European Journal of Combinatorics 32 (2011) 178–197

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Minimal paths in the commuting graphs of semigroups

João Araújo [a,b,1], Michael Kinyon [c], Janusz Konieczny [d]

[a] *Universidade Aberta, R. Escola Politécnica, 147, 1269-001 Lisboa, Portugal*

[b] *Centro de Álgebra, Universidade de Lisboa, 1649-003 Lisboa, Portugal*

[c] *Department of Mathematics, University of Denver, Denver, CO 80208, United States*

[d] *Department of Mathematics, University of Mary Washington, Fredericksburg, VA 22401, United States*

ARTICLE INFO

Article history:
Received 12 April 2010
Accepted 7 September 2010
Available online 14 October 2010

ABSTRACT

Let S be a finite non-commutative semigroup. The commuting graph of S, denoted $\mathcal{G}(S)$, is the graph whose vertices are the non-central elements of S and whose edges are the sets $\{a, b\}$ of vertices such that $a \neq b$ and $ab = ba$. Denote by $T(X)$ the semigroup of full transformations on a finite set X. Let J be any ideal of $T(X)$ such that J is different from the ideal of constant transformations on X. We prove that if $|X| \geq 4$, then, with a few exceptions, the diameter of $\mathcal{G}(J)$ is 5. On the other hand, we prove that for every positive integer n, there exists a semigroup S such that the diameter of $\mathcal{G}(S)$ is n.

We also study the left paths in $\mathcal{G}(S)$, that is, paths $a_1 - a_2 - \cdots - a_m$ such that $a_1 \neq a_m$ and $a_1a_i = a_ma_i$ for all $i \in \{1, \ldots, m\}$. We prove that for every positive integer $n \geq 2$, except $n = 3$, there exists a semigroup whose shortest left path has length n. As a corollary, we use the previous results to solve a purely algebraic old problem posed by B.M. Schein.

1. Introduction

The commuting graph of a finite non-abelian group G is a simple graph whose vertices are all non-central elements of G and two distinct vertices x, y are adjacent if $xy = yx$. Commuting graphs of various groups have been studied in terms of their properties (such as connectivity or diameter), for example in [4,6,8,15]. They have also been used as a tool to prove group theoretic results, for example in [5,12,13].

E-mail addresses: jaraujo@ptmat.fc.ul.pt (J. Araújo), mkinyon@math.du.edu (M. Kinyon), jkoniecz@umw.edu (J. Konieczny).

[1] Tel.: +351 21 790 47 00; fax: +351 21 795 42 88.

0195-6698/$ – see front matter
doi:10.1016/j.ejc.2010.09.004

The concept of the commuting graph carries over to semigroups. Let S be a finite non-commutative semigroup with center $Z(S) = \{a \in S : ab = ba \text{ for all } b \in S\}$. The *commuting graph* of S, denoted $\mathcal{G}(S)$, is the simple graph (that is, an undirected graph with no multiple edges or loops) whose vertices are the elements of $S - Z(S)$ and whose edges are the sets $\{a, b\}$ such that a and b are distinct vertices with $ab = ba$.

This paper initiates the study of commuting graphs of semigroups. Our main goal is to study the lengths of minimal paths. We shall consider two types of paths: ordinary paths and so called left paths.

We first investigate the semigroup $T(X)$ of full transformations on a finite set X, and determine the diameter of the commuting graph of every ideal of $T(X)$ (Section 2). We find that, with a few exceptions, the diameter of $\mathcal{G}(J)$, where J is an ideal of $T(X)$, is 5. This small diameter does not extend to semigroups in general. We prove that for every $n \geq 2$, there is a finite semigroup S whose commuting graph has diameter n (Theorem 4.1). To prove the existence of such a semigroup, we use our work on the *left paths* in the commuting graph of a semigroup.

Let S be a semigroup. A path $a_1 - a_2 - \cdots - a_m$ in $\mathcal{G}(S)$ is called a *left path* (or l-path) if $a_1 \neq a_m$ and $a_1 a_i = a_m a_i$ for every $i \in \{1, \ldots, m\}$. If there is any l-path in $\mathcal{G}(S)$, we define the *knit degree* of S, denoted $\mathrm{kd}(S)$, to be the length of a shortest l-path in $\mathcal{G}(S)$.

For every $n \geq 2$ with $n \neq 3$, we construct a band (semigroup of idempotents) of knit degree n (Section 3). It is an open problem if there is a semigroup of knit degree 3. In Section 4, the constructions presented in Section 3 also give a band S whose commuting graph has diameter n (for every $n \geq 4$). As another application of our work on the left paths, we settle a conjecture on bands formulated by B.M. Schein in 1978 (Section 5). Finally, we present some problems regarding the commuting graphs of semigroups (Section 6).

2. Commuting graphs of ideals of $T(X)$

Let $T(X)$ be the semigroup of full transformations on a finite set X, that is, the set of all functions from X to X with composition as the operation. We will write functions on the right and compose from left to right, that is, for $a, b \in T(X)$ and $x \in X$, we will write xa (not $a(x)$) and $x(ab) = (xa)b$ (not $(ba)(x) = b(a(x))$). In this section, we determine the diameter of the commuting graph of every ideal of $T(X)$. Throughout this section, we assume that $X = \{1, \ldots, n\}$.

Let Γ be a simple graph, that is, $\Gamma = (V, E)$, where V is a finite non-empty set of vertices and $E \subseteq \{\{u, v\} : u, v \in V, u \neq v\}$ is a set of edges. We will write $u - v$ to mean that $\{u, v\} \in E$. Let $u, w \in V$. A *path* in Γ from u to w is a sequence of pairwise distinct vertices $u = v_1, v_2, \ldots, v_m = w$ $(m \geq 1)$ such that $v_i - v_{i+1}$ for every $i \in \{1, \ldots, m-1\}$. If λ is a path $v_1, v_2, \ldots, v_m$, we will write $\lambda = v_1 - v_2 - \cdots - v_m$ and say that λ has *length* $m - 1$. We say that a path λ from u to w is a *minimal path* if there is no path from u to w that is shorter than λ.

We say that the *distance* between vertices u and w is k, and write $d(u, w) = k$, if a minimal path from u to w has length k. If there is no path from u to w, we say that the distance between u and w is infinity, and write $d(u, w) = \infty$. The maximum distance $\max\{d(u, w) : u, w \in V\}$ between vertices of Γ is called the *diameter* of Γ. Note that the diameter of Γ is finite if and only if Γ is connected.

If S is a finite non-commutative semigroup, then the commuting graph $\mathcal{G}(S)$ is a simple graph with $V = S - Z(S)$ and, for $a, b \in V$, $a - b$ if and only if $a \neq b$ and $ab = ba$.

For $a \in T(X)$, we denote by $\mathrm{im}(a)$ the image of a, by $\ker(a) = \{(x, y) \in X \times X : xa = ya\}$ the kernel of a, and by $\mathrm{rank}(a) = |\mathrm{im}(a)|$ the rank of a. It is well known (see [7, Section 2.2]) that in $T(X)$ the only element of $Z(T(X))$ is the identity transformation on X, and that $T(X)$ has exactly n ideals: $J_1, J_2, \ldots, J_n$, where, for $1 \leq r \leq n$,

$$J_r = \{a \in T(X) : \mathrm{rank}(a) \leq r\}.$$

Each ideal J_r is principal and any $a \in T(X)$ of rank r generates J_r. The ideal J_1 consists of the transformations of rank 1 (that is, constant transformations), and it is clear that $\mathcal{G}(J_1)$ is the graph with n isolated vertices.

Let S be a semigroup. We denote by $\mathcal{G}_E(S)$ the subgraph of $\mathcal{G}(S)$ induced by the non-central idempotents of S. The graph $\mathcal{G}_E(S)$ is said to be the *idempotent commuting graph* of S. We first determine the diameter of $\mathcal{G}_E(J_r)$. This approach is justified by the following lemma.

Lemma 2.1. *Let $2 \le r < n$ and let $a, b \in J_r$ be such that $ab \neq ba$. Suppose $a - a_1 - a_2 - \cdots - a_k - b$ ($k \ge 1$) is a minimal path in $\mathcal{G}(J_r)$ from a to b. Then there are idempotents $e_1, e_2, \ldots, e_k \in J_r$ such that $a - e_1 - e_2 - \cdots - e_k - b$ is a minimal path in $\mathcal{G}(J_r)$ from a to b.*

Proof. Since J_r is finite, there is an integer $p \ge 1$ such that $e_1 = a_1^p$ is an idempotent in J_r. Note that $e_1 \notin Z(J_r)$, since for any $x \in X - \mathrm{im}(e_1)$, e_1 does not commute with $c_x \in J_r$, where c_x is the constant transformation with $\mathrm{im}(c_x) = \{x\}$. Since a_1 commutes with a and a_2, the idempotent $e_1 = a_1^p$ also commutes with a and a_2, and so $a - e_1 - a_2 - \cdots - a_k - b$. Repeating the foregoing argument for $a_2, \ldots, a_k$, we obtain idempotents $e_2, \ldots, e_k$ in J_r such that $a - e_1 - e_2 - \cdots - e_k - b$. Since the path $a - a_1 - a_2 - \cdots - a_k - b$ is minimal, it follows that $a, e_1, e_2, \ldots, e_k, b$ are pairwise distinct and the path $a - e_1 - e_2 - \cdots - e_k - b$ is minimal. $\square$

It follows from Lemma 2.1 that if d is the diameter of $\mathcal{G}_E(J_r)$, then the diameter of $\mathcal{G}(J_r)$ is at most $d + 2$.

2.1. Idempotent commuting graphs

In this subsection, we assume that $n \ge 3$ and $2 \le r < n$. We will show that, with some exceptions, the diameter of $\mathcal{G}_E(J_r)$ is 3 (Theorem 2.8).

Let $e \in T(X)$ be an idempotent. Then there is a unique partition $\{A_1, A_2, \ldots, A_k\}$ of X and unique elements $x_1 \in A_1, x_2 \in A_2, \ldots, x_k \in A_k$ such that for every i, $A_i e = \{x_i\}$. The partition $\{A_1, \ldots, A_k\}$ is induced by the kernel of e, and $\{x_1, \ldots, x_k\}$ is the image of e. We will use the following notation for e:

$$e = (A_1, x_1\rangle (A_2, x_2\rangle \cdots (A_k, x_k\rangle. \tag{2.1}$$

Note that $(X, x\rangle$ is the constant idempotent with image $\{x\}$. The following result has been obtained in [1,9] (see also [2]).

Lemma 2.2. *Let $e = (A_1, x_1\rangle (A_2, x_2\rangle \cdots (A_k, x_k\rangle$ be an idempotent in $T(X)$ and let $b \in T(X)$. Then b commutes with e if and only if for every $i \in \{1, \ldots, k\}$, there is $j \in \{1, \ldots, k\}$ such that $x_i b = x_j$ and $A_i b \subseteq A_j$.*

We will use Lemma 2.2 frequently, not always mentioning it explicitly. The following lemma is an immediate consequence of Lemma 2.2.

Lemma 2.3. *Let $e, f \in J_r$ be idempotents and suppose there is $x \in X$ such that $x \in \mathrm{im}(e) \cap \mathrm{im}(f)$. Then $e - (X, x\rangle - f$.*

Lemma 2.4. *Let $e, f \in J_r$ be idempotents such that $\mathrm{im}(e) \cap \mathrm{im}(f) = \emptyset$. Suppose there is $(x, y) \in \mathrm{im}(e) \times \mathrm{im}(f)$ such that $(x, y) \in \ker(e) \cap \ker(f)$. Then there is an idempotent $g \in J_r$ such that $e - g - f$.*

Proof. Let $e = (A_1, x_1\rangle \cdots (A_k, x_k\rangle$ and $f = (B_1, y_1\rangle \cdots (B_m, y_m\rangle$. We may assume that $x = x_1$ and $y = y_1$. Since $(x, y) \in \ker(e) \cap \ker(f)$, we have $y \in A_1$ and $x \in B_1$. Let $g = (\mathrm{im}(e), x\rangle (X - \mathrm{im}(e), y\rangle$. Then g is in J_r since $\mathrm{rank}(g) = 2$ and $r \ge 2$. By Lemma 2.2, we have $eg = ge$ (since $y \in A_1$) and $fg = gf$ (since $\mathrm{im}(f) \subseteq X - \mathrm{im}(e)$ and $x \in B_1$). Hence $e - g - f$. $\square$

Lemma 2.5. *Let $e, f \in J_r$ be idempotents such that $\mathrm{im}(e) \cap \mathrm{im}(f) = \emptyset$. Then there are idempotents $g, h \in J_r$ such that $e - g - h - f$.*

Proof. Let $e = (A_1, x_1\rangle \cdots (A_k, x_k\rangle$ and $f = (B_1, y_1\rangle \cdots (B_m, y_m\rangle$. Since $\{A_1, \ldots, A_k\}$ is a partition of X, there is i such that $y_1 \in A_i$. We may assume that $y_1 \in A_1$. Let $g = (X - \{y_1\}, x_1\rangle (\{y_1\}, y_1\rangle$ and $h = (X, y_1\rangle$. Then g and h are in J_r (since $r \ge 2$). By Lemma 2.2, $eg = ge$, $gh = hg$, and $hf = fh$. Thus $e - g - h - f$. $\square$

Lemma 2.6. *Let m be a positive integer such that $2m \le n$, σ be an m-cycle on $\{1, \ldots, m\}$, and*

$$e = (A_1, x_1\rangle (A_2, x_2\rangle \cdots (A_m, x_m\rangle \quad \text{and} \quad f = (B_1, y_1\rangle (B_2, y_2\rangle \cdots (B_m, y_m\rangle$$

be idempotents in $T(X)$ such that $x_1, \ldots, x_m, y_1, \ldots, y_m$ are pairwise distinct, $y_i \in A_i$, and $x_{i\sigma} \in B_i$ ($1 \le i \le m$). Suppose that g is an idempotent in $T(X)$ such that $e - g - f$. Then:

(1) $x_j g = x_j$ and $y_j g = y_j$ for every $j \in \{1, \ldots, m\}$.
(2) If $1 \le i, j \le m$ are such that $A_i = \{x_i, y_i, z\}$, $B_j = \{y_j, x_{j\sigma}, z\}$ and $A_i \cap B_j = \{z\}$, then $zg = z$.

Proof. Since $eg = ge$, $x_1g = x_i$ for some i. Then $x_ig = x_i$ (since g is an idempotent). Thus, $e - g - f$ and Lemma 2.2 imply that $y_ig = y_i$. Since $x_i = x_{(i\sigma^{-1})\sigma} \in B_{i\sigma^{-1}}$ and g commutes with f, we have $y_{i\sigma^{-1}}g = y_{i\sigma^{-1}}$. But now, since $y_{i\sigma^{-1}} \in A_{i\sigma^{-1}}$ and g commutes with e, we have $x_{i\sigma^{-1}}g = x_{i\sigma^{-1}}$. Continuing this way, we obtain $x_{i\sigma^{-k}}g = x_{i\sigma^{-k}}$ and $y_{i\sigma^{-k}}g = y_{i\sigma^{-k}}$ for every $k \in \{1, \dots, m-1\}$. Since σ is an m-cycle, it follows that $x_jg = x_j$ and $y_jg = y_j$ for every $j \in \{1, \dots, m\}$. We have proved (1).

Suppose $A_i = \{x_i, y_i, z\}$, $B_j = \{y_j, x_{j\sigma}, z\}$, and $A_i \cap B_j = \{z\}$. Then $zg \in \{x_i, y_i, z\}$ (since $x_ig = x_i$ and $eg = ge$) and $zg \in \{y_j, x_{j\sigma}, z\}$ (since $y_jg = y_j$ and $fg = gf$). Since $A_i \cap B_j = \{z\}$, we have $zg = z$, which proves (2). □

Lemma 2.7. *Let $n \geq 4$. If $n \neq 5$ or $r \neq 4$, then for some idempotents $e, f \in J_r$, there is no idempotent $g \in J_r$ such that $e - g - f$.*

Proof. Let $n \neq 5$ or $r \neq 4$. Suppose that $r < n - 1$ or n is even. Then there is an integer m such that $m \leq r$ and $r < 2m \leq n$. Let e and f be idempotents from Lemma 2.6. Then $e, f \in J_r$ since $m \leq r$. But every idempotent $g \in T(X)$ such that $e - g - f$ fixes at least $2m$ elements, and so $g \notin J_r$ since $r < 2m$.

Suppose that $r = n - 1$ and $n = 2m + 1$ is odd. Then $n \geq 7$ since we are working under the assumption that $n \neq 5$ or $r \neq 4$. We again consider idempotents e and f from Lemma 2.6, which belong to J_r since $m < n - 1 = r$. Note that $X = \{x_1, \dots, x_m, y_1, \dots, y_m, z\}$. We may assume that $z \in A_m$ and $z \in B_1$. Since $n \geq 7$, we have $m \geq 3$. Thus, the intersection of $A_m = \{x_m, y_m, z\}$ and $B_1 = \{y_1, x_2, z\}$ is $\{z\}$, and so $zg = z$ by Lemma 2.6. Hence $g = \mathrm{id}_X \notin J_r$, which concludes the proof. □

Theorem 2.8. *Let $n \geq 3$ and let J_r be an ideal in $T(X)$ such that $2 \leq r < n$. Then:*

(1) *If $n = 3$ or $n = 5$ and $r = 4$, then the diameter of $\mathcal{G}_E(J_r)$ is 2.*
(2) *In all other cases, the diameter of $\mathcal{G}_E(J_r)$ is 3.*

Proof. Suppose $n = 3$ or $n = 5$ and $r = 4$. In these special cases, we obtained the desired result using GRAPE [16], which is a package for GAP [17].

Let $n \geq 4$ and suppose that $n \neq 5$ or $r \neq 4$. By Lemmas 2.3 and 2.5, the diameter of $\mathcal{G}_E(J_r)$ is at most 3. By Lemma 2.7, the diameter of $\mathcal{G}_E(J_r)$ is at least 3. Thus the diameter of $\mathcal{G}_E(J_r)$ is 3, which concludes the proof of (2). □

2.2. Commuting graphs of proper ideals of $T(X)$

In this subsection, we determine the diameter of every proper ideal of $T(X)$. The ideal J_1 consists of the constant transformations, so $\mathcal{G}(J_1)$ is the graph with n isolated vertices. Thus J_1 is not connected and its diameter is ∞. Therefore, for the remainder of this subsection, we assume that $n \geq 3$ and $2 \leq r < n$.

It follows from Lemma 2.1 and Theorem 2.8 that the diameter of $\mathcal{G}(J_r)$ is at most 5. We will prove that this diameter is in fact 5 except when $n = 3$ or $n \in \{5, 6, 7\}$ and $r = 4$. It also follows from Lemma 2.1 that if e and f are idempotents in J_r, then the distance between e and f in $\mathcal{G}(J_r)$ is the same as the distance between e and f in $\mathcal{G}_E(J_r)$. So no ambiguity will arise when we talk about the distance between idempotents in J_r.

For $a \in T(X)$ and $x, y \in X$, we will write $x \xrightarrow{a} y$ when $xa = y$.

Lemma 2.9. *Let $a, b \in T(X)$. Then $ab = ba$ if and only if for all $x, y \in X$, $x \xrightarrow{a} y$ implies $xb \xrightarrow{a} yb$.*

Proof. Suppose $ab = ba$. Let $x, y \in X$ with $x \xrightarrow{a} y$, that is, $y = xa$. Then, since $ab = ba$, we have $yb = (xa)b = x(ab) = x(ba) = (xb)a$, and so $xb \xrightarrow{a} yb$.

Conversely, suppose $x \xrightarrow{a} y$ implies $xb \xrightarrow{a} yb$ for all $x, y \in X$. Let $x \in X$. Since $x \xrightarrow{a} xa$, we have $xb \xrightarrow{a} (xa)b$. But this means that $(xb)a = (xa)b$, which implies $ab = ba$. □

Let $a \in T(X)$. Suppose $x_1, \dots, x_m$ are pairwise distinct elements of X such that $x_ia = x_{i+1}$ $(1 \leq i < m)$ and $x_ma = x_1$. We will then say that a contains a cycle $(x_1\, x_2 \dots x_m)$.

Lemma 2.10. *Let $a \in J_r$ be a transformation containing a unique cycle $(x_1\, x_2 \dots x_m)$. Let $e \in J_r$ be an idempotent such that $ae = ea$. Then $x_ie = x_i$ for every $i \in \{1, \dots, m\}$.*

Proof. Since a contains $(x_1\,x_2\ldots x_m)$, we have $x_1 \xrightarrow{a} x_2 \xrightarrow{a} \cdots \xrightarrow{a} x_m \xrightarrow{a} x_1$. Thus, by Lemma 2.9,

$$x_1e \xrightarrow{a} x_2e \xrightarrow{a} \cdots \xrightarrow{a} x_me \xrightarrow{a} x_1e.$$

Thus $(x_1e\,x_2e\ldots x_me)$ is a cycle in a, and is therefore equal to $(x_1\,x_2\ldots x_m)$. Hence, for every $i \in \{1,\ldots,m\}$, there exists $j \in \{1,\ldots,m\}$ such that $x_i = x_je$, and so $x_ie = (x_je)e = x_j(ee) = x_je = x_i$. □

To construct transformations $a, b \in J_r$ such that the distance between a and b is 5, it will be convenient to introduce the following notation.

Notation 2.11. Let $x_1,\ldots,x_m,z_1,\ldots,z_p$ be pairwise distinct elements of X, and let s be fixed such that $1 \le s < p$. We will denote by

$$a = (*\,z_s\rangle(z_p\,z_{p-1}\ldots z_1\,x_1\rangle(x_1\,x_2\ldots x_m) \tag{2.2}$$

the transformation $a \in T(X)$ such that

$$z_pa = z_{p-1},\quad z_{p-1}a = z_{p-2},\ldots,z_2a = z_1,\quad z_1a = x_1,$$
$$x_1a = x_2,\quad x_2a = x_3,\ldots,x_{m-1}a = x_m,\quad x_ma = x_1,$$

and $ya = z_s$ for all other $y \in X$. Suppose $w \in X$ such that $w \notin \{x_1,\ldots,x_m,z_1,\ldots,z_p\}$ and $1 \le t < p$ with $t \ne s$. We will denote by

$$b = (*\,z_s\rangle(w\,z_t\rangle(z_p\,z_{p-1}\ldots z_1\,x_1\rangle(x_1\,x_2\ldots x_m) \tag{2.3}$$

the transformation $b \in T(X)$ that is defined as a in (2.2) except that $wb = z_t$.

Lemma 2.12. *Let $a \in J_r$ be the transformation defined in (2.2) such that $m + p > r$. Let $e \in J_r$ be an idempotent such that $ae = ea$. Then:*

(1) *$x_ie = x_i$ for every $i \in \{1,\ldots,m\}$.*
(2) *$z_je = x_{m-j+1}$ for every $j \in \{1,\ldots,p\}$.*
(3) *$ye = x_{m-s}$ for every $y \in X - \{x_1,\ldots,x_m,z_1,\ldots,z_p\}$.*

(We assume that for every integer u, $x_u = x_v$, where $v \in \{1,\ldots,m\}$ and $u \equiv v (\mathrm{mod}\ m)$.)

Proof. Statement (1) follows from Lemma 2.10. By the definition of a, we have

$$z_p \xrightarrow{a} z_{p-1} \xrightarrow{a} \cdots \xrightarrow{a} z_1 \xrightarrow{a} x_1.$$

Thus, by Lemma 2.9,

$$z_pe \xrightarrow{a} z_{p-1}e \xrightarrow{a} \cdots \xrightarrow{a} z_1e \xrightarrow{a} x_1e = x_1.$$

Since $z_1e \xrightarrow{a} x_1$, either $z_1e = x_m$ or $z_1e \notin \{x_1,\ldots,x_m\}$. We claim that the latter is impossible. Indeed, suppose $z_1e \notin \{x_1,\ldots,x_m\}$. Then $z_je \notin \{x_1,\ldots,x_m\}$ for every $j \in \{1,\ldots,p\}$. Thus the set $\{x_1,\ldots,x_m,z_1e,\ldots,z_pe\}$ is a subset of $\mathrm{im}(e)$ with $m+p$ elements. But this implies that $e \notin J_r$ (since $m + p > r$), which is a contradiction. We proved the claim. Thus $z_1e = x_m$. Now, $z_2e \xrightarrow{a} z_1e = x_m$, which implies $z_2e = x_{m-1}$. Continuing this way, we obtain $z_3e = x_{m-2}$, $z_4e = x_{m-3},\ldots$. (A special argument is required when $j = qm + 1$ for some $q \ge 1$. Suppose $q = 1$, that is, $j = m + 1$. Then $z_je \xrightarrow{a} z_{j-1}e = z_me = x_1$, and so either $z_je = x_m$ or $z_je = z_1$. But the latter is impossible since we would have $x_m = z_1e = z_j(ee) = z_je = z_1$, which is a contradiction. Hence, for $j = m + 1$, we have $z_je = x_m$. Assuming, inductively, that $z_je = x_m$ for $j = qm + 1$, we prove by a similar argument that $z_je = x_m$ for $j = (q + 1)m + 1$.) This concludes the proof of (2).

Let $y \in X - \{x_1,\ldots,x_m,z_1,\ldots,z_p\}$. Then $y \xrightarrow{a} z_s$, and so $ye \xrightarrow{a} z_se = x_{m-s+1}$. Suppose s is not a multiple of m. Then $x_{m-s+1} \ne x_1$, and so $ye \xrightarrow{a} x_{m-s+1}$ implies $ye = x_{m-s}$. Suppose s is a multiple of m. Then $ye \xrightarrow{a} x_{m-s+1} = x_1$, and so either $ye = x_m$ or $ye = z_1$. But the latter is impossible since we would have $x_m = z_1e = y(ee) = ye = z_1$, which is a contradiction. Hence, for s that is a multiple of m, we have $ye = x_m$, which concludes the proof of (3). □

The proof of the following lemma is almost identical to the proof of Lemma 2.12.

Lemma 2.13. *Let $b \in J_r$ be the transformation defined in (2.3) such that $m + p > r$. Let $e \in J_r$ be an idempotent such that $be = eb$. Then:*

(1) $x_i e = x_i$ *for every* $i \in \{1, \ldots, m\}$.
(2) $z_j e = x_{m-j+1}$ *for every* $j \in \{1, \ldots, p\}$.
(3) $we = x_{m-t}$.
(4) $ye = x_{m-s}$ *for every* $y \in X - \{x_1, \ldots, x_m, z_1, \ldots, z_p, w\}$.

Lemma 2.14. *Let $n \in \{5, 6, 7\}$ and $r = 4$. Then there are $a, b \in J_4$ such that the distance between a and b in $\mathcal{G}(J_4)$ is at least 4.*

Proof. Let $a = (*\,4\rangle(3\,4\,1\rangle(1\,2)$ and $b = (*\,1\rangle(2\,1\,3\rangle(3\,4)$ (see Notation 2.11). Suppose e and f are idempotents in J_4 such that $a - e$ and $f - b$. Then, by Lemma 2.12, $e = (\{\ldots, 3, 1\}, 1\rangle(\{4, 2\}, 2\rangle$ and $f = (\{\ldots, 2, 3\}, 3\rangle(\{1, 4\}, 4\rangle$, where "…" denotes "5" (if $n = 5$), "5, 6" (if $n = 6$), and "5, 6, 7" (if $n = 7$). Then e and f do not commute, and so $d(e, f) \geq 2$. Thus $d(a, b) \geq 4$ by Lemma 2.1. □

Lemma 2.15. *Let $n \in \{6, 7\}$ and $r = 4$. Let $a \in J_4$ be a transformation that is not an idempotent. Then there is an idempotent $e \in J_4$ commuting with a such that $\mathrm{rank}(e) \neq 3$ or $\mathrm{rank}(e) = 3$ and $ye^{-1} = \{y\}$ for some $y \in \mathrm{im}(e)$.*

Proof. If a fixes some $x \in X$, then a commutes with $e = (X, x\rangle$ of rank 1. Suppose a has no fixed points. Let p be a positive integer such that a^p is an idempotent. If a contains a unique cycle $(x_1\, x_2)$, then $e = a^p$ has rank 2. If a contains a unique cycle $(x_1\, x_2\, x_3\, x_4)$ or two cycles $(x_1\, x_2)$ and $(y_1\, y_2)$ with $\{x_1, x_2\} \cap \{y_1, y_2\} = \emptyset$, then $e = a^p$ has rank 4.

Suppose a contains a unique cycle $(x_1\, x_2\, x_3)$. Define $e \in T(X)$ as follows. Set $x_i e = x_i$, $1 \leq i \leq 3$.

Suppose there are $y, z \in X - \{x_1, x_2, x_3\}$ such that $ya = z$ and $za = x_i$ for some i. We may assume that $za = x_1$. Define $ze = x_3$ and $ye = x_2$. Let u and w be the two remaining elements in X (only u remains when $n = 6$). Since $\mathrm{rank}(a) \leq 4$, we have $\{u, w\}a \subseteq \{z, x_1, x_2, x_3\}$. Suppose $ua = wa = z$. Define $ue = x_2$ and $we = x_2$. Then e is an idempotent of rank 3 such that $ae = ea$ and $x_1 e^{-1} = \{x_1\}$. Suppose ua or wa is in $\{x_1, x_2, x_3\}$, say $ua \in \{x_1, x_2, x_3\}$. Define $ue = u$, and $we = x_{i-1}$ (if $wa = x_i$), where $x_{i-1} = x_3$ if $i = 1$, or $we = x_2$ (if $wa = z$). Then e is an idempotent of rank 4 such that $ae = ea$.

Suppose that for every $y \in X - \{x_1, x_2, x_3\}$, $ya \in \{x_1, x_2, x_3\}$. Select $z \in X - \{x_1, x_2, x_3\}$ and define $ze = z$. For every $y \in X - \{z, x_1, x_2, x_3\}$, define $ye = x_{i-1}$ if $ya = x_i$. Then e is an idempotent of rank 4 such that $ae = ea$.

Since $a \in J_4$, we have exhausted all possibilities, and the result follows. □

Lemma 2.16. *Let $n \in \{6, 7\}$ and $r = 4$. Then for all $a, b \in J_4$, the distance between a and b in $\mathcal{G}(J_4)$ is at most 4.*

Proof. Let $a, b \in J_4$. If a or b is an idempotent, then $d(a, b) \leq 4$ by Lemma 2.1 and Theorem 2.8. Suppose a and b are not idempotents. By Lemma 2.15, there are idempotents $e, f \in J_4$ such that $ae = ea$, $bf = fb$, if $\mathrm{rank}(e) = 3$, then $ye^{-1} = \{y\}$ for some $y \in \mathrm{im}(e)$, and if $\mathrm{rank}(f) = 3$, then $yf^{-1} = \{y\}$ for some $y \in \mathrm{im}(f)$. We claim that there is an idempotent $g \in J_4$ such that $e - g - f$. If $\mathrm{im}(e) \cap \mathrm{im}(f) \neq \emptyset$, then such an idempotent g exists by Lemma 2.3. Suppose $\mathrm{im}(e) \cap \mathrm{im}(f) = \emptyset$. Then, since $n \in \{6, 7\}$, both $\mathrm{rank}(e) + \mathrm{rank}(f) \leq 7$. We may assume that $\mathrm{rank}(e) \leq \mathrm{rank}(f)$. There are six possible cases.

Case 1. $\mathrm{rank}(e) = 1$.

Then $e = (X, x\rangle$ for some $x \in X$. Let $y = xf$. Then $(x, y) \in \mathrm{im}(e) \times \mathrm{im}(f)$ and $(x, y) \in \ker(e) \cap \ker(f)$. Thus, by Lemma 2.4, there is an idempotent $g \in J_4$ such that $e - g - f$.

Case 2. $\mathrm{rank}(e) = 2$ and $\mathrm{rank}(f) = 2$.

We may assume that $e = (A_1, 1\rangle(A_2, 2\rangle$ and $f = (B_1, 3\rangle(B_2, 4\rangle$. If $\{1, 2\} \subseteq B_i$ or $\{3, 4\} \subseteq A_i$ for some i, then we can find $(x, y) \in \mathrm{im}(e) \times \mathrm{im}(f)$ such that $(x, y) \in \ker(e) \cap \ker(f)$, and so a desired idempotent g exists by Lemma 2.4. Otherwise, we may assume that $3 \in A_1$ and $4 \in A_2$. If $1 \in B_1$ or

$2 \in B_2$, then Lemma 2.4 can be applied again. So suppose $1 \in B_2$ and $2 \in B_1$. Now we have

$$e = (\{\dots, 3, 1\}, 1\rangle(\{\dots, 4, 2\}, 2\rangle \quad \text{and} \quad f = (\{\dots, 2, 3\}, 3\rangle(\{\dots, 1, 4\}, 4\rangle.$$

We define $g \in T(X)$ as follows. Set $xg = x$ for every $x \in \{1, 2, 3, 4\}$. Let $x \in \{5, 6, 7\}$ ($x \in \{5, 6\}$ if $n = 6$). If $x \in A_1 \cap B_1$, define $xg = 3$; if $x \in A_1 \cap B_2$, define $xg = 1$; if $x \in A_2 \cap B_1$, define $xg = 2$; finally, if $x \in A_2 \cap B_2$, define $xg = 4$. Then g is an idempotent of rank 4 and $e - g - f$.

Case 3. $\operatorname{rank}(e) = 2$ and $\operatorname{rank}(f) = 3$.

We may assume that $e = (A_1, 1\rangle(A_2, 2\rangle$ and $f = (B_1, 3\rangle(B_2, 4\rangle(B_3, 5\rangle$. If $\{3, 4, 5\} \subseteq A_1$ or $\{3, 4, 5\} \subseteq A_2$, then Lemma 2.4 applies. Otherwise, we may assume that $3, 4 \in A_1$ and $5 \in A_2$. If $1 \in B_1 \cup B_2$ or $2 \in B_3$, then Lemma 2.4 applies again. So suppose $1 \in B_3$ and $2 \in B_1 \cup B_2$. We may assume that $2 \in B_1$. Note that if $z \in \{6, 7\}$, then z cannot be in B_2 since $z \in B_2$ would imply that there is no $y \in \operatorname{im}(f)$ such that $yf^{-1} = \{y\}$. So now

$$e = (\{\dots, 3, 4, 1\}, 1\rangle(\{\dots, 5, 2\}, 2\rangle \quad \text{and} \quad f = (\{\dots, 2, 3\}, 3\rangle(\{4\}, 4\rangle(\{\dots, 1, 5\}, 5\rangle.$$

We define $g \in T(X)$ as follows. Set $xg = x$ for every $x \in \{1, 2, 3, 5\}$ and $4g = 3$. Let $z \in \{6, 7\}$. If $z \in A_1 \cap B_1$, define $zg = 3$; if $z \in A_1 \cap B_3$, define $zg = 1$; if $z \in A_2 \cap B_1$, define $zg = 2$; finally, if $z \in A_2 \cap B_3$, define $zg = 5$. Then g is an idempotent of rank 4 and $e - g - f$.

Case 4. $\operatorname{rank}(e) = 2$ and $\operatorname{rank}(f) = 4$.

We may assume that $e = (A_1, 1\rangle(A_2, 2\rangle$ and $f = (B_1, 3\rangle(B_2, 4\rangle(B_3, 5\rangle(B_4, 6\rangle$. If $\{3, 4, 5, 6\} \subseteq A_1$ or $\{3, 4, 5, 6\} \subseteq A_2$, then Lemma 2.4 applies. Otherwise, we may assume that $3, 4, 5 \in A_1$ and $6 \in A_2$ or $3, 4 \in A_1$ and $5, 6 \in A_2$.

Suppose $3, 4, 5 \in A_1$ and $6 \in A_2$. If $1 \in B_1 \cup B_2 \cup B_3$ or $2 \in B_4$, then Lemma 2.4 applies. So suppose $1 \in B_4$, and we may assume that $2 \in B_1$. Now we have

$$e = (\{\dots, 3, 4, 5, 1\}, 1\rangle(\{\dots, 6, 2\}, 2\rangle,$$
$$f = (\{\dots, 2, 3\}, 3\rangle(\{\dots, 4\}, 4\rangle(\{\dots, 5\}, 5\rangle(\{\dots, 1, 6\}, 6\rangle.$$

We define $g \in T(X)$ as follows. Set $xg = x$ for every $x \in \{1, 2, 3, 6\}$, $4g = 3$, and $5g = 3$. Define $7g = 3$ if $7 \in A_1$ and $7 \in B_1 \cup B_2 \cup B_3$; $7g = 1$ if $7 \in A_1$ and $7 \in B_4$; $7g = 2$ if $7 \in A_2$ and $7 \in B_1 \cup B_2 \cup B_3$; and $7g = 6$ if $7 \in A_2$ and $7 \in B_4$. Then g is an idempotent of rank 4 and $e - g - f$. The argument in the case when $3, 4 \in A_1$ and $5, 6 \in A_2$ is similar.

Case 5. $\operatorname{rank}(e) = 3$ and $\operatorname{rank}(f) = 3$.

Since both e and f have an element in their range whose preimage is the singleton, we may assume that $e = (A_1, 1\rangle(A_2, 2\rangle(\{3\}, 3\rangle$ and $f = (B_1, 4\rangle(B_2, 5\rangle(\{6\}, 6\rangle$. If $\{1, 2\} \subseteq B_i$ or $\{4, 5\} \subseteq A_i$ for some i, then Lemma 2.4 applies. Otherwise, we may assume that $4 \in A_1$ and $5 \in A_2$. If $1 \in B_1$ or $2 \in B_2$, then Lemma 2.4 applies again. So suppose $1 \in B_2$ and $2 \in B_1$. So now

$$e = (\{\dots, 4, 1\}, 1\rangle(\{\dots, 5, 2\}, 2\rangle(\{3\}, 3\rangle \quad \text{and} \quad f = (\{\dots, 2, 4\}, 4\rangle(\{\dots, 1, 5\}, 5\rangle(\{6\}, 6\rangle.$$

We define $g \in T(X)$ as follows. Set $xg = x$ for every $x \in \{1, 2, 4, 5\}$, $3g = 1$, and $6g = 4$. Define $7g = 4$ if $7 \in A_1$ and $7 \in B_1$; $7g = 1$ if $7 \in A_1$ and $7 \in B_2$; $7g = 2$ if $7 \in A_2$ and $7 \in B_1$; and $7g = 5$ if $7 \in A_2$ and $7 \in B_2$. Then g is an idempotent of rank 4 and $e - g - f$.

Case 6. $\operatorname{rank}(e) = 3$ and $\operatorname{rank}(f) = 4$.

We may assume that $e = (A_1, 1\rangle(A_2, 2\rangle(\{3\}, 3\rangle$ and $f = (B_1, 4\rangle(B_2, 5\rangle(B_3, 6\rangle(\{7\}, 7\rangle$. If $\{4, 5, 6\} \subseteq A_1$ or $\{4, 5, 6\} \subseteq A_2$, then Lemma 2.4 applies. So we may assume that $4, 5 \in A_1$ and $6 \in A_2$. If $1 \in B_1 \cup B_2$ or $2 \in B_3$, then Lemma 2.4 applies again. So we may assume that $1 \in B_3$ and $2 \in B_1$. So now

$$e = (\{\dots, 4, 5, 1\}, 1\rangle(\{\dots, 6, 2\}, 2\rangle(\{3\}, 3\rangle,$$
$$f = (\{\dots, 2, 4\}, 4\rangle(\{\dots, 5\}, 5\rangle(\{\dots, 1, 6\}, 6\rangle(\{7\}, 7\rangle.$$

We define $g \in T(X)$ as follows. Set $xg = x$ for every $x \in \{1, 2, 4, 6\}$ and $5g = 4$. Define $7g = 4$ if $7 \in A_1$; $7g = 6$ if $7 \in A_2$; $3g = 3$ if $3 \in B_1 \cup A_2$; and $3g = 1$ if $3 \in B_3$. Then g is an idempotent of rank 4 and $e - g - f$. $\square$

Theorem 2.17. *Let $n \geq 3$ and let J_r be an ideal in $T(X)$ such that $2 \leq r < n$. Then:*

(1) *If $n = 3$ or $n \in \{5, 6, 7\}$ and $r = 4$, then the diameter of $\mathcal{G}(J_r)$ is 4.*

(2) *In all other cases, the diameter of $\mathcal{G}(J_r)$ is 5.*

Proof. Let $n = 3$. Then the diameter of $\mathcal{G}(J_2)$ is at most 4 by Lemma 2.1 and Theorem 2.8. On the other hand, consider $a = (3\,1\rangle(1\,2)$ and $b = (2\,1\rangle(1\,3)$ in J_2. Suppose e and f are idempotents in J_2 such that $a - e$ and $f - b$. By Lemma 2.12, $e = (\{1\}, 1\rangle(\{3, 2\}, 2\rangle$ and $f = (\{1\}, 1\rangle(\{2, 3\}, 3\rangle$. Then e and f do not commute, and so $d(e, f) \geq 2$. Thus $d(a, b) \geq 4$ by Lemma 2.1, and so the diameter of $\mathcal{G}(J_2)$ is at least 4.

Let $n \in \{5, 6, 7\}$ and $r = 4$. If $n = 5$, then the diameter of $\mathcal{G}(J_4)$ is at least 4 (by Lemma 2.14) and at most 4 (by Lemma 2.1 and Theorem 2.8). If $n \in \{6, 7\}$, then the diameter of $\mathcal{G}(J_4)$ is at least 4 (by Lemma 2.14) and at most 4 (by Lemma 2.16). We have proved (1).

Let $n \geq 4$ and suppose that $n \notin \{5, 6, 7\}$ or $r \neq 4$. Then the diameter of $\mathcal{G}(J_r)$ is at most 5 by Lemma 2.1 and Theorem 2.8. It remains to find $a, b \in J_r$ such that the distance between a and b in $\mathcal{G}(J_r)$ is at least 5. We consider four possible cases.

Case 1. $r = 2m - 1$ for some $m \geq 2$.

Then $2 \leq m < r < 2m \leq n$. Let $x_1, \ldots, x_m, y_1, \ldots, y_m$ be pairwise distinct elements of X. Let

$$a = (*\,y_2\rangle(y_1\,y_2 \ldots y_m\,x_1\rangle(x_1\,x_2 \ldots x_m) \quad \text{and} \quad b = (*\,x_3\rangle(x_2\,x_3 \ldots x_{m-1}\,x_1\,y_1\rangle(y_1\,y_2 \ldots y_m)$$

(see Notation 2.11) and note that $a, b \in J_r$ and $ab \neq ba$. Then, by Lemma 2.1, there are idempotents $e_1, \ldots, e_k \in J_r$ $(k \geq 1)$ such that $a - e_1 - \cdots - e_k - b$ is a minimal path in $\mathcal{G}(J_r)$ from a to b. By Lemma 2.12,

$$e_1 = (A_1, x_1\rangle(A_2, x_2\rangle \cdots (A_m, x_m\rangle \quad \text{and} \quad e_k = (B_1, y_1\rangle(B_2, y_2\rangle \cdots (B_m, y_m\rangle,$$

where $y_i \in A_i$ $(1 \leq i \leq m)$, $x_{i+1} \in B_i$ $(1 \leq i < m)$, and $x_1 \in B_m$. Let $g \in T(X)$ be an idempotent such that $e_1 - g - e_k$. By Lemma 2.6, $x_j g = x_j$ and $y_j g = y_j$ for every $j \in \{1, \ldots, m\}$. Hence $\operatorname{rank}(g) \geq 2m > r$, and so $g \notin J_r$. It follows that the distance between e_1 and e_k is at least 3, and so the distance between a and b is at least 5.

Case 2. $r = 2m$ for some $m \geq 3$.

Then $3 \leq m < r = 2m < n$. Let $x_1, \ldots, x_m, y_1, \ldots, y_m, z$ be pairwise distinct elements of X. Let

$$a = (*\,y_2\rangle(z\,y_1\,y_2 \ldots y_m\,x_1\rangle(x_1\,x_2 \ldots x_m),$$
$$b = (*\,x_1\rangle(z\,x_3\rangle(x_2\,x_3 \ldots x_m\,x_1\,y_1\rangle(y_1\,y_2 \ldots y_m)$$

(see Notation 2.11) and note that $a, b \in J_r$ and $ab \neq ba$. Then, by Lemma 2.1, there are idempotents $e_1, \ldots, e_k \in J_r$ $(k \geq 1)$ such that $a - e_1 - \cdots - e_k - b$ is a minimal path in $\mathcal{G}(J_r)$ from a to b. By Lemma 2.12,

$$e_1 = (A_1, x_1\rangle(A_2, x_2\rangle \cdots (A_m, x_m\rangle \quad \text{and} \quad e_k = (B_1, y_1\rangle(B_2, y_2\rangle \cdots (B_m, y_m\rangle,$$

where $y_i \in A_i$ $(1 \leq i \leq m)$, $x_{i+1} \in B_i$ $(1 \leq i < m)$, $x_1 \in B_m$, $A_m = \{x_m, y_m, z\}$, and $B_1 = \{y_1, x_2, z\}$. Let $g \in T(X)$ be an idempotent such that $e_1 - g - e_k$. By Lemma 2.6, $x_j g = x_j$ and $y_j g = y_j$ for every $j \in \{1, \ldots, m\}$, and $zg = z$. Hence $\operatorname{rank}(g) \geq 2m + 1 > r$, and so $g \notin J_r$. It follows that the distance between e_1 and e_k is at least 3, and so the distance between a and b is at least 5.

Case 3. $r = 4$.

Since we are working under the assumption that $n \notin \{5, 6, 7\}$ or $r \neq 4$, we have $n \notin \{5, 6, 7\}$. Thus $n \geq 8$ (since $r \leq n - 1$). Let

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots & n \\ 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 1 & \cdots & 1 \end{pmatrix} \quad \text{and}$$
$$b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots & n \\ 5 & 6 & 7 & 8 & 6 & 7 & 8 & 5 & 5 & \cdots & 5 \end{pmatrix}.$$

Note that $a, b \in J_4$, $ab \neq ba$, $(1\,2\,3\,4)$ is a unique cycle in a, and $(5\,6\,7\,8)$ is a unique cycle in b. By Lemma 2.1, there are idempotents $e_1, \ldots, e_k \in J_4$ $(k \geq 1)$ such that $a - e_1 - \cdots - e_k - b$ is a minimal path in $\mathcal{G}(J_4)$ from a to b. By Lemma 2.10, $ie_1 = i$ and $(4 + i)e_k = 4 + i$ for every $i \in \{1, 2, 3, 4\}$. By Lemma 2.9, $5e_1 = 1$ or $5e_1 = 5$. But the latter is impossible since with $5e_1 = 5$ we would have $\operatorname{rank}(e_1) \geq 5$. Similarly, we obtain $6e_1 = 2$, $7e_1 = 3$, $8e_1 = 4$, $2e_k = 5$, $3e_k = 6$, $4e_k = 7$, and $1e_k = 8$. Let $g \in T(X)$ be an idempotent such that $e_1 - g - e_k$. By Lemma 2.6, $jg = j$ for every $j \in \{1, \ldots, 8\}$.

Hence $\operatorname{rank}(g) \geq 8 > r$, and so $g \notin J_4$. It follows that the distance between e_1 and e_k is at least 3, and so the distance between a and b is at least 5.

Case 4. $r = 2$.

In this case we let

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 2 & 1 & 2 & 1 & 1 & \cdots & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 3 & 4 & 4 & 3 & 3 & \cdots & 3 \end{pmatrix}.$$

Note that $a, b \in J_2$, $ab \neq ba$, $(1\,2)$ is a unique cycle in a, and $(3\,4)$ is a unique cycle in b. By Lemma 2.1, there are idempotents $e_1, \ldots, e_k \in J_2$ $(k \geq 1)$ such that $a - e_1 - \cdots - e_k - b$ is a minimal path in $\mathcal{G}(J_2)$ from a to b. By Lemma 2.10, $1e_1 = 1$, $2e_1 = 2$, $3e_k = 3$, and $4e_k = 4$. By Lemma 2.9, $3e_1 = 1$ or $3e_1 = 3$. But the latter is impossible since with $3e_1 = 3$ we would have $\operatorname{rank}(e_1) \geq 3$. Again, by Lemma 2.9, $4e_1 = 2$ or $4e_1 = y$ for some $y \in \{4, 5, \ldots, n\}$. But the latter is impossible since we would have $ye_1 = y$ and again $\operatorname{rank}(e_1)$ would be at least 3. Similarly, we obtain $2e_k = 3$, and $1e_k = 4$. Let $g \in T(X)$ be an idempotent such that $e_1 - g - e_k$. By Lemma 2.6, $jg = j$ for every $j \in \{1, \ldots, 4\}$. Hence $\operatorname{rank}(g) \geq 4 > r$, and so $g \notin J_2$. It follows that the distance between e_1 and e_k is at least 3, and so the distance between a and b is at least 5.

Thus the diameter of $\mathcal{G}(J_r)$ is at least 5, which concludes the proof of (2). □

2.3. *The commuting graph of $T(X)$*

Let X be a finite set with $|X| = n$. It has been proved in [8, Theorem 3.1] that if n and $n - 1$ are not prime, then the diameter of the commuting graph of $\operatorname{Sym}(X)$ is at most 5, and that the bound is sharp since the diameter of $\mathcal{G}(\operatorname{Sym}(X))$ is 5 when $n = 9$. In this subsection, we determine the exact value of the diameter of the commuting graph of $T(X)$ for every $n \geq 2$.

Throughout this subsection, we assume that X is a finite set with $n \geq 2$ elements.

Lemma 2.18. *Let $n \geq 4$ be composite. Let $a, f \in T(X)$ such that $a, f \neq \mathrm{id}_X$, $a \in \operatorname{Sym}(X)$, and f is an idempotent. Then $d(a, f) \leq 4$.*

Proof. Fix $x \in \operatorname{im}(f)$ and a cycle $(x_1 \ldots x_m)$ of a such that $x \in \{x_1, \ldots, x_m\}$. Consider three cases.

Case 1. a has a cycle $(y_1 \ldots y_k)$ such that k does not divide m.

Then a^m is different from id_X and it fixes x. Thus $a - a^m - (X, x\rangle - f$, and so $d(a, f) \leq 3$.

Case 2. a has at least two cycles and for every cycle $(y_1 \ldots y_k)$ of a, k divides m.

Suppose there is $z \in \operatorname{im}(f)$ such that $z \in \{y_1, \ldots, y_k\}$ for some cycle $(y_1 \ldots y_k)$ of a different from $(x_1 \ldots x_m)$. Since k divides m, there is a positive integer t such that $m = tk$. Define $e \in T(X)$ by:

$$x_1e = y_1, \ldots, x_ke = y_k, \quad x_{k+1}e = y_1, \ldots, x_{2k}e = y_k, \ldots, x_{(t-1)k+1}e = y_1, \ldots, x_{tk}e = y_k, \tag{2.4}$$

and $ye = y$ for all other $y \in X$. Then e is an idempotent such that $ae = ea$ and $z \in \operatorname{im}(e)$. Thus, by Lemma 2.3, $a - e - (X, z\rangle - f$, and so $d(a, f) \leq 3$.

Suppose that $\operatorname{im}(f) \subseteq \{x_1, \ldots, x_m\}$. Consider any cycle $(y_1 \ldots y_k)$ of a different from $(x_1 \ldots x_m)$. Since $\operatorname{im}(f) \subseteq \{x_1, \ldots, x_m\}$, $y_1 f = x_i$ for some i. We may assume that $y_1 f = x_1$. Define an idempotent e exactly as in (2.4). Then $\operatorname{im}(e) \cap \operatorname{im}(f) = \emptyset$, $(y_1, x_1) \in \operatorname{im}(e) \times \operatorname{im}(f)$, and $(y_1, x_1) \in \ker(e) \cap \ker(f)$. Thus, by Lemma 2.4, there is an idempotent $g \in T(X) - \{\mathrm{id}_X\}$ such that $e - g - f$. Hence $a - e - g - f$, and so $d(a, f) \leq 3$.

Case 3. a is an n-cycle.

Since n is composite, there is a divisor k of n such that $1 < k < n$. Then $a^k \neq \mathrm{id}_X$ is a permutation with $k \geq 2$ cycles, each of length $m = n/k$. By Case 2, $d(a^k, f) \leq 3$, and so $d(a, f) \leq 4$. □

Lemma 2.19. *Let $n \geq 4$ be composite. Let $a, b \in T(X)$ such that $a, b \neq \mathrm{id}_X$ and $a \in \operatorname{Sym}(X)$. Then $d(a, b) \leq 5$.*

Proof. Suppose $b \notin \operatorname{Sym}(X)$. Then b^k is an idempotent different from id_X for some $k \geq 1$. By Lemma 2.18, $d(a, b^k) \leq 4$, and so $d(a, b) \leq 5$.

Suppose $b \in \mathrm{Sym}(X)$. Suppose $n-1$ is not prime. Then, by [8, Theorem 3.1], there is a path from a to b in $\mathcal{G}(\mathrm{Sym}(X))$ of length at most 5. Such a path is also a path in $\mathcal{G}(T(X))$, and so $d(a,b) \le 5$. Suppose $p = n-1$ is prime. Then the proof of [8, Theorem 3.1] still works for a and b unless $a^p = \mathrm{id}_X$ or $b^p = \mathrm{id}_X$. (See also [8, Lemma 3.3] and its proof.) Thus, if $a^p \ne \mathrm{id}_X$ and $b^p \ne \mathrm{id}_X$, then there is a path from a to b in $\mathcal{G}(\mathrm{Sym}(X))$ of length at most 5, and so $d(a,b) \le 5$. Suppose $a^p = \mathrm{id}_X$ or $b^p = \mathrm{id}_X$. We may assume that $b^p = \mathrm{id}_X$. Then b is a cycle of length p, that is, $b = (x_1 \dots x_p)(x)$. Thus b commutes with the constant idempotent $f = (X, x\rangle$. By Lemma 2.18, $d(a,f) \le 4$, and so $d(a,b) \le 5$. $\square$

Lemma 2.20. *Let $X = \{x_1,\dots,x_m,y_1,\dots,y_k\}$, $a \in \mathrm{Sym}(X)$, and $b = (y_1 \dots y_k\, x_1\rangle(x_1 \dots x_m)$. If $ab = ba$ then $a = \mathrm{id}_X$.*

Proof. Suppose $ab = ba$. By Lemma 2.9,

$$x_1a \xrightarrow{b} x_2a \xrightarrow{b} \cdots \xrightarrow{b} x_ma \xrightarrow{b} x_1a \quad \text{and} \quad y_1a \xrightarrow{b} y_2a \xrightarrow{b} \cdots \xrightarrow{b} y_ka \xrightarrow{b} x_1a. \tag{2.5}$$

Since $(x_1\, x_2 \dots x_m)$ is a unique cycle in b, (2.5) implies that

$$x_1a = x_q, \quad x_2a = x_{q+1}, \dots, x_ma = x_{q+m-1}, \tag{2.6}$$

where $q \in \{1,\dots,m\}$ ($x_{q+i} = x_{q+i-m}$ if $q+i > m$). Thus $x_1a = x_j$ for some j. Since $y_k \xrightarrow{b} x_1$ and $x_m \xrightarrow{b} x_1$, we have $y_ka \xrightarrow{b} x_1a = x_j$ and $x_ma \xrightarrow{b} x_1a = x_j$. Suppose $j \ge 2$. Then $x_jb^{-1} = \{x_{j-1}\}$, and so $y_ka = x_{j-1} = x_ma$. But this implies $y_k = x_m$ (since a is injective), which is a contradiction. Hence $j = 1$, and so $x_1a = x_1$. But then $x_ia = x_i$ for all i by (2.6).

Since $y_ka \xrightarrow{b} x_1a = x_1$, we have $y_ka = y_k$ since $x_1b^{-1} = \{y_k, x_m\}$. Let $i \in \{1,\dots,k-1\}$ and suppose $y_{i+1}a = y_{i+1}$. Then $y_ia = y_i$ since $y_ia \xrightarrow{b} y_{i+1}a = y_{i+1}$ and $y_{i+1}b^{-1} = \{y_{i+1}\}$. It follows that $y_ia = y_i$ for all $i \in \{1,\dots,k\}$. $\square$

Lemma 2.21. *Let m be a positive integer such that $2m \le n$, σ be an m-cycle on $\{1,\dots,m\}$, $a \in \mathrm{Sym}(X)$, and*

$$e = (A_1, x_1\rangle(A_2, x_2\rangle \cdots (A_m, x_m\rangle \quad \text{and} \quad f = (B_1, y_1\rangle(B_2, y_2\rangle \cdots (B_m, y_m\rangle$$

be idempotents in $T(X)$ such that $x_1,\dots,x_m,y_1,\dots,y_m$ are pairwise distinct, $y_i \in A_i$, and $x_{i\sigma} \in B_i$ ($1 \le i \le m$). Then:

(1) *Suppose $X = \{x_1,\dots,x_m,y_1,\dots,y_m,z\}$ and $z \in A_i \cap B_j$ such that $A_i \cap B_j = \{z\}$. If $e - a - f$, then $a = \mathrm{id}_X$.*
(2) *Suppose $X = \{x_1,\dots,x_m,y_1,\dots,y_m,z,w\}$, $z \in A_i \cap B_j$ such that $A_i \cap B_j = \{z\}$, and $w \in A_s \cap B_t$ such that $A_s \cap B_t = \{w\}$, where $s \ne i$ and $t \ne j$. If $e - a - f$, then $a = \mathrm{id}_X$.*

Proof. To prove (1), suppose $e-a-f$ and note that $A_i = \{x_i, y_i, z\}$ and $B_j = \{y_j, x_{j\sigma}, z\}$. By Lemma 2.2, there is $p \in \{1,\dots,m\}$ such that $x_ia = x_p$ and $A_ia \subseteq A_p$. Suppose $p \ne i$. Then $A_p = \{x_p, y_p\}$, and so A_ia cannot be a subset of A_p since a is injective. It follows that $p = i$, that is, $x_ia = x_i$ and $A_ia \subseteq A_i$. Similarly, $y_ja = y_j$ and $B_ja \subseteq B_j$. Thus $za \in A_i \cap B_j = \{z\}$, and so $za = z$. Hence, since a is injective, $y_ia = y_i$.

We have proved that $x_ia = x_i$, $y_ia = y_i$, and $za = z$. We have $B_i = \{y_i, x_{i\sigma}\}$ or $B_i = \{y_i, x_{i\sigma}, z\}$. Since $y_ia = y_i$, we have $B_ia \subseteq B_i$ by Lemma 2.2. Since $za = z$ and a is injective, it follows that $x_{i\sigma}a = x_{i\sigma}$. By the foregoing argument applied to $A_{i\sigma} = \{x_{i\sigma}, y_{i\sigma}\}$, we obtain $y_{i\sigma}a = y_{i\sigma}$. Continuing this way, we obtain $x_{i\sigma^k}a = x_{i\sigma^k}$ and $y_{i\sigma^k}a = y_{i\sigma^k}$ for every $k \in \{1,\dots,m-1\}$. Since σ is an m-cycle, it follows that $x_ja = x_j$ and $y_jg = y_j$ for every $j \in \{1,\dots,m\}$. Hence $a = \mathrm{id}_X$. We have proved (1). The proof of (2) is similar. $\square$

Theorem 2.22. *Let X be a finite set with $n \ge 2$ elements. Then:*

(1) *If n is prime, then $\mathcal{G}(T(X))$ is not connected.*
(2) *If $n = 4$, then the diameter of $\mathcal{G}(T(X))$ is 4.*
(3) *If $n \ge 6$ is composite, then the diameter of $\mathcal{G}(T(X))$ is 5.*

Proof. Suppose $n = p$ is prime. Consider a p-cycle $a = (x_1\, x_2 \ldots x_p)$ and let $b \in T(X)$ be such that $b \neq \mathrm{id}_X$ and $ab = ba$. Let $x_q = x_1 b$. Then, by Lemma 2.9, $x_i b = x_{q+i}$ for every $i \in \{1, \ldots, p\}$ (where $x_{q+i} = x_{q+i-m}$ if $q + i > m$). Thus $b = a^q$, and so, since p is prime, b is also a p-cycle. It follows that if c is a vertex of $\mathcal{G}(T(X))$ that is not a p-cycle, then there is no path in $\mathcal{G}(T(X))$ from a to c. Hence $\mathcal{G}(T(X))$ is not connected. We have proved (1).

We checked the case $n = 4$ directly using GRAPE [16] through GAP [17]. We found that, when $|X| = 4$, the diameter of $\mathcal{G}(T(X))$ is 4.

Suppose $n \geq 6$ is composite. Let $a, b \in T(X)$ such that $a, b \neq \mathrm{id}_X$. If $a \in \mathrm{Sym}(X)$ or $b \in \mathrm{Sym}(X)$, then $d(a, b) \leq 5$ by Lemma 2.19. If $a, b \notin \mathrm{Sym}(X)$, then $a, b \in J_{n-1}$, and so $d(a, b) \leq 5$ by Theorem 2.17. Hence the diameter of $\mathcal{G}(T(X))$ is at most 5. It remains to find $a, b \in T(X) - \{\mathrm{id}_X\}$ such that $d(a, b) \geq 5$.

For $n \in \{6, 8\}$, we employed GAP [17]. When $n = 6$, we found that the distance between the 6-cycle $a = (1\,2\,3\,4\,5\,6)$ and $b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 1 & 2 & 4 \end{pmatrix}$ in $\mathcal{G}(T(X))$ is at least 5. And when $n = 8$, the distance between the 8-cycle $a = (1\,2\,3\,4\,5\,6\,7\,8)$ and $b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 1 & 1 & 4 & 8 & 6 & 5 \end{pmatrix}$ in $\mathcal{G}(T(X))$ is at least 5.

To verify this with GAP, we used the following sequence of arguments and computer calculations:

1. By Lemma 2.1, if there exists a path $a - c_1 - c_2 - \cdots - c_k - b$, then there exists a path $a - e_1 - e_2 - \cdots - e_k - b$, where each e_i is either an idempotent or a permutation;
2. Let E be the set idempotents of $T(X) - \{\mathrm{id}_X\}$ and let $G = \mathrm{Sym}(X) - \{\mathrm{id}_X\}$. For $A \subseteq T(X)$, let $C(A) = \{f \in E \cup G : (\exists_{a \in A}) af = fa\}$;
3. Calculate $C(C(\{a\}))$ and $C(\{b\})$;
4. Verify that for all $c \in C(C(\{a\}))$ and all $d \in C(\{b\})$, $cd \neq dc$;
5. If there were a path $a - c_1 - c_2 - c_3 - b$ from a to b, then we would have $c_2 \in C(C(\{a\}))$, $c_3 \in C(\{b\})$, and $c_2 c_3 = c_3 c_2$. But, by 4., there are no such c_2 and c_3, and it follows that the distance between a and b is at least 5.

Let $n \geq 9$ be composite. We consider two cases.

Case 1. $n = 2m + 1$ is odd ($m \geq 4$).

Let $X = \{x_1, \ldots, x_m, y_1, \ldots, y_m, z\}$. Consider

$$a = (z\, y_1\, y_2 \ldots y_m\, x_1\rangle (x_1\, x_2 \ldots x_m) \quad \text{and} \quad b = (x_2\, x_3 \ldots x_m\, x_1\, z\, y_2\rangle (y_1\, y_2 \ldots y_m).$$

Let λ be a minimal path in $\mathcal{G}(T(X))$ from a to b. By Lemma 2.20, there is no $g \in \mathrm{Sym}(X)$ such that $g \neq \mathrm{id}_X$ and $ag = ga$ or $bg = gb$. Thus, by the proof of Lemma 2.1, $\lambda = a - e_1 - \cdots - e_k - b$, where e_1 and e_k are idempotents. By Lemma 2.12,

$$e_1 = (A_1, x_1\rangle (A_2, x_2\rangle \cdots (A_m, x_m\rangle \quad \text{and} \quad e_k = (B_1, y_1\rangle (B_2, y_2\rangle \cdots (B_m, y_m\rangle,$$

where $y_i \in A_i$ $(1 \leq i \leq m)$, $x_{i+1} \in B_i$ $(1 \leq i < m)$, $x_1 \in B_m$, $A_m = \{x_m, y_m, z\}$, and $B_1 = \{y_1, x_2, z\}$. Since $m \geq 4$, $A_m \cap B_1 = \{z\}$. Thus, by Lemma 2.21, there is no $g \in \mathrm{Sym}(X)$ such that $g \neq \mathrm{id}_X$ and $e_1 - g - e_k$. Hence, if λ contains an element $g \in \mathrm{Sym}(X)$, then the length of λ is at least 5. Suppose λ does not contain any permutations. Then λ is a path in J_{n-1} and we may assume that all vertices in λ except a and b are idempotents (by Lemma 2.12). By Lemma 2.6, there is no idempotent $f \in J_{n-1}$ such that $e_1 - f - e_k$. (Here, the m-cycle that occurs in Lemmas 2.6 and 2.21 is $\sigma = (1\,2 \ldots m)$.) Hence the length of λ is at least 5.

Case 2. $n = 2m + 2$ is even ($m \geq 4$).

Let $X = \{x_1, \ldots, x_m, y_1, \ldots, y_m, z, w\}$. Consider

$$a = (z\, y_1\, y_2 \ldots y_m\, w\, x_2\rangle (x_1\, x_2 \ldots x_m) \quad \text{and} \quad b = (w\, x_2\, x_3 \ldots x_{m-2}\, x_m\, x_1\, x_{m-1}\, y_2\rangle (y_1\, y_2 \ldots y_m).$$

Let λ be a minimal path in $\mathcal{G}(T(X))$ from a to b. By Lemma 2.20, there is no $g \in \mathrm{Sym}(X)$ such that $g \neq \mathrm{id}_X$ and $ag = ga$ or $bg = gb$. Thus, by the proof of Lemma 2.1, $\lambda = a - e_1 - \cdots - e_k - b$, where e_1 and e_k are idempotents. By Lemma 2.12,

$$e_1 = (A_1, x_1\rangle (A_2, x_2\rangle \cdots (A_m, x_m\rangle \quad \text{and} \quad e_k = (B_1, y_1\rangle (B_2, y_2\rangle \cdots (B_m, y_m\rangle,$$

Table 1
Images of the generators.

$\mathrm{im}(a_1)$	y_0	x_1	y_1
$\mathrm{im}(a_2)$	y_1	x_2	y_2
$\mathrm{im}(a_3)$	y_2	x_3	y_3
$\mathrm{im}(a_4)$	y_3	x_4	y_4
$\mathrm{im}(b_1)$	y_4	u_1	v_1
$\mathrm{im}(b_2)$	v_1	u_2	v_2
$\mathrm{im}(b_3)$	v_2	u_3	v_3
$\mathrm{im}(b_4)$	v_3	u_4	v_4
$\mathrm{im}(e_1)$	v_4	r	s

where $y_i \in A_i$ $(1 \le i \le m)$, $x_{i+1} \in B_i$ $(1 \le i \le m-3)$, $x_m \in B_{m-2}$, $x_1 \in B_{m-1}$, $x_{m-1} \in B_m$, $A_1 = \{x_1, y_1, w\}$, $A_m = \{x_m, y_m, z\}$, $B_1 = \{y_1, x_2, z\}$, and $B_m = \{y_m, x_{m-1}, w\}$. Since $m \ge 4$, $A_m \cap B_1 = \{z\}$ and $A_1 \cap B_m = \{w\}$. Thus, by Lemma 2.21, there is no $g \in \mathrm{Sym}(X)$ such that $g \ne \mathrm{id}_X$ and $e_1 - g - e_k$. Hence, as in Case 1, the length of λ is at least 5. (Here, the m-cycle that occurs in Lemmas 2.6 and 2.21 is $\sigma = (1, 2, \ldots, m-3, m-2, m, m-1)$.)

Hence, if $n \ge 6$ is composite, then the diameter of $\mathcal{G}(T(X))$ is 5. This concludes the proof. □

3. Minimal left paths

In this section, we prove that for every integer $n \ge 4$, there is a band S with knit degree n. We will show how to construct such an S as a subsemigroup of $T(X)$ for some finite set X.

Let S be a finite non-commutative semigroup. Recall that a path $a_1 - a_2 - \cdots - a_m$ in $\mathcal{G}(S)$ is called a *left path* (or l-path) if $a_1 \ne a_m$ and $a_1 a_i = a_m a_i$ for every $i \in \{1, \ldots, m\}$. If there is any l-path in $\mathcal{G}(S)$, we define the *knit degree* of S, denoted $\mathrm{kd}(S)$, to be the length of a shortest l-path in $\mathcal{G}(S)$. We say that an l-path λ from a to b in $\mathcal{G}(S)$ is a *minimal l-path* if there is no l-path from a to b that is shorter than λ.

3.1. The even case

In this subsection, we will construct a band of knit degree n where $n \ge 4$ is even. For $x \in X$, we denote by c_x the constant transformation with image $\{x\}$. The following lemma is obvious.

Lemma 3.1. *Let $c_x, c_y, e \in T(X)$ such that e is an idempotent. Then:*
(1) *$c_x e = e c_x$ if and only if $x \in \mathrm{im}(e)$.*
(2) *$c_x e = c_y e$ if and only if $(x, y) \in \ker(e)$.*

Now, given an even $n \ge 4$, we will construct a band S such that $\mathrm{kd}(S) = n$. We will explain the construction using $n = 8$ as an example. The band S will be a subsemigroup of $T(X)$, where

$$X = \{y_0, y_1, y_2, y_3, y_4 = v_0, v_1, v_2, v_3, v_4, x_1, x_2, x_3, x_4, u_1, u_2, u_3, u_4, r, s\},$$

and it will be generated by idempotent transformations $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, e_1$, whose images are defined by Table 1.

We will define the kernels in such a way that the generators with the same subscript will have the same kernel. For example, $\ker(a_1) = \ker(b_1) = \ker(e_1)$ and $\ker(a_2) = \ker(b_2)$. Let $i \in \{2, 3, 4\}$. The kernel of a_i will have the following three classes (elements of the partition $X/\ker(a_i)$):

$$\text{Class-1} = \mathrm{im}(a_{i+1}) \cup \cdots \cup \mathrm{im}(a_4) \cup \mathrm{im}(b_1) \cup \cdots \cup \mathrm{im}(b_{i-1}),$$
$$\text{Class-2} = \mathrm{im}(b_{i+1}) \cup \cdots \cup \mathrm{im}(b_4) \cup \mathrm{im}(e_1) \cup \mathrm{im}(a_1) \cup \cdots \cup \mathrm{im}(a_{i-1}),$$
$$\text{Class-3} = \{x_i, u_i\}.$$

For example, $\ker(a_2)$ has the following classes:

$$\text{Class-1} = \{y_2, x_3, y_3, x_4, y_4, u_1, v_1\},$$
$$\text{Class-2} = \{v_2, u_3, v_3, u_4, v_4, r, s, y_0, x_1, y_1\},$$
$$\text{Class-3} = \{x_2, u_2\}.$$

We define the kernel of a_1 as follows:

$$\begin{aligned}
&\text{Class-1} = \mathrm{im}(a_2) \cup \mathrm{im}(a_3) \cup \mathrm{im}(a_4) \cup \{s\} = \{y_1, x_2, y_2, x_3, y_3, x_4, y_4, s\},\\
&\text{Class-2} = \mathrm{im}(b_2) \cup \mathrm{im}(b_3) \cup \mathrm{im}(b_4) \cup \{y_0\} = \{v_1, u_2, v_2, u_3, v_3, u_4, v_4, y_0\},\\
&\text{Class-3} = \{x_1, u_1, r\}.
\end{aligned}$$

Now the generators are completely defined since $\ker(b_i) = \ker(a_i)$, $1 \le i \le 4$, and $\ker(e_1) = \ker(a_1)$. Order the generators as follows:

$$a_1,\ a_2,\ a_3,\ a_4,\ b_1,\ b_2,\ b_3,\ b_4,\ e_1. \tag{3.1}$$

Let S be the semigroup generated by the idempotents listed in (3.1). Since the idempotents with the same subscript have the same kernel, they form a right-zero subsemigroup of S. For example, $\{a_1, b_1, e_1\}$ is a right-zero semigroup: $a_1a_1 = b_1a_1 = e_1a_1 = a_1$, $a_1b_1 = b_1b_1 = e_1b_1 = b_1$, and $a_1e_1 = b_1e_1 = e_1e_1 = e_1$. The product of any two generators with different subscripts is a constant transformation. For example, $a_2a_4 = c_{y_3}$, $a_4a_2 = c_{y_2}$, and $a_1b_3 = c_{v_3}$. The semigroup S consists of the nine generators listed in (3.1) and 10 constants:

$$S = \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, e_1, c_{y_0}, c_{y_1}, c_{y_2}, c_{y_3}, c_{y_4}, c_{v_1}, c_{v_2}, c_{v_3}, c_{v_4}, c_s\},$$

so S is a band. Note that $Z(S) = \emptyset$. Each idempotent in (3.1) commutes with the next idempotent, so $a_1 - a_2 - a_3 - a_4 - b_1 - b_2 - b_3 - b_4 - e_1$ is a path in $\mathcal{G}(S)$. Moreover, it is a unique l-path in $\mathcal{G}(S)$, so $\mathrm{kd}(S) = 8$.

We will now provide a general construction of a band S such that $\mathrm{kd}(S) = n$, where n is even.

Definition 3.2. Let $k \ge 2$ be an integer. Let

$$X = \{y_0, y_1, \ldots, y_k = v_0, v_1, \ldots, v_k, x_1, \ldots, x_k, u_1, \ldots, u_k, r, s\}.$$

We will define idempotents $a_1, \ldots, a_k, b_1, \ldots, b_k, e_1$ as follows. For $i \in \{1, \ldots, k\}$, let

$$\begin{aligned}
&\mathrm{im}(a_i) = \{y_{i-1}, x_i, y_i\},\\
&\mathrm{im}(b_i) = \{v_{i-1}, u_i, v_i\},\\
&\mathrm{im}(e_1) = \{v_k, r, s\}.
\end{aligned}$$

For $i \in \{2, \ldots, k\}$, define the $\ker(a_i)$-classes by:

$$\begin{aligned}
&\text{Class-1} = \mathrm{im}(a_{i+1}) \cup \cdots \cup \mathrm{im}(a_k) \cup \mathrm{im}(b_1) \cup \cdots \cup \mathrm{im}(b_{i-1}),\\
&\text{Class-2} = \mathrm{im}(b_{i+1}) \cup \cdots \cup \mathrm{im}(b_k) \cup \mathrm{im}(e_1) \cup \mathrm{im}(a_1) \cup \cdots \cup \mathrm{im}(a_{i-1}),\\
&\text{Class-3} = \{x_i, u_i\}.
\end{aligned}$$

(Note that for $i = k$, Class-1 $= \mathrm{im}(b_1) \cup \cdots \cup \mathrm{im}(b_{k-1})$ and Class-2 $= \mathrm{im}(e_1) \cup \mathrm{im}(a_1) \cup \cdots \cup \mathrm{im}(a_{i-1})$.)

Define the $\ker(a_1)$-classes by:

$$\begin{aligned}
&\text{Class-1} = \mathrm{im}(a_2) \cup \cdots \cup \mathrm{im}(a_k) \cup \{s\},\\
&\text{Class-2} = \mathrm{im}(b_2) \cup \cdots \cup \mathrm{im}(b_k) \cup \{y_0\},\\
&\text{Class-3} = \{x_1, u_1, r\}.
\end{aligned}$$

Let $\ker(b_i) = \ker(a_i)$ for every $i \in \{1, \ldots, k\}$, and $\ker(e_1) = \ker(a_1)$. Now, define the subsemigroup S_0^k of $T(X)$ by:

$$S_0^k = \text{the semigroup generated by } \{a_1, \ldots, a_k, b_1, \ldots, b_k, e_1\}. \tag{3.2}$$

We must argue that the idempotents $a_1, \ldots, a_k, b_1, \ldots, b_k, e_1$ are well defined, that is, for each of them, different elements of the image lie in different kernel classes. Consider a_i, where $i \in \{2, \ldots, k\}$. Then $\mathrm{im}(a_i) = \{y_{i-1}, x_i, y_i\}$. Thus y_i lies in Class-1 (see Definition 3.2) since $y_i \in \mathrm{im}(a_{i+1})$ (or $y_i \in \mathrm{im}(b_1)$ if $i = k$), y_{i-1} lies in Class-2 since $y_{i-1} \in \mathrm{im}(a_{i-1})$, and x_i lies in Class-3. Arguments for the remaining idempotents are similar.

For the remainder of this subsection, S_0^k will be the semigroup (3.2). Our objective is to prove that S_0^k is a band such that $\pi = a_1 - \cdots - a_k - b_1 - \cdots - b_k - e_1$ is a shortest l-path in S_0^k. Since π has length $2k = n$, it will follow that S_0^k is a band with knit degree n.

We first analyze products of the generators of S_0^k.

Lemma 3.3. *Let $1 \le i < j \le k$. Then:*

(1) $a_ib_i = b_i$, $b_ia_i = a_i$, $a_1e_1 = b_1e_1 = e_1$, $e_1a_1 = b_1a_1 = a_1$, *and* $e_1b_1 = a_1b_1 = b_1$.
(2) $a_ia_j = c_{y_{j-1}}$ *and* $a_ja_i = c_{y_i}$.
(3) $a_ib_j = c_{v_j}$ *and* $a_jb_i = c_{v_{i-1}}$.
(4) $b_ia_j = c_{y_j}$ *and* $b_ja_i = c_{y_{i-1}}$.
(5) $b_ib_j = c_{v_{j-1}}$ *and* $b_jb_i = c_{v_i}$.
(6) $e_1a_j = c_{y_{j-1}}$ *and* $a_je_1 = c_s$.
(7) $e_1b_j = c_{v_j}$ *and* $b_je_1 = c_{v_k}$.

Proof. Statement (1) is true because the generators of S_0^k are idempotents and the ones with the same subscript have the same kernel. By Definition 3.2, Class-2 of $\ker(a_j)$ contains both $\mathrm{im}(a_{j-1}) = \{y_{j-2}, x_{j-1}, y_{j-1}\}$ and $\mathrm{im}(a_i)$ (since $i < j$). Since $y_{j-1} \in \mathrm{im}(a_j) = \{y_{j-1}, x_j, y_j\}$, a_j maps all elements of Class-2 to y_{j-1}. Hence $a_ia_j = c_{y_{j-1}}$. Similarly, since $i < j$, Class-1 of $\ker(a_i)$ contains both $\mathrm{im}(a_{i+1}) = \{y_i, x_{i+1}, y_{i+1}\}$ and $\mathrm{im}(a_j)$. Since $y_i \in \mathrm{im}(a_i) = \{y_{i-1}, x_i, y_i\}$, a_i maps all elements of Class-1 to y_i. Hence $a_ja_i = c_{y_i}$. We have proved (2). Proofs of (3)–(7) are similar. For example, $b_je_1 = c_{v_k}$ because Class-2 of $\ker(e_1) = \ker(a_1)$ contains both $\mathrm{im}(b_j)$ and $\mathrm{im}(b_k) = \{v_{k-1}, u_k, v_k\}$, and $v_k \in \mathrm{im}(e_1)$. □

The following corollaries are immediate consequences of Lemma 3.3.

Corollary 3.4. *The semigroup S_0^k is a band. It consists of $2k + 1$ generators from Definition 3.2 and $2k + 2$ constant transformations:*

$$S_0^k = \{a_1, \dots, a_k, b_1, \dots, b_k, e_1, c_{y_0}, c_{y_1}, \dots, c_{y_k}, c_{v_1}, \dots, c_{v_k}, c_s\}.$$

Corollary 3.5. *Let $g, h \in S_0^k$ be generators from the list*

$$a_1, \dots, a_k, b_1, \dots, b_k, e_1. \tag{3.3}$$

Then $gh = hg$ if and only if g and h are consecutive elements in the list.

Lemma 3.3 gives a partial multiplication table for S_0^k. The following lemma completes the table.

Lemma 3.6. *Let $1 \le p \le k$ and $1 \le i < j \le k$. Then:*

(1) $c_{y_p}a_p = c_{y_p}$, $c_{y_p}b_p = c_{v_{p-1}}$, $c_{y_i}a_j = c_{y_{j-1}}$, $c_{y_j}a_i = c_{y_i}$, $c_{y_i}b_j = c_{v_j}$, $c_{y_j}b_i = c_{v_{i-1}}$, $c_{y_p}e_1 = c_s$, $c_{y_0}a_p = c_{y_{p-1}}$, $c_{y_0}b_p = c_{v_p}$, *and* $c_{y_0}e_1 = c_{v_k}$.
(2) $c_{v_p}a_p = c_{y_{p-1}}$, $c_{v_p}b_p = c_{v_p}$, $c_{v_i}a_j = c_{y_j}$, $c_{v_j}a_i = c_{y_{i-1}}$, $c_{v_i}b_j = c_{v_{j-1}}$, $c_{v_j}b_i = c_{v_i}$, *and* $c_{v_p}e_1 = c_{v_k}$.
(3) $c_sa_j = c_{y_{j-1}}$, $c_sb_j = c_{v_j}$, $c_sa_1 = c_{y_1}$, $c_sb_1 = c_{v_0}$, *and* $c_se_1 = c_s$.

Proof. We have $c_{y_p}a_p = c_{y_p}$ since $y_p \in \mathrm{im}(a_p)$. By Definition 3.2, Class-1 of $\ker(b_p)$ contains both $\mathrm{im}(a_{p+1})$ and $\mathrm{im}(b_{p-1})$. Since $y_p \in \mathrm{im}(a_{p+1})$ and $v_{p-1} \in \mathrm{im}(b_{p-1})$, both y_p and v_{p-1} are in Class-1. Hence $y_pb_p = v_{p-1}b_p = v_{p-1}$, where the last equality is true because $v_{p-1} \in \mathrm{im}(b_p)$. Thus $c_{y_p}b_p = c_{v_{p-1}}$. By Definition 3.2, y_p and s belong to Class-1 of $\ker(e_1)$, and $s \in \mathrm{im}(e_1)$. It follows that $c_{y_p}e_1 = c_s$. Again by Definition 3.2, y_0 and y_{p-1} belong to Class-2 of $\ker(a_p)$, and $y_{p-1} \in \mathrm{im}(a_p)$. Hence $c_{y_0}a_p = c_{y_{p-1}}$. Similarly, $c_{y_0}b_p = c_{v_p}$ and $c_{y_0}e_1 = c_{v_k}$. By Lemma 3.3,

$$c_{y_i}a_j = (c_{y_i}a_i)a_j = c_{y_i}(a_ia_j) = c_{y_i}c_{y_{j-1}} = c_{y_{j-1}},$$
$$c_{y_j}a_i = (c_{y_j}a_j)a_i = c_{y_j}(a_ja_i) = c_{y_j}c_{y_i} = c_{y_i},$$
$$c_{y_i}b_j = (c_{y_i}a_i)b_j = c_{y_i}(a_ib_j) = c_{y_i}c_{v_j} = c_{v_j},$$
$$c_{y_j}b_i = (c_{y_j}a_j)b_i = c_{y_j}(a_jb_i) = c_{y_j}c_{v_{i-1}} = c_{v_{i-1}}.$$

We have proved (1). Proofs of (2) and (3) are similar. □

Table 2
Cayley table for S_0^2.

	a_1	a_2	b_1	b_2	e_1	c_{y_0}	c_{y_1}	c_{y_2}	c_{v_1}	c_{v_2}	c_s
a_1	a_1	c_{y_1}	b_1	c_{v_2}	e_1	c_{y_0}	c_{y_1}	c_{y_2}	c_{v_1}	c_{v_2}	c_s
a_2	c_{y_1}	a_2	c_{y_2}	b_2	c_s	c_{y_0}	c_{y_1}	c_{y_2}	c_{v_1}	c_{v_2}	c_s
b_1	a_1	c_{y_2}	b_1	c_{v_1}	e_1	c_{y_0}	c_{y_1}	c_{y_2}	c_{v_1}	c_{v_2}	c_s
b_2	c_{y_0}	a_2	c_{v_1}	b_2	c_{v_2}	c_{y_0}	c_{y_1}	c_{y_2}	c_{v_1}	c_{v_2}	c_s
e_1	a_1	c_{y_1}	b_1	c_{v_2}	e_1	c_{y_0}	c_{y_1}	c_{y_2}	c_{v_1}	c_{v_2}	c_s
c_{y_0}	c_{y_0}	c_{y_1}	c_{v_1}	c_{v_2}	c_{v_2}	c_{y_0}	c_{y_1}	c_{y_2}	c_{v_1}	c_{v_2}	c_s
c_{y_1}	c_{y_1}	c_{y_1}	c_{y_2}	c_{v_2}	c_s	c_{y_0}	c_{y_1}	c_{y_2}	c_{v_1}	c_{v_2}	c_s
c_{y_2}	c_{y_1}	c_{y_2}	c_{y_2}	c_{v_1}	c_s	c_{y_0}	c_{y_1}	c_{y_2}	c_{v_1}	c_{v_2}	c_s
c_{v_1}	c_{y_0}	c_{y_2}	c_{v_1}	c_{v_1}	c_{v_2}	c_{y_0}	c_{y_1}	c_{y_2}	c_{v_1}	c_{v_2}	c_s
c_{v_2}	c_{y_0}	c_{y_1}	c_{v_1}	c_{v_2}	c_{v_2}	c_{y_0}	c_{y_1}	c_{y_2}	c_{v_1}	c_{v_2}	c_s
c_s	c_{y_1}	c_{y_1}	c_{y_2}	c_{v_2}	c_s	c_{y_0}	c_{y_1}	c_{y_2}	c_{v_1}	c_{v_2}	c_s

Table 2 presents the Cayley table for S_0^2.

Lemma 3.7. *Let $g, h, c_z \in S_0^k$ such that c_z is a constant and $g - c_z - h$ is a path in $\mathcal{G}(S_0^k)$. Then $gh = hg$.*

Proof. Note that g, h are not constants since different constants do not commute. Thus g and h are generators from list (3.3). We may assume that g is to the left of h in the list. Since c_z commutes with both g and h, $z \in \mathrm{im}(g) \cap \mathrm{im}(h)$ by Lemma 3.1. Suppose $g = a_i$, where $1 \le i \le k-1$. Then $h = a_{i+1}$ since a_{i+1} is the only generator to the right of a_i whose image is not disjoint from $\mathrm{im}(a_i)$. Similarly, if $g = a_k$ then $h = b_1$; if $g = b_i$ $(1 \le i \le k-1)$ then $h = b_{i+1}$; and if $g = b_k$ then $h = e_1$. Hence $gh = hg$ by Corollary 3.5. □

Lemma 3.8. *The paths*

(i) $\tau_1 = c_{y_0} - a_1 - \cdots - a_k - b_1 - \cdots - b_k - c_{v_k}$,
(ii) $\tau_2 = c_{y_1} - a_2 - \cdots - a_k - b_1 - \cdots - b_k - e_1 - c_s$

are the only minimal l-paths in $\mathcal{G}(S_0^k)$ with constants as the endpoints.

Proof. We have that τ_1 and τ_2 are l-paths by Lemmas 3.3 and 3.6. Suppose that $\lambda = c_z - \cdots - c_w$ is a minimal l-path in $\mathcal{G}(S_0^k)$ with constants c_z and c_w as the endpoints. Recall that $z, w \in \{y_0, y_1, \ldots, y_k, v_1, \ldots, v_k, s\}$. We may assume that z is to the left of w in the list $y_0, y_1, \ldots, y_k, v_1, \ldots, v_k, s$. Since λ is minimal, Lemma 3.7 implies that λ does not contain any constants except c_z and c_w. There are five cases to consider.

(a) $\lambda = c_{y_i} - \cdots - c_{y_j}$, where $0 \le i < j \le k$.
(b) $\lambda = c_{y_i} - \cdots - c_{v_j}$, where $0 \le i \le k$, $1 \le j \le k$.
(c) $\lambda = c_{y_i} - \cdots - c_s$, where $0 \le i \le k$.
(d) $\lambda = c_{v_i} - \cdots - c_{v_j}$, where $1 \le i < j \le k$.
(e) $\lambda = c_{v_i} - \cdots - c_s$, where $1 \le i \le k$.

Suppose (a) holds, that is, $\lambda = c_{y_i} - \cdots - h - c_{y_j}$, $0 \le i < j \le k$. Since $hc_{y_j} = c_{y_j}h$, either $h = a_j$ or $h = a_{j+1}$ (where $a_{k+1} = b_1$) (since a_j and a_{j+1} are the only generators that have y_j in their image). Suppose $h = a_{j+1}$. Then, by Corollary 3.5, either $\lambda = c_{y_i} - \cdots - a_j - a_{j+1} - c_{y_j}$ or $\lambda = c_{y_i} - \cdots - a_{j+2} - a_{j+1} - c_{y_j}$ (where $a_{j+2} = b_1$ if $j = k-1$, and $a_{j+2} = b_2$ if $j = k$). In the latter case,

$$\lambda = c_{y_i} - \cdots - a_1 - e_1 - b_k - \cdots - b_1 - a_k - \cdots - a_{j+2} - a_{j+1} - c_{y_j},$$

which is a contradiction since a_1 and e_1 do not commute. Thus either $\lambda = c_{y_i} - \cdots - a_j - c_{y_j}$ or $\lambda = c_{y_i} - \cdots - a_j - a_{j+1} - c_{y_j}$. In either case, λ contains a_j, and so $c_{y_i}a_j = c_{y_j}a_j$ (since λ is an l-path). But, by Lemma 3.6, $c_{y_i}a_j = c_{y_{j-1}}$ and $c_{y_j}a_j = c_{y_j}$. Hence $c_{y_{j-1}} = c_{y_j}$, which is a contradiction.

Suppose (b) holds, that is, $\lambda = c_{y_i} - g - \cdots - h - c_{v_j}$, $0 \le i \le k$ and $1 \le j \le k$. Then g is either a_i or a_{i+1} ($g = a_{i+1}$ if $i = 0$) and h is either b_j or b_{j+1} (where $b_{k+1} = e_1$). In any case, $\lambda = c_{y_i} - g - \cdots - a_k - b_1 - \cdots - h - c_{v_j}$. Suppose $i \ge 1$. Then, by Lemma 3.6 and the fact that λ is an l-path,

$c_{v_0} = c_{y_i} b_1 = c_{v_j} b_1 = c_{v_1}$, which is a contradiction. If $i = 0$ and $j < k$, then $c_{y_{k-1}} = c_{y_0} a_k = c_{v_j} a_k = c_{y_k}$, which is again a contradiction. If $i = 0$ and $j = k$, then $g = a_1$, and so $\lambda = \tau_1$.

Suppose (c) holds, that is, $\lambda = c_{y_i} - g - \cdots - a_k - b_1 - \cdots - b_k - e_1 - c_s$, $0 \le i \le k$, where g is either a_i or a_{i+1} ($g = a_{i+1}$ if $i = 0$). If $i > 1$, then $c_{v_{i-1}} = c_{y_i} b_i = c_s b_i = c_{v_i}$, which is a contradiction. If $i = 0$, then $c_{v_k} = c_{y_0} e_1 = c_s e_1 = c_s$, which is a contradiction. If $i = 1$ and $g = a_1$, then λ is not minimal since $c_{y_1} - a_2$, so a_1 can be removed. Finally, if $i = 1$ and $g = a_2$, then $\lambda = \tau_2$.

Suppose (d) holds, that is, $\lambda = c_{v_i} - g - \cdots - h - c_{v_j}$, $1 \le i < j \le k$, where g is either b_i or b_{i+1} and h is either b_j or b_{j+1} (where $b_{k+1} = e_1$). In any case, λ contains b_j, and so $c_{v_{j-1}} = c_{v_i} b_j = c_{v_j} b_j = c_{v_j}$, which is a contradiction.

Suppose (e) holds, that is, $\lambda = c_{v_i} - \cdots - e_1 - c_s$, $1 \le i \le k$. Then $c_{v_k} = c_{v_i} e_1 = c_s e_1 = c_s$, which is a contradiction.

We have exhausted all possibilities and obtained that λ must be equal to τ_1 or τ_2. The result follows. □

Lemma 3.9. *The path $\pi = a_1 - \cdots - a_k - b_1 - \cdots - b_k - e_1$ is a unique minimal l-path in $\mathcal{G}(S_0^k)$ with at least one endpoint that is not a constant.*

Proof. We have that π is an l-path by Lemmas 3.3 and 3.6. Suppose that $\lambda = e - \cdots - f$ is a minimal l-path in $\mathcal{G}(S_0^k)$ such that e or f is not a constant.

We claim that λ does not contain any constant c_z. By Lemma 3.7, there is no constant c_z such that $\lambda = e - \cdots - c_z - \cdots - f$ (since otherwise λ would not be minimal). We may assume that f is not a constant. But then e is not a constant either since otherwise we would have that ef is a constant and $ff = f$ is not a constant. But this is impossible since λ is an l-path, and so $ef = ff$. The claim has been proved.

Thus all elements in λ are generators from list (3.3). We may assume that e is to the left of f (according to the ordering in (3.3)). Since λ is an l-path, $e = ee = fe$. Hence, by Lemma 3.3, $e = a_p$ and $f = b_p$ (for some $p \in \{1, \ldots, k\}$) or $e = b_1$ and $f = e_1$ or $e = a_1$ and $f = e_1$.

Suppose that $e = a_p$ and $f = b_p$ for some p. Then, by Corollary 3.5, $\lambda = a_p - \cdots - a_k - b_1 - \cdots - b_p$. (Note that $\lambda = a_p - a_{p-1} - \cdots - a_1 - e_1 - b_k - \cdots - b_p$ is impossible since $a_1 e_1 \neq e_1 a_1$.) If $p > 1$ then, by Lemma 3.3, $c_{v_0} = a_p b_1 = b_p b_1 = c_{v_1}$, which is a contradiction. If $p = 1$, then $c_{y_{k-1}} = a_1 a_k = b_1 b_k = c_{y_k}$, which is again a contradiction.

Suppose that $e = b_1$ and $f = e_1$. Then $\lambda = b_1 - \cdots - b_k - e_1$, and so $c_{v_{k-1}} = b_1 b_k = e_1 b_k = c_{v_k}$, which is a contradiction.

Hence we must have $e = a_1$ and $f = e_1$. But then, by Corollary 3.5, $\lambda = a_1 - \cdots - a_k - b_1 - \cdots - b_k - e_1 = \pi$. The result follows. □

Theorem 3.10. *For every even integer $n \ge 2$, there is a band S with knit degree n.*

Proof. Let $n = 2$. Consider the band $S = \{a, b, c, d\}$ defined by the following Cayley table:

	a	b	c	d
a	a	b	c	d
b	b	b	b	b
c	a	b	c	d
d	d	d	d	d

It is easy to see that the center of S is empty and $a - b - c$ is a shortest l-path in $\mathcal{G}(S)$. Thus $\mathrm{kd}(S) = 2$.

Let $n = 2k$ where $k \ge 2$. Consider the semigroup S_0^k defined by (3.2). Then, by Corollary 3.4, S_0^k is a band. The paths τ_1, τ_2, and π from Lemmas 3.8 and 3.9 are the only minimal l-paths in $\mathcal{G}(S_0^k)$. Since τ_1 has length $2k + 1 = n + 1$, τ_2 has length $2k + 2 = n + 2$, and π has length $2k = n$, it follows that $\mathrm{kd}(S_0^k) = n$. □

3.2. The odd case

Suppose $n = 2k + 1 \ge 5$ is odd. We will obtain a band S of knit degree n by slightly modifying the construction of the band S_0^k from Definition 3.2. Recall that S_0^k has knit degree $2k$ (see the proof of

Theorem 3.10). We will obtain a band of knit degree $n = 2k + 1$ by simply removing transformations e_1 and c_s from S_0^k.

Definition 3.11. Let $k \geq 2$ be an integer. Consider the following subset of the semigroup S_k^0 from Definition 3.2:

$$S_k^1 = S_k^0 - \{e_1, c_s\} = \{a_1, \dots, a_k, b_1, \dots, b_k, c_{y_0}, c_{y_1}, \dots, c_{y_k}, c_{v_1}, \dots, c_{v_k}\}. \tag{3.4}$$

By Lemmas 3.3 and 3.6, S_k^1 is a subsemigroup of S_k^0.

Remark 3.12. Note that r and s, which still occur in the domain (but not the image) of each element of S_1^k, are now superfluous. We can remove them from the domain of each element of S_k^1 and view S_k^1 as a semigroup of transformations on the set

$$X = \{y_0, y_1, \dots, y_k = v_0, v_1, \dots, v_k, x_1, \dots, x_k, u_1, \dots, u_k\}.$$

It is clear from the definition of S_1^k that the multiplication table for S_1^k is the multiplication table for S_0^k (see Lemmas 3.3 and 3.6) with the rows and columns e_1 and c_s removed. This new multiplication table is given by Lemmas 3.3 and 3.6 if we ignore the multiplications involving e_1 or c_s. Therefore, the following lemma follows immediately from Corollary 3.4 and Lemmas 3.8 and 3.9.

Lemma 3.13. *Let S_1^k be the semigroups defined by (3.4). Then S_1^k is a band and $\tau = c_{y_0} - a_1 - \cdots - a_k - b_1 - \cdots - b_k - c_{v_k}$ is the only minimal l-path in $\mathcal{G}(S_1^k)$.*

Theorem 3.14. *For every odd integer $n \geq 5$, there is a band S of knit degree n.*

Proof. Let $n = 2k+1$ where $k \geq 2$. Consider the semigroup S_1^k defined by (3.4). Then, by Lemma 3.13, S_1^k is a band and $\tau = c_{y_0} - a_1 - \cdots - a_k - b_1 - \cdots - b_k - c_{v_k}$ is the only minimal l-path in $\mathcal{G}(S_1^k)$. Since τ has length $2k + 1 = n$, it follows that $\mathrm{kd}(S_1^k) = n$. $\square$

The case $n = 3$ remains unresolved.

Open question. Is there a semigroup of knit degree 3?

4. Commuting graphs with arbitrary diameters

In Section 2, we showed that, except for some special cases, the commuting graph of any ideal of the semigroup $T(X)$ has diameter 5. In this section, we use the constructions of Section 3 to show that there are semigroups whose commuting graphs have any prescribed diameter. We note that the situation is (might be) quite different in group theory: it has been conjectured that there is an upper bound for the diameters of the connected commuting graphs of finite non-abelian groups [8, Conjecture 2.2].

Theorem 4.1. *For every $n \geq 2$, there is a semigroup S such that the diameter of $\mathcal{G}(S)$ is n.*

Proof. Let $n \in \{2, 3, 4\}$. The commuting graph of the band S defined by the Cayley table in the proof of Theorem 3.10 is the cycle $a - b - c - d - a$. Thus the diameter of $\mathcal{G}(S)$ is 2. Consider the semigroup S defined by the following table:

	a	b	c	d
a	a	a	a	a
b	a	b	c	c
c	c	c	c	c
d	c	d	c	c

Note that $Z(S) = \emptyset$ and $\mathcal{G}(S)$ is the chain $a - b - c - d$. Thus the diameter of $\mathcal{G}(S)$ is 3. The diameter of $\mathcal{G}(J_4)$ is 4 (where J_4 is an ideal of $T(X)$ with $|X| = 5$).

Let $n \geq 5$. Suppose n is even. Then $n = 2k + 2$ for some $k \geq 2$. Consider the band S_0^k from Definition 3.2. Since c_{y_0} and a_1 are the only elements of S_0^k whose image contains y_0, they are the only elements of S_0^k commuting with c_{y_0} (see Lemma 3.1). Similarly, e_1 and c_s are the only elements commuting with c_s. Therefore, it follows from Corollary 3.5 that $c_{y_0} - a_1 - \cdots - a_k - b_1 - \cdots - b_k - e_1 - c_s$ is a shortest path in $\mathcal{G}(S_0^k)$ from c_{y_0} to c_s, that is, the distance between c_{y_0} and c_s is $2k + 2 = n$. Since $a_1 - \cdots - a_k - b_1 - \cdots - b_k - e_1$ is a path in $\mathcal{G}(S_0^k)$, $c_{y_i}a_i = a_i c_{y_i}$ and $c_{v_i}b_i = b_i c_{v_i}$ $(1 \le i \le k)$, it follows that the distance between any two vertices of $\mathcal{G}(S_0^k)$ is at most $2k + 2$. Hence the diameter of $\mathcal{G}(S_0^k)$ is n.

Suppose n is odd. Then $n = 2k + 1$ for some $k \ge 2$. Consider the band S_1^k from Definition 3.11. Then $c_{y_0} - a_1 - \cdots - a_k - b_1 - \cdots - b_k - c_{v_k}$ is a shortest path in $\mathcal{G}(S_1^k)$ from c_{y_0} to c_{v_k}, that is, the distance between c_{y_0} and c_{v_k} is $2k + 1 = n$. As for S_0^k, we have $c_{y_i}a_i = a_i c_{y_i}$ and $c_{v_i}b_i = b_i c_{v_i}$ $(1 \le i \le k)$. Thus the distance between any two vertices of S_1^k is at most $2k + 1$, and so the diameter of $\mathcal{G}(S_1^k)$ is n. $\square$

5. Schein's conjecture

The results obtained in Section 3 enable us to settle a conjecture formulated by Schein in 1978 [14, p. 12]. Schein stated his conjecture in the context of the attempts to characterize the r-semisimple bands.

A right congruence τ on a semigroup S is said to be modular if there exists an element $e \in S$ such that $(ex)\tau x$ for all $x \in S$. The radical R_r on a band S is the intersection of all maximal modular right congruences on S [11]. A band S is called *r-semisimple* if its radical R_r is the identity relation on S.

In 1969, Arendt announced a characterization of r-semisimple bands [3, Theorem 18]. In 1978, Schein pointed out that Arendt's characterization is incorrect and proved [14, p. 2] that a band S is r-semisimple if and only if it satisfies infinitely many quasi-identities: (1) and (A_n) for all integers $n \ge 1$, where

$$(1) \quad zx = zy \Rightarrow xy = yx,$$

$$(A_n) \quad x_1x_2 = x_2x_1 \wedge x_2x_3 = x_3x_2 \wedge \cdots \wedge x_{n-1}x_n = x_nx_{n-1}$$
$$\wedge\, x_1x_1 = x_nx_1 \wedge x_1x_2 = x_nx_2 \wedge \cdots \wedge x_1x_n = x_nx_n \Rightarrow x_1 = x_n.$$

Schein observed that (A_1) and (A_2) are true in every band, that (A_3) easily follows from (1), and that Arendt's characterization of r-semisimple bands is equivalent to (1). He used the last observation to show that Arendt's characterization is incorrect by providing an example of a band T for which (1) holds but (A_4) does not. We note that Schein's example is incorrect since the Cayley table in [14, p. 10], which is supposed to define T, does not define a semigroup because the operation is not associative: $(4 * 1) * 1 = 10 \neq 8 = 4 * (1 * 1)$. However, Schein was right that it is not true that condition (1) implies (A_n) for all n. The semigroup S_0^2 (see Table 2) satisfies (1) but it does not satisfy (A_5) since $a_1 - a_2 - b_1 - b_2 - e_1$ is an l-path (so the premise of (A_5) holds) but $a_1 \neq e_1$.

At the end of the paper, Schein formulates his conjecture [14, p. 12]:

Schein's conjecture. For every $n > 1$, (A_n) does not imply (A_{n+1}).

The reason that Section 3 enables us to settle Schein's conjecture is the following lemma.

Lemma 5.1. *Let $n \ge 1$ and let S be a band with no central elements. Then S satisfies (A_n) if and only if $\mathcal{G}(S)$ has no l-path of length $<n$.*

Proof. First note that (A_n) can be expressed as follows: for all $x_1, \ldots, x_n \in S$,

$$x_1 - \cdots - x_n \quad \text{and} \quad x_1x_i = x_nx_i \quad (1 \le i \le n) \Rightarrow x_1 = x_n. \tag{5.1}$$

(Here, we allow $x - x$ and do not require that $x_1, \ldots, x_n$ be distinct.)

Assume S satisfies (A_n). Suppose to the contrary that $\mathcal{G}(S)$ has an l-path $\lambda = x_1 - \cdots - x_k$ of length $<n$, that is, $k \le n$. Then $x_1 - \cdots - x_k - x_{k+1} - \cdots - x_n$, where $x_i = x_k$ for every $i \in \{k + 1, \ldots, n\}$, and so $x_1 = x_n = x_k$ by (5.1). This is a contradiction since λ is a path.

Conversely, suppose that $\mathcal{G}(S)$ has no l-path of length $<n$. Let $x_1 - \cdots - x_n$ and $x_1x_i = x_nx_i$ $(1 \le i \le n)$. Suppose to the contrary that $x_1 \neq x_n$. If there are i and j such that $1 \le i < j \le n$ and $x_i = x_j$, we can replace $x_1 - \cdots - x_i - \cdots - x_j - \cdots - x_n$ with $x_1 - \cdots - x_i - x_{j+1} - \cdots - x_n$. Therefore, we can assume that $x_1, \ldots, x_n$ are pairwise distinct. Recall that S has no central elements, so all x_i are vertices in $\mathcal{G}(S)$. Thus $x_1 - \cdots - x_n$ is an l-path in $\mathcal{G}(S)$ of length $n - 1$, which is a contradiction. $\square$

First, Schein's conjecture is false for $n = 3$.

Proposition 5.2. $(A_3) \Rightarrow (A_4)$.

Proof. Suppose a band S satisfies (A_3), that is,

$$x_1x_2 = x_2x_1 \wedge x_2x_3 = x_3x_2 \wedge x_1x_1 = x_3x_1 \wedge x_1x_2 = x_3x_2 \wedge x_1x_3 = x_3x_3 \Rightarrow x_1 = x_3. \tag{5.2}$$

To prove that S satisfies (A_4), suppose that

$$\begin{aligned} y_1y_2 &= y_2y_1 \wedge y_2y_3 = y_3y_2 \wedge y_3y_4 = y_4y_3 \wedge y_1y_1 = y_4y_1 \wedge y_1y_2 \\ &= y_4y_2 \wedge y_1y_3 = y_4y_3 \wedge y_1y_4 = y_4y_4. \end{aligned}$$

Take $x_1 = y_1$, $x_2 = y_2y_3$, and $x_3 = y_4$. Then x_1, x_2, x_3 satisfy the premise of (5.2):

$$\begin{aligned} &x_1x_2 = y_1y_2y_3 = y_1y_3y_2 = y_4y_3y_2 = y_3y_4y_2 = y_3y_1y_2 = y_3y_2y_1 = y_2y_3y_1 = x_2x_1, \\ &x_2x_3 = y_2y_3y_4 = y_2y_4y_3 = y_2y_1y_3 = y_1y_2y_3 = y_4y_2y_3 = x_3x_2, \\ &x_1x_1 = y_1y_1 = y_4y_1 = x_3x_1, \qquad x_1x_2 = y_1y_2y_3 = y_4y_2y_3 = x_3x_2, \\ &x_1x_3 = y_1y_4 = y_4y_4 = x_3x_3. \end{aligned}$$

Thus, by (5.2), $y_1 = x_1 = x_3 = y_4$, and so (A_4) holds. □

Second, Schein's conjecture is true for $n \neq 3$.

Proposition 5.3. *If $n > 1$ and $n \neq 3$, then (A_n) does not imply (A_{n+1}).*

Proof. Consider the band $S = \{e, f, 0\}$, where 0 is the zero, $ef = f$, and $fe = e$. Then $e - 0 - f$, $ee = fe$, $e0 = f0$, $ef = ff$, and $e \neq f$. Thus S does not satisfy (A_3). But S satisfies (A_2) since (A_2) is true in every band. Hence (A_2) does not imply (A_3).

Let $n \geq 4$. Then, by Theorems 3.10 and 3.14 and their proofs, the band S constructed in Definition 3.2 (if n is even) or Definition 3.11 (if n is odd) has knit degree n. By Lemmas 3.3 and 3.6, S has no central elements. Since $\mathrm{kd}(S) = n$, there is an l-path in $\mathcal{G}(S)$ of length n and there is no l-path in $\mathcal{G}(S)$ of length $<n$. Hence, by Lemma 5.1, S satisfies (A_n) and S does not satisfy (A_{n+1}). Thus (A_n) does not imply (A_{n+1}). □

6. Problems

We finish this paper with a list of some problems concerning commuting graphs of semigroups.

(1) Is there a semigroup with knit degree 3? Our guess is that such a semigroup does not exist.
(2) Classify the semigroups whose commuting graph is Eulerian (proposed by M. Volkov). The same problem for Hamiltonian and planar graphs.
(3) Classify the commuting graphs of semigroups.
(4) Is it true that for all natural numbers $n \geq 3$, there is a semigroup S such that the clique number (girth, chromatic number) of $\mathcal{G}(S)$ is n?
(5) Classify the semigroups S such that the clique and chromatic numbers of $\mathcal{G}(S)$ coincide.
(6) Calculate the clique and chromatic numbers of the commuting graphs of $T(X)$ and $\mathrm{End}(V)$, where X is a finite set and V is a finite-dimensional vector space over a finite field.
(7) Let $\mathcal{G}(S)$ be the commuting graph of a finite non-commutative semigroup S. An *rl-path* is a path $a_1 - \cdots - a_m$ in $\mathcal{G}(S)$ such that $a_1 \neq a_m$ and $a_1a_ia_1 = a_ma_ia_m$ for all $i = 1, \ldots, m$. For *rl*-paths, prove the results analogous to the results for l-paths contained in this paper.
(8) Find classes of finite non-commutative semigroups such that if S and T are two semigroups in that class and $\mathcal{G}(S) \cong \mathcal{G}(T)$, then $S \cong T$.

Acknowledgements

We are pleased to acknowledge the assistance of the automated deduction tool Prover9 and the finite model builder Mace4, both developed by McCune [10]. We also thank the developers of GAP [17], Soicher for GRAPE [16], and Aedan Pope and Kyle Pula for their suggestions after carefully reading the manuscript.

The first author was partially supported by FCT and FEDER, Project POCTI-ISFL-1-143 of Centro de Algebra da Universidade de Lisboa, by FCT and PIDDAC through the project PTDC/MAT/69514/2006, by PTDC/MAT/69514/2006 Semigroups and Languages, and by PTDC/MAT/101993/2008 Computations in groups and semigroups.

References

[1] J. Araújo, J. Konieczny, Automorphism groups of centralizers of idempotents, J. Algebra 269 (2003) 227–239.
[2] J. Araújo, J. Konieczny, Semigroups of transformations preserving an equivalence relation and a cross-section, Comm. Algebra 32 (2004) 1917–1935.
[3] B.D. Arendt, Semisimple bands, Trans. Amer. Math. Soc. 143 (1969) 133–143.
[4] C. Bates, D. Bundy, S. Perkins, P. Rowley, Commuting involution graphs for symmetric groups, J. Algebra 266 (2003) 133–153.
[5] E.A. Bertram, Some applications of graph theory to finite groups, Discrete Math. 44 (1983) 31–43.
[6] D. Bundy, The connectivity of commuting graphs, J. Combin. Theory Ser. A 113 (2006) 995–1007.
[7] A.H. Clifford, G.B. Preston, The Algebraic Theory of Semigroups, vol. I, in: Mathematical Surveys and Monographs, vol. 7, American Mathematical Society, Providence, RI, 1964.
[8] A. Iranmanesh, A. Jafarzadeh, On the commuting graph associated with the symmetric and alternating groups, J. Algebra Appl. 7 (2008) 129–146.
[9] J. Konieczny, Semigroups of transformations commuting with idempotents, Algebra Colloq. 9 (2002) 121–134.
[10] W. McCune, Prover9 and Mace4, version LADR-2009-11A. http://www.cs.unm.edu/~mccune/prover9/.
[11] R.H. Oehmke, On maximal congruences and finite semisimple semigroups, Trans. Amer. Math. Soc. 125 (1966) 223–237.
[12] A.S. Rapinchuk, Y. Segev, Valuation-like maps and the congruence subgroup property, Invent. Math. 144 (2001) 571–607.
[13] A.S. Rapinchuk, Y. Segev, G.M. Seitz, Finite quotients of the multiplicative group of a finite dimensional division algebra are solvable, J. Amer. Math. Soc. 15 (2002) 929–978.
[14] B.M. Schein, On semisimple bands, Semigroup Forum 16 (1978) 1–12.
[15] Y. Segev, The commuting graph of minimal nonsolvable groups, Geom. Dedicata 88 (2001) 55–66.
[16] L.H. Soicher, The GRAPE package for GAP, Version 4.3, 2006. http://www.maths.qmul.ac.uk/~leonard/grape/.
[17] The GAP group, GAP—Groups, Algorithms, and Programming, Version 4.4.12, 2008. http://www.gap-system.org.