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SHORT COMMUNICATION

Analytical approximations for a conservative nonlinear singular oscillator in plasma physics

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Abstract A modified variational approach and the coupled homotopy perturbation method with variational formulation are exerted to obtain periodic solutions of a conservative nonlinear singular oscillator in plasma physics. The frequency–amplitude relations for the oscillator which the restoring force is inversely proportional to the dependent variable are achieved analytically. The approximate frequency obtained using the coupled method is more accurate than the modified variational approach and ones obtained using other approximate methods and the discrepancy between the approximate frequency using this coupled method and the exact one is lower than 0.31% for the whole range of values of oscillation amplitude. The coupled method provides a very good accuracy and is a promising technique to a lot of practical engineering and physical problems.

1. Introduction

There is a large variety of approximate methods usually used for solving nonlinear oscillatory systems including Linstedt–Poincaré [1,2], the multiple scales [3–5], variational iteration [6,7], homotopy perturbation [8–11], energy balance [12–15], Krylov–Bogoliubov–Mitropolsky [16,17], harmonic balance [18–22], the max–min [23–25] methods, and so on [26–33].

In this paper, we apply a modified variational approach (MVA) and the coupled homotopy perturbation method with variational formulation (CHV) to obtain analytic approximate solutions for a nonlinear oscillator in which the restoring force is inversely proportional to the dependent variable. This singular nonlinear oscillator has been recently studied by some researchers using a modified generalized, rational harmonic balance method [20], the standard harmonic balance method [18,22], and the homotopy perturbation method [26]. the results indicate that the coupled method [32] has a higher accuracy and provides a better solution. The approximate frequency derived by the coupled method is more accurate and closer to the exact

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solution and the relative error in the frequency is reduced and the maximum relative error is less than 0.31%.

In the next sections of the paper, the modified variational approach [33] and the coupled method are applied to an important and interesting conservative nonlinear singular oscillator in plasma physics in the following one:

$$\ddot{u} + \frac{1}{u} = 0, \quad u(0) = A, \quad \dot{u}(0) = 0, \tag{1}$$

where it can be shown that the exact value for the angular frequency is given as:

$$\omega_{ex} = \frac{2\pi}{2\sqrt{2}\int_0^A \frac{du}{\sqrt{\ln A - \ln u}}} = \frac{1}{A}\sqrt{\frac{\pi}{2}} = \frac{1.253314}{\mathrm{A}}. \tag{2}$$

The results demonstrate that by using both methods a solution with good accuracy is obtained. As we will see, the relative error of the coupled method is lower than the relative error obtained by the modified variational approach and other approximate methods which utilized for the oscillator till now.

2. Modified variational approach

In this section, the modified variational approach is utilized. we can rewrite Eq. (1) in the following form:

$$u \cdot \ddot{u} + 1 = 0, \tag{3}$$

based on the procedure, the minimization problem is:

$$\text{Minimize } E(\dot{u}, u, t) = \int_0^T (u \cdot \ddot{u} + 1)^2 dt, \quad T = \frac{2\pi}{\omega}. \tag{4}$$

For the first order approximation, We begin the procedure with the simplest trial function:

$$u_1(t) = A\cos(\omega t). \tag{5}$$

Substituting Eq. (5) into Eq. (4) yields:

$$\text{Minimize } E(\dot{u}_1, u_1, t) = \frac{\pi(3A^4\omega^4 - 8A^2\omega^2 + 8)}{4\omega}. \tag{6}$$

The solution of Eq. (6) could be found through

$$\frac{\partial E(\dot{u}_1, u_1, t)}{\partial \omega} = 0, \tag{7}$$

by some simplifications, the following equation is obtained:

$$9A^4\omega^4 - 8A^2\omega^2 - 8 = 0, \tag{8}$$

by solving Eq. (8), the approximate frequency is achieved as:

$$\omega_{MVA1} = \frac{1}{3}\frac{\sqrt{4 + 2\sqrt{22}}}{A} = \frac{1.219327}{A}. \tag{9}$$

For the second order approximation, consider the trial function as:

$$u_2(t) = b\cos(\omega t) + (A - b)\cos(3\omega t). \tag{10}$$

By substituting Eq. (10) into Eq. (4) yields:

$$\begin{aligned}\text{Minimize } E(\dot{u}_2, u_2, t) = \frac{\pi}{4\omega}[&243A^4\omega^4 - 972A^3\omega^4 \\ &+ 1694A^2b^2\omega^4 - 1424Ab^3\omega^4 \\ &+ 462b^4\omega^4 - 72A^2\omega^2 + 144Ab\omega \\ &- 80b^2\omega^2 + 8].\end{aligned} \tag{11}$$

The solution of Eq. (11) could be found through

$$\frac{\partial E(\dot{u}_2, u_2, t)}{\partial \omega} = 0, \quad \frac{\partial E(\dot{u}_2, u_2, t)}{\partial b} = 0. \tag{12}$$

After some simplifications, the solution for stationary conditions yields:

$$\omega_{MVA2} = \frac{1.227214}{A}. \tag{13}$$

For the third order approximation, consider the trial function as follows:

$$u_3(t) = b\cos(\omega t) + c\cos(3\omega t) + (A - b - c)\cos(5\omega t). \tag{14}$$

By substituting Eq. (14) in Eq. (4), The solution could be found through

$$\frac{\partial E(\dot{u}_3, u_3, t)}{\partial \omega} = 0, \quad \frac{\partial E(\dot{u}_3, u_3, t)}{\partial b} = 0, \quad \frac{\partial E(\dot{u}_3, u_3, t)}{\partial c} = 0. \tag{15}$$

After some simplifications, the approximate frequency is achieved as:

$$\omega_{MVA3} = \frac{1.245689}{A}. \tag{16}$$

3. Coupled homotopy perturbation method with variational formulation

In this section, the coupled homotopy perturbation method with variational formulation is employed. we can rewrite Eq. (1) in the following form:

$$uu''^2 + u'' = 0, \tag{17}$$

based on the coupled method, we construct the following homotopy:

$$u'' + \omega^2 u + p[uu''^2 - \omega^2 u] = 0, \quad p \in [0, 1]. \tag{18}$$

In Eq. (18), u is assumed to be

$$u = u_0 + p^1 u_1 + p^2 u_2 + \ldots \tag{19}$$

Substituting Eq. (19) into Eq. (18) and collecting terms of the same power of p, gives:

$$u_0'' + \omega^2 u_0 = 0, \quad u_0(0) = A, \quad u_0'(0) = 0, \tag{20}$$

And

$$u_1'' + \omega^2 u_1 + u_0 u_0''^2 - \omega^2 u_0 = 0, \quad u_1(0) = 0, \quad u_1'(0) = 0. \tag{21}$$

From Eq. (20), we find:

$$u_0(t) = A\cos\omega t, \tag{22}$$

for determining ω, we applied the variational formulation for u_1 in Eq. (21), which reads

$$J(u_1) = \int_0^T \left(-\frac{1}{2}u_1'^2 + \frac{1}{2}\omega^2 u_1^2 + +u_0 u_0''^2 u_1 - \omega^2 u_0 u_1\right)dt,$$
$$T = \frac{2\pi}{\omega}. \tag{23}$$

For first order approximation, we consider the trial function for u_1 as follows:

$$u_1(t) = B(\cos\omega t - \frac{1}{3}\cos 3\omega t + \frac{1}{3}\cos 5\omega t - \cos 7\omega t). \tag{24}$$

Substituting Eq. (24) into Eq. (23) yields:

Table 1 A comparison between the accuracy of the achieved frequencies.

Frequencies	ω_{MVA1}	ω_{MVA2}	ω_{MVA3}	ω_{CHV1}	ω_{CHV2}
Relative error (%)	2.7117	2.0825	0.6084	2.2795	0.3017

$$J(A,B,\omega)=\frac{2}{3}\pi A^3B\omega^3-\pi AB\omega+\frac{10}{9}\pi B^2\omega. \tag{25}$$

The stationary condition of Eq. (25) requires that

$$\frac{\partial J}{\partial B}=0,\quad \frac{\partial J}{\partial \omega}=0. \tag{26}$$

Then we have:

$$\frac{\partial J}{\partial B}=\frac{2}{3}\pi A^3\omega^3-\pi A\omega+\frac{20}{9}\pi B\omega=0, \tag{27}$$

$$\frac{\partial J}{\partial \omega}=2\pi A^3B\omega^2-\pi AB+\frac{10}{9}\pi B^2=0. \tag{28}$$

Solving above equations, the frequency obtained as:

$$\omega_{CHV1}=\frac{1}{2}\frac{\sqrt{6}}{A}=\frac{1.224745}{A}. \tag{29}$$

For the second order approximation, we consider the trial function for u_1 as follows:

$$u_1(t)=B\left(\cos\omega t-\frac{1}{3}\cos 3\omega t+\frac{1}{3}\cos 5\omega t-\cos 7\omega t\right)+C(\cos(\omega t)-\cos(3\omega t)). \tag{30}$$

Substituting Eq. (30) into Eq. (23) yields:

$$J(A,B,C,\omega)=\frac{1}{2}\pi A^3B\omega^3+\frac{2}{3}\pi A^3C\omega^3-\pi AB\omega+\pi B^2\omega+\frac{4}{3}\pi BC\omega-\pi AC\omega+\frac{10}{9}\pi C^2\omega. \tag{31}$$

The stationary condition of Eq. (31) requires that

$$\frac{\partial J}{\partial B}=0,\quad \frac{\partial J}{\partial C}=0,\quad \frac{\partial J}{\partial \omega}=0. \tag{32}$$

Then we have:

$$\frac{\partial J}{\partial B}=\frac{1}{2}\pi A^3\omega^3-\pi A\omega+2\pi B\omega+\frac{4}{3}\pi C\omega, \tag{33}$$

$$\frac{\partial J}{\partial C}=\frac{2}{3}\pi A^3\omega^3+\frac{4}{3}\pi B\omega-\pi A\omega+\frac{20}{9}\pi C\omega, \tag{34}$$

$$\frac{\partial J}{\partial \omega}=\frac{3}{2}\pi A^3B\omega^2+2\pi A^3C\omega^2-\pi AB+\pi B^2+\frac{4}{3}\pi BC-\pi AC+\frac{10}{9}\pi C^2. \tag{35}$$

Solving above equations, the frequency is obtained as:

$$\omega_{CHV2}=\frac{1}{5}\frac{\sqrt{24+\sqrt{226}}}{A}=\frac{1.249533}{A}. \tag{36}$$

Accuracy of the achieved frequencies for the above mentioned methods is shown in Table 1.

4. Conclusion

A modified variational approach and the coupled homotopy perturbation method with variational formulation have been applied to obtain analytical approximate solutions for nonlinear oscillators in which the restoring force is inversely proportional to the dependent variable. The major conclusion is that the coupled method provides excellent approximations to the solution of equation (1), with high accuracy. The analytical representations obtained using this technique give excellent approximations to the exact solutions for the whole range of values of oscillation amplitude. The second order approximation of the coupled method is better than the approximate solutions obtained using other approximate methods presented in the literature. For the second order approximation, the relative error of the analytical approximate frequency obtained using the coupled method is lower than 0.31%. The general conclusion is that the coupled method provides an easy and direct procedure for determining the second-order analytical approximate solution of these nonlinear systems. This procedure also gives a very accurate estimate for the frequency. Finally, in comparison to the modified variational approach, the coupled method produces a better solution.

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