Therefore, for $\varepsilon=\pm 1$, one defines a non-zero distribution with holomorphic parameter $\nu\in\mathbb{C}$ by

$$\psi_{\nu,\varepsilon}(x):=\frac{1}{\Gamma\left(\frac{2\nu+3-\varepsilon}{4}\right)}|x|^\nu\chi_\varepsilon(\operatorname{sgn}x), \tag{3.9.2}$$

where the Gamma factor cancels exactly every pole. For example, a residue computation shows that $\psi_{-1,1}(x)=\delta(x)$, Dirac's delta function.

We are now ready to define the Knapp–Stein intertwining operator

$$A_{\lambda,\varepsilon}:\ \operatorname{Ind}_{P^{\max}}^G(\varepsilon\otimes\mathbb{C}_\lambda)\to\operatorname{Ind}_{P^{\max}}^G(\varepsilon\otimes\mathbb{C}_{-\lambda})$$

by

$$(A_{\lambda,\varepsilon}f)(x):=\int_M\psi_{\lambda-\rho,\varepsilon}([x,b])\,f(b)\,db \quad (x\in M). \tag{3.9.3}$$

Here, $\rho=\frac{p+q-2}{2}$ and db is the Riemannian measure on $M\simeq S^{p-1}\times S^{q-1}$, a double cover of $G/P^{\max}$.

In view of the K-type formula of the degenerate principal series representation

$$\operatorname{Ind}_{P^{\max}}^G(\varepsilon\otimes\mathbb{C}_\lambda)=\bigoplus_{\substack{a,b\in\mathbb{N}\\(-1)^{a-b}=\varepsilon \bmod 2}}\mathscr{H}^a(\mathbb{R}^p)\boxtimes\mathscr{H}^b(\mathbb{R}^q),$$

we have the following spectral decomposition of $A_{\lambda,\varepsilon}$ that intertwines G-actions, especially, K-actions:

Theorem 3.9.1. *Let $a, b \in \mathbb{N}$ and $\varepsilon = \pm 1$ such that $(-1)^{a-b} = \varepsilon$. On the subspace $\mathscr{H}^a(\mathbb{R}^p) \boxtimes \mathscr{H}^b(\mathbb{R}^q)$, the intertwining operator $A_{\lambda,\varepsilon}$ acts as a scalar:*

$$\frac{4\pi^{\frac{p+q-2}{2}}(-1)^{[\frac{a-b}{2}]}\Gamma(\lambda)\Gamma(-B_\lambda^{++})}{\Gamma(\frac{-2\lambda+p+q-1-\varepsilon}{4})\Gamma(1+B_\lambda^{--})\Gamma(1+B_\lambda^{+-})\Gamma(1+B_\lambda^{-+})}, \tag{3.9.4}$$

where for $\varepsilon_1, \varepsilon_2 = \pm$, we define

$$B_\lambda^{\varepsilon_1,\varepsilon_2} \equiv B_\lambda^{\varepsilon_1,\varepsilon_2}(a,b) := \frac{1}{2}\left(\lambda - 1 - \varepsilon_1\left(a + \frac{p}{2} - 1\right) - \varepsilon_2\left(b + \frac{q}{2} - 1\right)\right). \tag{3.9.5}$$

We remark that the above functions $B_\lambda^{\varepsilon_1,\varepsilon_2}(a,b)$ define "barriers" which determine irreducible subquotients of non-unitary degenerate principal series representations $\mathrm{Ind}_{P^{\max}}^G(\varepsilon \otimes \mathbb{C}_\lambda)$, as in the diagrams of the paper of Howe and Tan [11].

Though the statement of Theorem 3.9.1 itself concerns only with the degenerate principal series representations, we take a new approach for the proof, which is based on analysis on an affine symmetric space (a hyperboloid). An (elementary) setup for hyperboloids will be given in Part II [23], and so, we shall postpone the proof of Theorem 3.9.1 until Part II, Section 8.3.

Since the integration on $G/P^{\max} \simeq M/\sim\mathbb{Z}_2$ gives a G-invariant non-degenerate sesquilinear form

$$\mathrm{Ind}_{P^{\max}}^G(\varepsilon \otimes \mathbb{C}_1) \times \mathrm{Ind}_{P^{\max}}^G(\varepsilon \otimes \mathbb{C}_{-1}) \to \mathbb{C},$$

we have the following corollary by applying Theorem 3.9.1 to the case $\lambda = 1$.

Corollary 3.9.2. *Let $p+q \in 2\mathbb{N}$, $p \geqslant 2$, $q \geqslant 2$, $(p,q) \neq (2,2)$ and $\varepsilon = (-1)^{\frac{p-q}{2}}$.*

(1) *The Knapp–Stein intertwining operator*

$$A_{1,\varepsilon}: \mathrm{Ind}_{P^{\max}}^G(\varepsilon \otimes \mathbb{C}_1) \to \mathrm{Ind}_{P^{\max}}^G(\varepsilon \otimes \mathbb{C}_{-1})$$

is non-zero exactly on the submodule $(\varpi_K^{p,q}, V_K^{p,q})$.

(2) *$A_{1,\varepsilon}$ acts on the subspace $\mathscr{H}^a(\mathbb{R}^p) \boxtimes \mathscr{H}^b(\mathbb{R}^q)$ $(a + \frac{p}{2} = b + \frac{q}{2})$ of $V_K^{p,q}$ as a scalar*

$$\frac{(-1)^{[\frac{q-p}{4}]}c_1}{a + \frac{p}{2} - 1} = \frac{(-1)^{[\frac{q-p}{4}]}c_1}{b + \frac{q}{2} - 1},$$

where we define a constant c_1 (independent of a and b) by

$$c_1 := \frac{1}{2\pi^{\frac{p+q-1}{2}}}\Gamma\left(\frac{p+q-3-(-1)^{\frac{p-q}{2}}}{4}\right). \tag{3.9.6}$$

(3) *The* $(\mathfrak{g}, K)$*-module* $(\varpi_K^{p,q}, V_K^{p,q})$ *is unitarizable with the inner product*:

$$(f_1, f_2)_M := \int_M (D_p f_1)\overline{D_p f_2}\, d\omega = \int_M (D_q f_1)\overline{D_q f_2}\, d\omega, \quad f_1, f_2 \in V_K^{p,q}, \tag{3.9.7}$$

where $d\omega$ *is the standard Riemannian measure on* M. *Namely, if* $F = \sum_a F_{a,b} \in V_K^{p,q}$ *with* $F_{a,b} \in \mathscr{H}^a(\mathbb{R}^p) \boxtimes \mathscr{H}^b(\mathbb{R}^q)$ *and* $b = a + \frac{p-q}{2}$, *then* $D_p F = \sum_a \sqrt{a + \frac{p-2}{2}}\, F_{a,b}$ *and*

$$||F||_M^2 = \sum_{\mathbb{N} \ni a \geqslant \max(0, \frac{p-q}{2})} \left(a + \frac{p-2}{2}\right) ||F_{a,b}||_{L^2(M)}^2. \tag{3.9.8}$$

We denote by $\overline{V^{p,q}}$ the Hilbert completion of $V^{p,q}$ with respect to the above inner product $(\,,\,)$. On $\overline{V^{p,q}}$, we can define an (irreducible) unitary representation of G, for which we use the same notation $\varpi^{p,q}$.

In view of Section 3.8, we can describe the Hilbert space $\overline{V^{p,q}}$ more explicitly as follows: Let $\mathscr{V}$ be the Hilbert space of the completion of $C^\infty(M)$ by the norm defined by

$$||F||_{L^2(M)}^2 + ||(D_p + D_q)F||_{L^2(M)}^2 \quad \text{for } F \in C^\infty(M).$$

Then, $\mathscr{V}$ is a dense subspace of $L^2(M)$ and

$$\mathscr{V} = \mathrm{Dom}(D_p) \cap \mathrm{Dom}(D_q).$$

With this notation, the closure $\overline{V^{p,q}}$ is characterized directly by the following:

Theorem 3.9.3. *Let p and q as in Corollary* 3.9.2. *The minimal* (*unitary*) *representation* $\varpi^{p,q}$ *of* $\mathrm{O}(p,q)$ *is defined on the Hilbert space* $\overline{V^{p,q}}$ *which is given by*

$$\overline{V^{p,q}} = \{f \in \mathscr{V} : D_p f = D_q f\} = \{f \in \mathscr{V} : \tilde{\Delta}_M f = 0\},$$

where $\tilde{\Delta}_M f = 0$ *is in the distribution sense.*

Remark. By comparing the construction of [2], (3.9.8) coincides with the formula obtained by Binegar–Zierau by a different method (see Remark 3.7.3(1)). They defined a similar operator $\mathscr{D}_n$ of [2, p. 249]; we remark that there is a typographical error in their definition of $\mathscr{D}_n$; $n-2$ should read $(n-2)^2$. Then, the square D_p^2 of our operator corresponds to $\mathscr{D}_p$ if $p \neq 2$; $|\mathscr{D}_p|$ if $p = 2$, with the notation in [2].

3.10. The following lemma is rather weak, but is clear from the Sobolev estimate.

Lemma 3.10. *Suppose W is an open set of M such that the measure of $M \setminus W$ is zero. Suppose F is a C^∞ function on W satisfying $\tilde{\Delta}_M F = 0$ on W. If $F \in L^2(M)$ and if $YY'F \in L^1(M)$ for any $Y, Y' \in \mathfrak{X}(M)$ (differentiation in the sense of the Schwartz distributions), then $F \in \overline{V^{p,q}}$.*

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