Computers and Mathematics with Applications 56 (2008) 2429–2443

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Higher-order three-point boundary value problem on time scales

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ARTICLE INFO

Article history:
Received 18 June 2007
Received in revised form 24 April 2008
Accepted 27 May 2008

Keywords:
Time scale
Dynamic equation
Cone
Fixed-point theorem
Positive solution

ABSTRACT

In this paper, we consider a higher-order three-point boundary value problem on time scales. We study the existence of solutions of a non-eigenvalue problem and of at least one positive solution of an eigenvalue problem. Later we establish the criteria for the existence of at least two positive solutions of a non-eigenvalue problem. Examples are also included to illustrate our results.

1. Introduction

We are concerned with the dynamic three-point boundary value problem (TPBVP)

$$\begin{cases} (-1)^n y^{\Delta^{2n}}(t) = f(t, y^{\sigma}(t)), & t \in [a, b], \\ \alpha_{i+1} y^{\Delta^{2i}}(\eta) + \beta_{i+1} y^{\Delta^{2i+1}}(a) = y^{\Delta^{2i}}(a), \qquad \gamma_{i+1} y^{\Delta^{2i}}(\eta) = y^{\Delta^{2i}}(\sigma(b)), & 0 \le i \le n-1, \end{cases} \tag{1.1}$$

and the eigenvalue problem $(-1)^n y^{\Delta^{2n}}(t) = \lambda f(t, y^{\sigma}(t))$ with the same boundary conditions where λ is a positive parameter, $n \ge 1$, $a < \eta < \sigma(b)$, and $f : [a, \sigma(b)] \times \mathbb{R} \to \mathbb{R}$ is continuous. We assume that $\sigma(b)$ is right dense so that $\sigma^j(b) = \sigma(b)$ for $j \ge 1$ and that for each $1 \le i \le n$, α_i, β_i, γ_i coefficients satisfy the following condition;

$$\text{(H)}\ 0 \le \alpha_i < \frac{\sigma(b) - \gamma_i \eta + (\gamma_i - 1)(a - \beta_i)}{\sigma(b) - \eta}, \qquad \beta_i \ge 0, \qquad 0 < \gamma_i < \frac{\sigma(b) - a + \beta_i}{\eta - a + \beta_i}.$$

Throughout this paper we let $\mathbb{T}$ be any time scale (non-empty closed subset of $\mathbb{R}$) and $[a, b]$ be a subset of $\mathbb{T}$ such that $[a, b] = \{t \in \mathbb{T}, a \le t \le b\}$.

Some preliminary definitions and theorems on time scales can be found in books [1,2] which are excellent references for calculus of time scales.

Second-order, three-point boundary value problems for dynamic equations on time scales have been studied in recent years [3–12]. Anderson and Avery [13] have been interested in an even-order three-point boundary value problem on time scales with a delta-nabla differential operator. Their problem is an extension of the works [3,7,14] on positive solutions of a linear three-point boundary value problem.

$2n$th-order two-point boundary value problems have attracted considerable attention in recent years [14–16]. Cetin and Topal [17] were interested in the following TPBVP,

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doi:10.1016/j.camwa.2008.05.018

$$\begin{cases}(-1)^n y^{\Delta^{2n}}(t) = f(t, y^\sigma(t)), & t \in [0,1] \subset \mathbb{T},\\ y^{\Delta^{2i}}(0) = y^{\Delta^{2i}}(\sigma(1)) = 0, & 0 \le i \le n-1.\end{cases} \tag{1.2}$$

They have studied the existence of solutions and of at least one positive solution to TPBVP (1.2). For this purpose, they used the Schauder fixed-point theorem, the monotone method and the Krasnosel'skii fixed-point theorem.

In this paper, existence results of bounded solutions of a non-eigenvalue problem are first established as a result of the Schauder fixed-point theorem. Second, the monotone method is discussed to ensure the existence of solutions of TPBVP (1.1). Third, we establish criteria for the existence of at least one positive solution of the eigenvalue problem by using the Krasnosel'skii fixed-point theorem. Later, we investigate the existence of at least two positive solutions of TPBVP (1.1) by using the Avery–Henderson fixed-point theorem. Finally, as an application, we also give some examples to demonstrate our results. Our results extend the problem (1.2). Moreover, our problem is more general than some in the existing literature on three-point boundary value problems [5,18,19].

2. The preliminary lemmas

To state and prove the main results of this paper, we need the following lemmas.

For $1 \le i \le n$, let $G_i(t,s)$ be Green's function for the boundary value problems

$$\begin{cases}-y^{\Delta^2}(t) = 0, & t \in [a,b],\\ \alpha_i y(\eta) + \beta_i y^\Delta(a) = y(a), & \gamma_i y(\eta) = y(\sigma(b)).\end{cases} \tag{2.1}$$

First, we need a few results on the related second-order homogeneous problem (2.1).

Lemma 2.1. *For $1 \le i \le n$, let*

$$d_i = (\gamma_i - 1)(a - \beta_i) + (1 - \alpha_i)\sigma(b) + \eta(\alpha_i - \gamma_i).$$

The homogeneous boundary value problem (2.1) has only the trivial solution if and only if $d_i \neq 0$.

Proof. A general solution of $-y^{\Delta^2}(t) = 0$ is $y(t) = At + B$. The boundary conditions at a, η, and $\sigma(b)$ lead to two equations

$$A(\alpha_i\eta + \beta_i - a) + B(\alpha_i - 1) = 0,$$
$$A(\gamma_i\eta - \sigma(b)) + B(\gamma_i - 1) = 0,$$

for $1 \le i \le n$. The determinant of the coefficients for this system is d_i. It follows that $A = B = C = 0$ if and only if $d_i \neq 0$. This implies the given boundary value problem (2.1) has only a trivial solution if and only if $d_i \neq 0$. □

Lemma 2.2. *Let $G_i(t,s)$ be Green's function for the boundary value problem (2.1).Then, for $1 \le i \le n$,*

$$G_i(t,s) = \begin{cases}G_{i_1}(t,s), & a \le s \le \eta,\\ G_{i_2}(t,s), & \eta < s \le b,\end{cases} \tag{2.2}$$

where

$$G_{i_1}(t,s) = \frac{1}{d_i}\begin{cases}[\gamma_i(t-\eta) + \sigma(b) - t](\sigma(s) + \beta_i - a), & \sigma(s) \le t,\\ [\gamma_i(\sigma(s) - \eta) + \sigma(b) - \sigma(s)](t + \beta_i - a) + \alpha_i(\eta - \sigma(b))(t - \sigma(s)), & t \le s,\end{cases}$$

and

$$G_{i_2}(t,s) = \frac{1}{d_i}\begin{cases}[\sigma(s)(1-\alpha_i) + \alpha_i\eta + \beta_i - a](\sigma(b) - t) + \gamma_i(\eta - a + \beta_i)(t - \sigma(s)), & \sigma(s) \le t,\\ [t(1-\alpha_i) + \alpha_i\eta + \beta_i - a](\sigma(b) - \sigma(s)), & t \le s.\end{cases}$$

Proof. It is easy to see that $G_i(t,s)$ satisfies the boundary conditions

$$\alpha_i y(\eta) + \beta_i y^\Delta(a) = y(a), \qquad \gamma_i y(\eta) = y(\sigma(b)),$$

for all $(t,s) \in [a,\sigma(b)] \times [a,b]$. For $t \in [a,\eta]$,

$$y^\Delta(t) = \frac{1}{d_i}\int_a^t (\gamma_i - 1)(\sigma(s) + \beta_i - a) f(s, y^\sigma(s))\Delta s + \frac{1}{d_i}\int_t^\eta [(1-\alpha_i)(\sigma(b) - \eta) + (1-\gamma_i)(\eta - \sigma(s))]$$
$$\times f(s, y^\sigma(s))\Delta s + \frac{1}{d_i}\int_\eta^{\sigma(b)} (1-\alpha_i)(\sigma(b) - \sigma(s)) f(s, y^\sigma(s))\Delta s$$

so that $-y^{\Delta^2}(t) = f(t, y^\sigma(t))$. Likewise for $t \in [\eta, \sigma(b)]$, we get $-y^{\Delta^2}(t) = f(t, y^\sigma(t))$. Therefore G_i as given in (2.2) is Green's function for (2.1). □

Lemma 2.3. *Assume that condition* (H) *is satisfied. Then, Green's function satisfies the following inequality.*

$$G_i(t,s) \geq \left(\frac{t-a}{\sigma(b)-a}\right) G_i(\sigma(b),s), \quad (t,s) \in (a,\sigma(b)) \times (a,b).$$

Proof. We proceed sequentially on the branches of Green's function.

(i) Fix $s \in [a,\eta]$ and $\sigma(s) \leq t$. Then

$$G_i(t,s) = \frac{1}{d_i}[\gamma_i(t-\eta)+\sigma(b)-t](\sigma(s)+\beta_i-s).$$

For $0 < \gamma_i < \frac{\sigma(b)-a}{\eta-a}$, we have the inequality

$$\gamma_i[(t-a)(\sigma(b)-\eta)-(\sigma(b)-a)(t-\eta)] < (\sigma(b)-a)(\sigma(b)-t).$$

Hence we get

$$\frac{G_i(t,s)}{G_i(\sigma(b),s)} = \frac{\gamma_i(t-\eta)+\sigma(b)-t}{\gamma_i(\sigma(b)-\eta)} > \frac{t-a}{\sigma(b)-a}$$

for $0 < \gamma_i < \frac{\sigma(b)-a}{\eta-a}$. Since the inequality $\frac{\sigma(b)-a+\beta_i}{\eta-a+\beta_i} < \frac{\sigma(b)-a}{\eta-a}$ holds, we have

$$G_i(t,s) > \frac{t-a}{\sigma(b)-a} G_i(\sigma(b),s)$$

for $0 < \gamma_i < \frac{\sigma(b)-a+\beta_i}{\eta-a+\beta_i}$.

(ii) Fix $s \in [a,\eta]$ and $t \leq s$. Then

$$G_i(t,s) = \frac{1}{d_i}[\gamma_i(\sigma(s)-\eta)+\sigma(b)-\sigma(s)](t+\beta_i-a)+\alpha_i(\eta-\sigma(b))(t-\sigma(s)).$$

Using the inequalities $0 < \gamma_i < \frac{\sigma(b)-a+\beta_i}{\eta-a+\beta_i}$ and $\alpha_i(\sigma(s)-t)(\eta-a+\beta_i)(\sigma(b)-a)+\beta_i(\sigma(b)-t)(\sigma(s)-a+\beta_i) > 0$, we obtain

$$\begin{aligned}\frac{G_i(t,s)}{G_i(\sigma(b),s)} &= \frac{[\gamma_i(\sigma(s)-\eta)+\sigma(b)-\sigma(s)](t+\beta_i-a)+\alpha_i(\eta-\sigma(b))(t-\sigma(s))}{\gamma_i(\sigma(b)-\eta)(\sigma(s)+\beta_i-a)} \\ &> \frac{(\sigma(s)-a+\beta_i)(t-a+\beta_i)+\alpha_i(\sigma(s)-t)(\eta-a+\beta_i)}{(\sigma(b)-a+\beta_i)(\sigma(s)-a+\beta_i)} \\ &> \frac{t-a}{\sigma(b)-a}.\end{aligned}$$

(iii) Take $s \in [\eta,b]$ and $\sigma(s) \leq t$. Then

$$\begin{aligned}G_i(t,s) &= [\sigma(s)(1-\alpha_i)+\alpha_i\eta+\beta_i-a](\sigma(b)-t)+\gamma_i(\eta-a+\beta_i)(t-\sigma(s)) \\ &= G_i(\sigma(b),s)+\frac{1}{d_i}[(\gamma_i-1)(a-\beta_i)+(1-\alpha_i)\sigma(s)+\eta(\alpha_i-\gamma_i)](\sigma(b)-t).\end{aligned}$$

Since $(\gamma_i-1)(a-\beta_i)+(1-\alpha_i)\sigma(s)+\eta(\alpha_i-\gamma_i) > 0$, we get

$$\frac{t-a}{\sigma(b)-a} G_i(\sigma(b),s) < G_i(t,s).$$

(iv) Take $s \in [\eta,b]$ and $t \leq s$. Then

$$G_i(t,s) = [t(1-\alpha_i)+\alpha_i\eta+\beta_i-a](\sigma(b)-\sigma(s)).$$

Since the inequality $(t-a)d_i+(\sigma(b)-t)(\alpha_i(\eta-a)+\beta_i) > 0$ holds, we have

$$\frac{G_i(t,s)}{G_i(\sigma(b),s)} = \frac{t(1-\alpha_i)+\alpha_i\eta+\beta_i-a}{\gamma_i(\eta-a+\beta_i)} > \frac{t-a}{\sigma(b)-a}. \quad \square$$

Lemma 2.4. *Under condition* (H), *for* $1 \leq i \leq n$, *Green's function* $G_i(t,s)$ *in* (2.2) *possesses the following property;*

$$G_i(t,s) > 0, \quad (t,s) \in (a,\sigma(b)) \times (a,b).$$

Proof. By Lemma 2.3, it suffices to show that $G_i(\sigma(b), s) > 0$ for $s \in (a, b)$. For $s \in (a, \eta]$,

$$G_i(\sigma(b), s) = \frac{1}{d_i}\gamma_i(\sigma(b) - \eta)(\sigma(s) + \beta_i - a) > 0,$$

and for $s \in [\eta, b)$,

$$G_i(\sigma(b), s) = \frac{1}{d_i}\gamma_i(\eta - a + \beta_i)(\sigma(b) - \sigma(s)) > 0. \quad \square$$

Lemma 2.5. *Assume* (H) *holds. Then, for* $1 \le i \le n$*, Green's function* $G_i(t, s)$ *in* (2.2) *satisfies*

$$G_i(t, s) \le \max\left\{G_i(a, s), G_i(\sigma(s), s), \frac{1}{d_i}(\eta - a + \beta_i)(\sigma(b) - \sigma(s))\right\}, \quad (t, s) \in [a, \sigma(b)] \times [a, b],\ 0 < \gamma_i \le 1,$$

and

$$G_i(t, s) \le \max\{G_i(\sigma(b), s), G_i(\sigma(s), s)\}, \qquad (t, s) \in [a, \sigma(b)] \times [a, b], \qquad 1 < \gamma_i < \frac{\sigma(b) - a + \beta_i}{\eta - a + \beta_i}.$$

Proof. We again deal with the branches of Green's function.

(i) Let $s \in [a, \eta]$ and take $\sigma(s) \le t \le \sigma(b)$. Here $G_i(t, s)$ is non-increasing in t if $0 < \gamma_i \le 1$, so that $G_i(t, s) \le G_i(\sigma(s), s)$. If $1 < \gamma_i < \frac{\sigma(b)-a+\beta_i}{\eta-a+\beta_i}$, however, the function is non-decreasing in t and $G_i(t, s) \le G_i(\sigma(b), s)$.

(ii) Fix $s \in [a, \eta]$ and consider any t with $a \le t \le s$. Then $G_i(t, s)$ is increasing in t for all $t \in [a, s]$, for any $\gamma_i \in (0, \frac{\sigma(b)-a+\beta_i}{\eta-a+\beta_i})$. Therefore $G_i(t, s) \le G_i(\sigma(s), s)$.

(iii) Take $s \in [\eta, b]$, $\sigma(s) \le t \le \sigma(b)$. Here $G_i(t, s)$ is non-increasing in t if $0 < \gamma_i \le 1$, so that $G_i(t, s) \le G_i(\sigma(s), s)$. Let $\gamma_i \in (1, \frac{\sigma(b)-a+\beta_i}{\eta-a+\beta_i})$. So $\alpha_i < 1$. Our analysis depends on the placement of s. If $s \in [\eta, \frac{\gamma_i(\eta-a+\beta_i)-\alpha_i\eta-\beta_i+a}{1-\alpha_i})$, then $G_i(t, s)$ is non-decreasing in t and $G_i(t, s) \le G_i(\sigma(b), s)$. Otherwise, for $s \in (\frac{\gamma_i(\eta-a+\beta_i)-\alpha_i\eta-\beta_i+a}{1-\alpha_i}, \sigma(b)]$, $G_i(t, s)$ is non-increasing in t and $G_i(t, s) \le G_i(\sigma(s), s)$.

(iv) Take $s \in [\eta, b]$, $a \le t \le s \le b$. Let $\gamma_i \in (0, 1]$. If $\alpha_i \in (0, 1)$, then $G_i(t, s)$ is non-decreasing in t and $G_i(t, s) \le G_i(\sigma(s), s)$. For $\alpha_i > 1$, $G_i(t, s)$ is non-increasing in t and $G_i(t, s) \le G_i(a, s)$. If $\alpha_i = 1$, then $G_i(t, s)$ is constant in t and $G_i(t, s) = \frac{1}{d}(\eta - a + \beta_i)(\sigma(b) - \sigma(s))$. If $1 < \gamma_i < \frac{\sigma(b)-a+\beta_i}{\eta-a+\beta_i}$, we get $\alpha_i < 1$. Thus $G_i(t, s)$ is non-decreasing in t, so that $G_i(t, s) \le G_i(\sigma(s), s)$. $\square$

Lemma 2.6. *Assume* (H) *holds. For* $1 \le i \le n$ *and fixed* $s \in [a, b]$ *Green's function* $G_i(t, s)$ *in* (2.2) *satisfies*

$$\min_{t \in [\eta, \sigma(b)]} G_i(t, s) \ge m_i \|G_i(., s)\| \tag{2.3}$$

where

$$m_i := \min\left\{\frac{\gamma_i(\sigma(b) - \eta)}{\sigma(b) - a + \gamma_i(a - \eta)}, \frac{\gamma_i(\eta - a + \beta_i)}{\sigma(b)(1 - \alpha_i) + \alpha_i\eta + \beta_i - a}, \frac{\gamma_i(\eta - a + \beta_i)}{\alpha_i(\eta - a) + \beta_i}, \frac{\gamma_i}{\eta - a + \beta_i}, \frac{\eta - a + \beta_i}{\sigma(b) - a + \beta_i}\right\} \tag{2.4}$$

and $\|.\|$ *is defined by* $\|x\| = \max\{|x(t)| : t \in [a, \sigma(b)]\}$.

Proof. First consider the case where $0 < \gamma_i \le 1$. From Lemma 2.5,

$$\|G_i(., s)\| = \max\left\{G_i(a, s), G_i(\sigma(s), s), \frac{1}{d_i}(\eta - a + \beta_i)(\sigma(b) - \sigma(s))\right\}.$$

By the second boundary condition we know that $G(\eta, s) \ge G_i(\sigma(b), s)$, so that

$$\min_{t \in [\eta, \sigma(b)]} G_i(t, s) = G_i(\sigma(b), s).$$

For $s \in [a, \eta]$ we have from the branches in (2.2) that

$$G_i(\sigma(b), s) \ge \frac{\gamma_i(\sigma(b) - \eta)}{\sigma(b) - a + \gamma_i(a - \eta)} G_i(\sigma(s), s).$$

Let $s \in [\eta, b]$. If $\alpha_i < 1$, then the inequality

$$G_i(\sigma(b), s) \ge \frac{\gamma_i(\eta - a + \beta_i)}{\sigma(b) - a + \beta_i + \alpha_i(\eta - \sigma(b))} G_i(\sigma(s), s)$$

holds. If $\alpha_i > 1$, we have

$$G_i(\sigma(b), s) = \frac{\gamma_i(\eta - a + \beta_i)}{\alpha_i(\eta - a) + \beta_i} G_i(a, s).$$

If $\alpha_i = 1$, we get

$$G_i(\sigma(b), s) \geq \frac{\gamma_i}{\eta - a + \beta_i} \cdot \frac{1}{d_i}(\eta - a + \beta_i)(\sigma(b) - \sigma(s)).$$

Next consider the case $1 < \gamma_i < \frac{\sigma(b)-a+\beta_i}{\eta-a+\beta_i}$. The second boundary condition this time implies

$$\min_{t\in[\eta,\sigma(b)]} G_i(t, s) = G_i(\eta, s);$$

using Lemma 2.5, we have

$$\|G_i(., s)\| = \max\{G_i(\sigma(b), s), G_i(\sigma(s), s)\}.$$

By using (2.2) and the cases in the proof of Lemma 2.5, we see that

$$G_i(\eta, s) \geq \frac{\eta - a + \beta_i}{\sigma(b) - a + \beta_i} G_i(\sigma(b), s)$$

for $s \in [a, \frac{\gamma_i(\eta-a+\beta_i)-\alpha_i\eta-\beta_i+a}{1-\alpha_i})$, and

$$G_i(\eta, s) \geq \frac{\eta - a + \beta_i}{\sigma(b) - a + \beta_i + \alpha_i(\eta - \sigma(b))} G_i(\sigma(s), s)$$

for $s \in [\frac{\gamma_i(\eta-a+\beta_i)-\alpha_i\eta-\beta_i+a}{1-\alpha_i}, b]$. □

Lemma 2.7. *Assume that condition* (H) *is satisfied. For G as in* (2.2)*, take $H_1(t, s) := G_1(t, s)$, and recursively define*

$$H_j(t, s) = \int_a^{\sigma(b)} H_{j-1}(t, r) G_j(r, s) \Delta r$$

for $2 \leq j \leq n$. Then $H_n(t, s)$ is Green's function for the homogeneous problem

$$\begin{cases} (-1)^n y^{\Delta^{2n}}(t) = 0, & t \in [a, b], \\ \alpha_{i+1} y^{\Delta^{2i}}(\eta) + \beta_{i+1} y^{\Delta^{2i+1}}(a) = y^{\Delta^{2i}}(a), \quad \gamma_{i+1} y^{\Delta^{2i}}(\eta) = y^{\Delta^{2i}}(\sigma(b)), & 0 \leq i \leq n-1. \end{cases}$$

Lemma 2.8. *Assume* (H) *holds. If we define*

$$K = \Pi_{j=1}^{n-1} K_j, \qquad L = \Pi_{j=1}^{n-1} m_j L_j$$

then Green's function $H_n(t, s)$ in Lemma 2.7 satisfies

$$0 \leq H_n(t, s) \leq K \|G_n(., s)\|, \quad (t, s) \in [a, \sigma(b)] \times [a, b]$$

and

$$H_n(t, s) \geq m_n L \|G_n(., s)\|, \quad (t, s) \in [\eta, \sigma(b)] \times [a, b],$$

where m_n is given in (2.4)*,*

$$K_j := \int_a^{\sigma(b)} \|G_j(., s)\| \Delta s > 0, \quad 1 \leq j \leq n \tag{2.5}$$

and

$$L_j := \int_\eta^{\sigma(b)} \|G_j(., s)\| \Delta s > 0, \quad 1 \leq j \leq n. \tag{2.6}$$

Proof. Use induction on n and Lemma 2.6. □

3. Existence of solutions

In this section, first we obtain the existence of bounded solutions to the TPBVP (1.1). The proof of this result is based on an application of the Schauder fixed-point theorem. Later we prove the existence theorem for solutions of the TPBVP (1.1) which lie between the lower and upper solutions when they are given in the well order i.e.; the lower solution is under the upper solution.

Let $\mathcal{B}$ denote the Banach space $\mathcal{C}[a, \sigma(b)]$ with the norm $\|y\| = \max_{t\in[a,\sigma(b)]} |y(t)|$.

Theorem 3.1. *Suppose that condition* (H) *holds and that the function* $f(t, \xi)$ *is continuous with respect to* $\xi \in \mathbb{R}$. *If* $R > 0$ *satisfies* $Q\prod_{j=1}^{n} K_j \le R$, *where* $Q > 0$ *satisfies*

$$Q \ge \max_{\|y\|\le R} |f(t, y^\sigma)|,$$

for $t \in [a, \sigma(b)]$ *and* K_j *is as in* (2.5), *then TPBVP* (1.1) *has a solution* $y(t)$.

Proof. Let $\mathcal{P} := \{y \in \mathcal{B} : \|y\| \le R\}$. Note that $\mathcal{P}$ is a closed, bounded and convex subset of $\mathcal{B}$ to which the Schauder fixed-point theorem is applicable. Define $A : \mathcal{P} \to \mathcal{B}$

$$Ay(t) = \int_a^{\sigma(b)} H_n(t, s) f(s, y^\sigma(s))\Delta s,$$

for $t \in [a, \sigma(b)]$. Obviously the solutions of problem (1.1) are the fixed points of operator A. It can be shown that $A : \mathcal{P} \to \mathcal{B}$ is continuous.

Claim that $A : \mathcal{P} \to \mathcal{P}$. Let $y \in \mathcal{P}$. By using Lemma 2.8, we get

$$\begin{aligned}
|Ay(t)| &= \left| \int_a^{\sigma(b)} H_n(t, s) f(s, y^\sigma(s))\Delta s \right| \\
&\le \int_a^{\sigma(b)} |H_n(t, s)||f(s, y^\sigma(s))|\Delta s \\
&\le QK \int_a^{\sigma(b)} \|G_n(., s)\|\Delta s \\
&\le Q \prod_{j=1}^{n} K_j \le R
\end{aligned}$$

for every $t \in [a, \sigma(b)]$. This implies that $\|Ay\| \le R$.

It can be shown that $A : \mathcal{P} \to \mathcal{P}$ is a compact operator by the Arzela–Ascoli theorem. Hence A has a fixed point in $\mathcal{P}$ by the Schauder fixed-point theorem. □

Corollary 3.1. *Assume that condition* (H) *is satisfied. If* f *is continuous and bounded on* $[a, b] \times \mathbb{R}$, *then the TPBVP* (1.1) *has a solution.*

Proof. Since the function $f(t, y^\sigma)$ is bounded, it has a supremum for $t \in [a, \sigma(b)]$ and $y \in \mathbb{R}$. Let us choose $P > \sup\{|f(t, y^\sigma)| : (t, y^\sigma) \in [a, \sigma(b)] \times \mathbb{R}\}$. Pick R large enough such that $P < R$. Then there is a number $Q > 0$ such that

$$P > Q, \quad \text{where } Q \ge \max\{|f(t, y^\sigma)| : t \in [a, \sigma(b)], |y| \le R\}.$$

Hence

$$1 < \frac{R}{P} \le \frac{R}{Q},$$

and thus the TPBVP (1.1) has a solution by Theorem 3.1. □

Now, we give the existence of solutions by the monotone method, and we define the set

$$D := \{y : y^{\Delta^{2n}} \text{ is continuous on } [a, \sigma(b)]\}.$$

For any $u, v \in D$, we define the sector $[u, v]$ by

$$[u, v] := \{\omega \in D : u \le \omega \le v\}.$$

Definition 3.1. A real valued function $u(t) \in D$ on $[a, \sigma(b)]$ is a lower solution for TPBVP (1.1) if

$$(-1)^n u^{\Delta^{2n}}(t) \le f(t, u^\sigma(t)), \quad t \in [a, b]$$
$$(-1)^i [u^{\Delta^{2i}}(a) - \alpha_{i+1} u^{\Delta^{2i}}(\eta) - \beta_{i+1} u^{\Delta^{2i+1}}(a)] \le 0, \qquad (-1)^i [y^{\Delta^{2i}}(\sigma(b)) - \gamma_{i+1} y^{\Delta^{2i}}(\eta)] \le 0, \quad 0 \le i \le n-1.$$

Similarly, real valued function $v(t) \in D$ on $[a, \sigma(b)]$ is an upper solution for TPBVP (1.1) if

$$(-1)^n v^{\Delta^{2n}}(t) \ge f(t, v^\sigma(t)), \quad t \in [a, b]$$
$$(-1)^i [v^{\Delta^{2i}}(a) - \alpha_{i+1} v^{\Delta^{2i}}(\eta) - \beta_{i+1} v^{\Delta^{2i+1}}(a)] \ge 0, \qquad (-1)^i [v^{\Delta^{2i}}(\sigma(b)) - \gamma_{i+1} v^{\Delta^{2i}}(\eta)] \ge 0, \quad 0 \le i \le n-1.$$

Lemma 3.1. *Let condition* (H) *hold. Assume that $u(t) \in \mathcal{C}^2[a, b]$ and that u satisfies*

$$-u^{\Delta\Delta}(t) \ge 0, \quad t \in [a, b]$$
$$u(a) - \alpha_i u(\eta) - \beta_i u^\Delta(a) \ge 0, \qquad u(\sigma(b)) - \gamma_i u(\eta) \ge 0, \quad 1 \le i \le n.$$

Then $u(t) \ge 0$ on $[a, \sigma(b)]$.

Proof. For $1 \le i \le n$, let

$$\begin{cases} -u^{\Delta\Delta}(t) = h(t), & t \in [a, b], \\ u(a) - \alpha_i u(\eta) - \beta_i u^\Delta(a) = t_1, & u(\sigma(b)) - \gamma_i u(\eta) = t_2, \end{cases}$$

where $t_1 \ge 0$, $t_2 \ge 0$, $h \ge 0$.

It is easy to check that u can be given by the expression

$$u(t) = R_i(t) + \int_a^{\sigma(b)} G_i(t, s) h(s) \Delta s$$

where

$$R_i(t) = \frac{1}{d_i} \{[(\gamma_i - 1)t - \gamma_i \eta + \sigma(b)]t_1 + [(1 - \alpha_i)t + \alpha_i \eta + \beta_i - a]t_2\}$$

and $G_i(t, s)$ is as in (2.2). Since $0 \le \frac{\sigma(b)-t}{\sigma(b)-a+\beta_i} \alpha_i(\eta - a + \beta_i) < (1 - \alpha_i)t + \alpha_i \eta + \beta_i - a$, $0 \le \frac{t-a}{\sigma(b)-a} \gamma_i(\sigma(b) - \eta) < t(\gamma_i - 1) + \sigma(b) - \gamma_i \eta$, we get $R_i(t) \ge 0$, for $t \in [a, \sigma(b)]$. From (2.2), $G_i(t, s) \ge 0$ for $(t, s) \in [a, \sigma(b)] \times [a, b]$. Therefore we get $u(t) \ge 0$ for $t \in [a, \sigma(b)]$. The proof is completed. □

Lemma 3.2. *Let condition* (H) *hold. Assume that $u \in \mathcal{C}^{2n}[a, \sigma(b)]$ and u satisfies*

$$\begin{cases} (-1)^n u^{\Delta^{2n}}(t) \ge 0, & t \in [a, b], \\ (-1)^i [u^{\Delta^{2i}}(a) - \alpha_{i+1} u^{\Delta^{2i}}(\eta) - \beta_{i+1} u^{\Delta^{2i+1}}(a)] \ge 0, & \\ (-1)^i [u^{\Delta^{2i}}(\sigma(b)) - \gamma_{i+1} u^{\Delta^{2i}}(\eta)] \ge 0, & 0 \le i \le n-1. \end{cases} \tag{3.1}$$

Then $u(t) \ge 0$ on $[a, \sigma(b)]$.

Proof. Let $v_{n-1}(t) := (-1)^{n-1} u^{\Delta^{2(n-1)}}(t)$. Then $-v_{n-1}^{\Delta\Delta}(t) \ge 0$ on $[a, b]$ and

$$v_{n-1}(a) - \alpha_n v_{n-1}(\eta) - \beta_n v_{n-1}^\Delta = (-1)^{n-1} [u^{\Delta^{2(n-1)}}(a) - \alpha_n u^{\Delta^{2(n-1)}}(\eta) - \beta_n u^{\Delta^{2n-1}}(a)] \ge 0$$
$$v_{n-1}(\sigma(b)) - \gamma_n v_{n-1}(\eta) = (-1)^{n-1} [u^{\Delta^{2(n-1)}}(\sigma(b)) - \gamma_n u^{\Delta^{2(n-1)}}(\eta)] \ge 0.$$

Then it follows from Lemma 3.1 that $v_{n-1}(t) \ge 0$ on $[a, \sigma(b)]$.

Similarly let $v_{n-2}(t) := (-1)^{n-2} u^{\Delta^{2(n-2)}}(t)$. Then $-v_{n-2}^{\Delta\Delta}(t) \ge 0$ on $[a, b]$ and

$$v_{n-2}(a) - \alpha_{n-1} v_{n-2}(\eta) - \beta_{n-1} v_{n-2}^\Delta = (-1)^{n-2} [u^{\Delta^{2(n-2)}}(a) - \alpha_{n-1} u^{\Delta^{2(n-2)}}(\eta) - \beta_{n-1} u^{\Delta^{2n-3}}(a)] \ge 0$$
$$v_{n-2}(\sigma(b)) - \gamma_{n-1} v_{n-2}(\eta) = (-1)^{n-2} [u^{\Delta^{2(n-2)}}(\sigma(b)) - \gamma_{n-1} u^{\Delta^{2(n-2)}}(\eta)] \ge 0.$$

Then it follows from Lemma 3.1 that $v_{n-2}(t) \ge 0$ on $[a, \sigma(b)]$.

The conclusion of the lemma follows by an induction argument. □

Theorem 3.2. *Let condition* (H) *hold and let f be continuous on $[a, \sigma(b)] \times \mathbb{R}$. Assume that there exist a lower solution u and an upper solution v for TPBVP* (1.1) *such that $u \le v$ on $[a, \sigma(b)]$. Then the TPBVP* (1.1) *has a solution $y \in [u, v]$ on $[a, \sigma(b)]$.*

Proof. Consider the TPBVP

$$\begin{cases}(-1)^n y^{\Delta^{2n}}(t) = F(t, y^\sigma(t)), & t \in [a, b],\\ \alpha_{i+1} y^{\Delta^{2i}}(\eta) + \beta_{i+1} y^{\Delta^{2i+1}}(a) = y^{\Delta^{2i}}(a), \quad \gamma_{i+1} y^{\Delta^{2i}}(\eta) = y^{\Delta^{2i}}(\sigma(b)), & 0 \le i \le n-1,\end{cases} \tag{3.2}$$

where

$$F(t, \xi) = \begin{cases} f(t, v^\sigma(t)) - \dfrac{\xi - v^\sigma(t)}{1 + |y^\sigma(t)|}, & \xi \ge v^\sigma(t),\\ f(t, \xi), & u^\sigma(t) \le \xi \le v^\sigma(t),\\ f(t, u^\sigma(t)) - \dfrac{\xi - u^\sigma(t)}{1 + |\xi|}, & \xi \le v^\sigma(t),\end{cases}$$

for $t \in [a, b]$. Clearly, the function F is bounded for $t \in [a, b]$ and $\xi \in \mathbb{R}$, and is continuous in ξ. Thus, by Corollary 3.1 there exists a solution $y(t)$ of the TPBVP (3.2). We claim that $y(t) \le v(t)$ for $t \in [a, \sigma(b)]$. If not, we know that $y^\sigma(t) - v^\sigma(t) \ge 0$ for $t \in [a, b]$ and

$$\begin{aligned}(-1)^n y^{\Delta^{2n}}(t) &= F(t, y^\sigma(t))\\ &= f(t, v^\sigma(t)) - \frac{y^\sigma(t) - v^\sigma(t)}{1 + |\xi|}\\ &\le f(t, v^\sigma(t))\\ &\le (-1)^n v^{\Delta^{2n}}(t).\end{aligned}$$

Hence, we have

$$(-1)^n (v - y)^{\Delta^{2n}}(t) \ge 0$$

and from the boundary conditions we get

$$(-1)^i[(v - y)^{\Delta^{2i}}(a) - \alpha_{i+1}(v - y)^{\Delta^{2i}}(\eta) - \beta_{i+1}(v - y)^{\Delta^{2i}}(a)] \ge 0$$

and

$$(-1)^i[(v - y)^{\Delta^{2i}}(\sigma(b)) - \gamma_{i+1}(v - y)^{\Delta^{2i}}(\eta)] \ge 0, \quad 0 \le i \le n-1.$$

Using Lemma 3.2 we obtain that

$$v - y \ge 0 \quad \text{on } [a, \sigma(b)]$$

which is a contradiction. It follows that $y(t) \le v(t)$ on $[a, \sigma(b)]$.

Similarly, $u \le y$ on $[a, \sigma(b)]$. Thus y is a solution of TPBVP (1.1) and lies between u and v. □

4. Existence of one positive solution

In this section we consider the following TPBVP with parameter λ,

$$\begin{cases}(-1)^n y^{\Delta^{2n}}(t) = \lambda f(t, y^\sigma(t)), & t \in [a, b],\\ \alpha_{i+1} y^{\Delta^{2i}}(\eta) + \beta_{i+1} y^{\Delta^{2i+1}}(a) = y^{\Delta^{2i}}(a), \quad \gamma_{i+1} y^{\Delta^{2i}}(\eta) = y^{\Delta^{2i}}(\sigma(b)), & 0 \le i \le n-1.\end{cases} \tag{4.1}$$

We need the following fixed-point theorem to prove the existence at least one positive solution to TPBVP (4.1).

Theorem 4.1 ([20]). *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone. Assume Ω_1 and Ω_2 are open bounded subsets of $\mathcal{B}$ with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that either

(i) $\|Au\| \le \|u\|$, $\quad u \in \mathcal{P} \cap \partial\Omega_1$, $\quad \|Au\| \ge \|u\|$, $\quad u \in \mathcal{P} \cap \partial\Omega_2$; *or*

(ii) $\|Au\| \ge \|u\|$, $\quad u \in \mathcal{P} \cap \partial\Omega_1$, $\quad \|Au\| \le \|u\|$, $\quad u \in \mathcal{P} \cap \partial\Omega_2$,

holds. Then A has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let

$$M = m_n \prod_{j=1}^{n-1} \frac{m_j L_j}{K_j}. \tag{4.2}$$

We assume that $f \in \mathcal{C}([a,\sigma(b)]\times\mathbb{R}^+,\mathbb{R}^+)$, and the limits

$$f_0 := \lim_{y\to 0^+}\frac{f(t,y)}{y}, \qquad f_\infty := \lim_{y\to\infty}\frac{f(t,y)}{y}$$

exist uniformly in the extended reals. The case $f_0 = 0$ and $f_\infty = \infty$ is called the superlinear case, and the case $f_0 = \infty$ and $f_\infty = 0$ is called the sublinear case.

In [7], in the case f is sublinear or superlinear, the existence of at least one positive solution to TPBVP (1.2) has been studied.

Theorem 4.2. *Assume that condition* (H) *is satisfied. Then for λ satisfying*

$$\text{(a)}\ \frac{1}{Mm_n\prod_{j=1}^{n}L_jf_\infty} < \lambda < \frac{1}{\prod_{j=1}^{n}K_jf_0}, \tag{4.3}$$

or

$$\text{(b)}\ \frac{1}{Mm_n\prod_{j=1}^{n}L_jf_0} < \lambda < \frac{1}{\prod_{j=1}^{n}K_jf_\infty}, \tag{4.4}$$

there exists at least one positive solution of the TPBVP (4.1) where m_n, L_j, K_j, M are as in (2.4)–(2.6) and (4.2), respectively. Moreover, in the case f is superlinear (sublinear), then Eq. (4.3) (Eq. (4.4)) becomes $0 < \lambda < \infty$.

Proof. Define $\mathcal{B}$ to be Banach space of all continuous functions on $[a,\sigma(b)]$ equipped with the norm $\|.\|$ defined by

$$\|y\| = \max_{t\in[a,\sigma(b)]}|y(t)|.$$

Define the cone $\mathcal{P}\subset\mathcal{B}$ by

$$\mathcal{P} = \{y\in\mathcal{B} : y(t)\geq 0,\ \min_{t\in[\eta,\sigma(b)]}y(t)\geq M\|y\|\},$$

where M is as in (4.2). Define an operator A_λ by

$$A_\lambda y(t) = \lambda\int_a^{\sigma(b)}H_n(t,s)f(s,y^\sigma(s))\Delta s$$

for $t\in[a,\sigma(b)]$. The solutions of the TPBVP (4.1) are the fixed points of the operator A_λ.

Firstly, we show that $A_\lambda : \mathcal{P}\to\mathcal{P}$. Note that $y\in\mathcal{P}$ implies that $A_\lambda y(t)\geq 0$ on $[a,\sigma(b)]$ and

$$\begin{aligned}\min_{t\in[\eta,\sigma(b)]}A_\lambda y(t) &= \lambda\int_a^{\sigma(b)}\min_{t\in[\eta,\sigma(b)]}H_n(t,s)f(s,y^\sigma(s))\Delta s\\ &\geq M\lambda\int_a^{\sigma(b)}\max_{t\in[a,\sigma(b)]}|H_n(t,s)|f(s,y^\sigma(s))\Delta s\end{aligned}$$

by Lemma 2.8. It follows that

$$\min_{t\in[\eta,\sigma(b)]}A_\lambda y(t)\geq M\|A_\lambda y\|.$$

Hence $A_\lambda y\in\mathcal{P}$ and so $A_\lambda : \mathcal{P}\to\mathcal{P}$ which is what we want to prove. Therefore A_λ is completely continuous.

Assume that (a) holds. Since $\lambda < \frac{1}{\prod_{j=1}^{n}K_jf_0}$, there exists $\epsilon_1 > 0$ so that $0<\lambda\leq 1/\prod_{j=1}^{n}K_j(f_0+\epsilon_1)$.

Using the definition of f_0, there is an $r_1 > 0$, sufficiently small, so that

$$f(t,y) < (f_0+\epsilon_1)y \quad \text{for } 0<y\leq r_1,\ t\in[a,\sigma(b)].$$

If $y\in\mathcal{P}$, with $\|y\| = r_1$, then

$$\begin{aligned}A_\lambda y(t) &= \lambda\int_a^{\sigma(b)}H_n(t,s)f(s,y^\sigma(s))\Delta s\\ &< \lambda(f_0+\epsilon_1)\int_a^{\sigma(b)}H_n(t,s)y^\sigma(s)\Delta s\\ &\leq \lambda(f_0+\epsilon_1)\|y\|K\int_a^{\sigma(b)}\|G_n(.,s)\|\Delta s\end{aligned}$$

$$\begin{aligned} &\le \lambda(f_0+\epsilon_1)\prod_{j=1}^{n}K_j\|y\| \\ &\le \|y\| \end{aligned}$$

for $t\in[a,\sigma(b)]$. So, if we set $\Omega_1:=\{y\in\mathcal{P}:\|y\|\le r_1\}$, then $\|A_\lambda y\|\le\|y\|$ for $y\in\mathcal{P}\cap\partial\Omega_1$.

Now, we use assumption $\frac{1}{Mm_n\prod_{j=1}^{n}L_jf_\infty}<\lambda$.

First, we consider the case when $f_\infty<\infty$. In this case pick an $\epsilon_2>0$ so that

$$\lambda Mm_n\prod_{j=1}^{n}L_j(f_\infty-\epsilon_2)\ge 1.$$

Using the definition f_∞, there exists $\bar r_2>r_1$, sufficiently large, so that

$$f(t,y)>(f_\infty-\epsilon_2)y\quad\text{for } y\ge\bar r_2,\ t\in[a,\sigma(b)].$$

We now show that there exists $r_2\ge\bar r_2$ such that if $y\in\partial\mathcal{P}_{r_2}$, then $\|A_\lambda y\|>\|y\|$. Let $r_2=\max\{2r_1,\frac{1}{M}\bar r_2\}$ and set $\Omega_2:=\{y\in\mathcal{P}:\|y\|\le r_2\}$. If $y\in\mathcal{P}\cap\partial\Omega_2$, then

$$\min_{t\in[\eta,\sigma(b)]}y(t)\ge M\|y\|=Mr_2\ge\bar r_2,$$

and so

$$\begin{aligned} A_\lambda y(t) &= \lambda\int_a^{\sigma(b)}H_n(t,s)f(s,y^\sigma(s))\Delta s \\ &> \lambda(f_\infty-\epsilon_2)\int_a^{\sigma(b)}H_n(t,s)y^\sigma(s)\Delta s \\ &\ge \lambda(f_\infty-\epsilon_2)\int_\eta^{\sigma(b)}H_n(t,s)y^\sigma(s)\Delta s \\ &\ge \lambda(f_\infty-\epsilon_2)M\|y\|m_nL\int_\eta^{\sigma(b)}\|G_n(.,s)\|\Delta s \\ &\ge \lambda(f_\infty-\epsilon_2)Mm_n\prod_{j=1}^{n}L_j\|y\| \\ &\ge \|y\|=r_2. \end{aligned}$$

Consequently, $\|A_\lambda y(t)\|\le\|y(t)\|$, for $t\in[a,\sigma(b)]$.

Finally, we consider the case $f_\infty=\infty$. In this case the hypothesis becomes $\lambda>0$. Choose $N>0$ sufficiently large so that

$$\lambda NMm_n\prod_{j=1}^{n}L_j\ge 1.$$

Hence there exists $\bar r_2>r_1$ so that $f(t,y)>Ny$ for $y\ge\bar r_2$ and for all $t\in[a,\sigma(b)]$. Now define r_2 as before and assume $y\in\partial\mathcal{P}_{r_2}$. Then

$$\begin{aligned} A_\lambda y(t) &> \lambda N\int_a^{\sigma(b)}H_n(t,s)y^\sigma(s)\Delta s \\ &\ge \lambda NM\|y\|m_nL\int_a^{\sigma(b)}\|G_n(.,s)\|\Delta s \\ &= \lambda NMm_n\prod_{j=1}^{n}L_j\|y\| \\ &\ge \|y\|=r_2 \end{aligned}$$

for $t\in[a,\sigma(b)]$. Hence $\|A_\lambda y\|\ge\|y\|$ for $y\in\mathcal{P}\cap\partial\Omega_1$ and $\|A_\lambda y\|\le\|y\|$ for $y\in\mathcal{P}\cap\partial\Omega_2$ hold. Then A_λ has a fixed point in $\mathcal{P}\cap(\overline{\Omega}_2\setminus\Omega_1)$.

Now we show (b). Since $\frac{1}{Mm_n\Pi_{j=1}^{n}L_jf_0}<\lambda$, there exists $\epsilon_3>0$ so that $\lambda Mm_n\prod_{j=1}^{n}L_j(f_0-\epsilon_3)\ge 1$.

From the definition of f_0, there exists an $r_3>0$ such that $f(t,y)\ge(f_0-\epsilon_3)y$ for $0<y\le r_3$. If $y\in\mathcal{P}$ with $\|y\|=r_3$, then

$$\begin{aligned}A_\lambda y(t) &= \lambda \int_a^{\sigma(b)} H_n(t,s) f(s, y^\sigma(s)) \Delta s \\ &\geq \lambda(f_0 - \epsilon_3) \int_\eta^{\sigma(b)} H_n(t,s) y^\sigma(s) \Delta s \\ &\geq \lambda M(f_0 - \epsilon_3)\|y\| m_n L \int_\eta^{\sigma(b)} \|G_n(.,s)\| \Delta s \\ &\geq \lambda(f_0 - \epsilon_3) M m_n \prod_{j=1}^n L_j \|y\| \\ &\geq \|y\| = r_3.\end{aligned}$$

Hence $\|A_\lambda y\| \geq \|y\|$. So, if we set $\Omega_3 := \{y \in \mathcal{P} : \|y\| \leq r_3\}$, then $\|Ay\| \leq \|y\|$ for $y \in \mathcal{P} \cap \partial\Omega_3$.

Now, we use assumption $\frac{1}{\prod_{j=1}^n K_j f_\infty} > \lambda$. Pick an $\epsilon_4 > 0$ so that

$$\lambda \prod_{j=1}^n K_j (f_\infty + \epsilon_4) \leq 1.$$

Using the definition of f_∞, there exists an $\bar{r}_4 > 0$ such that $f(t,y) \leq (f_\infty + \epsilon_4) y$ for all $y \geq \bar{r}_4$. We consider the two cases.

Case I. Suppose $f(t,y)$ is bounded on $[a, \sigma(b)] \times (0,\infty)$. In this case, there is $N > 0$ such that $f(t,y) \leq N$ for $t \in [a,\sigma(b)],\ y \in (0,\infty)$. Let $r_4 = \max\{2r_3, \lambda N \prod_{j=1}^n K_j\}$. Then for $y \in \mathcal{P}$ with $\|y\| = r_4$,

$$\begin{aligned}A_\lambda y(t) &= \lambda \int_a^{\sigma(b)} H_n(t,s) f(s, y^\sigma(s)) \Delta s \\ &\leq \lambda N K \int_a^{\sigma(b)} \|G_n(.,s)\| \Delta s \\ &\leq \lambda N \prod_{j=1}^n K_j \\ &\leq \|y\| = r_4,\end{aligned}$$

so that $\|A_\lambda y\| \leq \|y\|$.

Case II. Suppose $f(t,y)$ is unbounded on $[a,\sigma(b)] \times (0,\infty)$. In this case,

$$g(r) := \max\{f(t,y) : t \in [a,\sigma(b)], 0 \leq y \leq r\}$$

satisfies

$$\lim_{r\to\infty} g(r) = \infty.$$

We can therefore choose

$$r_4 = \max\{2r_3, \bar{r}_4\}$$

such that

$$g(r_4) \geq g(r)$$

for $0 \leq r \leq r_4$ and hence for $y \in \mathcal{P}$ and $\|y\| = r_4$, we have

$$\begin{aligned}A_\lambda y(t) &= \lambda \int_a^{\sigma(b)} H_n(t,s) f(s, y^\sigma(s)) \Delta s \\ &\leq \lambda \int_a^{\sigma(b)} H_n(t,s) g(r_4) \Delta s \\ &\leq \lambda(f_\infty + \epsilon_4) r_4 K \int_a^{\sigma(b)} \|G_n(.,s)\| \Delta s \\ &= \lambda(f_\infty + \epsilon_4) \prod_{j=1}^n K_j r_4 \\ &\leq r_4 = \|y\|,\end{aligned}$$

and again we hence have $\|A_\lambda y\| \leq \|y\|$ for $y \in \mathcal{P} \cup \partial\Omega_4$, where $\Omega_4 = \{y \in \mathcal{B} : \|y\| \leq H_4\}$ in both cases. It follows from part (ii) of Theorem (4.1) that A has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_4 \setminus \Omega_3)$, such that $r_3 \leq \|y\| \leq r_4$. The proof of part (b) of this theorem is complete. Therefore, the TPBVP (4.1) has at least one positive solution. □

5. Existence of two positive solutions

In this section, using Theorem 5.1 (Avery-Henderson fixed-point theorem) we prove the existence of at least two positive solutions of the TPBVP (1.1).

Theorem 5.1 (*[21]*). *Let $\mathcal{P}$ be a cone in a real Banach space S. If φ and ψ are increasing, non-negative continuous functionals on $\mathcal{P}$, let θ be a non-negative continuous functional on $\mathcal{P}$ with $\theta(0) = 0$ such that, for some positive constants r and M,*

$$\psi(u) \leq \theta(u) \leq \varphi(u) \quad \text{and} \quad \|u\| \leq M\psi(u)$$

for all $u \in \overline{\mathcal{P}(\psi, r)}$. Suppose that there exist positive numbers $p < q < r$ such that

$$\theta(\lambda u) \leq \lambda\theta(u), \quad \text{for all } 0 \leq \lambda \leq 1 \quad \text{and} \quad u \in \partial P(\theta, q).$$

If $A : \overline{\mathcal{P}(\psi, r)} \to \mathcal{P}$ is a completely continuous operator satisfying

(i) $\psi(Au) > r$ *for all* $u \in \partial\mathcal{P}(\psi, r)$,
(ii) $\theta(Au) < q$ *for all* $u \in \partial\mathcal{P}(\theta, q)$,
(iii) $\mathcal{P}(\varphi, p) \neq \{\}$ *and* $\varphi(Au) > p$ *for all* $u \in \partial\mathcal{P}(\varphi, p)$,

then A has at least two fixed points u_1 and u_2 such that

$$p < \varphi(u_1) \quad \text{with } \theta(u_1) < q \quad \text{and} \quad q < \theta(u_2) \quad \text{with } \psi(u_2) < r.$$

Let the Banach space $\mathcal{B} = \mathcal{C}[a, \sigma(b)]$ with the norm $\|.\|$ defined by $\|y\| = \max_{t\in[a,\sigma(b)]} |y(t)|$. Again define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \{y \in \mathcal{B} : y(t) \geq 0, \quad \min_{t\in[\eta,\sigma(b)]} y(t) \geq M\|y\|\}$$

where M is as in (4.2), and the operator $A : \mathcal{P} \to \mathcal{B}$ by

$$Ay(t) = \int_a^{\sigma(b)} H_n(t, s)f(s, y^\sigma(s))\Delta s.$$

Let the non-negative, increasing, continuous functionals ψ, θ, and φ be defined on the cone $\mathcal{P}$ by

$$\psi(y) := \min_{t\in[\eta,\sigma(b)]} y(t), \qquad \theta(y) := \max_{t\in[\eta,\sigma(b)]} y(t), \qquad \varphi(y) := \max_{t\in[a,\sigma(b)]} y(t) \tag{5.1}$$

and let $\mathcal{P}(\psi, r) := \{y \in \mathcal{P} : \psi(y) < r\}$.

In the next theorem, we will assume

(H1) $f \in \mathcal{C}([a, \sigma(b)] \times [0, \infty), [0, \infty))$.

Theorem 5.2. *Assume* (H) *and* (H1) *hold. Suppose there exist positive numbers $0 < p < q < r$ such that the function f satisfies the following conditions:*

(D1) $f(t, y) > p/(m_n \prod_{j=1}^n L_j)$ *for $t \in [\eta, \sigma(b)]$ and $y \in [Mp, p]$,*
(D2) $f(t, y) < q/\prod_{j=1}^n K_j$ *for $t \in [a, \sigma(b)]$ and $y \in [0, q/M]$,*
(D3) $f(t, y) > r/(Mm_n \prod_{j=1}^n L_j)$ *for $t \in [\eta, \sigma(b)]$ and $y \in [r, r/M]$,*

where m_n, L_j, K_j, M are as defined in (2.4)–(2.6) and (4.2) respectively. Then the TPBVP (1.1) has at least two positive solutions y_1 and y_2 such that

$$p < \max_{t\in[a,\sigma(b)]} y_1(t) \quad \text{with} \quad \max_{t\in[\eta,\sigma(b)]} y_1(t) < q,$$
$$q < \max_{t\in[\eta,\sigma(b)]} y_2(t) \quad \text{with} \quad \min_{t\in[\eta,\sigma(b)]} y_2(t) < r.$$

Proof. From (H), Lemma 2.4 and Lemma 2.8, $A\mathcal{P} \subset \mathcal{P}$. Moreover, A is completely continuous. From (5.1), for each $y \in \mathcal{P}$ we have

$$\psi(y) \le \theta(y) \le \varphi(y), \tag{5.2}$$

$$\|y\| \le \frac{1}{M} \min_{t\in[\eta,\sigma(b)]} y(t) = \frac{1}{M}\psi(y) \le \frac{1}{M}\theta(y) \le \frac{1}{M}\varphi(y). \tag{5.3}$$

For any $y \in \mathcal{P}$, (5.2) and (5.3) imply

$$\psi(y) \le \theta(y) \le \varphi(y), \qquad \|y\| \le \frac{1}{M}\psi(y).$$

For all $y \in \mathcal{P},\ \lambda \in [0,1]$ we have

$$\theta(\lambda y) = \max_{t\in[\eta,\sigma(b)]} (\lambda y)(t) = \lambda \max_{t\in[\eta,\sigma(b)]} y(t) = \lambda\theta(y).$$

It is clear that $\theta(0) = 0$.

We now show that the remaining conditions of Theorem 5.1 are satisfied.

Firstly, we shall verify that condition (iii) of Theorem 5.1 is satisfied. Since $0 \in \mathcal{P}$ and $p > 0$, $\mathcal{P}(\varphi, p) \ne \{\}$. Since $y \in \partial\mathcal{P}(\varphi, p)$, $Mp \le y(t) \le \|y\| = p$ for $t \in [\eta, \sigma(b)]$. Therefore,

$$\begin{aligned}
\varphi(Ay) &= \max_{t\in[a,\sigma(b)]} Ay(t) \\
&\ge Ay(t) \\
&= \int_a^{\sigma(b)} H_n(t,s) f(s, y^\sigma(s))\Delta s \\
&\ge \frac{p}{m_n \prod_{j=1}^n L_j} m_n L \int_\eta^{\sigma(b)} \|G_n(.,s)\|\Delta s \\
&\ge p
\end{aligned}$$

using hypothesis (D1).

Now we shall show that condition (ii) of Theorem 5.1 is satisfied. Since $y \in \partial\mathcal{P}(\theta, q)$, from (5.3) we have that $0 \le y(t) \le \|y\| \le q/M$ for $t \in [a, \sigma(b)]$. Thus

$$\begin{aligned}
\theta(Ay) &= \max_{t\in[\eta,\sigma(b)]} Ay(t) \\
&= \max_{t\in[\eta,\sigma(b)]} \int_a^{\sigma(b)} H_n(t,s) f(s, y^\sigma(s))\Delta s \\
&\le \frac{q}{\prod_{j=1}^n K_j} K \int_a^{\sigma(b)} \|G_n(.,s)\|\Delta s = q
\end{aligned}$$

by hypothesis (D2).

Finally using hypothesis (D3), we shall show that condition (i) of Theorem 5.1 is satisfied. Since $y \in \partial\mathcal{P}(\psi, r)$, from (5.3) we have that $\min_{t\in[\eta,\sigma(b)]} y(t) = r$ and $r \le \|y\| \le r/M$. Then

$$\begin{aligned}
\psi(Ay(t)) &= \min_{[\eta,\sigma(b)]} \int_a^{\sigma(b)} H_n(t,s) f(s, y^\sigma(s))\Delta s \\
&= \int_a^{\sigma(b)} \min_{[\eta,\sigma(b)]} H_n(t,s) f(s, y^\sigma(s))\Delta s \\
&\ge M \int_\eta^{\sigma(b)} \|H_n(.,s)\| f(s, y^\sigma(s))\Delta s \\
&\ge M \frac{r}{M m_n \prod_{j=1}^n L_j} m_n L \int_\eta^{\sigma(b)} \|G_n(.,s)\|\Delta s = r.
\end{aligned}$$

This completes the proof. □

6. Examples

Example 6.1. We illustrate Theorem 4.2 with a specific time scale

$$\mathbb{T} = \left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\} \cup \{1\} \cup [2, 3].$$

Consider the TPBVP:

$$\begin{cases} (-1)^n y^{\Delta^{2n}}(t) = e^{-y^2} = 0, & t \in \left[1, \frac{3}{2}\right] \subset \mathbb{T}, \\ y^{\Delta^{2i}}\left(\frac{4}{3}\right) + \frac{1}{3} y^{\Delta^{2i+1}}(1) = y^{\Delta^{2i}}(1), \quad \frac{1}{2} y^{\Delta^{2i}}\left(\frac{4}{3}\right) = y^{\Delta^{2i}}(2), & 0 \le i \le n-1. \end{cases} \tag{6.1}$$

Then $a = 1,\ \eta = \frac{4}{3}, b = \frac{3}{2}, \alpha_i = 1, \beta_i = \frac{1}{3}, \gamma_i = \frac{1}{2}, (0 \le i \le n-1)$ and

$$f(t, y) = f(y) = e^{-y^2}, \quad y \in [0, \infty).$$

Since $\lim_{y\to 0^+}(f(y)/y) = +\infty, \quad \lim_{y\to +\infty}(f(y)/y) = 0.$

We can also see that for $0 \le i \le n-1$,

$$0 \le \alpha_i(\sigma(b) - \eta) = \frac{2}{3} \le \sigma(b) - \gamma_i\eta + (\gamma_i - 1)(a - \beta_i) = 1,$$

$$0 < \gamma_i(\eta - a + \beta_i) = \frac{2}{3} < \sigma(b) - a + \beta_i = \frac{4}{3}.$$

Thus the TPBVP (6.1) has at least one positive solution by Theorem 4.2.

Example 6.2. Let us introduce an example to illustrate the usage of Theorem 5.2. Let $n = 2$, $\mathbb{T} = \{(\frac{2}{5})^n : n \in \mathbb{N}_0\} \cup \{0\} \cup [1, 2]$, $f(t, y) = f(y) = \frac{100(y+1)^4}{16(y^2+999)}$, $a = 8/125,\ \eta = 4/25,\ b = 2/5,\ \alpha_1 = \beta_2 = 1/2,\ \beta_1 = 1/8,\ \gamma_1 = 3/2, \alpha_2 = 1/10,\ \gamma_2 = 2$. Then condition (H) is satisfied. Green's function $G_1(t, s)$ in Lemma 2.2 is

$$G_1(t, s) = \begin{cases} G_{1_1}(t, s), & 8/125 \le s \le 4/25, \\ G_{1_2}(t, s), & 4/25 < s \le 2/5, \end{cases}$$

where

$$G_{1_1}(t, s) = \frac{2000}{619}\begin{cases} (19/25 - t/2)(5s/2 + 61/1000), & 5/2s \le t, \\ (19/25 + 5s/4)(t + 61/1000) - 21/50(t - 5s/2), & t \le s, \end{cases}$$

and

$$G_{1_2}(t, s) = \frac{2000}{619}\begin{cases} (5s/4 + 141/1000)(1 - t) + 663/2000(t - 5s/2), & 5s/2 \le t, \\ (t/2 + 141/1000)(1 - 5s/2), & t \le s. \end{cases}$$

Green's function $G_2(t, s)$ in Lemma 2.2 is

$$G_2(t, s) = \begin{cases} G_{2_1}(t, s), & 8/125 \le s \le 4/25, \\ G_{2_2}(t, s), & 4/25 < s \le 2/5, \end{cases}$$

where

$$G_{2_1}(t, s) = \frac{25}{4}\begin{cases} (17/25 + t)(5s/2 + 109/250), & 5/2s \le t, \\ (17/25 + 5s/2)(t + 109/250) - 21/250(t - 5s/2), & t \le s, \end{cases}$$

and

$$G_{2_2}(t, s) = \frac{25}{4}\begin{cases} (9s/4 + 113/250)(1 - t) + 149/125(t - 5s/2), & 5s/2 \le t, \\ (9t/10 + 113/250)(1 - 5s/2), & t \le s. \end{cases}$$

From Lemma 2.5 and (2.4)–(2.6), we get

$$m_1 = 221/1061, \qquad K_1 = 465\,426/1934\,375, \qquad L_1 = 12\,276/77\,375$$
$$m_2 = 149/359, \qquad K_1 = 52\,299/31\,250, \qquad L_2 = 1341/1250.$$

Clearly f is continuous and increasing on $[0, \infty)$. If we take $p = 0.001$, $q = 0.06$ and $r = 19$ then

$$0 < p < q < r.$$

It is clear that (D1), (D2) and (D3) of Theorem 5.2, are satisfied. Thus the TPBVP (1.1) has at least two positive solutions y_1 and y_2 satisfying

$$0.001 < \max_{t\in[8/125,2/5]} y_1(t) \quad \text{with} \quad \max_{t\in[4/25,1]} y_1(t) < 0.06$$
$$0.06 < \max_{t\in[4/25,1]} y_2(t) \quad \text{with} \quad \max_{t\in[4/25,1]} y_2(t) < 19.$$

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