On spacelike hypersurfaces with constant scalar curvature in the de Sitter space

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Abstract
We classify spacelike hypersurfaces of the de Sitter space $S^{n+1}_1(c)$ with constant scalar curvature and with two principal curvatures. Moreover, we prove that if $M^n$ is a complete spacelike hypersurface with constant scalar curvature $n(n-1)R$ and with two distinct principal curvatures such that the multiplicity of one of the principal curvatures is $n-1$, then $R < (n-2)c/n$. Additionally, we prove several rigidity theorems for such hypersurfaces.

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1. Introduction

Let $\mathbb{L}^{n+2}$ be the $(n+2)$-dimensional Lorentz–Minkowski space, that is, the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentzian metric

$$\langle v, w \rangle := -v_0 w_0 + \sum_{i=1}^{n+1} v_i w_i$$

(1.1)

for $v, w \in \mathbb{R}^{n+2}$. Then, for $c > 0$, the $(n+1)$-dimensional de Sitter space $S^{n+1}_1(c)$ can be defined as the following hyperquadric of $\mathbb{L}^{n+2}$

$$S^{n+1}_1(c) = \left\{ x \in \mathbb{L}^{n+2} : \langle x, x \rangle = \frac{1}{c} \right\}.$$ 

(1.2)
For $n \geq 2$, the de Sitter space $S_{1}^{n+1}(c)$ inherits from $(\cdot, \cdot)$ a metric which makes it the standard simply connected Lorentzian space form of constant sectional curvature $c$. For the theory of indefinite Riemannian manifolds, we refer to B. O’Neill [22]. A smooth immersion $\varphi : M^n \rightarrow S_{1}^{n+1}(c) \subset \mathbb{L}^{n+2}$ of an $n$-dimensional connected manifold $M^n$ is said to be a spacelike hypersurface if the induced metric via $\varphi$ is a Riemannian metric on $M^n$. As is usual, the spacelike hypersurface is said to be complete if the Riemannian induced metric is a complete metric on $M^n$. By endowing $M^n$ with the induced metric we can suppose $M^n$ to be Riemannian and $\varphi$ to be an isometric spacelike immersion.

Spacelike hypersurfaces of $S_{1}^{n+1}(c)$ with constant mean curvature have been under very extensive study since Goddard [12] posed the conjecture that every complete spacelike hypersurface in $S_{1}^{n+1}(c)$ with constant mean curvature $H$ must be totally umbilic. It is well known that this conjecture is false in general and holds only for some special cases. See Akutagawa [2], Ramanathan [24] and Montiel [18] for details. For a more closely study of spacelike hypersurfaces in Lorentzian space forms with constant scalar curvature, we refer to [6,10,14,16,18–21,25], among many others.

Remarks on causality. The study of spacelike hypersurfaces of $S_{1}^{n+1}(c)$ with constant scalar curvature is of substantial interest from geometric and mathematical cosmology points of view. For the second realm the causal structure of spacetime models is of great importance:

If $p, q \in M$, then

(a) $p \ll q$ means there is a future-pointing timelike curve in $M$ from $p$ to $q$,
(b) $p < q$ means there is a future-pointing causal curve in $M$ from $p$ to $q$.

Furthermore a subset $A \subset M$ is achronal if the relation $p \ll q$ never holds for $p, q \in A$ and acausal if the relation $p < q$ never holds for $p, q \in A$.

The question if achronal hypersurfaces are acausal is interesting and of physical relevance. It can be answered as follows (see [22]):

An achronal spacelike hypersurface is acausal.

So the knowledge of spacelike hypersurfaces in de Sitter spaces gives information about the causal structure in this interesting class of spacetimes.

An interesting result of Cheng and Ishikawa [9] states that the totally umbilical round spheres are the only compact spacelike hypersurfaces in $S_{1}^{n+1}(c)$ with normalized constant scalar curvature $R < c$. Some other authors, such as Aledo–Alias–Romero [3], Brasil–Colares–Palmas [4,5], Liu [17] and Zheng [26] have also worked on the related problem of characterizing the totally umbilical round spheres as the only compact spacelike hypersurfaces in $S_{1}^{n+1}(c)$ with constant scalar curvature.

In this paper, we will focus on spacelike hypersurfaces of $S_{1}^{n+1}(1)$ with constant scalar curvature and with two principal curvatures. To state our main result, we recall the following example, constructed by S. Montiel [18] for the purpose of showing that Goddard’s conjecture is false for $n > 2$ if the mean curvature $H$ satisfies $H^2 \geq 4(n-1)/n^2$.

Example 1.1. (Cf. [18].) Consider the spacelike hypersurface embedded into $S_{1}^{n+1}(1)$ given by

$$T_{k,r} = \{ x \in S_{1}^{n+1}(1) | -x_0^2 + x_1^2 + \cdots + x_k^2 = -\sinh^2 r \}$$

with $r$ a positive real number and $1 \leq k \leq n-1$. $T_{k,r}$ is complete and isometric to the Riemannian product $\mathbb{H}^k(1-\coth^2 r) \times S^{n-k}(1-\tanh^2 r)$ of a $k$-dimensional hyperbolic space and an $(n-k)$-dimensional sphere of constant sectional curvatures $1-\coth^2 r$ and $1-\tanh^2 r$, respectively. For $x \in T_{k,r}$, the unit timelike normal field of $T_{k,r} \hookrightarrow S_{1}^{n+1}(1)$, unique up to orientation, is given by

$$N(x) = \tanh r x + (\sinh r \cosh r)^{-1}(x_0, x_1, \ldots, x_k, 0, \ldots, 0).$$

It is easy to verify that $T_{k,r} \hookrightarrow S_{1}^{n+1}(1)$ has two principal curvatures, namely $\coth r$ and $\tanh r$ with multiplicities $k$ and $n-k$, respectively. Therefore $T_{k,r}$ has constant mean curvature $H = \frac{1}{n}[k \coth r + (n-k) \tanh r]$ and constant scalar curvature $n(n-1)R = k(k-1)(1-\coth^2 r) + (n-k)(n-k-1)(1-\tanh^2 r)$, where $R$ denotes the normalized
scalar curvature. \( H \) satisfies the inequality
\[
H^2 \geq \frac{1}{n^2} \left( \coth r + (n - 1) \tanh r \right)^2 \geq \frac{4(n - 1)}{n^2}; \tag{1.5}
\]
equality is attained for \( k = 1 \) and \( \coth^2 r = n - 1 \) which forces \( n > 2 \).

In addition, if \( k = 1 \) and \( n > 2 \), then \( H^2 \) takes all possible values in the range \( \left[ \frac{4(n - 1)}{n^2}, \infty \right) \) and \( 0 < R = \frac{n - 2}{n} (1 - \tan^2 r) < \frac{n^2}{n^2} \); similarly, if \( k = n - 1 \geq 2 \), we see that \( H^2 \) takes all possible values in the range \( \left[ \frac{4(n - 1)}{n^2}, \infty \right) \) and \( R = \frac{n - 2}{n} (1 - \coth^2 r) < 0 \).

On the other hand, the squared length \( S \) of the second fundamental form of \( T_{k,r} \) is given by \( S = k \coth^2 r + (n - k) \tanh^2 r \). In particular, if \( n > 2 \), for both \( k = 1 \) and \( k = n - 1 \), one can show that \( S \) and \( R \) are related by the same identity
\[
S = \frac{(n - 1)(n - 2 - nR)}{n - 2} + \frac{n - 2}{n - 2 - nR}; \tag{1.6}
\]
Analogously, \( S \) and \( H \) are related by
\[
S = -n + \frac{n^3 H^2}{2(n - 1)} \pm \frac{n(n - 2)}{2(n - 1)} H \sqrt{n^2 H^2 - 4(n - 1)}, \tag{1.7}
\]
where, for \( k = 1 \), the sign + (-) respectively is taken if \( \coth r \geq \sqrt{n - 1} \) (\( \coth r < \sqrt{n - 1} \), respectively); and for \( k = n - 1 \), the sign + (resp. -) is taken if \( \tanh r \geq \sqrt{n - 1} \) (resp. \( \tanh r < \sqrt{n - 1} \)).

Moreover, a direct calculation shows that (1.6) holds for the hypersurface
\[
T_{k,r} = \mathbb{H}^k (1 - \coth^2 r) \times S^{n-k} (1 - \tanh^2 r) \hookrightarrow S_1^{n+1} (1)
\]
if and only if \( k \) and \( r \) are related by
\[
(k - 1)(n - k - 1)(\tanh^2 r - 1)^2 \left[ (n - k) \tanh^2 r + k \right]^2 = 0;
\]
therefore, (1.6) can hold for \( T_{k,r} \) only in the case \( k = 1 \) or \( k = n - 1 \).

Now we can state our main theorem as follows.

**Main Theorem.** Let \( M^n \) (\( n \geq 3 \)) be an \( n \)-dimensional complete spacelike hypersurface in \( S_1^{n+1} (1) \) with two distinct principal curvatures and with constant scalar curvature. Then:

(i) If the multiplicities of both principal curvatures are constant and greater than one, then \( M^n \) is isoparametric and isometric to the Riemannian product \( \mathbb{H}^m(c_1) \times S^{n-m}(c_2) \) for some \( 2 \leq m \leq n - 2 \), where \( c_1 < 0 \) and \( c_2 > 0 \) are constants and satisfy \( \frac{1}{c_1} + \frac{1}{c_2} = 1 \).

(ii) There exist infinitely many spacelike hypersurfaces with constant scalar curvature and with two distinct principal curvatures such that the multiplicity of one of the principal curvatures is \( n - 1 \).

(iii) Assume that \( M^n \) has constant scalar curvature \( n(n - 1)R \) and that the multiplicity of one of the principal curvatures is \( n - 1 \). Then \( R < \frac{n - 2}{n} \). Moreover, if we assume that \( R \neq 0 \) and that the squared length \( S \) of the second fundamental form of \( M^n \) satisfies
\[
S \geq \frac{(n - 1)(n - 2 - nR)}{n - 2} + \frac{n - 2}{n - 2 - nR}; \tag{1.9}
\]
on \( M^n \), then \( M^n \) is isometric either to the Riemannian product
\[
\mathbb{H}^1 (-nR/(n - 2 - nR)) \times S^{n-1}(nR/(n - 2)) \quad \text{for} \ R > 0,
\]
or to the Riemannian product
\[
\mathbb{H}^{n-1}(nR/(n - 2)) \times S^1(-nR/(n - 2 - nR)) \quad \text{for} \ R < 0.
\]
(iv) Assume that \( M^n \) has constant scalar curvature \( n(n - 1)R > 0 \) and that the multiplicity of one of the principal curvatures is \( n - 1 \); if additionally the square of the length \( S \) of the second fundamental form of \( M^n \) satisfies

\[
S \leq \frac{(n-1)(n - 2 - nR)}{n-2} + \frac{n-2}{n-2 - nR} \tag{1.10}
\]
on \( M^n \), then it is isometric to \( \mathbb{H}^1(nR/(n - 2 - nR)) \times S^{n-1}(nR/(n - 2)) \).

**Remark 1.1.** In [7,8], Q.M. Cheng proved results similar to the Main Theorem for constant scalar curvature hypersurfaces in the Euclidean unit sphere \( S^{n+1}(1) \) and the Euclidean space \( \mathbb{E}^{n+1} \). In [13], we established a counterpart of the Main Theorem for constant scalar curvature hypersurfaces in the hyperbolic space \( \mathbb{H}^{n+1}(-1) \). We also refer to S.Y. Cheng and S.T. Yau [11] and H. Li [15] for related results concerning hypersurfaces with constant scalar curvature in space forms.

**Remark 1.2.** The calculations in Example 1.1, for both cases of \( k = 1 \) and \( k = n - 1 \), and also conclusion (iii) of our Main Theorem, show that Liu’s claim (see [17]), namely that the identity (1.6) characterizes \( \mathbb{H}^1(1 - \coth^2 r) \times S^{n-1}(1 - \tanh^2 r) \), is not correct. In fact, even for the case of having two distinct principal curvatures, the identity (1.6) characterizes both \( T_{1,r} \) and \( T_{n-1,r} \); see Proposition 5.1 for a proof. Our calculations also show that the normalized scalar curvature of \( T_{1,r} \) satisfies \( R < \frac{n^2}{n} < 1 \), thus the second case of Theorem 1.1 in the papers [4] and [5] does not occur. To explain the reason for their incorrect conclusions, we point out that an identical mistake appeared in [4,5,17]: the inequalities (3.14) in [4], (14) in [5] and (41) in [17] are not correct.

This paper is organized as follows: In Section 2 we recall the structure equations and basic formulas as well as a local theorem on the integrability of the distributions of the principal curvature vectors of a spacelike hypersurface in the de Sitter space. In Section 3 we focus on proving (i) and (ii) of the Main Theorem by establishing the structure theorems for spacelike hypersurfaces in \( S^{n+1}_1(c) \) with constant scalar curvature and with two distinct principal curvatures. Section 4 is devoted to a more detailed discussion for those spacelike hypersurfaces of the type in (ii). In Section 5 we give the proof of (iii) and (iv) of the Main Theorem.

**2. Structure equations of spacelike hypersurfaces in \( S^{n+1}_1(c) \)**

Let \( M^n \) be a spacelike hypersurface of \( S^{n+1}_1(c) \). We choose a local field of pseudo-Riemannian orthonormal frames \( \{e_1, \ldots, e_{n+1}\} \) in \( S^{n+1}_1(c) \), with dual coframe \( \{\omega_1, \ldots, \omega_{n+1}\} \), such that, at each point of \( M^n \), \( e_1, \ldots, e_n \) are tangent to \( M^n \) and \( e_{n+1} \) is the unit timelike normal vector. We adopt the following notational convention: Indices \( A, B, C \) will range from 1 to \( n+1 \) while indices \( i, j, k \) will range from 1 to \( n \). Then the structure equations of \( S^{n+1}_1(c) \) are given by

\[
d\omega_A = \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \quad \varepsilon_i = 1, \quad \varepsilon_{n+1} = -1, \tag{2.1}
\]

\[
d\omega_{AB} = \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D, \tag{2.2}
\]

\[
K_{ABCD} = \varepsilon_A \varepsilon_B (\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}) c. \tag{2.3}
\]

When restricted to \( M^n \), we have \( \omega_{n+1} = 0 \) and

\[
0 = d\omega_{n+1} = \sum_i \omega_{n+1} \wedge \omega_i. \tag{2.4}
\]

By Cartan’s lemma, there are \( h_{ij} \) such that

\[
\omega_{n+1} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji}. \tag{2.5}
\]
This gives the second fundamental form of $M^n$, $B = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j$. The mean curvature $H$ is defined by $H = \frac{1}{n} \sum_i h_{ii}$. From (2.1)–(2.5) we obtain the structure equations of $M^n$

\begin{align}
  d\omega_i &= \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \\
  d\omega_{ij} &= \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,
\end{align}

and the Gauss equation

\begin{equation}
  R_{ijkl} = c(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) - (h_{ik} h_{jl} - h_{il} h_{jk}).
\end{equation}

Let $h_{ijk}$ denote the covariant derivative of $h_{ij}$. Then we have

\begin{equation}
  \sum_k h_{ijk} \omega_k = dh_{ij} + \sum_k h_{kj} \omega_{ki} + \sum_k h_{ik} \omega_{kj}.
\end{equation}

Then, by exterior differentiation of (2.5), we obtain the Codazzi equation

\begin{equation}
  h_{ijk} = h_{ikj}.
\end{equation}

Now we state a theorem that can be proved using the method of Otsuki [23].

**Theorem 2.1.** Let $M^n$ be a spacelike hypersurface in $S^{n+1}(c)$ such that the multiplicities of principal curvatures are all constant. Then the distribution of the space of principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of principal vectors.

3. Spacelike hypersurfaces of constant scalar curvature with two distinct principal curvatures

Now, let us consider the case in Theorem 2.1 that $M^n \hookrightarrow S^{n+1}(c)$ has constant scalar curvature $n(n-1)R$ and two distinct principal curvatures $\lambda$ and $\mu$ of multiplicities $m$ and $n-m$, respectively. We further adopt the notational convention that indices $a, b, c$ range from 1 to $m$ and indices $\alpha, \beta, \gamma$ from $m+1$ to $n$. We may choose $\{e_A\}_{1 \leq A \leq n+1}$ such that

\begin{align}
  h_{ab} &= \lambda \delta_{ab}, \quad h_{a\beta} = \mu \delta_{a\beta}, \quad h_{a\alpha} = 0.
\end{align}

Since $M^n$ has constant scalar curvature $n(n-1)R$, from (2.8) we have

\begin{equation}
  n(n-1)(c-R) = m(m-1)\lambda^2 + (n-m)(n-m-1)\mu^2 + 2m(n-m)\lambda\mu.
\end{equation}

By Theorem 2.1, let us denote the integral submanifold through $x \in M^n$, corresponding to $\lambda$ and $\mu$, by $M^m_1(x)$ and $M^{n-m}_2(x)$, respectively. We write

\begin{align}
  d\lambda &= \sum_i \lambda_{,i} \omega_i, \quad d\mu = \sum_i \mu_{,i} \omega_i.
\end{align}

Then Theorem 2.1 implies

\begin{equation}
  \lambda_{,1} = \cdots = \lambda_{,m} = 0, \quad \text{if } m \geq 2
\end{equation}

and

\begin{equation}
  \mu_{,m+1} = \cdots = \mu_{,n} = 0, \quad \text{if } n-m \geq 2.
\end{equation}

In the following, we separate our discussion into two cases.

**Case (i).** $2 \leq m \leq n-2$. 

In this case, using (3.2)–(3.4) we see that \( \lambda \) and \( \mu \) must be constant on \( M^n \). From the formula
\[
\sum_k h_{abk} \omega_k = dh_{ab} + \sum_k h_{kb} \omega_{ka} + \sum_k h_{ak} \omega_{kb}
\]
and (3.1), we get \( h_{abk} = 0 \). Similarly, from
\[
\sum_k h_{abk} \omega_k = dh_{ab} + \sum_k h_{kb} \omega_{ka} + \sum_k h_{ak} \omega_{kb}
\]
and (3.1), we get \( h_{abk} = 0 \). It follows from the symmetry of \( h_{ijk} \) that
\[
h_{ijk} \equiv 0.
\]
Thus we have
\[
dh_{aa} + \sum_b h_{ba} \omega_{ba} + \sum_\beta h_{\beta a} \omega_{\beta a} + \sum_b h_{ab} \omega_{ba} + \sum_\beta h_{a\beta} \omega_{a\beta} = \sum_k h_{aak} \omega_k = 0.
\]
Using (3.1) and (3.8) we get
\[
\omega_a^{aa} = 0.
\]
Hence, from (2.7), (2.8) and (3.9), we have
\[
d\omega_{ab} = \sum_c \omega_{ac} \wedge \omega_{cb} - (c - \lambda^2) \omega_a \wedge \omega_b
\]
and analogously
\[
d\omega_{ab} = \sum_\gamma \omega_{a\gamma} \wedge \omega_{b\gamma} - (c - \mu^2) \omega_a \wedge \omega_b.
\]
(3.10) and (3.11) show that \( M^1_1(x) \) and \( M^{n-m}_2(x) \) are of constant curvatures \( c - \lambda^2 \) and \( c - \mu^2 \), respectively. From (2.8), (3.9) and the structure equations
\[
0 = d\omega_{i\gamma} = \sum_i \omega_{ai} \wedge \omega_{i\gamma} - (c - \lambda \mu) \omega_a \wedge \omega_\gamma = -(c - \lambda \mu) \omega_a \wedge \omega_\gamma
\]
we get \( \lambda \mu = c \).

Let us consider the orthonormal frame \( \{ x; e_1, \ldots, e_{n+2} \} \) in \( \mathbb{R}^{n+2} \) such that \( e_{n+2} = \sqrt{c} x \). Then from the formulas
\[
d e_{n+2} = \sqrt{c} dx = \sum_i \sqrt{c} \omega_i e_i,
\]
we have
\[
d e_{n+2} = \sum_b \omega_{ab} e_b + \omega_{an+1} e_{n+1} + \sqrt{c} \omega_a e_{n+2},
\]
\[
d e_{n+2} = \sum_\beta \omega_{a\beta} e_\beta + \omega_{an+1} e_{n+1} + \sqrt{c} \omega_a e_{n+2}.
\]
Since \( \lambda \mu = c \), we have
\[
d(\lambda e_{n+1} - \sqrt{c} e_{n+2}) = -\lambda \sum_i \omega_{n+1 i} e_i - c \sum_i \omega_i e_i = (\lambda^2 - c) \sum_i \omega_i e_i,
\]
\[
d(\mu e_{n+1} - \sqrt{c} e_{n+2}) = -\mu \sum_i \omega_{n+1 i} e_i - c \sum_i \omega_i e_i = (\mu^2 - c) \sum_i \omega_i e_i.
\]
Then, we get
\[
d(e_1 \wedge \cdots \wedge e_m \wedge (\lambda e_{n+1} - \sqrt{c} e_{n+2})) = 0,
\]
(3.12)
\[ d(e_{m+1} \wedge \cdots \wedge e_n \wedge (\mu e_{n+1} - \sqrt{c} e_{n+2})) = 0. \]  \hspace{1cm} (3.13)

Without loss of generality, we assume $\lambda > \sqrt{c} > \mu > 0$. Then we see from (3.10)–(3.13) that $M^n_1(x)$ and $M^n_{1-\mu}(x)$ lie in the intersections of $S^{n+1}_1(c)$ with the linear subspaces $L^{m+1}(x)$ and $\mathbb{R}^{n-m+1}(x)$ of dimension $(m+1)$ and $(n-m+1)$, respectively. Furthermore, for every $x$, $L^{m+1}(x)$ and $\mathbb{R}^{n-m+1}(x)$ are parallel to some fixed $L^{m+1}$ and $\mathbb{R}^{n-m+1}$, where $L^{m+1} \perp \mathbb{R}^{n-m+1}$.

These intersections are $m$-dimensional hyperbolic spaces of curvature $c - \lambda^2 < 0$ and $(n-m)$-dimensional spheres of curvature $c - \mu^2 > 0$. Hence $M^n$ is locally a Riemannian product $M^n \cong \mathbb{H}^m(c_1) \times S^{n-m}(c_2)$, where $c_1 = c - \lambda^2 < 0$ and $c_2 = c - \mu^2 > 0$ satisfy $1/c_1 + 1/c_2 = 1/c$. If we suppose that $M^n$ is a complete spacelike hypersurface in $S^{n+1}_1(c)$ with constant scalar curvature $n(n-1)R$, then

\[ M^n = \mathbb{H}^m(c_1) \times S^{n-m}(c_2), \]

where $c_1$, $c_2$ and $R$ are related by

\[ \frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}, \quad R = \frac{m(m-1)c_1 + (n-m)(n-m-1)c_2}{n(n-1)}. \]  \hspace{1cm} (3.14)

This proves the following result:

**Theorem 3.1.** Let $M^n$ ($n \geq 4$) be an $n$-dimensional spacelike hypersurface of $S^{n+1}_1(c)$ with constant scalar curvature $n(n-1)R$ and with two distinct principal curvatures. If the multiplicities $m$ and $n-m$ of the principal curvatures $\lambda$ and $\mu$ (with $\lambda^2 > \mu^2$) are greater than 1, then we have:

(i) Both $\lambda$ and $\mu$ are constant and they satisfy $\lambda \mu = c$. In addition, $M^n$ is locally the Riemannian product $\mathbb{H}^m(c_1) \times S^{n-m}(c_2)$;

(ii) If $M^n$ is assumed to be complete in $S^{n+1}_1(c)$, then

\[ M^n = \mathbb{H}^m(c_1) \times S^{n-m}(c_2), \]

where $c_1 < 0$ and $c_2 > 0$ are determined by (3.14).

**Case (ii).** $m = n - 1$, $m \geq 2$.

For simplicity we further assume that $M^n$ is complete. In this case, (3.2) reduces to

\[ n(c - R) = [(n-2)\lambda + 2\mu] \lambda. \]  \hspace{1cm} (3.2)'

If $\lambda = 0$ at some point, then $R = c$, and by continuity we see that $\lambda \equiv 0$ on $M^n$. From the Gauss equation (2.8), the sectional curvature of $M^n$ is not less than $c$. Therefore, $M^n$ is compact and has constant scalar curvature $n(n-1)c$. In this case, we can assume $\mu > 0$ on $M^n$. Then, according to Proposition 3.3 of [26], we know that $M^n$ is totally umbilic, a contradiction.

Now we assume that $\lambda > 0$ on $M^n$. Then (3.2)' yields

\[ \mu = \frac{n(c - R)}{2\lambda} - \frac{n - 2}{2}\lambda, \]  \hspace{1cm} (3.15)

and from the formula

\[ \lambda - \mu = \frac{n(\lambda^2 + R - c)}{2\lambda} \neq 0, \]

we know that $\lambda^2 + R - c \neq 0$.

From (3.1), (3.2)', (3.3) and (3.5), we have

\[ \sum_i h_{ab} \omega_i = dh_{ab} = d\lambda \delta_{ab} = \lambda_n \delta_{ab} \omega_n; \quad \sum_i h_{n\mu} \omega_i = dh_{nn} = d\mu = \mu_n \omega_n, \]

it follows that

\[ h_{abc} = 0, \quad h_{ab} = \lambda_n \delta_{ab}; \quad h_{nna} = 0, \quad h_{nnn} = \mu_n. \]  \hspace{1cm} (3.17)
Combining this with (2.10) and the formula
\[ \sum_i h_{an} \omega_i = dh_{an} + \sum_i h_{in} \omega_{ia} + \sum_i h_{ai} \omega_{in} = (\lambda - \mu) \omega_{an}, \]  
we obtain
\[ \omega_{an} = \frac{\lambda_n}{\lambda - \mu} \omega_a. \]  

Therefore, we have \( d\omega_n = \sum_a \omega_{na} \wedge \omega_a = 0 \). Notice that we may consider \( \lambda \) to be locally a function of the parameter \( s \), where \( s \) is the arc length of an orthogonal trajectory of the family of the integral submanifolds corresponding to \( \lambda \). We set
\[ \omega_n = ds, \]  
then, for \( \lambda = \lambda(s) \), we have \( \lambda_n = \lambda'(s) \).

From the expression for \( \lambda - \mu \) and (3.19), we get
\[ \omega_{an} = (\log |\lambda^2 + R - c|^{1/n})' \omega_a, \]  
which shows that the integral submanifold \( M^{n-1}_n(s) \) corresponding to \( \lambda \) and \( s \) is umbilical in \( M^n \) and \( \mathbb{S}^{n+1}_1(c) \). Then, according to (2.6), (2.7), (2.8) and (3.21), we compute
\[ d\omega_{an} = \sum_b \omega_{ab} \wedge \omega_{bn} + (\lambda \mu - c) \omega_a \wedge \omega_n, \]
\[ d\omega_{an} = \left( (\log |\lambda^2 + R - c|^{1/n})' \right) \sum_b \omega_{ab} \wedge \omega_b + (\lambda \mu - c) \omega_a \wedge \omega_n, \]
\[ d\omega_{an} = d\left[ (\log |\lambda^2 + R - c|^{1/n})' \omega_a \right] \]
\[ = (\log |\lambda^2 + R - c|^{1/n})'' ds \wedge \omega_a + (\log |\lambda^2 + R - c|^{1/n})' d\omega_a \]
\[ = [- (\log |\lambda^2 + R - c|^{1/n})''] + \left( (\log |\lambda^2 + R - c|^{1/n})' \right)^2 \omega_a \wedge \omega_n \]
\[ + (\log |\lambda^2 + R - c|^{1/n})' \sum_b \omega_{ab} \wedge \omega_b. \]

By comparison and using (3.15) we get
\[ (\log |\lambda^2 + R - c|^{1/n})'' = \left( (\log |\lambda^2 + R - c|^{1/n})' \right)^2 - \frac{1}{2} (n-2) (\lambda^2 + R - c) - R = 0. \]  

As in the previous case, we consider the orthonormal frame \( \{x; e_1, \ldots, e_{n+1}, e_{n+2}\} \) in \( \mathbb{R}^{n+2} \) with \( e_{n+2} = \sqrt{c} \). Let us denote
\[ W = e_1 \wedge e_2 \wedge \cdots \wedge e_{n-1} \wedge \{ (\log |\lambda^2 + R - c|^{1/n})' e_n + \lambda \epsilon_{n+1} - \sqrt{c} \epsilon_{n+2} \}, \]
then, by a long but straightforward calculation we can show, that
\[ dW = (\log |\lambda^2 + R - c|^{1/n})' W ds. \]  

Let us define a positive function \( w(s) \) over \( s \in (-\infty, +\infty) \) by
\[ w = \begin{cases} (\lambda^2 + R - c)^{-1/n}, & \text{for } \lambda^2 + R - c > 0, \\ (-\lambda^2 + R - c)^{-1/n}, & \text{for } \lambda^2 + R - c < 0, \end{cases} \]
then (3.22) reduces to
\[ \frac{d^2 w}{ds^2} + w \left( \frac{n-2}{2} \frac{1}{w^n} + R \right) = 0, \quad \text{for } \lambda^2 + R - c > 0. \]  

or

\[
\frac{d^2w}{ds^2} - w\left(\frac{n-2}{2} \frac{1}{w^n} - R\right) = 0, \quad \text{for } \lambda^2 + R - c < 0.
\]

Integrating (3.26), we obtain

\[
\left(\frac{dw}{ds}\right)^2 = C_1 - \frac{1}{w^{n-2}} - Rw^2, \quad \text{for } \lambda^2 + R - c > 0,
\]

or

\[
\left(\frac{dw}{ds}\right)^2 = C_2 - \frac{1}{w^{n-2}} - Rw^2, \quad \text{for } \lambda^2 + R - c < 0.
\]

where \(C_1\) and \(C_2\) are integration constants. It follows from (3.27) that

\[
\left(\frac{\lambda}{c}\right)^2 + 2 \lambda (\lambda - c) w^2 + \left(\frac{C}{w^2}\right)^2 = 0
\]

which depends on \(\lambda\).

Let \(M_k\) have constant scalar curvature \(n(n - 1)R\). It follows from (3.27) that

\[
\left(\frac{\lambda}{c}\right)^2 + 2 \lambda (\lambda - c) w^2 + \left(\frac{C}{w^2}\right)^2 = 0
\]

where \(C = C_1\) if \(\lambda^2 + R - c > 0\) and \(C = C_2\) if \(\lambda^2 + R - c < 0\). This shows that the vector \((\log |\lambda^2 + R - c|^{1/n})' e_n + \lambda e_{n+1} - \sqrt{c} e_{n+2}\)

is spacelike (resp. timelike or lightlike) if \(C > 0\) (resp. \(C < 0\) or \(C = 0\)).

(3.24) shows that the \(n\)-vector \(W\) in \(\mathbb{L}^{n+2}\) is constant along \(M_k^{n-1}(s)\). Hence there exists an \(n\)-dimensional linear space \(E^n(s)\) in \(\mathbb{L}^{n+2}\) containing \(M_k^{n-1}(s)\). (3.24) also implies that the \(n\)-space field \(W\) depends only on \(s\) and integration gives

\[
W(s) = \frac{\lambda^2(s) + R - c}{\lambda^2(s_0) + R - c} \frac{1}{n} W(s_0).
\]

Hence we see that \(E^n(s)\) is parallel to the fixed subspace \(E^n(s_0)\) in \(\mathbb{L}^{n+2}\) for every \(s\).

From the calculation

\[
d\omega_{ab} - \sum_c \omega_{ac} \wedge \omega_{cb} = \omega_{an} \wedge \omega_{nb} + (\lambda^2 - c) \omega_a \wedge \omega_b = -\{(\log |\lambda^2 + R - c|^{1/n})' + \lambda \omega_a \wedge \omega_b \}
\]

\[
= -\frac{C}{w^2} \omega_a \wedge \omega_b,
\]

we see that \(M_k^{n-1}(s)\) is of constant sectional curvature \(C/w^2\) which has the same sign for all \(s\).

Moreover, if \((\log |\lambda^2 + R - c|^{1/n})' + \lambda \omega_a \wedge \omega_b \) is not identically equal to \(0\), we can prove that \(M_k^{n-1}(s)\) has a center which depends on \(s\) and is given by

\[
q = x + \frac{(\log |\lambda^2 + R - c|^{1/n})' e_n + \lambda e_{n+1} - \sqrt{c} e_{n+2}}{(\log |\lambda^2 + R - c|^{1/n})' + \lambda^2}.
\]

It is a direct calculation to check that \(q = q(s)\) forms a plane curve located in a fixed plane \(E^2\) through the origin of \(\mathbb{L}^{n+2}\) and that \(E^2\) is orthogonal to \(E^n(s_0)\).

This proves the following result:

**Theorem 3.2.** Let \(M^n\) be an \(n\)-dimensional complete spacelike hypersurface of \(S_1^{n+1}(c)\) for \(n \geq 3\). Assume that \(M^n\) has constant scalar curvature \(n(n - 1)R\) and that \(M^n\) has two distinct principal curvatures such that, for one of them, the associated space of principal curvature vectors has dimension 1. Then:

(i) \(M^n\) is the locus of a family of moving \((n - 1)\)-dimensional submanifolds \(M_k^{n-1}(s)\). The principal curvature \(\lambda\) of multiplicity \(n - 1\) is constant along each of the submanifolds \(M_k^{n-1}(s)\). The manifolds \(M_k^{n-1}(s)\) have constant curvature \((\log |\lambda^2 + R - c|^{1/n})' + \lambda^2\), which does not change sign. Here the parameter \(s\) is the arc length of an
orthogonal trajectory of the family $M_1^{n-1}(s)$, and $\lambda = \lambda(s)$ satisfies the ordinary second order differential equation given by (3.22).

(ii) $M_1^{n-1}(s)$ is a totally umbilic submanifold $E^n(s) \cap \mathcal{S}_1^{n+1}(c)$ of the intersection of $\mathcal{S}_1^{n+1}(c)$ with an $n$-dimensional linear space $E^n(s)$ in $\mathbb{L}^{n+2}$ which is parallel to a fixed subspace $E^n$. If $(|\lambda^2 + R - c|^{1/n})^2 + c - \lambda^2 \neq 0$, the center $q$ of $E^n(s) \cap \mathcal{S}_1^{n+1}(c)$, given by (3.29), moves on a plane curve in a plane $E^2$ which goes through the origin of $\mathbb{L}^{n+2}$ and is orthogonal to $E^n$.

**Corollary 3.1.** In $\mathcal{S}_1^{n+1}(c)$, there exist infinitely many spacelike hypersurfaces with constant scalar curvature that are not congruent to each other.

In the next section, we will give a more detailed description of the spacelike hypersurfaces of $\mathcal{S}_1^{n+1}(1)$ that appeared in Theorem 3.2. As the understanding of the umbilical submanifolds $M_1^{n-1}(s)$ of Theorem 3.2 is crucial, we therefore recall the following umbilical spacelike hypersurfaces of $\mathcal{S}_1^{n+1}(1)$.

**Example 3.1.** (Cf. [18].) Let $E^n+1(a, \tau) = \{x \in \mathbb{L}^{n+2}: \langle x, a \rangle = \tau \}$ be an $(n+1)$-dimensional linear space in $\mathbb{L}^{n+2}$, define

$$M^n_{a,\tau} = E^n+1(a, \tau) \cap \mathcal{S}_1^{n+1}(1) = \{x \in \mathcal{S}_1^{n+1}: \langle x, a \rangle = \tau \},$$

(3.30)

where $a = (a_0, a_1, \ldots, a_{n+1}) \in \mathbb{L}^{n+2}$, $\langle a, a \rangle = \sigma = 1, 0, -1$ and $\tau^2 > \sigma$. Then $M^n_{a,\tau}$ is a spacelike hypersurface embedded into $\mathcal{S}_1^{n+1}(1)$ with unit timelike normal field being given by

$$N(x) = (\tau^2 - \sigma)^{-1/2}(a - \tau x), \quad x \in M^n_{a,\tau}.$$  

(3.31)

It can be verified that $M^n_{a,\tau}$ is umbilic with principal curvature $\frac{\tau}{\sqrt{\tau^2 - \sigma}}$. In fact, we have the following three cases:

(a) $M^n_{a,\tau}$ is isometric to an $n$-dimensional hyperbolic space of constant sectional curvature $-1/(\tau^2 - 1)$;
(b) $M^n_{0,\tau}$ is isometric to the Euclidean space $\mathbb{R}^n$;
(c) $M^n_{-1,\tau}$ is isometric to an $n$-dimensional sphere with constant sectional curvature $1/(\tau^2 + 1)$.

4. On spacelike hypersurfaces of $\mathcal{S}_1^{n+1}(1)$ of the type in Theorem 3.2

The discussions for the three cases $R = 0$, $R > 0$ and $R < 0$ are similar. For simplicity, we will consider only the case of $R < 0$ in this section.

For $R < 0$, the realization of the spacelike hypersurfaces in $\mathcal{S}_1^{n+1}(1)$ of the type in Theorem 3.2 is as follows. Let us consider $\mathcal{S}_1^{n+1}(1)$ as

$$\mathcal{S}_1^{n+1}(1) \subset \mathbb{L}^{n+2} = \mathbb{L}^n \times \mathbb{R}^2$$

and denote the standard immersion by $x : \mathbb{H}^{n-1}(-1) \hookrightarrow \mathbb{L}^n$, with $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$ being a local orthonormal frame field in $\mathbb{L}^n$ such that $\{\tilde{e}_1, \ldots, \tilde{e}_{n-1}\}$ is tangent to $\mathbb{H}^{n-1}(-1)$ and $x = \tilde{e}_n$ is the timelike normal vector field. Consider

$$d\tilde{e}_n = \sum_a \tilde{\omega}_a \tilde{e}_a, \quad d\tilde{e}_a = \sum_b \tilde{\omega}_{ab} \tilde{e}_b, \quad \tilde{\omega}_{ab} + \tilde{\omega}_{ba} = 0.$$  

(4.1)

Then we have

$$d\tilde{\omega}_a = \sum_b \tilde{\omega}_{ab} \wedge \tilde{\omega}_b, \quad d\tilde{\omega}_{ab} = \sum_c \tilde{\omega}_{ac} \wedge \tilde{\omega}_{cb} + \tilde{\omega}_{ab} \wedge \tilde{\omega}_b.$$  

(4.2)

We take a plane curve $\xi$ in $\mathbb{R}^2 = \mathbb{C}$ with a given supporting function $h(\theta) \geqslant 0$. The generic point $q(\theta)$ of $C$ is expressed as

$$q(\theta) = e^{i(\theta - \pi/2)}(h(\theta) + ih'(\theta)).$$  

(4.3)
The Frenet frame of $\xi$ is given by
\[ \bar{e}_{n+1} = e^{i\theta}, \quad \bar{e}_{n+2} = e^{i(\theta + \pi/2)} \] (4.4)
and the arc length $u$ of $\xi$ is given by
\[ du = \{ h(\theta) + h''(\theta) \} d\theta. \] (4.5)
Using $\bar{e}_{n+1}$ and $\bar{e}_{n+2}$, we have
\[ q = h'\bar{e}_{n+1} - h\bar{e}_{n+2}, \quad dq = \bar{e}_{n+1} du. \] (4.6)
Supposing $\xi$ is in the outside of the unit circle, we define a function $\rho(\theta) > 0$ by
\[ \rho^2 = \| q \|^2 - 1 = h^2 + (h')^2 - 1. \] (4.7)
From now on we assume $h + h'' > 0$. Then, a spacelike hypersurface $M^n$ of the type in Theorem 3.2 can be defined as
\[ \varphi : \mathbb{H}^{n-1}(-1) \times \mathbb{R} \to S^{n+1}_1(1) \subset \mathbb{L}^{n+2} \]
with
\[ \varphi = \rho\bar{e}_n + q = \rho\bar{e}_n + h'\bar{e}_{n+1} - h\bar{e}_{n+2}. \] (4.8)
By means of (4.1) and (4.6), we have
\[ d\varphi = \rho \sum_a \bar{\omega}_a \bar{e}_a + \rho' d\theta \bar{e}_n + (h + h'') d\theta \bar{e}_{n+1}. \]
Since $h + h'' > 0$, from (4.7) it is easily checked that $(h + h'')^2 - (\rho')^2 > 0$. If we define
\[ e_n = \bar{e}_n, \quad e_a = (\rho'\bar{e}_a + (h + h'')\bar{e}_{n+1})/\sqrt{(h + h'')^2 - (\rho')^2}; \]
\[ \omega_a = \rho\bar{\omega}_a, \quad \omega_n = \sqrt{(h + h'')^2 - (\rho')^2} d\theta, \] (4.9)
then $\{e_i\}_{1 \leq i \leq n}$ is an orthonormal frame of $\varphi$ with $\{\omega_i\}_{1 \leq i \leq n}$ its dual co-frame.
From the calculations of $\rho\rho' = h'(h + h'')$, $d\omega_a = \sum_b \omega_{ab} \wedge \omega_b + \omega_{an} \wedge \omega_n$ and
\[ d\omega_a = d(\rho\bar{\omega}_a) = \rho' d\theta \wedge \bar{\omega}_a + \rho d\bar{\omega}_a = \sum_b \bar{\omega}_{ab} \wedge \omega_b + \frac{\rho'}{\sqrt{(h + h'')^2 - \rho^2}} \omega_n \wedge \bar{\omega}_a, \]
we get
\[ \omega_{ab} = \bar{\omega}_{ab}, \quad \omega_{an} = -\frac{\rho' |h'|}{\sqrt{\rho' \sqrt{h^2 - 1}}} \bar{\omega}_a = -\frac{h'}{\sqrt{h^2 - 1}} \bar{\omega}_a. \] (4.10)
From (2.7), (4.2) and (4.10), we have
\[ -\frac{1}{2} \sum_{k,l} R_{abkl} \omega_k \wedge \omega_l = d\omega_{ab} - \sum_c \omega_{ac} \wedge \omega_{cb} - \omega_{an} \wedge \omega_{nb} \]
\[ = d\bar{\omega}_{ab} - \sum_c \bar{\omega}_{ac} \wedge \bar{\omega}_{cb} + \frac{h^2}{h^2 - 1} \bar{\omega}_a \wedge \bar{\omega}_b = \frac{1}{h^2 - 1} \omega_a \wedge \omega_b. \] (4.11)
Then we obtain
\[ R_{abkl} = -\frac{1}{h^2 - 1} (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}). \] (4.12)
Analogously, by using (2.7), (4.2), (4.10) and the formula $\rho\rho' = h'(h + h'')$ we get
\[ R_{ankl} = -\frac{h''(h^2 - 1) - hh'^2}{(h^2 - 1)^2(h + h'')} (\delta_{ak}\delta_{nl} - \delta_{al}\delta_{nk}). \] (4.13)
As $M^n$ has constant scalar curvature $n(n - 1)R$, we see that

$$n(n - 1)R = \sum_{i,j} R_{ijij} = \sum_{a,b} R_{abab} + 2\sum_a R_{anan},$$

or, equivalently, by (4.12) and (4.13),

$$n(h^2 - 1)[R(h^2 - 1) + 1] \frac{d^2 h}{d\theta^2} - 2h \left(\frac{dh}{d\theta}\right)^2 + h(h^2 - 1)[nR(h^2 - 1) + n - 2] = 0. \quad (4.14)$$

Conversely, for a constant $R < 0$, if a function $h(\theta)$ satisfies (4.14) and it defines a plane curve with curvature $k > 0$ in $\mathbb{R}$ by (4.3) that is in the outside of the unit circle, then by (4.8) we get a spacelike hypersurface $M^n \hookrightarrow S_{1}^{n+1}(1)$ with constant scalar curvature $n(n - 1)R$. Since the properties of this constant scalar curvature spacelike hypersurface $M^n$ are determined only by $h(\theta)$, we need to investigate the properties of the ordinary differential equation (4.14).

**Remark 4.1.** In terms of extrinsic geometry, we may also use Gauss equation (2.8) to compute the scalar curvature of the spacelike hypersurface $\varphi : \mathbb{H}^{n+1}(-1) \times \mathbb{R} \rightarrow S_{1}^{n+1}(1)$. We may write down a unit timelike normal vector field $e_{n+1}$ and the corresponding principal curvatures $\{\lambda_i\}_{1 \leq i \leq n}$ of $\varphi : \mathbb{H}^{n+1}(-1) \times \mathbb{R} \rightarrow S_{1}^{n+1}(1)$ as follows:

$$e_{n+1} = -\frac{h}{\sqrt{h^2 - 1}}(\rho \tilde{e}_n + h' \tilde{e}_{n+1} + \sqrt{h^2 - 1}\tilde{e}_{n+2}), \quad (4.15)$$

$$\lambda_1 = \cdots = \lambda_{n-1} = \frac{h}{\sqrt{h^2 - 1}}, \quad \lambda_n = \frac{h}{\sqrt{h^2 - 1}} - \frac{\rho^2}{(h + h'\sqrt{(h^2 - 1)^3}}. \quad (4.16)$$

First of all we note that a unique constant solution of (4.14) is given by

$$h(\theta) = \sqrt{\frac{nR - n + 2}{nR}}. \quad (4.17)$$

The spacelike hypersurface generated by this solution is

$$M^n = \mathbb{H}^{n+1}\left(\frac{nR}{n - 2}\right) \times S^1\left(\frac{nR}{nR - n + 2}\right). \quad (4.18)$$

Secondly, we state the following easy result.

**Lemma 4.1.** Let $h(\theta)$ be a solution of (4.14), then either we have $h^2(\theta) < 1 - \frac{1}{R}$ for all $\theta$; or $h^2(\theta) > 1 - \frac{1}{R}$ for all $\theta$. Moreover, if a solution $h(\theta)$ of (4.14) satisfies $h(\theta_0) = 1$ for some $\theta_0$, then we have $h(\theta) \equiv 1$.

**Proof.** If there exists $\theta_0$ such that $h^2(\theta_0) = 1 - \frac{1}{R}$, then (4.14) implies that

$$\left(h'(\theta_0)\right)^2 = \frac{1}{2}\left(h(\theta_0^2) - 1\right)[nR(h^2 - 1) + n - 2] = \frac{1}{R} < 0,$$

a contradiction. Moreover, if for some $\theta_0$ the equation $h(\theta_0) = 1$ holds, then from (4.14) we have $h'(\theta_0) = 0$. Since $h(\theta) \equiv 1$ is indeed a solution of (4.14) and a solution $h(\theta)$ of (4.14) depends only on the value of $h(\theta_0)$ and $h'(\theta_0)$, our second claim follows. \qed

From the viewpoint of our problem, we will consider only solutions of (4.14) such that

$$h(\theta) > 1, \quad \left(h(\theta)\right)^2 + \left(h'(\theta)\right)^2 > 1 \quad (4.19)$$

hold.

By Lemma 4.1, the solution of (4.14) and (4.19) can be separated into two types. We consider each type separately in what follows.

**Type I:** $h^2(\theta) > 1 - \frac{1}{R}$ for all $\theta$. 

Consider $f = h^2 + (h')^2$. From (4.14) we have

\[ \frac{1}{2} \frac{df}{d\theta} = hh' + h'' = \frac{2hh'(f - 1)}{n(h^2 - 1)[R(h^2 - 1) + 1]} . \]

Hence we get

\[ \frac{df}{f - 1} = \frac{2}{n} \left[ \frac{1}{h^2 - 1} - \frac{1}{h^2 - 1 + \frac{1}{R}} \right] dh^2 - 1) . \quad (4.20) \]

Integrating the above equation, we get

\[ f - 1 = \sigma \left( \frac{h^2 - 1}{h^2 - 1 + \frac{1}{R}} \right)^{\frac{2}{n}} , \quad \sigma = \text{const.} > 0 , \]

that is

\[ \left( \frac{dh}{d\theta} \right)^2 = 1 - h^2 + \sigma \left( \frac{h^2 - 1}{h^2 - 1 + \frac{1}{R}} \right)^{\frac{2}{n}} . \quad (4.21) \]

Now we need to know the information about the critical points of nonconstant solutions $h(\theta)$ to (4.14) which are of type I. For such a solution, its range is given by

\[ 1 - h^2 + \sigma \left( \frac{h^2 - 1}{h^2 - 1 + \frac{1}{R}} \right)^{\frac{2}{n}} \geq 0 . \quad (4.22) \]

Since $(h^2 - 1)/(h^2 - 1 + 1/R) \to 1$ if $h^2 \to +\infty$, (4.22) implies that the range of $h(\theta)$ should be bounded. Thus, we can assume the existence of constants $A > B > 1$ so that $A > h(\theta) > B$ for all $\theta$.

**Proposition 4.1.** Every nonconstant solution of (4.14) which is of type I can have at most one critical point.

**Proof.** Suppose on the contrary, that a nonconstant type I solution $h(\theta)$ to (4.14) has two critical points $\theta_1, \theta_2$, then we can assume that $h(\theta_2) = a_2 > a_1 = h(\theta_1)$. From (4.21), we find that $a_1, a_2$ are solutions of the equation

\[ \left( \frac{h^2 - 1}{h^2 - 1 + \frac{1}{R}} \right)^{\frac{2}{n}} = \frac{h^2 - 1}{\sigma} , \]

that is

\[ \sigma^n = (h^2 - 1)^{n-2} \left( h^2 - 1 + \frac{1}{R} \right)^2 . \quad (4.23) \]

Since the right hand side of (4.23) is a strictly increasing function of $h^2$ for $h^2 > 1 - 1/R$, we get a contradiction. \(\square\)

Note that a solution $h(\theta)$ of (4.14) is invariant under the reflection $\theta_0 + \theta \to \theta_0 - \theta$ if $\theta_0$ satisfies $h'(\theta_0) = 0$; it is easy to see that if a nonconstant solution to (4.14) has two critical values, then it must be a periodic function in $\theta$. From Proposition 4.1 we obtain

**Corollary 4.1.** Eq. (4.14) does not possess a periodic solution of type I. Therefore, every immersion $\varphi: \mathbb{H}^{n-1}(-1) \times \mathbb{R} \to S^{n+1}_1(1)$, as defined by (4.8) and corresponding to the type I solution to (4.14), cannot produce a warped product immersion from $\mathbb{H}^{n-1} \times S^1$ into $S^{n+1}_1(1)$.

**Type II:** $1 < h^2(\theta) < 1 - \frac{1}{R}$ for all $\theta$.

Integrating Eq. (4.20) again, we now have

\[ f - 1 = \sigma \left( \frac{h^2 - 1}{1 - \frac{1}{R} - h^2} \right)^{\frac{2}{n}} , \quad \sigma = \text{const.} > 0 , \]
that is
\[
\left( \frac{dh}{d\theta} \right)^2 = 1 - h^2 + \sigma \left( \frac{h^2 - 1}{1 - \frac{1}{R} - h^2} \right)^{\frac{2}{n}}.
\]  
\hspace{1cm} (4.24)

Note that the constant function in (4.17) is a solution of (4.24) with \( \sigma = \frac{n-2}{nR} \left( \frac{2}{n-2} \right)^{2/n} \).

Similar to the type I solutions, we have

**Proposition 4.2.** Every nonconstant type II solution of (4.14) can have at most one critical point.

**Proof.** From (4.24), every nonconstant type II solution \( h(\theta) \) to (4.14) has its range given by
\[
1 - h^2 + \sigma \left( \frac{h^2 - 1}{1 - \frac{1}{R} - h^2} \right)^{\frac{2}{n}} \geq 0.
\]  
\hspace{1cm} (4.25)

Notice that the range of \( x \) such that
\[
1 - x + \sigma \left( \frac{x - 1}{1 - \frac{1}{R} - x} \right)^{\frac{2}{n}} \geq 0, \hspace{1cm} 1 < x < 1 - \frac{1}{R},
\]  
\hspace{1cm} (4.26)
is given by the set of points of the curve \( \Gamma: y = \left( \frac{x-1}{1-\frac{1}{R}-x} \right)^{2/n} \), located above the line \( L_\sigma: y = \frac{x-1}{\sigma} \) in the interval \( 1 < x < 1 - \frac{1}{R} \).

An analysis of the curve \( \Gamma \) shows that, for given \( \sigma > 0 \), if (4.26) holds, then either we have
\[
1 - x + \sigma \left( \frac{x - 1}{1 - \frac{1}{R} - x} \right)^{\frac{2}{n}} > 0, \hspace{1cm} \text{for all } 1 < x < 1 - \frac{1}{R},
\]  
\hspace{1cm} (4.27)
or, there exist constants \( a_1, a_2 > a_1 \) satisfying
\[
1 < a_1 \leq \frac{nR - n + 2}{nR} \leq a_2 < 1 - \frac{1}{R}, \hspace{1cm} 1 - a_i + \sigma \left( \frac{a_i - 1}{1 - \frac{1}{R} - a_i} \right)^{\frac{2}{n}} = 0, \hspace{1cm} i = 1, 2,
\]
such that the first inequality in (4.26) is equivalent to either \( 1 < x \leq a_1 \); or \( 1 - \frac{1}{R} > x \geq a_2 \). Therefore, considering that \( h(\theta) \) is nonconstant and the above facts, if \( a_1 \) and \( a_2 \) are distinct critical values of \( h^2(\theta) \), then \( h^2(\theta) \notin (a_1, a_2) \) for every \( \theta \). This contradicts the continuity of the function \( h(\theta) \). \( \Box \)

From Proposition 4.2 we obtain

**Corollary 4.2.** Eq. (4.14) does not possess a nontrivial periodic solution of type II. Therefore, every immersion \( \varphi: \mathbb{H}^{n-1}(-1) \times \mathbb{R} \rightarrow S_1^{n+1}(1) \), as defined by (4.8), corresponding to the type II solution of (4.14), except the unique constant solution (4.17), cannot produce a warped product immersion from \( \mathbb{H}^{n-1} \times S^1 \) into \( S_1^{n+1}(1) \).

**Remark 4.2.** Corollary 4.1 and 4.2 show that, for \( R < 0 \), the standard product immersion \( \mathbb{H}^{n-1}(1-\coth^2 r) \times S^1(1-\tanh^2 r) \rightarrow S_1^{n+1}(1) \) is rigid for spacelike hypersurfaces of the type in Theorem 3.2. This property is unexpected if we compare it with the result of T. Otsuki [23], where infinitely many minimal immersions of \( S^{n-1} \times S^1 \) with the warped product metric into \( S^{n+1} \) are produced from immersions of \( S^{n-1} \times \mathbb{R} \) into \( S^{n+1} \), see Section 4 and Theorem 5 of [23] for details.

5. Characterizations of the Riemannian products \( T_{1,r} \) and \( T_{n-1,r} \)

In this section, we assume that \( M^n \) is an \( n \)-dimensional complete spacelike hypersurface with constant scalar curvature \( n(n-1)R \) and with two distinct principal curvatures in \( S_1^{n+1}(c) \), and \( \lambda \) is a principal curvature of multiplicity \( n-1 \). Recall that \( \lambda^2 + R - c \neq 0 \), thus we have two possibilities: \( \lambda^2 + R - c > 0 \) on \( M^n \), or \( \lambda^2 + R - c < 0 \) on \( M^n \).
Theorem 5.1. Let $M^n$ be an $n$-dimensional complete spacelike hypersurface in $S^{n+1}_1(c)$ with constant scalar curvature $n(n - 1)R$ and with two distinct principal curvatures, one of which is simple. Then $R < \frac{n-2}{n}c$.

Proof. Suppose on the contrary that $R \geq \frac{n-2}{n}c$. Then, after rewriting (3.26)± and because of $R \geq (n-2)c/n$ and $\lambda^2 > 0$, we have

\[
\frac{d^2w}{ds^2} = w\left(\frac{n-2}{2} \pm R\right) = -w\left(\frac{n-2}{2} \lambda^2 + \frac{nR - (n-2)c}{2}\right) < 0.
\]

Therefore, $dw(s)/ds$ is a strictly monotone decreasing function of $s$ and thus it has at most one zero point for $s \in (-\infty, +\infty)$. If $dw(s)/ds$ has no zero point in $(-\infty, +\infty)$, then $w(s)$ is a monotone function of $s$ in $(-\infty, +\infty)$. If $dw(s)/ds$ has exactly one zero point $s_0$ in $(-\infty, +\infty)$, then $w(s)$ is a monotone function of $s$ in both $(-\infty, s_0]$ and $[s_0, +\infty)$.

On the other hand, from the assumption $R \geq \frac{n-2}{n}c > 0$ and (3.27)±, we see that the positive function $w(s)$ must be bounded from above. Combining this fact with the monotonicity of $w(s)$ near infinity, we find that both $\lim_{s \to -\infty} w(s)$ and $\lim_{s \to +\infty} w(s)$ exist and this implies that

\[
\lim_{s \to -\infty} \frac{dw(s)}{ds} = \lim_{s \to +\infty} \frac{dw(s)}{ds} = 0.
\]

This is impossible because $dw(s)/ds$ is a strictly monotone decreasing function of $s$. \(\square\)

Proposition 5.1. Let $M^n$ be a complete spacelike hypersurface in $S^{n+1}_1(c)$ with constant scalar curvature $n(n - 1)R$ and with two distinct principal curvatures. Assume that $\lambda$ is the principal curvature of multiplicity $n - 1$. If

\[
S = \frac{(n - 1)(n(2)c - nR)}{n - 2} + \frac{(n-2)c^2}{(n-2)c - nR}, \quad \text{on } M^n,
\]

then $R \neq 0$ and $M^n$ is isometric either to the Riemannian product

\[
\mathbb{H}^1\left(\frac{nRc}{nR - (n-2)c}\right) \times S^{n-1}\left(\frac{R}{n-2}\right) \quad \text{for } R > 0,
\]

or to the Riemannian product

\[
\mathbb{H}^{n-1}\left(\frac{nR}{n-2}\right) \times S^1\left(\frac{nRc}{nR - (n-2)c}\right) \quad \text{for } R < 0.
\]

Proof. From the definition of $S$ and (3.15), we have

\[
S = (n-1)\lambda^2 + \mu^2 = (n-1)\lambda^2 + \left(\frac{n(c-R)}{2\lambda} - \frac{n-2}{2}\right)^2
\]

\[
= \frac{n^2}{4}\lambda^2 + \frac{n^2(c-R)^2}{4\lambda^2} - \frac{1}{2}n(n-2)(c-R).
\]

Comparing (5.1) with (5.2), we see that $\lambda^2$ is constant and satisfying

\[
\lambda^2 + R - c = -\frac{2R}{n-2}, \quad \text{or} \quad \lambda^2 + R - c = \frac{2R(c-R)}{(n-2)c - nR}.
\]

From (3.22), it must be the case that $\lambda^2 + R - c = -\frac{2R}{n-2} \neq 0$. Substituting this into Theorem 3.2, we see that $M^n_{\lambda-1}(s)$ is of constant sectional curvature $c - \lambda^2 = nR/(n-2)$, whereas $c - \mu^2 = -nRc/(n-2)c - nR)$. Thus $M^n$ is isoparametric and, according to the congruence theorem in [1], our conclusion follows. \(\square\)

Lemma 5.1. Let $M^n$ be a complete spacelike hypersurface in $S^{n+1}_1(c)$ with constant scalar curvature $n(n - 1)R$ and with two distinct principal curvatures. Assume that $\lambda$ is the principal curvature of multiplicity $n - 1$. Then

\[
S \geq \frac{(n-1)(n(2)c - nR)}{n - 2} + \frac{(n-2)c^2}{(n-2)c - nR}
\]

(5.3)
holds if and only if
\[ \lambda^2 + R - c \geq \begin{cases} -2R/(n-2), & \text{if } R \leq 0, \\ 2R(c - R)/((n - 2)c - nR), & \text{if } 0 < R < (n - 2)c/n, \end{cases} \] 
(5.4)

or
\[ \lambda^2 + R - c \leq \begin{cases} -2R/(n-2), & \text{if } 0 < R < (n - 2)c/n, \\ 2R(c - R)/((n - 2)c - nR), & \text{if } R \leq 0. \end{cases} \] 
(5.5)

**Proof.** Using (5.2), we have the calculation that
\[ S = \left( \frac{(n - 1)((n - 2)c - nR)}{n - 2} + \frac{(n - 2)c^2}{(n - 2)c - nR} \right) \frac{\lambda^2 + R - c + \frac{2R}{n - 2}}{\lambda^2 + R - c - \frac{2R(c - R)}{(n - 2)c - nR}}. \] 
(5.6)

Notice that, according to Theorem 5.1, we have \( R < \frac{n-2}{n}c \). Then it is easy to find that
\[ -\frac{2R}{n-2} \geq \frac{2R(c - R)}{(n - 2)c - nR} \]
holds if and only if \( R \leq 0 \), and
\[ -\frac{2R}{n-2} \leq \frac{2R(c - R)}{(n - 2)c - nR} \]
holds if and only if \( 0 \leq R < \frac{n-2}{n}c \). From these facts and (5.6), we get the conclusion of Lemma 5.1. \( \Box \)

**Theorem 5.2.** Let \( M^n \) be an \( n \)-dimensional complete spacelike hypersurface in \( S^{n+1}_1(c) \) with constant scalar curvature \( n(n - 1)R \) \((R \neq 0)\) and with two distinct principal curvatures, one of which is simple. If
\[ S \geq \frac{(n - 1)((n - 2)c - nR)}{n - 2} + \frac{(n - 2)c^2}{(n - 2)c - nR} \] 
(5.7)
holds on \( M^n \), then \( M^n \) is isometric either to the Riemannian product
\[ \mathbb{H}^1 \left( \frac{nRc}{nR - (n - 2)c} \right) \times S^{n-1} \left( \frac{nR}{n - 2} \right), \quad \text{for } R > 0, \]
or to the Riemannian product
\[ \mathbb{H}^{n-1} \left( \frac{nR}{n - 2} \right) \times S^1 \left( \frac{nRc}{nR - (n - 2)c} \right), \quad \text{for } R < 0. \]

**Proof.** Eq. (3.26)\(^+\) and (3.26)\(^-\) can be rewritten as
\[ \frac{d^2w}{ds^2} = -w \left( \frac{n - 2}{2} (\lambda^2 + R - c) + R \right). \] 
(5.8)

From Lemma 5.1, one of the following four cases holds:
(a) \( R < 0 \) and \( \lambda^2 + R - c \geq \frac{2R}{n-2} > 0 \) on \( M^n \);
(b) \( R < 0 \) and \( \lambda^2 + R - c \leq \frac{2R(c - R)}{(n - 2)c - nR} < 0 \) on \( M^n \);
(c) \( 0 < R < \frac{(n-2)c}{n} \) and \( \lambda^2 + R - c \leq -\frac{2R}{n-2} < 0 \) on \( M^n \);
(d) \( 0 < R < \frac{(n-2)c}{n} \) and \( \lambda^2 + R - c \geq \frac{2R(c - R)}{(n - 2)c - nR} > 0 \) on \( M^n \).
In each case, we see from (3.25) that the positive function \( w(s) \) is bounded from above and, by (5.8), \( d^2w(s)/d^2s \) does not change sign on \( M^n \), thus \( dw(s)/ds \) is a monotonic function of \( s \in (-\infty, +\infty) \). Therefore, \( w(s) \) must be monotonic if \( s \) tends to infinity.

Since \( w(s) \) is bounded and is monotone if \( s \) tends to infinity, we find that both \( \lim_{s \to -\infty} w(s) \) and \( \lim_{s \to +\infty} w(s) \) exist and due to the monotonicity of \( dw(s)/ds \) we have

\[
\lim_{s \to -\infty} \frac{dw(s)}{ds} = \lim_{s \to +\infty} \frac{dw(s)}{ds} = 0.
\]

By the monotonicity of \( dw(s)/ds \) we see that \( dw(s)/ds \equiv 0 \) and \( w(s) \) is a constant. Then Theorem 3.2 implies that \( M^{n-1}_1(s) \) is of constant sectional curvature \( c - \lambda^2 = nR/(n - 2) \neq 0 \), whereas \( c - \mu^2 = -nRc/(n - 2)c - nR \neq 0 \). Thus \( M^n \) is isoparametric and, according to the congruence theorem in [1], our conclusion follows. \( \square \)

**Theorem 5.3.** Let \( M^n \) be an \( n \)-dimensional complete spacelike hypersurface in \( S^{n+1}_1(c) \) with constant scalar curvature \( n(n - 1)R \) \((R > 0)\) and with two distinct principal curvatures, one of which is simple. If

\[
S \leq \frac{(n - 1)(n - 2)c - nR}{n - 2} + \frac{(n - 2)c^2}{(n - 2)c - nR}
\]

holds on \( M^n \), then \( M^n \) is isometric to the Riemannian product

\[
\mathbb{H}^1\left(\frac{nRc}{nR - (n - 2)c}\right) \times S^{n-1}\left(\frac{nR}{n - 2}\right).
\]

**Proof.** From the calculation (5.6), the conditions \( R > 0 \) and (5.9) imply that one of the following two cases holds:

(a) \( 0 < \lambda^2 + R - c \leq \frac{2R(c - R)}{(n - 2)c - nR} \) on \( M^n \);

(b) \( -\frac{2R}{n - 2} \leq \lambda^2 + R - c < 0 \) on \( M^n \).

From the condition \( R > 0 \) and \( (3.27)^{\pm} \) we see that the positive function \( w(s) \) is bounded from above. In each case, (5.8) implies that \( d^2w(s)/d^2s \) does not change sign on \( M^n \). Then, using the same argument as in the proof of Theorem 5.2, one finds that \( w(s) \) is a constant function. Then Theorem 3.2 implies that \( M^{n-1}_1(s) \) is of constant sectional curvature \( c - \lambda^2 = nR/(n - 2) > 0 \), whereas \( c - \mu^2 = -nRc/(n - 2)c - nR < 0 \). Thus \( M^n \) is isoparametric and, according to the congruence theorem in [1], \( M^n \) is isometric to the Riemannian product

\[
\mathbb{H}^1\left(\frac{nRc}{nR - (n - 2)c}\right) \times S^{n-1}\left(\frac{nR}{n - 2}\right).
\]

This proves Theorem 5.3. \( \square \)

**Remark 5.1.** In contrast to the discussion given in Liu [17], we find that although Liu’s claim in [17] cannot be fully correct, a partial version of it does hold, at least for the situation in Theorem 5.3.

Remarks on the physical relevance. As mentioned in the introduction spacelike hypersurfaces play an important role in the analysis of the causal structure. The hypersurfaces here have constant scalar curvature, so they are simple in a special kind and in the case of \( S^4_1(c) \) we have an interesting class of spacetimes with additional properties. Because of the result for achronal subsets (Section 1) we know something about the important causal structure. But thinking about the physical relevance of our spacetimes we need some more information:

If a manifold \( M \) contains no timelike closed curves we say that the chronology condition holds; physically this is a natural requirement in order to avoid paradoxes. So we want to give a criterion in order to guarantee the absence of timelike closed curves in the class of spacetimes given here, where the \( g_{ik} \) are the components of the metric tensor (signature here: \( -2; \alpha, \beta \in \{0, 1, 2\}, k \in \{0, 1, 2, 3\} \)):

If there exist timelike closed curves then the time \( t \) will be a periodic function of the parameter \( s \) of the curve. Hence there is a maximum and a minimum for \( t(s) \) and there exist points for which \( \frac{dt}{ds} = 0 \) holds. Let \( x^i(s) \) with
0 ≤ s ≤ 1 and \( x^i(0) = x^i(1) \) be a timelike closed curve on a chart of the underlying manifold, i.e.

\[
g_{ik} \frac{dx^i}{ds} \frac{dx^k}{ds} > 0
\]

holds for all \( s \). If \( s_0 \) is chosen such that \( \frac{dt}{ds} = 0 \) holds (which is always possible by the arguments above) then

\[
g_{ik} \frac{dx^i}{ds} \frac{dx^k}{ds} \bigg|_{s=s_0} = g_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}
\]

is negative if the matrix consisting of the \( g_{\alpha\beta} \) is negative definite and this exactly is the condition for the absence of timelike closed curves.

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**References**


